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On the boundary conditions in estimating $\nabla\omega$ by $\operatorname{div}\omega$ and $\operatorname{curl}\omega$

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In this paper, we study under what boundary conditions the inequality

$$\|\nabla\omega\|^2_{L^2(\Omega)} \leqslant C(\|\operatorname{curl}\omega\|^2_{L^2(\Omega)} + \|\operatorname{div}\omega\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)})$$

holds true. It is known that such an estimate holds if either the tangential or normal component of ω vanishes on the boundary $\partial\Omega$. We show that the vanishing tangential component condition is a special case of a more general one. In two dimensions, we give an interpolation result between these two classical boundary conditions.

Keywords: Gaffney inequality; divergence and curl operators; tangential and normal components; boundary conditions

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1. Introduction

In this paper, we study the estimate

$$\|\nabla\omega\|^2_{L^2(\Omega)} \leqslant C(\|\operatorname{curl}\omega\|^2_{L^2(\Omega)} + \|\operatorname{div}\omega\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}), \tag{1}$$

where $\omega \in H^1(\Omega)^n$ is a vector field ($n = 2, 3$ in most applications) and C is a constant independent of ω. $H^1(\Omega)^n$ denotes the Sobolev space of vector fields, whose components and all its derivatives are L^2 integrable. It is well known that such an estimate holds true if either the tangential or normal component of ω vanishes on the boundary $\partial\Omega$, which we shall call the classical boundary conditions. More precisely, if ν is the unit exterior normal vector on $\partial\Omega$, then (1) holds true if

$$\nu \times \omega = 0 \quad \text{on } \partial\Omega \quad \text{or} \quad \langle \nu; \omega \rangle = 0 \quad \text{on } \partial\Omega. \tag{2}$$

These boundary conditions have been studied in great detail and the literature on it and its applications to physical systems, mainly Maxwell's equations and Navier–Stokes equations, is very large. Our aim is to show that some of these classical boundary conditions can be extended to much more general ones. A particular case of our main result gives in two dimensions an interpolation between the two classical boundary conditions (cf. remark 3.3 (ii)).

Let us first mention that inequality (1) cannot hold true without further restrictions on ω. To see this, take any domain $\Omega \subset \mathbb{R}^2$ and define for $n \in \mathbb{N}$

$$\omega_n(x) = (e^{nx_1} \cos(nx_2), -e^{nx_1} \sin(nx_2)).$$

Then one easily verifies that $\operatorname{div} \omega_n = 0$, $\operatorname{curl} \omega_n = 0$,

$$|\nabla \omega_n(x)|^2 = 2n^2 e^{2nx_1} \quad \text{and} \quad |\omega_n(x)|^2 = e^{2nx_1}.$$

Hence there can be no constant C independent of n such that for all n

$$2n^2 \int_\Omega \mathrm{e}^{2nx_1} \leqslant C \int_\Omega \mathrm{e}^{2nx_1}. \tag{3}$$

A similar example works also in higher dimensions.

Some standard references on (1) and its applications are Amrouche-Bernardi-Dauge-Girault [**1**], Costabel [**8**], Dautray–Lions [**15**], and Grisvard [**21**]. The inequality (1) has also been studied in the more general context of differential forms, where curl is replaced by the d operator, respectively, div is replaced by δ. In this setting, it is called Gaffney-Friedrichs inequality after Gaffney [**17**, **18**], but for domains with boundary and the classical boundary conditions, it is due to Morrey [**28**], [**30**] or Friedrichs [**16**]. Proofs of this general version can also be found in Csató–Dacorogna–Kneuss [**13**], Iwaniec–Martin [**23**], Morrey [**29**], Schwarz [**31**], Taylor [**32**]. Therefore, we will call also (1) Gaffney inequality henceforth.

The first and simplest generalization of the boundary conditions (cf. theorem 2.1) is by mixing the classical ones, namely requiring that on some parts of the boundary the tangential part vanishes and on other parts, the normal part vanishes. This result already seems to be known, see for instance Goldsthein–Mitrea [**20**] or Jakab–Mitrea–Mitrea [**25**] and the references therein. We state and indicate a very simple proof of this result for completeness (cf. theorem 2.1), since it does not appear explicitly in the references.

