Robotica (2010) volume 28, pp. 811–819. © Cambridge University Press 2009
doi:10.1017/S0263574709990518

Determination of singularities of some 4-DOF parallel manipulators by translational/rotational Jacobian matrices

Yi Lu†,*, Yan Shi† and Jianping Yu‡

†*Robotics Research Center, College of Mechanical Engineering, Yanshan University, Qinhuangdao, Hebei 066004, P.R. China*
‡*College of Foreign Studies, Yanshan University, Qinhuangdao, Hebei 066004, P.R. China*

(Received in Final Form: August 26, 2009. First published online: September 21, 2009)

SUMMARY
A novel analytic approach is proposed for determining the singularities of some four degree of freedom (DOF) parallel manipulators (PMs). First, the constraint and displacement of a general 4-DOF PM are analyzed. Second, a common 3×4 translational Jacobian matrix $\mathbf{J}_v$ and a common 3×4 rotational Jacobian matrix $\mathbf{J}_\omega$ are derived, and a 4×4 general Jacobian matrix $\mathbf{J}$ of the 4-DOF PMs is derived from $\mathbf{J}_v$ and $\mathbf{J}_\omega$. Since a complicated process to determine singularities from the 4×6 Jacobian matrix is transformed into a simple process to determine singularity from $\mathbf{J}$, the singularities of the some 4-DOF PMs with 3 translations and 1 rotation, or with 3 rotations and 1 translation, or with combined translation–rotations are analyzed and determined easily by this approach.

KEYWORDS: Parallel manipulator; Singularity; Jacobian matrix.

1. Introduction

The four degree of freedom (DOF) parallel manipulators (PMs) have been found having larger workspace and more flexibility than the 3-DOF PMs, so they have attracted much attention.[1–3] However, when the singularities of PMs occur at some poses, the DOF of a PM may vary, and this results in some motion errors and malfunctions. A singularity of PMs must be avoided in path planning or designing better PMs and parallel machine tools. The determination of singularity of PMs has been attracted much attention in order to evaluate the characteristics of PMs and parallel machine tools.[1–7] In this aspect, Huang *et al*. derived a geometry condition for discriminating singularity and proposed a singularity principle of PMs.[2–4] Merlet[5] introduced Grassmann line into geometry to find the singularity for Stewart PM. Gosselin *et al*. analyzed the singularity loci of a spherical 3-DOF PM and a 5-DOF PM, and studied an uncertainty singularity of PMs and singularity of a 3-leg 6-DOF PM by line geometry.[6–10] Sandipan and Ashitava analyzed singularity space of PMs and a geometric characterization and parametric representation of the singularity manifold of a 6–6 Stewart PM.[11,12] Gallardo-Alvarado *et al*. analyzed singularity of a 4-DOF PM by screw theory.[13] Gregorio explored forward problem singularity in PMs which generate SX–YS–ZS structures.[14] Zhao *et al*. analyzed the singularity of PM with terminal constraints.[15] Ider presented inverse dynamics of PMs in the presence of drive singularities.[16] Wolf *et al*. analyzed singularity of a 3-DOF CaPaMan PM by line geometry and linear complex.[17] Anjan *et al*. studied singularity-free path-planning of some PMs.[18–21] Lu *et al*. determined singularities of some PMs by Computer Aided Design (CAD) variation geometry.[22] Each of the above mentioned approaches has its merit for determining singularity. However, since the Jacobian matrix of the 4-DOF PMs is a 4×6 rectangle matrix and includes partial differential items which are difficult to be transformed into algebra item, the determinant of the Jacobian matrix is hard to solve when determining the singularity of the 4-DOF PMs.

This paper focuses on a new analytic approach for analyzing singularities of the 4-DOF PMs by using a 3×4 translational Jacobian matrix and a 3×4 rotational Jacobian matrix. A 4×6 Jacobian matrix is transformed into a 4×4 general Jacobian matrix from translational/rotational Jacobian matrices. Since each of the partial differential items in these Jacobian matrices can be easily transformed into an algebra item, the singularity analyses of the 4-DOF PMs with 3 translations and 1 rotation, or 3 rotations and 1 translation, or combinations of translations and rotations can be are simplified obviously and some singularities can be determined easily.

2. Unified Kinematics Analyses of 4-DOF PMs

2.1. *Inverse displacement*

The 4-DOF PMs may be classified as the 4-DOF PM with 1T3R, the 4-DOF PM with 2T2R, and the 4-DOF PM with 3T1R according to its motion characteristics. Here, T is translation and R is rotation. According to the number of legs, 4-DOF PMs may be classified as the 4-DOF PM with 4 linear active legs[24,25] and the 4-DOF PM with 3 linear active legs and a rotational actuator.[26]

A general 4-DOF PM includes a fixed base B, a moving platform m, and 3 or 4 linear active legs $r_i (i = 1, \ldots, 3 \text{ or } 4)$ with the linear actuators, as shown in Fig. 1. Where m is regular polygon with 3 or 4 vertices b_i, 3 or 4 sides l_i, and a central point o. B is regular polygon with 3 or 4 vertices B_i, 3 or 4 sides L_i, and a central point O. Each of r_i connects m at

* Corresponding author. E-mail: luyi@ysu.edu.cn

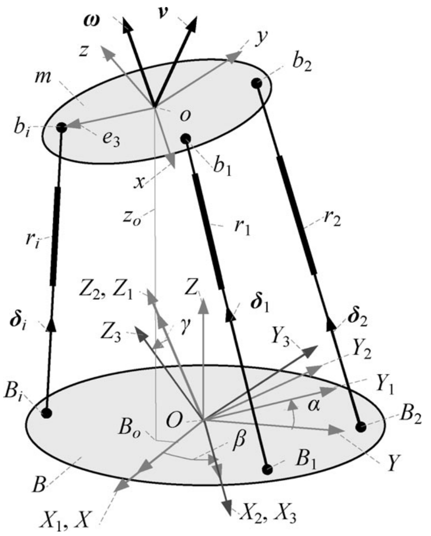

Fig. 1. A general 4-DOF PM with Euler rotations *XZX*.

a point b_i with B at point B_i. Let $\{m\}$ be a coordinate o-xyz fixed on m at central point o. Let $\{B\}$ be a coordinate O-XYZ fixed on B at central point O. Let $\|$ be the parallel constraint, $\perp$ be the perpendicular constraint.

Before determining the singularities of some 4-DOF PMs, their common inverse displacement should be analyzed. The position vector $\boldsymbol{B}_i$ of point B_i in $\{B\}$ and the position vectors ${}^m\boldsymbol{b}_i$ and $\boldsymbol{b}_i$ of point b_i in $\{m\}$ and $\{B\}$ are expressed in ref. [23] as follows:

$$\boldsymbol{B}_i = \begin{bmatrix} X_{Bi} \\ Y_{Bi} \\ Z_{Bi} \end{bmatrix}, {}^m\boldsymbol{b}_i = \begin{bmatrix} x_{bi} \\ y_{bi} \\ z_{bi} \end{bmatrix}, \boldsymbol{b}_i = \begin{bmatrix} X_{bi} \\ Y_{bi} \\ Z_{bi} \end{bmatrix}, \boldsymbol{o} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix},$$

$${}^B_m\mathbf{R} = \begin{bmatrix} x_l & y_l & z_l \\ x_m & y_m & z_m \\ x_n & y_n & z_n \end{bmatrix}, \boldsymbol{b}_i = {}^B_m\mathbf{R}\,{}^m\boldsymbol{b}_i + \boldsymbol{o}. \tag{1}$$

Here, $\boldsymbol{o}$ is a position vector of point o on m in $\{B\}$, (X_o, Y_o, Z_o) are the components of $\boldsymbol{o}$; ${}^B_m\mathbf{R}$ is a rotation transformation matrix from $\{m\}$ to $\{B\}$; $(x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n)$ are the orientation parameters of m, their constrained equations can be obtained from refs. [23, 24].

