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ON THE STEADY-STATE DISTRIBUTION OF NUMBER IN THE QUEUE Geom/G/1

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Discrete-time queues are frequently used in telecommunication networks. This paper proposes a new method which is based only on probabilistic arguments. It derives the distributions of numbers of customers in the discrete-time systems Geom/G/1 and Geom/G/1/M. Though the derivation does not involve use of the transforms, the transforms may be obtained, if desired. Another advantage of this is that the numerical results obtained are stable as there are no negative signs involved in summations. Further, the method can be easily used to solve more complex problems in discrete- and continuous-time queues.

1. INTRODUCTION

In recent years, discrete-time queues have gained importance due to digitization of signal processing devices, microcomputer and computer networks. In discrete-time queues, events (arrivals or departures) are assumed to occur only around slot boundaries. Besides, we need to specify the order in which the arrivals and departures take place in case of simultaneity. Essentially, there are two cases: (i) late arrival system with delayed access and (ii) early arrival system. For more details on this, see Hunter [3]. In the present paper, we discuss case (i).

Let the time axis be marked by $0, \delta, 2\delta, \ldots, m\delta, \ldots,$ with fixed-length intervals of magnitude δ. For the sake of simplicity, we assume $\delta = 1$. Consider the epoch m and assume that the arrivals occur in $(m-, m)$ and departures in $(m, m+)$. The model under discussion may also be viewed through a self-explanatory figure (Fig. 1).

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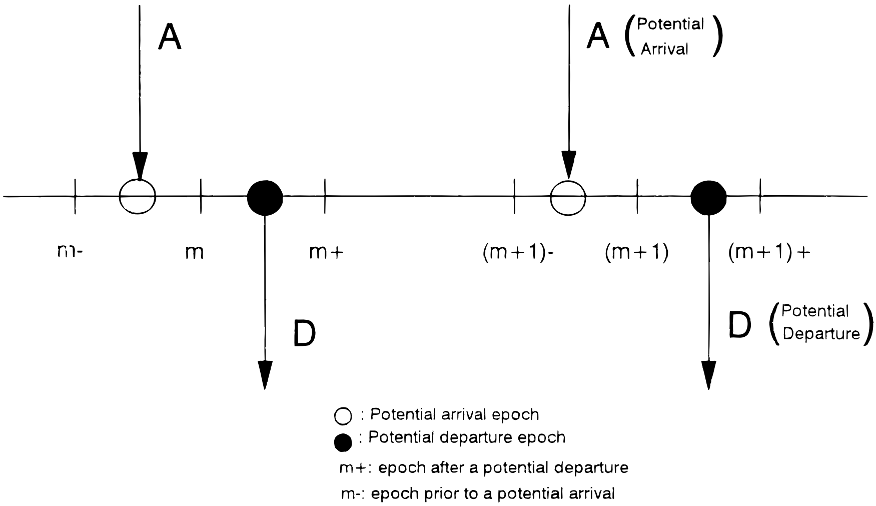

FIGURE 1. Various epochs in late arrival system with delayed access (LAS-DA).

The model Geom/G/1 has been discussed by many authors who have used different techniques such as imbedded Markov chain (IMC) and supplementary variable. For numerical work, methods such as roots and Fast Fourier Transform (FFT) have been used, for example, see Gouweleeuw and Tijms [2], who use FFT to compute state probabilities for the infinite–space bulk–arrival model Geom^X/G/1. The purpose of this paper is to give another method which not only uses elementary probabilistic arguments but is elegant, avoids the use of roots or FFT for numerical work, and gives transform free results. Transform results, however, can be obtained, if desired.

In the queue Geom/G/1, it is assumed that arrivals are by a Bernoulli process with mean $1/\lambda$, service–time distribution is general with probability mass function (p.m.f.) b_n, where $\sum_{n=0}^{\infty} b_n = 1$ with $b_0 = 0$; probability generating function $b(z) = \sum_{n=0}^{\infty} b_n z^n$ and mean service time $b = b^{(1)}(1) = \sum_{n=1}^{\infty} nb_n$, where $b^{(1)}(1)$ is the derivative of $b(z)$ evaluated at $z = 1$.