First attempts to give more general boundary conditions have been obtained in Csató–Dacorogna [**12**], see also Csató [**11**] for a general version on Riemannian manifolds. There the authors have proven, in particular, that in three dimensions, if λ is a given fixed vector field, then there exists a constant $C = C(\Omega, \lambda)$ such that (1) holds true if

$$\nu \times \omega = \lambda \langle \nu; \omega \rangle \quad \text{on } \partial\Omega.$$

This generalizes the classical condition of vanishing tangential component by setting $\lambda = 0$.

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Our first main result is an even simpler generalization of this classical boundary condition (vanishing tangential component) which additionally has an obvious geometric interpretation. Namely, theorem 3.2 asserts that (1) holds true if

$$\lambda \times \omega = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where again C will depend on λ and Ω. Geometrically, this means that Gaffney inequality holds true whenever the vector fields ω are collinear with a given fixed vector field on $\partial\Omega$. This time, setting $\lambda = \nu$ gives the classical boundary condition. We will prove Gaffney inequality under the condition (4) for Lipschitz domains as long as λ is C^1. Thus, if Ω is not C^2 (and thus ν is not C^1), this result does not include the classical boundary condition $\nu \times \omega = 0$. However, we will give in the case of domains in $\mathbb{R}^2$ a better result which does not even require λ to be globally Lipschitz on $\partial\Omega$, see theorem 4.2. A special case of this theorem is, for instance, Gaffney inequality on polygonial domains with either of the classical boundary conditions on different parts of the polygon. This is the first step in providing more general Gaffney inequalities, with simple proof, to be applicable in numerical analysis. We refer to Arnold–Falk–Winther [**2**] (§ 7.7) and Bonizzoni–Buffa–Nobile [**5**] for a discussion on vector-valued finite element methods and applications of Gaffney inequality in that setting.

We do not require in any of our results that Ω is convex. This is because we assume that our vector fields ω are at least in $H^1(\Omega)^n$. A weaker formulation of the classical Gaffney inequality for Lipschitz domains requires Ω to be convex. By the weak formulation, we mean that we assume

$$\begin{aligned}\omega \in H_T(\operatorname{div}, \operatorname{curl}; \Omega) &= \{\omega \in L^2(\Omega)^n | \operatorname{div}\omega \in L^2(\Omega),\\ &\operatorname{curl}\omega \in L^2(\Omega)^n, \quad \nu \times \omega = 0 \quad \text{on } \partial\Omega\}.\end{aligned}$$

Under this hypothesis on ω, Gaffney inequality becomes a regularity result and states that $\omega \in H^1(\Omega)^n$ and satisfies the corresponding estimate (1). The same result holds true if we replace H_T with H_N, the space with vanishing normal component. The usual approach to prove such regularity results is to use Gaffney inequality for an approximating sequence $\{\omega_k\}$ in H^1. The difficulty consists in establishing $\nu \times \omega_k = 0$ on $\partial\Omega$, using the assumption that $\nu \times \omega = 0$ on $\partial\Omega$ in a weak sense. This approximation fails for nonconvex domains, which are only Lipschitz and the regularity statement does not hold true. See, for instance, the remark following the proof of Theorem 5.1 in Mitrea [**26**]. This is essentially the same example as the one for the Laplace equation: it is well known that the solution u of $\Delta u = f$, $f \in L^2$, is in general only in $H^{3/2}$ if Ω is a nonconvex polygonial domain, cf. Grisvard [**22**]. For more details on these approximation theorems and regularity results, we refer to Amrouche–Bernardi-Dauge-Girault [**1**], Belgacem-Bernardi-Costabel-Dauge [**4**], Ciarlet-Hazard-Lohrengel [**6**], Costabel [**7**], Costabel-Dauge [**9**] and Girault-Raviart [**19**]. For a different approach in proving the classical Gaffney inequality for non-smooth domains see Mitrea [**26**], where the inequality is obtained using existence and regularity of an elliptic boundary value system established in Mitrea [**27**].

Note that the proof of Theorem 3.2 (Gaffney inequality with condition (4)) would not simplify if we assumed Ω to be smooth.