The length $r_i (i = 1, 2, 3 \text{ or } 4)$ and the unit vectors $\boldsymbol{\delta}_i$ of active legs in $\{B\}$, and the vectors $\boldsymbol{e}_i$ from the center o of m to the joint b_i on m in $\{B\}$ can be solved from Eq. (1) as follows:

$$r_i = |\boldsymbol{b}_i - \boldsymbol{B}_i|, \boldsymbol{\delta}_i = (\boldsymbol{b}_i - \boldsymbol{B}_i)/r_i = [\delta_{ix} \quad \delta_{iy} \quad \delta_{iz}]^T,$$
$$\boldsymbol{e}_i = \boldsymbol{b}_i - \boldsymbol{o}_i = [e_{ix} \quad e_{iy} \quad e_{iz}]^T. \tag{2}$$

Let $s_\varphi = \sin\varphi$, $c_\varphi = \cos\varphi$, $t_\varphi = \tan\varphi$, here φ may be one of Euler angles (α, β, γ) of m. Let $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$ be the unit vectors of rotational axes of (α, β, γ), respectively. Each of $(x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n, C_1, C_2 \text{ and } C_3)$ can be expressed by (α, β, γ) in ref. [23]. Let x_i $(i = 1, 2, 3, 4)$ be 4 independent pose parameters of m, $x_i \in (X_o, Y_o, Z_o, \alpha, \beta, \gamma)$.

Generally, the platform m of a limited-DOF PM with $(n \le 6)$ DOFs is applied by n active wrenches from n actuators and is exerted 6-n passive constraints from mechanism structures.[23,24] Hence, the platform m of a 4-DOF PM is applied by 4 active wrenches and is exerted 2 structure (passive) constraints. Based on 2 structure (passive) constraints, 2 independent constraint equations can be derived. After that, x_i can be determined from the 2 independent constrained equations. Thus, extension r_i and the unit vector $\boldsymbol{\delta}_i$ of active legs, and the vectors $\boldsymbol{e}_i$ from the central point o to the joints b_i on m can be expressed by x_i. Each of $(X_o, Y_o, Z_o, \alpha, \beta, \gamma)$ can be expressed by x_i as follows:

$$X_o = X_o(x_1, x_2, x_3, x_4), \quad Y_o = Y_o(x_1, x_2, x_3, x_4),$$
$$Z_o = Z_o(x_1, x_2, x_3, x_4), \quad \alpha = \alpha(x_1, x_2, x_3, x_4), \tag{3}$$
$$\beta = \beta(x_1, x_2, x_3, x_4), \quad \gamma = \gamma(x_1, x_2, x_3, x_4).$$

2.2. *Inverse/forward velocity and Jacobian matrix*

Differentiating each of $(X_o, Y_o, Z_o, \alpha, \beta, \gamma)$ with respect to time t, it leads to

$$\dot{X}_o = \sum_{i=1}^{4} \frac{\partial X_o}{\partial x_i}\dot{x}_i, \quad \dot{Y}_o = \sum_{i=1}^{4} \frac{\partial Y_o}{\partial x_i}\dot{x}_i, \quad \dot{Z}_o = \sum_{i=1}^{4} \frac{\partial Z_o}{\partial x_i}\dot{x}_i,$$
$$\dot{\alpha} = \sum_{i=1}^{4} \frac{\partial \alpha}{\partial x_i}\dot{x}_i, \quad \dot{\beta} = \sum_{i=1}^{4} \frac{\partial \beta}{\partial x_i}\dot{x}_i, \quad \dot{\gamma} = \sum_{i=1}^{4} \frac{\partial \gamma}{\partial x_i}\dot{x}_i. \tag{4}$$

Let $\boldsymbol{V}$ be a general forward velocity of m at o, $\boldsymbol{v}$, and $\boldsymbol{\omega}$ be a linear velocity and an angular velocity of m at its center, respectively. Let $\boldsymbol{v}_e$ be an equivalent velocity of $x_i (i = 1, \ldots, 4)$. They can be expressed as follows:

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} \dot{X}_o \\ \dot{Y}_o \\ \dot{Z}_o \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \boldsymbol{v}_e = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}. \tag{5}$$

$\boldsymbol{v}$ can be expressed from Eqs. (4) and (5) as follows:

$$\boldsymbol{v} = \mathbf{J}_v \boldsymbol{v}_e,$$
$$\mathbf{J}_v = [\boldsymbol{J}_{v1} \ \boldsymbol{J}_{v2} \ \boldsymbol{J}_{v3} \ \boldsymbol{J}_{v4}] \tag{6}$$
$$= \begin{bmatrix} \partial X_o/\partial x_1 & \partial X_o/\partial x_2 & \partial X_o/\partial x_3 & \partial X_o/\partial x_4 \\ \partial Y_o/\partial x_1 & \partial Y_o/\partial x_2 & \partial Y_o/\partial x_3 & \partial Y_o/\partial x_4 \\ \partial Z_o/\partial x_1 & \partial Z_o/\partial x_2 & \partial Z_o/\partial x_3 & \partial Z_o/\partial x_4 \end{bmatrix}.$$

Here $\mathbf{J}_v$ is a 3×4 translational Jacobian matrix.

https://doi.org/10.1017/S0263574709990518 Published online by Cambridge University Press

$\boldsymbol{\omega}$ can be expressed from Eq. (4) as follows:

$$\boldsymbol{\omega} = \mathbf{C}\begin{bmatrix}\dot{\alpha}\\ \dot{\beta}\\ \dot{\gamma}\end{bmatrix} = \mathbf{J}_{\omega}\boldsymbol{v}_e,$$

$$\mathbf{C} = [\boldsymbol{C}_1 \quad \boldsymbol{C}_2 \quad \boldsymbol{C}_3] = \begin{bmatrix} c_{1x} & c_{2x} & c_{3z}\\ c_{1y} & c_{2y} & c_{3z}\\ c_{1z} & c_{2z} & c_{3z}\end{bmatrix}, \tag{7}$$

$$\mathbf{J}_{\omega} = [\, \boldsymbol{J}_{\omega 1}\ \boldsymbol{J}_{\omega 2}\ \boldsymbol{J}_{\omega 3}\ \boldsymbol{J}_{\omega 4}\,] = \mathbf{C}\begin{bmatrix} \partial\alpha/\partial x_1 & \partial\alpha/\partial x_2 & \partial\alpha/\partial x_3 & \partial\alpha/\partial x_4\\ \partial\beta/\partial x_1 & \partial\beta/\partial x_2 & \partial\beta/\partial x_3 & \partial\beta/\partial x_4\\ \partial\gamma/\partial x_1 & \partial\gamma/\partial x_2 & \partial\gamma/\partial x_3 & \partial\gamma/\partial x_4\end{bmatrix}.$$

Here $\mathbf{J}_{\omega}$ is a 3×4 rotational Jacobian matrix. Combining Eq. (6) with Eq. (7), it leads to

$$\boldsymbol{V} = \mathbf{J}_e\boldsymbol{v}_e,$$
$$\mathbf{J}_e = \begin{bmatrix}\mathbf{J}_v\\ \mathbf{J}_{\omega}\end{bmatrix} = \begin{bmatrix}\boldsymbol{J}_{v1} & \boldsymbol{J}_{v2} & \boldsymbol{J}_{v3} & \boldsymbol{J}_{v4}\\ \boldsymbol{J}_{\omega 1} & \boldsymbol{J}_{\omega 2} & \boldsymbol{J}_{\omega 3} & \boldsymbol{J}_{\omega 4}\end{bmatrix}_{6\times 4}. \tag{8}$$

Here $\mathbf{J}_e$ is a 6×4 equivalent Jacobian matrix.