Let $\mathbf{N}_m$ be the number of customers in the system at an epoch $m-$, an epoch before a potential arrival, with $P_n = \lim_{m\to\infty} P(\mathbf{N}_m = n)$, $n \geq 1$, as its steady-state distribution. We assume that the distribution $\{P_n\}$ exists provided that $\rho = \lambda b < 1$. In this model, the distributions such as P_n, P_n^- = prearrival-epoch probability and P_n^+ = post-departure epoch probability are identical, though the latter two are conditional distributions. Whereas in Section 2 we consider only the derivation of the sequence $\{P_n\}$ for Geom/G/1, in Section 3 we derive $\{P_n\}$ for Geom/G/1/M. Sections 4, 5, and 6 deal with numerical aspects, continuous-time results, and extensions, respectively.

2. DERIVATION OF THE DISTRIBUTION $\{P_n\}$ FOR Geom/G/1

Though the distribution $\{P_n\}$ is independent of the queue discipline, in our derivation it is easier to visualize the situation if we assume that the queue discipline is last-come, first-served (LCFS). In steady state, let the random variable $\mathbf{N}$ represent number in system at an arbitrary epoch with distribution $\{P_n\}$ given above and $P(\mathbf{N} > 0) = \rho$.

Assuming that the system is busy at an arbitrary epoch, let $\mathbf{R}$ be the elapsed service time of the customer in service at the arbitrary epoch under consideration and $\mathbf{N}^-$ the number of customers in the system as seen by an arrival that is undergoing service at the arbitrary epoch. Then the limiting p.m.f. for $\mathbf{R}$ (see, e.g., Chaudhry and Templeton [1]) is given by

$$P(\mathbf{R} = j) = \left[1 - \sum_{i=1}^{j} b_i\right] \Big/ b, \qquad j \geq 0. \tag{2.1}$$

If $\mathbf{A}$ represents the number that arrive during $\mathbf{R}$, then

$$\begin{aligned} k_j &= \sum_{k=0}^{\infty} P(\mathbf{A} = j \,|\, \mathbf{R} = k) P(\mathbf{R} = k) \\ &= \frac{1}{b} \sum_{k=j}^{\infty} \binom{k}{j} \lambda^j (1-\lambda)^{k-j} \left[1 - \sum_{i=1}^{k} b_i\right], \qquad 0 \leq j \leq k. \end{aligned} \tag{2.2}$$

Now the event $\{\mathbf{N} = n\}_{n=1}^{\infty}$ being the union of two mutually exclusive events $\{\mathbf{N} > 0, \mathbf{N}^- = 0, \mathbf{A} = n-1\}$ and $\{\mathbf{N} > 0, \mathbf{N}^- = r, \mathbf{A} = n-r\}$, $r \geq 1$, gives

$$\begin{aligned} P(\mathbf{N} = n) &= P(\mathbf{N} > 0, \mathbf{N}^- = 0, \mathbf{A} = n-1) + \sum_{r=1}^{n} P(\mathbf{N} > 0, \mathbf{N}^- = r, \mathbf{A} = n-r) \\ &= P(\mathbf{N} > 0) \Bigg[P(\mathbf{N}^- = 0, \mathbf{A} = n-1 \,|\, \mathbf{N} > 0) \\ &\qquad + \sum_{r=1}^{n} P(\mathbf{N}^- = r, \mathbf{A} = n-r \,|\, \mathbf{N} > 0) \Bigg], \qquad n \geq 1. \end{aligned}$$

Since the two events in $P(\cdot)$ within brackets $[\cdot]$ are conditionally independent conditioned on the fact that the server is busy and P_n and P_n^- are identical, we have the distribution

$$P_n = \rho \left(P_0 k_{n-1} + \sum_{r=1}^{n} P_r k_{n-r} \right), \qquad n \geq 1, \tag{2.3}$$

where k_n is given by Eq. (2.2).