In view of the condition (4), one might expect that the classical condition $\langle \nu; \omega \rangle = 0$ can be generalized too, by replacing ν with a nonvanishing vector field λ. This is, however, not true if $n \geqslant 3$ as can be seen by a simple counterexample. It is also not true that condition (4) generalizes to differential forms of higher order. We give these counterexamples at the end of this paper.

2. Mixed classical boundary conditions

If Ω is a bounded $C^{1,1}$ open set with unit exterior normal ν on its boundary $\partial\Omega$ and ω is some vector field, we shall decompose it as

$$\omega = \omega_T + \omega_N, \quad \text{where} \quad \omega_N = \langle \omega; \nu \rangle \nu \quad \text{and} \quad \omega_T = \omega - \omega_N.$$

Throughout this paper, for vectors fields ω, λ in $\mathbb{R}^n$, the curl and cross product are defined as vectors in $\mathbb{R}^{\binom{n}{2}}$ defined by

$$(\operatorname{curl} \omega)_{ij} = \frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} \quad \text{and} \quad (\omega \times \lambda)_{ij} = \omega_i \lambda_j - \omega_j \lambda_i \,, \quad 1 \leqslant i < j \leqslant n.$$

We now state a theorem whose proof is essentially the same as the one presented in Csató–Dacorogna–Kneuss [13] for the classical Gaffney inequality.

Theorem 2.1. *Let $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded open $C^{1,1}$ set with exterior unit normal ν on $\partial\Omega$. Then there exists a constant $C = C(\Omega)$ such that*

$$\|\nabla \omega\|_{L^2(\Omega)}^2 \leqslant C \left(\|\operatorname{curl} \omega\|_{L^2(\Omega)}^2 + \|\operatorname{div} \omega\|_{L^2(\Omega)}^2 + \|\omega\|_{L^2(\Omega)}^2 \right),$$

for all $\omega \in H^1(\Omega)^n$ satisfying

$$\omega_T = 0 \text{ or } \omega_N = 0 \quad \text{on } \Gamma_i, \qquad \partial\Omega = \bigcup_{i=1}^{M} \overline{\Gamma}_i \,,$$

and Γ_i are open sets in $\partial\Omega$ and $M \in \mathbb{N}$.

Remark 2.2. If $M = 1$ (classical boundary conditions, $\Gamma_1 = \partial\Omega$) and Ω is contractible, then one easily obtains the better estimate

$$\|\nabla \omega\|_{L^2(\Omega)}^2 \leqslant C \left(\|\operatorname{curl} \omega\|_{L^2(\Omega)}^2 + \|\operatorname{div} \omega\|_{L^2(\Omega)}^2 \right),$$

see Csató–Dacorogna–Kneuss [13] theorems 6.5 and 6.7 (Step 1 of the proof). A precise treatment of the optimal topological assumptions on the domain for such an estimate to hold true is carried out in von Wahl [33].

Proof. We will not give a detailed proof. The result follows from [13] theorem 5.7 (see also [21] theorem 3.1.1.1) in the same way as the classical Gaffney inequality:

Indeed, as in the proof of Theorem 5.16 in [**13**], one obtains that

$$\int_{\Omega}((|\operatorname{curl}\omega|^2+|\operatorname{div}\omega|^2)\geqslant\int_{\Omega}|\nabla\omega|^2-C\int_{\partial\Omega}|\omega|^2,$$

and one concludes similarly. The above-mentioned references treat C^2 domains but remain valid without any change for $C^{1,1}$ domains. See also Step 3 in the first proof of Proposition 3.4. □

3. The $\boldsymbol{\lambda}\times\boldsymbol{\omega}=\mathbf{0}$ condition

We now state our first main result. We will distinguish the case $n=2$ as we will give in § 4 in the two-dimensional case an improvement of the theorem by weakening the regularity assumptions . To state the theorem, we need the following definition (which we will use actually only for $C^{r,\alpha}=C^{1,0}$ or $C^{0,1}$).

Definition 3.1. Let $r\geqslant 0$ be an integer and $0\leqslant\alpha\leqslant 1$. If Ω is a Lipschitz set (meaning that $\partial\Omega$ is Lipschitz), we say that a function $\lambda:\partial\Omega\to\mathbb{R}$ is in $C^{r,\alpha}(\partial\Omega)$ if there exists an extension of λ to $\mathbb{R}^n$ such that $\lambda\in C^{r,\alpha}(\mathbb{R}^n)$. We make the convention that $C^{r,0}=C^r$.