The scalar velocities v_{ri} of r_i along r_i can be obtained from ref. [23] as follows:

$$v_{ri} = [\,\boldsymbol{\delta}_i^T \ (\boldsymbol{e}_i \times \boldsymbol{\delta}_i)^T\,]\boldsymbol{V}. \tag{9}$$

When given 4 input translational displacements $r_i (i = 1, 2, 3, 4)$ along the active legs of the 4-DOF PM with 4 active legs, an input displacement vector $\boldsymbol{X}_{in}$ and its velocity vector $\boldsymbol{v}_{in}$ are expressed as follows:

$$\boldsymbol{X}_{in} = [r_1 \quad r_2 \quad r_3 \quad r_4]^T,$$
$$\boldsymbol{v}_{in} = [v_{r1} \quad v_{r2} \quad v_{r3} \quad v_{r4}]^T. \tag{10}$$

When given $\boldsymbol{V}$, $\boldsymbol{v}_{in}$ of the 4-DOF PM with 4 active legs is derived from Eq. (8) to Eq. (10) as follows:

$$\boldsymbol{v}_{in} = {}^4\mathbf{J}_r\boldsymbol{V} = {}^4\mathbf{J}_r\mathbf{J}_e\boldsymbol{v}_e = \mathbf{J}\boldsymbol{v}_e,$$
$$\boldsymbol{v}_e = \mathbf{J}^{-1}\boldsymbol{v}_{in}, \quad \mathbf{J} = [{}^4\mathbf{J}_r]_{4\times 6}[\mathbf{J}_e]_{6\times 4},$$
$$\boldsymbol{v}_{in} = \begin{bmatrix} v_{r1}\\ v_{r2}\\ v_{r3}\\ v_{r4}\end{bmatrix}, {}^4\mathbf{J}_r = \begin{bmatrix}\boldsymbol{\delta}_1^T & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T\\ \boldsymbol{\delta}_2^T & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T\\ \boldsymbol{\delta}_3^T & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T\\ \boldsymbol{\delta}_4^T & (\boldsymbol{e}_4 \times \boldsymbol{\delta}_4)^T\end{bmatrix}_{4\times 6}. \tag{11}$$

Here $\mathbf{J}$ is a 4×4 general Jacobian matrix, and $\mathbf{J}^{-1}$ exists.

$\mathbf{J}$ for the 4-DOF PM with 4 active legs can be expanded from Eqs. (8) and (11) as follows:

$$\mathbf{J} = {}^4\mathbf{J}_r\mathbf{J}_e = ({}^4\mathbf{J}_r)_{4\times 6}\begin{bmatrix}\boldsymbol{J}_{v1} & \boldsymbol{J}_{v2} & \boldsymbol{J}_{v3} & \boldsymbol{J}_{v4}\\ \boldsymbol{J}_{\omega 1} & \boldsymbol{J}_{\omega 2} & \boldsymbol{J}_{\omega 3} & \boldsymbol{J}_{\omega 4}\end{bmatrix}_{6\times 4}$$
$$= \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14}\\ J_{21} & J_{22} & J_{23} & J_{24}\\ J_{31} & J_{32} & J_{33} & J_{34}\\ J_{41} & J_{42} & J_{43} & J_{44}\end{bmatrix}, \quad J_{ij} = \boldsymbol{\delta}_i^T\boldsymbol{J}_{vj} + (\boldsymbol{e}_i \times \boldsymbol{\delta}_i)^T\boldsymbol{J}_{\omega j}, \quad i = 1, \ldots, 4; \quad j = 1, \ldots, 4. \tag{12}$$

When given an input rotational angle α about $\boldsymbol{C}_1$ and 3 input extensions r_i $(i = 1, 2, 3)$ of active legs for the 4-DOF PMs with 3 active legs, $\boldsymbol{X}_{in}$ and $\boldsymbol{v}_{in}$ are expressed from Eq. (7) as follows:

$$\boldsymbol{X}_{in} = [r_1 \quad r_2 \quad r_3 \quad \alpha\,]^T,$$
$$v_{in} = [v_{r1} \quad v_{r2} \quad v_{r3} \quad \dot{\alpha}\,]^T,$$
$$\begin{bmatrix}\dot{\alpha}\\ \dot{\beta}\\ \dot{\gamma}\end{bmatrix} = \mathbf{C}^{-1}\boldsymbol{\omega} = \begin{bmatrix} c_{1x} & c_{2x} & c_{3x}\\ c_{1y} & c_{2y} & c_{3y}\\ c_{1z} & c_{2z} & c_{3z}\end{bmatrix}^{-1}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}, \tag{13}$$
$$\dot{\alpha} = [(c_{2y}c_{3z} - c_{3y}c_{2z})\boldsymbol{\omega}_x + (c_{3x}c_{2z} - c_{2x}c_{3z})\boldsymbol{\omega}_y + (c_{2x}c_{3y} - c_{3x}c_{2y})\boldsymbol{\omega}_z]/\,|\mathbf{C}|\,.$$

When given $\boldsymbol{V}$, $\boldsymbol{v}_{in}$ of the 4-DOF PM with 3 active legs is derived from Eqs. (9) and (13) as follows:

$$\boldsymbol{v}_{in} = {}^3\mathbf{J}_r\boldsymbol{V} = {}^3\mathbf{J}_r\mathbf{J}_e\boldsymbol{v}_e = \mathbf{J}\boldsymbol{v}_e,$$
$$\boldsymbol{v}_e = \mathbf{J}^{-1}\boldsymbol{v}_{in}, \quad \mathbf{J} = [{}^3\mathbf{J}_r]_{4\times 6}[\mathbf{J}_e]_{6\times 4}.$$
$${}^3\mathbf{J}_r = \begin{bmatrix}\boldsymbol{\delta}_1^T & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T\\ \boldsymbol{\delta}_2^T & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T\\ \boldsymbol{\delta}_3^T & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T\\ \mathbf{0} & \boldsymbol{K}^T\end{bmatrix}, \tag{14}$$
$$K = \frac{1}{|\mathbf{C}|}\begin{bmatrix} c_{2y}c_{3z} - c_{3y}c_{2z}\\ c_{3x}c_{2z} - c_{2x}c_{3z}\\ c_{2x}c_{3y} - c_{3x}c_{2y}\end{bmatrix}.$$

The general Jacobian matrix $\mathbf{J}$ for the 4-DOF PM with 3 active legs can be expanded from Eqs. (8) and (14) as follows:

$$\mathbf{J} = {}^3\mathbf{J}_r\mathbf{J}_e = ({}^3\mathbf{J}_r)_{4\times 6}\begin{bmatrix}\boldsymbol{J}_{v1} & \boldsymbol{J}_{v2} & \boldsymbol{J}_{v3} & \boldsymbol{J}_{v4}\\ \boldsymbol{J}_{\omega 1} & \boldsymbol{J}_{\omega 2} & \boldsymbol{J}_{\omega 3} & \boldsymbol{J}_{\omega 4}\end{bmatrix}_{6\times 4}$$
$$= \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14}\\ J_{21} & J_{22} & J_{23} & J_{24}\\ J_{31} & J_{32} & J_{33} & J_{34}\\ J_{41} & J_{42} & J_{43} & J_{44}\end{bmatrix}, \tag{15}$$
$$J_{ij} = \boldsymbol{\delta}_i^T\boldsymbol{J}_{vj} + (\boldsymbol{e}_i \times \boldsymbol{\delta}_i)^T\boldsymbol{J}_{\omega j},$$
$$J_{4j} = \boldsymbol{K}^T\boldsymbol{J}_{\omega j},$$
$$i = 1, 2, 3,$$
$$j = 1, \ldots, 4.$$

https://doi.org/10.1017/S0263574709990518 Published online by Cambridge University Press

When given $\boldsymbol{v}_{in}$, $\boldsymbol{V}$ can be derived from Eqs. (8), (11) and (14) as follows:

$$\boldsymbol{V} = \mathbf{J}_e\mathbf{J}^{-1}\boldsymbol{v}_{in}, \quad \boldsymbol{v}_{in} = \mathbf{J}\boldsymbol{v}_e. \tag{16}$$

Thus, a complicated process to determine singularities of the 4×6 Jacobian matrix $\mathbf{J}_r$ is transformed into a simple process to determine singularity of the 4×4 general Jacobian matrix $\mathbf{J}$.