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Equation (2.3) gives the desired transform-free result. If we wish to find a transform of Eq. (2.3), viz. $P(z) = \sum_{n=0}^{\infty} P_n z^n$, it is obtained as follows

$$\begin{aligned} P(z) - P_0 &= \rho\left[P_0 zk(z) + \sum_{n=1}^{\infty}\sum_{r=1}^{n} P_r k_{n-r} z^n\right] \\ &= \rho\left[P_0 zk(z) + \sum_{n=r}^{\infty}\sum_{r=1}^{\infty} P_r k_{n-r} z^n\right] \\ &= \rho[P_0 zk(z) + (P(z) - P_0)k(z)]. \end{aligned} \tag{2.4}$$

Solving for $P(z)$ we finally get

$$P(z) = \frac{(1-\rho)(1-z)b(\lambda z + 1 - \lambda)}{b(\lambda z + 1 - \lambda) - z}, \tag{2.5}$$

where $k(z)$ being the transform of k_j given in Eq. (2.2) is given by

$$k(z) = \sum_{j=0}^{\infty} k_j z^j = \frac{1}{\rho}\,\frac{1 - b(\lambda z + 1 - \lambda)}{1 - z}.$$

Equation (2.5) is the standard form given in several books (see, e.g., Takagi [4]).

3. DERIVATION OF THE DISTRIBUTION $\{P_n\}$ FOR Geom/G/1/M

The chief aim of this paper was to get the sequence $\{P_n\}_0^{\infty}$ for the infinite-space model Geom/G/1 easily, as these probabilities are difficult to obtain. Once they are known, the sequence $\{P_n\}_0^{M-1}$ for the finite-space model Geom/G/1/M can be easily obtained. In fact, this model has been discussed by several authors (e.g., [4]). Here, we give the results for the sake of completeness.

First, note that since the post-departure probabilities $\{P_n^+\}$ for the infinite-space system Geom/G/1 are the same as the random epoch probabilities, the sequence $\{P_n^+\}_0^{M-1}$ can be obtained by truncating the sequence $\{P_n^+\}_0^{\infty}$ (Eq. (2.3)) and is given by

$$P_n^+ = \rho\left(P_0^+ k_{n-1} + \sum_{r=1}^{n} P_r^+ k_{n-r}\right), \qquad 1 \le n \le M-1.$$

Assuming $P_0^+ = 1$, we solve the above equation and then normalize the results to get the sequence $\{P_n^+\}_0^{M-1}$. Once the sequence $\{P_n^+\}_0^{M-1}$ is known, the sequence $\{P_n\}_0^{M}$ can be obtained using the following relations:

$$P_n = (1 - P_M)P_n^+, \qquad 0 \le n \le M-1$$

and

$$\rho(1 - P_M) = 1 - P_0.$$

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Using these equations, one can get

$$P_M = 1 - \frac{1}{\rho + P_0^+}.$$

Now, the first equation together with the last equation leads to the final results. These relations are given in several places, for example, [4]. From this, we not only get the blocking probability P_M, but also the whole sequence $\{P_n\}_0^M$. Besides, the results are good for any ρ.

4. COMPUTATIONS

For computational purposes, we rewrite Eq. (2.3) as

$$P_n = \rho\left(P_0 k_{n-1} + \sum_{r=1}^{n-1} P_r k_{n-r}\right)\Big/(1-\rho k_0), \qquad n \geq 1. \tag{4.1}$$

Equation (4.1) along with (2.2) can be solved recursively. The order of the complexity of k_j and P_j is $O(M^3)$ and $O(jM)$, respectively, where M is the maximum size of the service-time distribution. Since all the terms involved in the calculations are nonnegative, the numerical results obtained are stable though they are not being presented here.

5. CONTINUOUS-TIME RESULTS

Using appropriate limits, we can get results corresponding to Eqs. (2.1) and (2.2) or directly corresponding to Eq. (2.5), for the continuous-time model M/G/1.

6. EXTENSIONS

The method used here can be applied to discuss the distributions of numbers in system in other queues such as discrete- and continuous-time bulk-arrival queues.

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