Theorem 3.2. *Let* $n\geqslant 2$, $\Omega\subset\mathbb{R}^n$ *be a bounded open Lipschitz set and* $\lambda\in C^1(\partial\Omega)^n$ *be such that*

$$\lambda\neq 0\quad on\ \partial\Omega.$$

Then there exists a constant $C=C(\Omega,\lambda)$ *such that*

$$\|\nabla\omega\|^2_{L^2(\Omega)}\leqslant C(\|\operatorname{curl}\omega\|^2_{L^2(\Omega)}+\|\operatorname{div}\omega\|^2_{L^2(\Omega)}+\|\omega\|^2_{L^2(\Omega)}),$$

for all $\omega\in H^1(\Omega)^n$ *which satisfy*

$$\lambda\times\omega=0\quad on\ \partial\Omega.$$

If $n=2$ *then the same conclusion holds under the weaker regularity assumptions* $\lambda\in C^{0,1}(\partial\Omega)^2$.

Remark 3.3.

(i) Note that if Ω is a C^2 set, then the unit outward normal vector ν is C^1 and the theorem implies the classical boundary condition $\nu\times\omega=0$.

(ii) If $n=2$, then this theorem interpolates between the two classical boundary conditions $\omega_T=0$, respectively, $\omega_N=0$. To see this take $\lambda=\nu=(\nu_1,\nu_2)$, respectively, $\lambda=(\nu_2,-\nu_1)$.

(iii) Recall (see remark 2.2) that if Ω is contractible and $\lambda=\nu$, then in the above theorem the inequality can be replaced by

$$\|\nabla\omega\|^2_{L^2(\Omega)}\leqslant C(\|\operatorname{curl}\omega\|^2_{L^2(\Omega)}+\|\operatorname{div}\omega\|^2_{L^2(\Omega)}),$$

This is not true for general λ. To see this just notice that one can take $\lambda\in C^\infty(\overline{\Omega})^n$ equal to a harmonic field (i.e. $\operatorname{curl}\lambda=0$ and $\operatorname{div}\lambda=0$) that

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never vanishes on the boundary. Then $\omega = \lambda$ trivially satisfies $\lambda \times \omega = 0$ on $\partial\Omega$. Such non-constant harmonic fields exist, for example, take $\omega = (x_2, x_1)$ and a domain $\Omega \subset \mathbb{R}^2$ such that $0 \notin \partial\Omega$, so that $\lambda = \omega \neq 0$ on the boundary.

(iv) If λ is constant then $C = 1$, see lemma 3.10 or proof of Proposition 3.4. $C = 1$ also if $\lambda = \nu$ is normal to $\partial\Omega$ and Ω is convex (actually $n - 1$ convex is sufficient), see [**14**], but this requires a different proof.

Proof of Theorem 3.2. We first prove the result for C^1 vector fields ω, respectively, Lipschitz vector fields if $n = 2$ (cf. proposition 3.4). Theorem 3.2 will then follow by approximation (cf. proposition 3.15). □

PROPOSITION 3.4. *Let $n \geqslant 2$, $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set and $\lambda \in C^{0,1}(\partial\Omega)^n$ be such that $\lambda \neq 0$ on $\partial\Omega$. Then there exists a constant $C = C(\Omega, \lambda)$ such that*

$$\|\nabla\omega\|^2_{L^2(\Omega)} \leqslant C\left(\|\operatorname{curl}\omega\|^2_{L^2(\Omega)} + \|\operatorname{div}\omega\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}\right),$$

for all $\omega \in C^1(\overline{\Omega})^n$ which satisfy $\lambda \times \omega = 0$ on $\partial\Omega$. If $n = 2$, then the same holds true if $\omega \in C^{0,1}(\overline{\Omega})^2$.

REMARK 3.5. Note that in this proposition, we require that λ is only Lipschitz. The loss of regularity compared with the main theorem 3.2 arises in the approximation, see proposition 3.15.