3. Singularity Analyses

Let $|\mathbf{J}|$ denote the determinant of the general Jacobian matrix $\mathbf{J}$. The singularities of the 4-DOF PMs have been classed into following 3 types according to Eq. (16):[1]

(1) When $|\mathbf{J}| = 0$, the boundary singularities of the 4-DOF PMs occur;
(2) When $|\mathbf{J}| \to \infty$, the local singularities of the 4-DOF PMs occur;
(3) When $|\mathbf{J}| \to 0/0$, the structure singularities of the 4-DOF PMs occur.

Based on Eqs. (12) and (15) and the above three types of singularities, some singularities of the 4-DOF PMs with 4 legs or 3 legs can be analyzed and determined as follows.

3.1. *General Jacobian matrix* J *of 3T1R 4-DOF PM*

When the platform of the 4-DOF PM in $\{B\}$ has 3 translations (X_o, Y_o, Z_o) and a rotation α about Z, (x_1, x_2, x_3, x_4) and their velocity vector $\boldsymbol{v}_e$ are expressed as follows:

$$x_1 = X_o, x_2 = Y_o, x_3 = Z_o, x_4 = \alpha,$$
$$\boldsymbol{v}_e = [\dot{X}_o \quad \dot{Y}_o \quad \dot{Z}_o \quad \dot{\alpha}]^T. \tag{17}$$

The 3×4 translational Jacobian matrix $\mathbf{J}_v$ is derived from Eqs. (6) and (17) as below,

$$\mathbf{J}_v = [\boldsymbol{J}_{v1} \quad \boldsymbol{J}_{v2} \quad \boldsymbol{J}_{v3} \quad \boldsymbol{J}_{v4}] = \begin{bmatrix} \partial X_o/\partial X_o & \partial X_o/\partial Y_o & \partial X_o/\partial Z_o & \partial X_o/\partial \alpha \\ \partial Y_o/\partial X_o & \partial Y_o/\partial Y_o & \partial Y_o/\partial Z_o & \partial Y_o/\partial \alpha \\ \partial Z_o/\partial X_o & \partial Z_o/\partial Y_o & \partial Z_o/\partial Z_o & \partial Z_o/\partial \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{18}$$

Suppose that platform of the 4-DOF PM in $\{B\}$ rotate in an order ZXY, the 3×4 rotational Jacobian matrix $\mathbf{J}_\omega$ is derived from Eqs. (7) and (17) as follows:

$$\mathbf{J}_\omega = [\boldsymbol{J}_{\omega 1} \quad \boldsymbol{J}_{\omega 2} \quad \boldsymbol{J}_{\omega 3} \quad \boldsymbol{J}_{\omega 4}], = \mathbf{C}\begin{bmatrix} \partial\alpha/\partial X_o & \partial\alpha/\partial Y_o & \partial\alpha/\partial Z_o & \partial\alpha/\partial\alpha \\ \partial\beta/\partial X_o & \partial\beta/\partial Y_o & \partial\beta/\partial Z_o & \partial\beta/\partial\alpha \\ \partial\gamma/\partial X_o & \partial\gamma/\partial Y_o & \partial\gamma/\partial Z_o & \partial\gamma/\partial\alpha \end{bmatrix}, = \begin{bmatrix} 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{19}$$

The general Jacobian matrix $\mathbf{J}$ for the 4-DOF PM with 4 active legs is derived from Eqs. (12), (18), and (19) as,

$$\mathbf{J} = {}^4\mathbf{J}_r\mathbf{J}_e = {}^4\mathbf{J}_r\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \delta_{1x} & \delta_{1y} & \delta_{1z} & e_{1x}\delta_{1y} - e_{1y}\delta_{1x} \\ \delta_{2x} & \delta_{2y} & \delta_{2z} & e_{2x}\delta_{2y} - e_{2y}\delta_{2x} \\ \delta_{3x} & \delta_{3y} & \delta_{3z} & e_{3x}\delta_{3y} - e_{3y}\delta_{3x} \\ \delta_{4x} & \delta_{4y} & \delta_{4z} & e_{4x}\delta_{4y} - e_{4y}\delta_{4x} \end{bmatrix}. \tag{20}$$

The general Jacobian matrix $\mathbf{J}$ for the 4-DOF PM with 3 active legs is derived from Eqs. (15), (18), and (19) as,

$$\mathbf{J} = {}^3\mathbf{J}_r\mathbf{J}_e = {}^3\mathbf{J}_r\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \delta_{1x} & \delta_{1y} & \delta_{1z} & e_{1x}\delta_{1y} - e_{1y}\delta_{1x} \\ \delta_{2x} & \delta_{2y} & \delta_{2z} & e_{2x}\delta_{2y} - e_{2y}\delta_{2x} \\ \delta_{3x} & \delta_{3y} & \delta_{3z} & e_{3x}\delta_{3y} - e_{3y}\delta_{3x} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$
$$K = [0\ 0\ 1]^T.$$

3.2. *Singularity analyses of a 2UPU+RRPU PM*

A 4-DOF overconstrained 2UPU+RRPU PM[24] is developed in Yanshan University, as shown in Fig. 2. It has 4 DOFs corresponding to 3 translations and 1 rotation about Z. Hence, this PM is one type of the 3T1R 4-DOF PMs with 3 active legs. This PM includes a moving platform m, a fixed base B, 2 (UPU) (universal joint-active prismatic joint-universal joint)

https://doi.org/10.1017/S0263574709990518 Published online by Cambridge University Press

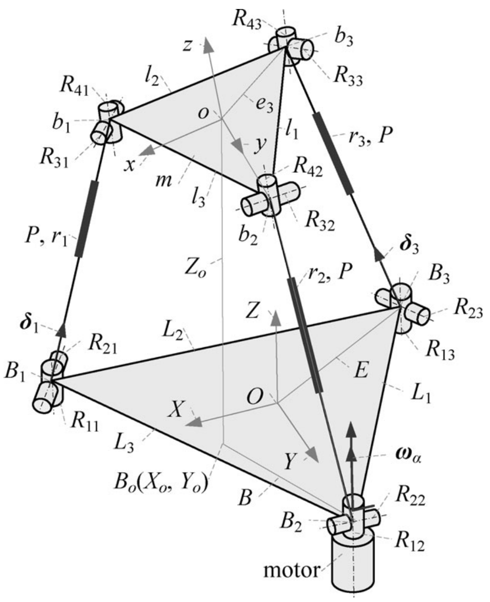

Fig. 2. A overconstrained 2UPU+RRPU PM.

legs with a linear actuator, and an RRPU (active revolute joint-revolute joint-active prismatic joint-universal joint) leg with a rotational actuator and a linear actuator. Here, m is a regular triangle with 3 vertices $b_i(i = 1, 2, 3)$ and 3 sides $l_i = l$ and a central point o; B is a regular triangle with 3 vertices B_i and 3 sides $L_i = L$ and a central point O. Since this PM has 3 legs and 4 DOFs corresponding to 1 rotation and 3 translations, it is simpler in structure and has less potential interference than some 4-DOF PMs with 4 active legs.

The determination $|\mathbf{J}|$ can be derived from Eq. (21) as follow:

$$|\mathbf{J}| = \begin{vmatrix} \delta_{1x} & \delta_{1y} & \delta_{1z} & e_{1x}\delta_{1y} - e_{1y}\delta_{1x} \\ \delta_{2x} & \delta_{2y} & \delta_{2z} & e_{2x}\delta_{2y} - e_{2y}\delta_{2x} \\ \delta_{3x} & \delta_{3y} & \delta_{3z} & e_{3x}\delta_{3y} - e_{3y}\delta_{3x} \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \delta_{1x} & \delta_{1y} & \delta_{1z} \\ \delta_{2x} & \delta_{2y} & \delta_{2z} \\ \delta_{3x} & \delta_{3y} & \delta_{3z} \end{vmatrix}$$
$$= 0. \qquad (22)$$

Some singularities of the 2UPU+RRPU PM can be determined from Eq. (22) as follows:

(1) When $\delta_{ix} = 0(i = 1, 2, 3)$, $|\mathbf{J}| = 0$ is satisfied. In this case, when $l_2 = L_2$, each of the active legs $r_i(i = 1, 2, 3)$ may locate in the planes parallel with *O-YZ*, a singularity occurs. Similarly, when $l_1 = L_1$ or $l_3 = L_3$, other two symmetry singularities can be determined.