We give two proofs of this proposition. The first one is simpler, following the ideas of Csató–Dacorogna [**12**]. However, we do not use the identity established in [**12**] and which is used in establishing the classical Gaffney inequality, respectively theorem 2.1. The second proof that we give is a generalization of Morrey's original proof of Gaffney inequality (see Morrey [**28**], Morrey–Eells [**30**] or Iwaniec–Scott–Stroffolini [**24**] for an L^p version) for the boundary condition $\nu \times \omega = 0$. It is longer, but several of the intermediate steps are of interest on their own right, cf. lemma 3.10, and also lemmas 3.8 and 3.12 which are independent of the boundary conditions. In the first proof, we will use the following abbreviation, f being a function defined on a neighborhood of $\partial\Omega$:

$$\partial_{ij}[f] := \nu_j \frac{\partial f}{\partial x_i} - \nu_i \frac{\partial f}{\partial x_j},$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward unit normal vector on $\partial\Omega$. It can be easily seen that $\partial_{ij}[f]$ is a tangential derivative and depends only on the values of f on $\partial\Omega$. Therefore, if f is Lipschitz then $\partial_{ij}[f]$ is well defined $\mathcal{H}^{n-1}$ almost everywhere on any Lipschitz boundary $\partial\Omega$, see for instance lemma 3.6 equations (3.6)–(3.7) for the case $n = 2$ (if $n \geqslant 3$, the argument is similar by composing f with a local parametrization of $\partial\Omega$). Moreover, by the product rule of derivation:

$$\partial_{ij}[fg] = \partial_{ij}[f]g + f\partial_{ij}[g]. \tag{3.1}$$

Throughout the proof, we will frequently use that any Lipschitz function defined on a subset of $\mathbb{R}^n$ can be extended to a Lipschitz function on the whole space, and conversely, that restrictions of Lipschitz functions to any subset are still Lipschitz.

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First Proof of Proposition 3.4.

Step 1. Let us assume first that $\omega \in C^2(\overline{\Omega})^n$. A direct calculation gives the identity

$$|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 - |\nabla\omega|^2 = 2\sum_{i<j}\left(\frac{\partial\omega_i}{\partial x_i}\frac{\partial\omega_j}{\partial x_j} - \frac{\partial\omega_i}{\partial x_j}\frac{\partial\omega_j}{\partial x_i}\right).$$

So we obtain by partial integration that

$$\int_\Omega \left(|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 - |\nabla\omega|^2\right) = -\sum_{i<j}\int_{\partial\Omega}\omega_i\partial_{ij}[\omega_j] + \sum_{i<j}\int_{\partial\Omega}\omega_j\partial_{ij}[\omega_i]. \tag{3.2}$$

Note that (3.2) involves only the first derivatives of ω. Therefore, by approximation one directly deduces that (3.2) remains true for any $\omega \in C^1(\overline{\Omega})^n$. To see this, note that standard convolution in the whole space works since the derivatives of ω are uniformly continuous, and the derivatives of the approximating sequence will converge also uniformly on $\partial\Omega$. If $n = 2$ we apply lemma 3.6 to obtain that (3.2) remains true if $\omega \in C^{0,1}(\overline{\Omega})^2$.

Step 2. Since $\lambda = (\lambda_1, \dots, \lambda_n) \neq 0$ on $\partial\Omega$, there exist open sets $W_1, \dots, W_M$, integers $1 \leqslant k(1), \dots, k(M) \leqslant n$ and $\epsilon > 0$ such that

$$\partial\Omega \subset \bigcup_{l=1}^{M} W_l \quad \text{and} \quad |\lambda_{k(l)}| \geqslant \epsilon \text{ in } W_l \quad \text{for } 1 \leqslant l \leqslant M.$$

We now define inductively

$$S_1 = W_1 \cap \partial\Omega, \quad S_2 = (W_2 \cap \partial\Omega)\backslash S_1, \dots,$$
$$S_j = (W_j \cap \partial\Omega)\backslash\left(\bigcup_{m=1}^{j-1} S_m\right),$$

for $j = 1, \dots, M$. Thus the S_j form a disjoint union of $\partial\Omega$ and we can write

$$\int_{\partial\Omega}(-\omega_i\partial_{ij}[\omega_j] + \omega_j\partial_{ij}[\omega_i]) = \sum_{l=1}^{M}\int_{S_l}(-\omega_i\partial_{ij}[\omega_j] + \omega_j\partial_{ij}[\omega_i]), \tag{3.3}$$

for any $i < j$. We now claim that for each $l = 1, \dots, M$ and each $i < j$, there exists a constant $C = C(\Omega, \lambda) > 0$ such that

$$\left|\int_{S_l}(-\omega_i\partial_{ij}[\omega_j] + \omega_j\partial_{ij}[\omega_i])\right| \leqslant C\int_{S_l}|\omega|^2 \tag{3.4}$$

for any ω satisfying $\lambda \times \omega = 0$ on $\partial\Omega$. Indeed, fix l and assume without loss of generality that $k(l) = 1$. Then we obtain from the boundary

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condition on ω that $\lambda_1\omega_i - \lambda_i\omega_1 = 0$ for $i = 1, \ldots, n$ on $\partial\Omega$. Thus we first obtain that for $i = 1, \ldots, n$

$$\omega_i = \mu_i\omega_1 \quad \text{and} \quad \text{where } \mu_i = \frac{\lambda_i}{\lambda_1} \in C^{0,1}(\overline{S_i}).$$

This gives, using (3.1) that on S_l, we have

$$(-\omega_i\partial_{ij}[\omega_j] + \omega_j\partial_{ij}[\omega_i]) = -\omega_1^2\left(\mu_i\partial_{ij}[\mu_j] - \mu_j\partial_{ij}[\mu_i]\right).$$

From this indentity, we obtain (3.4).

Step 3. From (3.2), (3.3) and (3.4) it follows that

$$\int_\Omega (|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 - |\nabla\omega|^2) \geqslant -C_1\int_{\partial\Omega}|\omega|^2$$

for some constant $C_1 = C_1(\Omega, \lambda) > 0$. We now recall that there exists a constant $C_2 = C_2(\Omega)$ such that (see for instance [**21**] theorem 1.5.1.10 or [**13**] proposition 5.15) for any $0 < \epsilon < 1$

$$\int_{\partial\Omega}|\omega|^2 \leqslant \epsilon\int_\Omega|\nabla\omega|^2 + \frac{C_2}{\epsilon}\int_\Omega|\omega|^2.$$

Choose ϵ such that $\epsilon C_1 \leqslant 1/2$ and then the theorem follows. □

We have used in the proof of Proposition 3.4, in the case $n = 2$, the following lemma. In this case one cannot prove (3.2) for Lipschitz vectors by approximation, since the standard convolution by some smoothing kernels $\{\eta_k\}_{k\in\mathbb{N}}$ in the whole space does not imply any kind of convergence of $\{\eta_k * \partial\omega_i/\partial x_j\}_{k\in\mathbb{N}}$ on $\partial\Omega$ to the required function.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open Lipschitz set with unit outward normal ν and assume that $\omega_1, \omega_2 \in W^{1,\infty}(\Omega)$. Then the following identity holds*

$$\int_{\partial\Omega}\omega_1\left(\frac{\partial\omega_2}{\partial x_2}\nu_1 - \frac{\partial\omega_2}{\partial x_1}\nu_2\right) = \int_\Omega\left(\frac{\partial\omega_1}{\partial x_1}\frac{\partial\omega_2}{\partial x_2} - \frac{\partial\omega_2}{\partial x_1}\frac{\partial\omega_1}{\partial x_2}\right). \tag{3.5}$$

Proof.

Step 1. Clearly, (3.5) holds true for $(\omega_1, \omega_2) \in C^2(\overline{\Omega})^2$, by partial integration. Let us first show that (3.5) holds true if $\omega_1 \in C^2(\overline{\Omega})$ and ω_2 is Lipschitz. Let us first assume that $\partial\Omega$ is connected and hence there exists a Lipschitz curve φ and some interval $[0, L]$ such that

$$\varphi : [0, L] \to \partial\Omega, \quad \varphi(0) = \varphi(L) \tag{3.6}$$

is a parametrization of $\partial\Omega$. We obtain that $\omega_2 \circ \varphi \in W^{1,\infty}([0, L])$, as it is the composition of two Lipschitz functions, and it is differentiable