(2) When $\delta_{iy} = 0(i = 1, 2, 3)$, $|\mathbf{J}| = 0$ is satisfied. In this case, when a line from b_2 to middle point of l_2 is the same as a line from B_2 to middle point of L_2, the two active legs (r_1 and r_3) may locate in one plane parallel with *O-XZ*, and leg r_2 may locate in the other plane parallel with *O-XZ*, and a singularity occurs. Similarly, when a line from b_i to middle point of l_i is the same as a line from B_i to middle point of $L_i(i = 1, 3)$, other two singularities can be determined.

(3) When $\delta_{iz} = 0(i = 1, 2, 3)$, $|\mathbf{J}| = 0$ is satisfied. In this case, when the platform and the base are coplanar, a singularity occurs.

(4) When $\delta_1 = \delta_3$, $|\mathbf{J}| = 0$ is satisfied. In this case, when $\alpha = 0°$ and $l_2 = L_2$, a singularity occurs. Similarly, when $\alpha = 0°$ and $l_1 = L_1$, or $\alpha = 0°$ and $l_3 = L_3$, other two singularities occur.

(5) When $\delta_1 = \delta_1 = \delta_3$, $|\mathbf{J}| = 0$ is satisfied. In this case, when $l_i = L_i(i = 1, 2, 3)$, a singularity occurs.

(6) When $\boldsymbol{e}_i = 0(i = 1, 2, 3)$, $|\mathbf{J}| = 0$ is satisfied. In this case, when platform becomes a point o, a singularity occurs.

3.3. General Jacobian matrix J of 3R1T 4-DOF PMs

When a 4-DOF PM such as 4 SPS (spherical joint-active prismatic joint-spherical joint) + PS (prismatic joint-spherical joint) PM has 3 Euler rotations (α, β, γ) and a translation along Z, (x_1, x_2, x_3, x_4) and their velocity vector $\boldsymbol{v}_e$ are expressed as follows:

$$x_1 = Z_o, \quad x_2 = \alpha, \quad x_3 = \beta, \quad x_4 = \gamma,$$
$$\boldsymbol{v}_e = [\dot{Z}_o \quad \dot{\alpha} \quad \dot{\beta} \quad \dot{\gamma}]^T. \qquad (23)$$

The 3×4 translational Jacobian matrix $\mathbf{J}_v$ is derived from Eqs. (6) and (23) as follows:

$$\mathbf{J}_v = [\boldsymbol{J}_{v1} \quad \boldsymbol{J}_{v2} \quad \boldsymbol{J}_{v3} \quad \boldsymbol{J}_{v4}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \qquad (24)$$

The 3×4 translational Jacobian matrix $\mathbf{J}_\omega$ is derived from Eqs. (7) and (23) as follows:

$$\mathbf{J}_\omega = \mathbf{C} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{C}_1 & \boldsymbol{C}_2 & \boldsymbol{C}_3 \end{bmatrix}. \qquad (25)$$

The general Jacobian matrix $\mathbf{J}$ for the 3R1T 4-DOF PM with 4 active legs is derived from Eqs. (12), (24), (25) as follows:

$$\mathbf{J} = {}^4\mathbf{J}_r\mathbf{J}_e = {}^4\mathbf{J}_r \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & c_{1x} & c_{2x} & c_{3x} \\ 0 & c_{1y} & c_{2y} & c_{3y} \\ 0 & c_{1z} & c_{2z} & c_{3z} \end{bmatrix}$$
$$= \begin{bmatrix} \delta_{1z} & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T\boldsymbol{C}_1 & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T\boldsymbol{C}_2 & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T\boldsymbol{C}_3 \\ \delta_{2z} & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T\boldsymbol{C}_1 & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T\boldsymbol{C}_2 & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T\boldsymbol{C}_3 \\ \delta_{3z} & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T\boldsymbol{C}_1 & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T\boldsymbol{C}_2 & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T\boldsymbol{C}_3 \\ \delta_{4z} & (\boldsymbol{e}_4 \times \boldsymbol{\delta}_4)^T\boldsymbol{C}_1 & (\boldsymbol{e}_4 \times \boldsymbol{\delta}_4)^T\boldsymbol{C}_2 & (\boldsymbol{e}_4 \times \boldsymbol{\delta}_4)^T\boldsymbol{C}_3 \end{bmatrix}.$$
$$(26)$$

The general Jacobian matrix $\mathbf{J}$ of the 3R1T 4-DOF PM with 3 active legs is derived from Eqs. (15), (24), and (25) as

https://doi.org/10.1017/S0263574709990518 Published online by Cambridge University Press

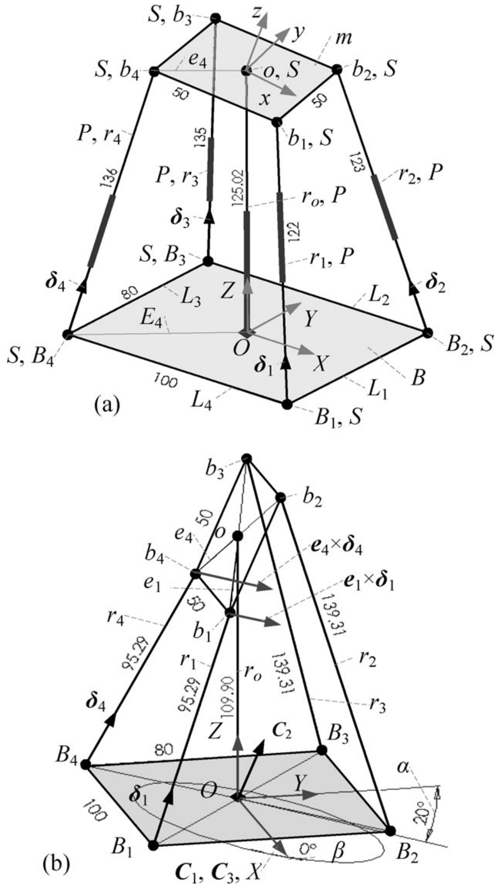

Fig. 3. A 4 SPS+PS PM and it's a singularity.

follows:

$$\mathbf{J} = {}^3\mathbf{J}_r\mathbf{J}_e = {}^3\mathbf{J}_r \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & c_{1x} & c_{2x} & c_{3x} \\ 0 & c_{1y} & c_{2y} & c_{3y} \\ 0 & c_{1z} & c_{2z} & c_{3z} \end{bmatrix}$$

$$= \begin{bmatrix} \delta_{1z} & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T \boldsymbol{C}_1 & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T \boldsymbol{C}_2 & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T \boldsymbol{C}_3 \\ \delta_{2z} & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \boldsymbol{C}_1 & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \boldsymbol{C}_2 & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \boldsymbol{C}_3 \\ \delta_{3z} & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_1 & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_2 & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_3 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{27}$$

3.4. Singularity analyses of a 4 SPS+PS PM

A 4-DOF 4 SPS+PS PM is one type of the 3R1T 4-DOF PM with 4 active legs (see Fig. 3a).

It includes a moving platform m, a fixed base B, and 4 SPS legs with a linear actuator, and a PS passive constrained leg. Here, m is a rectangle quaternary with a short side l_1, a long side l_2, 4 vertices b_i, and a central point o. B is a square with side $L_i = L$, 4 vertices B_i, and a central point O. Each of r_i connects m with B by spherical joint S on m at b_i, a leg r_i with an active prismatic joint P, and S on B at B_i. r_o connects m with B by a S on m at o, a passive prismatic joint P on B at O, and a geometric constraint $r_o \perp B$ is satisfied.

Some singularities of the 4 SPS+PS PM can be determined from Eq. (26) as follows:

(1) When $\delta_{iz} = 0 (i = 1, 2, 3, 4)$, $|\mathbf{J}| = 0$ is satisfied. In this case, when platform and the base are coplanar, a singularity occurs.
(2) When $e_i = 0$ and $\delta_{iz} = 0$ ($i = 1$ or 2 or 3 or 4), $|\mathbf{J}| = 0$ is satisfied. In this case, when one of vertices b_i coincides with o and locates in the base, a singularity occurs.
(3) When $e_i = 0$ or $E_i = 0$ $(i = 1, 2, 3, 4)$, $\delta_i (i = 1, \ldots, 4)$ become dependent each other, $|\mathbf{J}| = 0$ is satisfied. In this case, when platform become one point o or B becomes one point o, δ_i must intersect to one point, two singularities occur.
(4) When $\delta_i (i = 1, \ldots, 4)$ interest at one point, and $|\mathbf{J}| = 0$ is satisfied. In this case, when $(l_i = l, L_i = L$, and $m || B$, $i = 1, \ldots, 4)$, a singularity occurs.
(5) When one of $\delta_i (i = 1, \ldots, 4)$ is zero, $|\mathbf{J}| = 0/0$ is satisfied. In this case, one of r_i is zero, a singularity occurs.
(6) When $\boldsymbol{e}_i \times \delta_i = 0 (\boldsymbol{e}_i \neq 0)$, and $\delta_{iz} = 0$ ($i =$ 1 or 2 or 3 or 4), $|\mathbf{J}| = 0$ is satisfied. In this case, when one of $r_i (i = 1, \ldots, 4)$ pass through the center point O of base, a singularity occurs.
(7) When $\boldsymbol{e}_i \times \delta_i \neq 0$ and $(\boldsymbol{e}_i \times \delta_i)^T \boldsymbol{C}_j = 0$ ($i = 1, 4$ or $2, 3$; $j = 1, 2, 3$), $|\mathbf{J}| = 0$ is satisfied. As $i = 1, 4$, from Eq. (26), it leads to

$$|\mathbf{J}| = \begin{vmatrix} \delta_{1z} & 0 & 0 & 0 \\ \delta_{2z} & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \boldsymbol{C}_1 & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \boldsymbol{C}_2 & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \boldsymbol{C}_3 \\ \delta_{3z} & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_1 & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_2 & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_3 \\ \delta_{4z} & 0 & 0 & 0 \end{vmatrix} = 0. \tag{28}$$

In this case, the plane including $\boldsymbol{e}_i$ and $\delta_i (i = 1, 4)$ must be perpendicular to the 3 vectors $\boldsymbol{C}_j$ $(j = 1, 2, 3)$. Suppose that the platform m rotates by Euler order XZX, thus $\boldsymbol{C}_1$ and X being collinear, $\boldsymbol{C}_2$ and Z_1 (formed from Z about X by an angle α) being collinear, and $\boldsymbol{C}_3 || x$ are satisfied. Hence, when (B_1, b_1, b_4, B_4) locate a plane of m and is parallel with a plane including $\boldsymbol{C}_j$ $(j = 1, 2, 3)$, a singularity occurs, as shown in Fig. 3b. Similarly, other 3 symmetry singularities can be determined.

3.5. Singularity analyses of a 4 SPS+SP PM

A 4-DOF 4 SPS+SP PM[25] is similar to the 4-DOF 4 SPS+PS PM, except that the platform m and the base B are exchanged each other in their functions and positions. However, the position workspace of the platform is enlarged obviously. A prototype of the 4-DOF 4 SPS+SP PM is built in Yanshan University, as shown in Fig. 4.

Since the configuration of the 4-DOF 4 SPS+SP PM is an inverse configuration of the 4 SPS+PS PM, some singularity configurations of the 4-DOF 4 SPS+SP PM must be the inverse singularity configurations of the 4 SPS+PS PM. Thus, when sizes of m and B are exchanged

https://doi.org/10.1017/S0263574709990518 Published online by Cambridge University Press

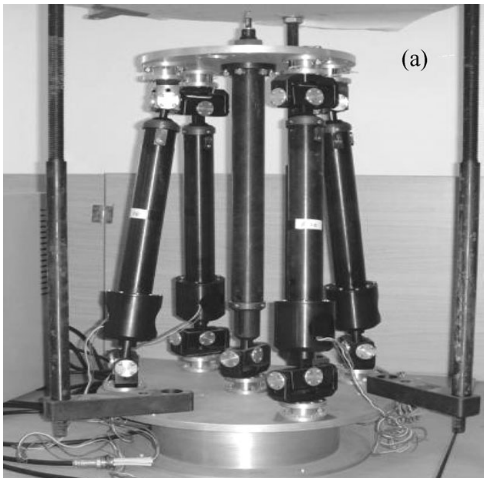

Fig. 4. A prototype of a 4 SPS+SP PM.

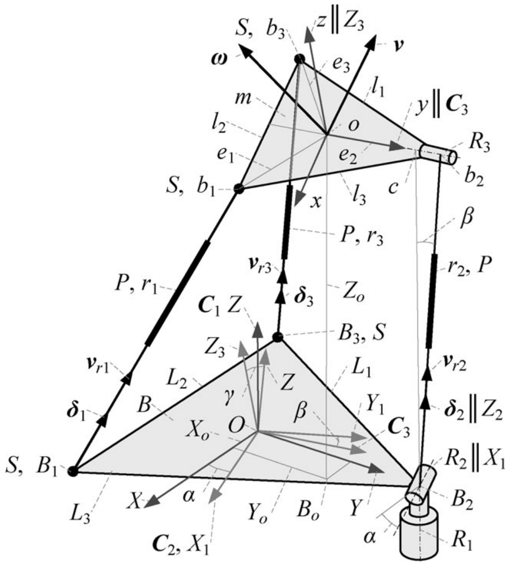

Fig. 5. A 2 SPS+RRPR PM.

each other, the singularities of the 4-DOF 4 SPS+SP PM are similar to that of the 4-DOF 4 SPS+SP PM, except that when (B_1, b_1, b_4, B_4) locate a plane of base B and parallel with planes including $\boldsymbol{C}_j$ $(j = 1, 2, 3)$, a singularity occurs. Similarly, other 3 symmetry singularities can be determined.

3.6. Singularity analyses of 4-DOF 2 SPS+RRPR PM

A 2 SPS+RRPR PM[26] is one type of 4-DOF PMs with 3 active legs, as shown in Fig. 5.

This PM is composed of a moving platform m, a fixed base B, and 2 SPS active legs $r_i (i = 1, 3)$ with the linear actuators, and an RRPR (active revolute joint-revolute joint-active prismatic joint-revolute joint) constrained active leg r_2 with a rotational actuator and a linear actuator. Here, m and B are the same as that of 4-DOF overconstrained 2UPU+RRPU PM. Each of $r_i (i = 1, 3)$ connects m to B by a spherical joint S at b_i, an active leg r_i with a prismatic joint P, and a spherical joint S at B_i. The RRPR constrained active leg r_2 connects m to B by a revolute joint R_3 attached to m at b_2, a constrained active leg r_2 with a prismatic joint P, and a universal joint U attached to B at B_2. The universal joint U at B_2 is composed of two cross-revolute joints R_1 and R_2. In structure, there are following geometric constraints: R_1 and the axis of motor are collinear, R_3 and y are collinear; $R_1 || Z$, $R_1 \perp B$, $R_2 \perp R_1$, $R_2 \perp R_3$, $R_2 \perp r_2$, and $R_3 \perp r_2$. Since each of the SPS active legs $r_i (i = 1, 3)$ only bears the active force along r_i, it obviously has relative larger capacity of load bearing and is simple in structure.

The vector B_i of B_i in $\{B\}$ and the vectors $\boldsymbol{b}_i$ of b_i in $\{B\}$ are expressed in ref. [26] as follows:

$$\boldsymbol{b}_1 = \frac{1}{2}\begin{bmatrix} qex_l - ey_l + 2X_o \\ qex_m - ey_m + 2Y_o \\ qex_n - ey_n + 2Z_o \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} ey_l + X_o \\ ey_m + Y_o \\ ey_n + Z_o \end{bmatrix},$$

$$\boldsymbol{b}_3 = \frac{1}{2}\begin{bmatrix} -qex_l - ey_l + 2X_o \\ -qex_m - ey_m + 2Y_o \\ -qex_n - ey_n + 2Z_o \end{bmatrix}, \quad q = \sqrt{3}, \tag{29}$$

$$\boldsymbol{B}_1 = \frac{E}{2}\begin{bmatrix} q \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ E \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_3 = \frac{E}{2}\begin{bmatrix} -q \\ -1 \\ 0 \end{bmatrix}.$$

When a 2 SPS+RRPR PM has 3 rotations (a, β, γ) and a translation along Z, there are $x_1 = Z_o$, $x_2 = \alpha$, $x_3 = \beta$, $x_4 = \gamma$, and their velocity vector $\boldsymbol{v}_e$ are expressed as Eq. (23).