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almost everywhere in $[0, L]$ with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\omega_2 \circ \varphi)(t) &= \frac{\partial \omega_2}{\partial x_1}(\varphi(t))\varphi_1'(t) + \frac{\partial \omega_2}{\partial x_2}(\varphi(t))\varphi_2'(t) \\ &= \left(\frac{\partial \omega_2}{\partial x_1}\nu_1 - \frac{\partial \omega_2}{\partial x_2}\nu_2\right)(\varphi(t))|\varphi'(t)|. \end{aligned} \tag{3.7}$$

We have assumed here that φ turns around the domain counterclockwise. Thus we obtain, using that $\varphi(0) = \varphi(L)$, $\omega_1 \in C^2(\overline{\Omega})$ (and hence its second derivatives commute)

$$\begin{aligned}\int_{\partial\Omega} \omega_1 \left(\frac{\partial \omega_2}{\partial x_2}\nu_1 - \frac{\partial \omega_2}{\partial x_1}\nu_2\right) &= \int_0^L \omega_1(\varphi(t))\frac{\mathrm{d}}{\mathrm{d}t}\left[\omega_2(\varphi(t))\right] \mathrm{d}t \\ &= -\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}\left[\omega_1(\varphi(t))\right]\omega_2(\varphi(t))\,\mathrm{d}t \\ &= -\int_{\partial\Omega} \omega_2 \left(\frac{\partial \omega_1}{\partial x_2}\nu_1 - \frac{\partial \omega_1}{\partial x_1}\nu_2\right) \\ &= \int_{\Omega} \left(\frac{\partial \omega_1}{\partial x_1}\frac{\partial \omega_2}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1}\frac{\partial \omega_1}{\partial x_2}\right).\end{aligned}$$

This proves the claim of the present step, in case $\partial\Omega$ is connected. If $\partial\Omega$ is not connected then we first show that on each connected component S_i of $\partial\Omega$ ($i = 1, \ldots, K$ for some $K \in \mathbb{N}$)

$$\int_{S_i} \omega_1 \left(\frac{\partial \omega_2}{\partial x_2}\nu_1 - \frac{\partial \omega_2}{\partial x_1}\nu_2\right) = -\int_{S_i} \omega_2 \left(\frac{\partial \omega_1}{\partial x_2}\nu_1 - \frac{\partial \omega_1}{\partial x_1}\nu_2\right),$$

as before, taking periodic paramterizations φ_i of S_i. Then we take the sum over these integrals and can proceed in the same way. This proves the claim of Step 1.

Step 2. Let us now assume that ω_1, ω_2 are both Lipschitz. Take a sequence $\{\omega_1^k\} \in C^\infty(\overline{\Omega})$, $k \in \mathbb{N}$, such that

$$\omega_1^k \to \omega \quad \text{in } W^{1,2}(\Omega) \quad \text{for } k \to \infty.$$

By Step 1, we have for each k

$$\int_{\partial\Omega} \omega_1^k \left(\frac{\partial \omega_2}{\partial x_2}\nu_1 - \frac{\partial \omega_2}{\partial x_1}\nu_2\right) = \int_{\Omega} \left(\frac{\partial \omega_1^k}{\partial x_1}\frac{\partial \omega_2}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1}\frac{\partial \omega_1^k}{\partial x_2}\right).$$

By the trace theorem $\omega_1^k \to \omega_1$ in $L^2(\partial\Omega)$, and by (3.7)

$$\left(\frac{\partial \omega_2}{\partial x_2}\nu_1 - \frac{\partial \omega_2}{\partial x_1}\nu_2\right) \in L^\infty(\partial\Omega).$$

So by letting $k \to \infty$, we obtain (3.5). □

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We split the second proof of Propostion 3.4 (which requires λ to be $C^{1,1}$) into several intermediate steps. We first recall the definition of the pushforward of a vector field.

Definition 3.7. Let $U, V \subset \mathbb{R}^n$ be two open sets and $\Phi \in \mathrm{Diff}^1(U; V)$. Then for any $\omega \in C(U)^n$ we define its pushforward $\Phi_*(\omega) \in C(V)^n$ by

$$\Phi_*(\omega)(x) = \nabla\Phi(\Phi^{-1}(x))\, \omega(\Phi^{-1}(x)),$$

where $A\, b$ is the usual multiplication of a (column) vector b by a matrix A.