Suppose that platform rotate in order ZXY, ${}^B_m\mathbf{R}$ is expressed in ref. [26] as below,

$${}^B_m\mathbf{R} = \begin{bmatrix} -s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & -s_\alpha c_\beta & s_\alpha s_\beta c_\gamma + c_\alpha s_\gamma \\ c_\alpha s_\beta s_\gamma + s_\alpha c_\gamma & c_\alpha c_\beta & -c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ -c_\beta s_\gamma & s_\beta & c_\beta c_\gamma \end{bmatrix}. \tag{30}$$

Two pose parameters X_o and Y_o can be expressed by (Z_o, α, β) and have been derived in ref. [26] as follows:

$$X_o = s_\alpha(e + Z_o s_\beta)/c_\beta, \quad Y_o = E - c_\alpha(e + Z_o s_\beta)/c_\beta. \tag{31}$$

The 3×4 translational Jacobian matrix $\mathbf{J}_v$ is derived from Eqs. (6), (23), and (30) as below,

$$\mathbf{J}_v = \begin{bmatrix} s_\alpha t_\beta & c_\alpha(e + Z_o s_\beta)/c_\beta & s_\alpha(Z_o + e s_\beta)/c_\beta^2 & 0 \\ -c_\alpha t_\beta & s_\alpha c_\beta(e + Z_o s_\beta)/c_\beta & -c_\alpha(Z_o + e s_\beta)/c_\beta^2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

Since $\boldsymbol{C}_1$ and Z being collinear, $\boldsymbol{C}_2$ and X_1 (formed from X about Z by an angle α) being collinear, and $C_3 || y_2$ are satisfied, the matrix $\mathbf{C}$ and the vector $\boldsymbol{K}$ have been derived in

https://doi.org/10.1017/S0263574709990518 Published online by Cambridge University Press

ref. [26] as follows:

$$\mathbf{C} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 & \boldsymbol{C}_3 \end{bmatrix} = \begin{bmatrix} 0 & c_\alpha & -s_\alpha c_\beta \\ 0 & s_\alpha & c_\alpha c_\beta \\ 1 & 0 & s_\beta \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} s_\alpha t_\beta \\ -c_\alpha t_\beta \\ 1 \end{bmatrix}. \quad (33)$$

A 3×4 translational Jacobian matrix $\mathbf{J}_\omega$ is derived from Eqs. (7), (23), and (33) as below,

$$\mathbf{J}_\omega = \mathbf{C} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & c_\alpha & -s_\alpha c_\beta \\ 0 & 0 & s_\alpha & c_\alpha c_\beta \\ 0 & 1 & 0 & s_\beta \end{bmatrix}. \quad (34)$$

The general 4×4 Jacobian matrix $\mathbf{J}$ can be derived from Eqs. (15), (31), and (33) as below,

$$\mathbf{J} = {}^3\mathbf{J}_r \mathbf{J}_e$$
$$= {}^3\mathbf{J}_r \begin{bmatrix} s_\alpha t_\beta & \frac{c_\alpha}{c_\beta}(e + Z_o s_\beta) & \frac{s_\alpha}{c_\beta^2}(Z_o + e s_\beta) & 0 \\ -c_\alpha t_\beta & \frac{s_\alpha}{c_\beta}(e + Z_o s_\beta) & -\frac{c_\alpha}{c_\beta^2}(Z_o + e s_\beta) & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & c_\alpha & -s_\alpha c_\beta \\ 0 & 0 & s_\alpha & c_\alpha c_\beta \\ 0 & 1 & 0 & s_\beta \end{bmatrix}. \quad (35)$$

Some items in $\mathbf{J}$ can be expressed as follows:

$$\begin{aligned} J_{11} &= (\delta_{1x} s_\alpha - \delta_{1y} c_\alpha) t_\beta + \delta_{1z}, \\ J_{13} &= (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T \boldsymbol{C}_2 + (\delta_{1x} s_\alpha - \delta_{1y} c_\alpha)(Z_o + e s_\beta)/c_\beta^2, \\ J_{14} &= (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T \boldsymbol{C}_3, \quad J_{21} = (\delta_{2x} s_\alpha - \delta_{2y} c_\alpha) t_\beta + \delta_{2z}, \\ J_{23} &= (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \boldsymbol{C}_2 + (\delta_{2x} s_\alpha - \delta_{2y} c_\alpha)(Z_o + e s_\beta)/c_\beta^2, \\ J_{24} &= (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \boldsymbol{C}_3 = 0, \quad J_{31} = (\delta_{3x} s_\alpha - \delta_{3y} c_\alpha) t_\beta + \delta_{3z}, \\ J_{33} &= (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_2 + (\delta_{3x} s_\alpha - \delta_{3y} c_\alpha)(Z_o + e s_\beta)/c_\beta^2, \\ J_{34} &= (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_3, \quad J_{41} = J_{43} = J_{44} = 0, \quad J_{42} = 1. \end{aligned} \quad (36)$$

The length r_i $(i = 1, 2, 3)$ and the unit vectors δ_i of active legs in $\{B\}$, and the vectors $\boldsymbol{e}_i$ from the center o of m to the joint b_i on m in $\{B\}$ have been derived in ref [26].

Determinant $|\mathbf{J}|$ can be simplified from Eqs. (35) and (36) as below,

$$|\mathbf{J}| = \begin{vmatrix} J_{11} & J_{13} & J_{14} \\ J_{21} & J_{23} & J_{24} \\ J_{31} & J_{33} & J_{34} \end{vmatrix} = \begin{vmatrix} J_{11} & J_{13} & J_{14} \\ J_{21} & J_{23} & 0 \\ J_{31} & J_{33} & J_{34} \end{vmatrix}. \quad (37)$$

Some singularities of the 2 SPS+RRPR PM can be determined from Eq. (35)–(37) as follows:

(1) When $\beta = 90°$, $|\mathbf{J}| \to \infty$ is satisfied. In this case, $X_o \to \infty$, $Y_o \to \infty$, a local singularity occurs.
(2) When $\delta_i = 0$ $(i = 1$ or 2 or $3)$, $|\mathbf{J}| = 0$ is satisfied. In this case, $r_i = 0$ $(i = 1$ or 2 or $3)$, a singularity occurs.
(3) When $\boldsymbol{e}_1 = \boldsymbol{e}_3 = 0$, $|\mathbf{J}| = 0$ is satisfied. In this case, platform m becomes a line ob_2, a singularity occurs.
(4) When $(\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T \boldsymbol{C}_3 = (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \boldsymbol{C}_3 = 0$, $|\mathbf{J}| = 0$ is satisfied. In this case, side $l_2 = 0$, a singularity occurs.
(5) When $Z_o = -e s_\beta$ and $(\boldsymbol{e}_i \times \boldsymbol{\delta}_i)^T \boldsymbol{C}_2 = 0$ $(i = 1, 2, 3)$, $|\mathbf{J}| = 0$ is satisfied. In this case, a singularity occurs.
(6) When $(\delta_{iy} c_\alpha - \delta_{ix} s_\alpha) t_\beta = \delta_{iz} (i = 1, 2, 3)$, $|\mathbf{J}| = 0$ is satisfied. In this case, a singularity occurs.

These singularities may be only a part of the whole singularities of the 2 SPS+RRPR PM. Other singularities of this PM can be also determined from Eqs. (34)–(37).

4. Conclusions

A common 3×4 translational Jacobian matrix $\mathbf{J}_v$ without partial differential items and a common 3×4 rotational Jacobian matrix $\mathbf{J}_\omega$ without partial differential items for the 4-DOF PMs can be derived separately. A 4×6 Jacobian matrix $\mathbf{J}$ with partial differential items can be transformed into a 4×4 general Jacobian matrix $\mathbf{J}$ without partial differential items by means of $\mathbf{J}_v$ and $\mathbf{J}_\omega$. Thus, the singularities of the 4-DOF PMs can be determined easily by $\mathbf{J}$.

When the 4-DOF PMs have 3 translations and 1 rotation about Z, $\mathbf{J}_v$ is a 3×4 sum matrix of a 3×3 unit matrix and a zero vector; $\mathbf{J}_\omega$ is a 3×4 sum matrix of a 3×3 zero matrix and a Z vector; and $\mathbf{J}$ is a 4×4 matrix only including the unit vectors along active legs and the vectors from central point to vertices on platform. Since $(\mathbf{J}_v, \mathbf{J}_\omega, \mathbf{J})$ have no any partial differential items, the determination of the singularity of the 4-DOF PMs with 3 translations and 1 rotation is easiest.

When the 4-DOF PMs have 1 translation along Z and 3 Euler rotations, $\mathbf{J}_v$ is a 3×4 sum matrix of a 3×1 zero vector and a 3×3 unit matrix; $\mathbf{J}_\omega$ is a 3×4 sum matrix of a Z vector and a 3×3 zero matrix; and $\mathbf{J}$ is a 4×4 matrix including the unit vectors along active legs, the vectors from central point to vertices on platform, and the unit vectors of rotational axes of Euler angles. Since $(\mathbf{J}_v, \mathbf{J}_\omega, \mathbf{J})$ have no any partial differential items, the determination of the singularity of the 4-DOF PMs with 1 rotation and 3 translations is quite easy.

When the 4-DOF PMs have 3 active legs, the rank of the determinant of the 4×4 general Jacobian matrix $\mathbf{J}$ may be reduced from 4 to 3. Thus, the determination of the singularity of the 4-DOF PMs with 3 active legs becomes quite easy.

In fact, the singularity analyses of the 4-DOF PMs are far from being exhaustive. There may be other singularities or the singular space needed to be determined from $\mathbf{J}$. Since $\mathbf{J}$ is simplified and its determinant rank is reduced to 3, other singularities or the singular space can be determined easily.

Acknowledgments

The authors would like to acknowledge the financial support of the Natural Sciences Foundation Council of China (NSFC) 50575198. The authors would like to acknowledge the financial support of Doctoral Fund from National Education Ministry No. 20060216006.

References

1. Z. Huang, Y.-S. Zhao and T.-S. Zhao, *Advanced Spatial Mechanism* (Press of Advanced Education, Beijing, China, 2006).
2. Z. Huang, L. H. Chen and Y. W. Li, "The singularity principle and property of Stewart parallel manipulator," *J. Robot. Syst.* **20**(4), 163–176 (2003).
3. Z. Huang, Y. Cao, Y.-W. Li and L.-H. Chen, "Structure and property of the singularity loci of the 3/6-Stewart–Gough platform for general orientations," *Robotica* **24**, 75–84 (2006).
4. Z. Huang and Y. Cao, "Property identification of the singularity loci of a class of Gough–Stewart manipulators," *Int. J. Robot. Res.* **24**(8), 675–685 (2005).
5. J. P. Merlet, "Singular configurations of parallel manipulators and Grassmann geometry," *Int. J. Robot. Res.* **8**(5), 45–56 (1989).
6. J. Wang and C. M. Gossalin, "Singularity analysis and representation of spatial five-degree-of-freedom parallel mechanisms," *J. Robot. Syst.* **14**(12), 851–869 (1997).
7. J. Wang and C. M. Gossalin, "Singularity loci of a special class of spherical 3-DOF parallel mechanisms with pridmatic actuators," *ASME J. Mech. Des.* **126**(2), 319–326 (2004).
8. G. Alıcı and B. Shirinzadeh, "Loci of singular configurations of a 3-DOF spherical parallel manipulator," *Robot. Auton. Syst.* **48**(2–3). 77–91 (2004).
9. X.-W. Kong and C. M. Gosselin, "Uncertainty singularity analysis of parallel manipulators based on the instability analysis of structures," *Int. J. Robot. Res.* **20**(11), 847–856 (2001).
10. B. Monsarrat and C. M. Gossalin, "Singularity analysis of a three-leg six-degree-of-freedom parallel platform mechanism based on Grassmann line geometry," *Int. J. Robot. Res.* **20**(40), 312–326 (2001).
11. S. Bandyopadhyay and A. Ghosal, "Geometric characterization and parametric representation of the singularity manifold of a 6–6 Stewart platform manipulator," *Mech. Mach. Theory* **41**(11),1377–1400 (2006).
12. S. Bandyopadhyay and A. Ghosal, "Analysis of configuration space singularities of closed-loop mechanisms and parallel manipulators," *Mech. Mach. Theory* **39**(5), 519–544 (2004).
13. J. Gallardo-Alvarado, J. María Rico-Martínez and G. Alici, "Kinematics and singularity analyses of a 4-dof parallel manipulator using screw theory," *Mech. Mach. Theory* **41**(9), 1048–1061 (2006).
14. R. Di Gregorio, "Forward problem singularities in parallel manipulators which generate SX–YS–ZS structures," *Mech. Mach. Theory* **40**(5), 600–612 (2005).
15. J.-S. Zhao, Z.-J. Feng, K. Zhou and J.-X. Dong, "Analysis of the singularity of spatial parallel manipulator with terminal constraints," *Mech. Mach. Theory* **40**(3), 275–284 (2005).
16. S. K. Ider, "Inverse dynamics of parallel manipulators in the presence of drive singularities," *Mech. Mach. Theory* **40**(1), 33–44 (2005).
17. A. Wolf, E. Ottaviano, M. Shoham and M. Ceccarelli, "Application of line geometry and linear complex approximation to singularity analysis of the 3-DOF CaPaMan parallel manipulator," *Mech. Mach. Theory* **39**(1), 75–95 (2004).
18. A. K. Dash, I.-M. Chen, H.-Y. Song and G.-L. Yang, "Workspace generation and planning singularity-free path for parallel manipulators," *Mech. Mach. Theory* **40**(7), 776–805 (2005).
19. S. Sen, B. Dasgupta and A. K. Mallik, "Variational approach for singularity-free path-planning of parallel manipulators," *Mech. Mach. Theory* **38**(11), 1165–1183 (2003).
20. H. Zhou and K.-L. Ting, "Path generation with singularity avoidance for five-bar slider-crank parallel manipulators," *Mech. Mach. Theory* **40**(3), 371–384 (2005).
21. O. Ma and J. Angeles, "Architecture Singularity of Platform Manipulator," *IEEE International Conference on Robotics and Automation*, Sacramento, CA (1991) pp. 1542–1547.
22. Y. Lu and B. Hu, "Determining singularity of parallel manipulators with n linear active legs by CAD variation geometry," *Int. J. Robot. Autom.* **23**(3), 160–167 (2008).
23. Y. Lu and B. Hu, "Unification and simplification of velocity/acceleration of limited-dof parallel manipulators with linear active legs," *Mech. Mach. Theory* **43**(9), 1112–1128 (2008).
24. Y. Lu, Y. Shi, Z. Huang, J. P. Yu, S. H. Li and X. B. Tian, "Kinematics/statics of a 4-DOF over-constrained parallel manipulator with 3 legs," *Mech. Mach. Theory* **44**(6), 1497–1506 (2009).
25. Y. Lu and B. Hu, "Unified solving Jacobian/Hessian matrices of some parallel manipulators with n SPS active legs and a constrained leg," *ASME J. Mech. Des.* **129**(11), 1161–1169 (2007).
26. Y. Lu, B. Hu and Y. Shi, "Kinematics analysis and statics of a 2SPS+UPR parallel manipulator," *Multibody Syst. Dyn.* **18**(4), 619–636 (2007).