We will use several times the following elementary properties: $(\Phi \circ \Psi)_*(\omega) = \Phi_*(\Psi_*(\omega))$ and

$$\alpha \times \beta = 0 \quad \text{at } x \quad \Leftrightarrow \quad \Phi_*(\alpha) \times \Phi_*(\beta) = 0 \quad \text{at } \Phi(x). \tag{3.8}$$

The proof of the next lemma is a straightforward algebraic calculation. The analogous result for the pullback of general k-forms can be found in [**10**], lemma B.13. However, in the present case of vector fields, the proof is much simpler. $O(n)$ shall denote the set of orthogonal matrices.

Lemma 3.8. *Let $U, V \subset \mathbb{R}^n$ be open sets, $A \in O(n)$, $b \in \mathbb{R}^n$, and $\psi : U \to V = \psi(U)$ defined by $\psi(u) = Au + b$. Then for all $\omega \in C^{0,1}(U)^n$ and almost every $u \in U$ the following three identities hold true:*

$$|\nabla\omega(u)|^2 = |\nabla(\psi_*(\omega))|^2(\psi(u)) \tag{3.9}$$

$$|\operatorname{curl}\omega(u)|^2 = |\operatorname{curl}(\psi_*(\omega))|^2(\psi(u)) \tag{3.10}$$

$$|\operatorname{div}\omega(u)|^2 = |\operatorname{div}(\psi_*(\omega))|^2(\psi(u)). \tag{3.11}$$

Remark 3.9. This lemma holds true by the specific algebraic properties of ∇, curl and div and is not valid in general for an arbitrary linear combination of derivatives of ω. In case of div, we have actually something stronger: $\operatorname{div}\omega(u) = \operatorname{div}(\psi_*(\omega))(\psi(u))$ for any invertible matrix A.

Proof. We first prove (3.9). Let a_{ij} denote the entries of the matrix A. Since $A^t = A^{-1}$, we have that for any $k, l = 1, \ldots, n$,

$$\sum_{i=1}^{n} a_{ik}a_{il} = \delta_{kl}. \tag{3.12}$$

We can assume that $b = 0$. Let $x = \psi(u) = Au$, and hence $\psi_*(\omega)(x) = A\omega(A^t x)$. So the components of $\psi_*(\omega)$, respectively, their derivatives are

$$(\psi_*(\omega))_i\,(x) = \sum_{k=1}^{n} a_{ik}\omega_k(A^t x) \quad \text{and} \quad \frac{\partial\, (\psi_*(\omega))_i}{\partial x_j}(x) = \sum_{k,l=1}^{n} a_{ik}a_{jl}\frac{\partial \omega_k}{\partial u_l}(A^t x).$$

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We, therefore, obtain

$$|\nabla\psi_*(\omega)|^2(x) = \sum_{i,j=1}^n \left(\sum_{k,l=1}^n a_{ik}a_{jl}\frac{\partial\omega_k}{\partial u_l}(u)\right)^2$$
$$= \sum_{i,j=1}^n \sum_{k,l=1}^n \sum_{r,s=1}^n a_{ik}a_{jl}a_{ir}a_{js}\frac{\partial\omega_k}{\partial u_l}(u)\frac{\partial\omega_r}{\partial u_s}(u).$$

Using now (3.12) gives the desired result. To prove (3.10), we use that

$$|\operatorname{curl}(\psi_*(\omega))|^2 = \sum_{i<j}\left(\frac{\partial\left(\psi_*(\omega)\right)_j}{\partial x_i} - \frac{\partial\left(\psi_*(\omega)\right)_i}{\partial x_j}\right)^2$$
$$= \frac{1}{2}\sum_{i,j=1}^n\left(\frac{\partial\left(\psi_*(\omega)\right)_j}{\partial x_i} - \frac{\partial\left(\psi_*(\omega)\right)_i}{\partial x_j}\right)^2$$
$$= \frac{1}{2}\sum_{i,j=1}^n\left(\sum_{k,l=1}^n a_{ik}a_{jl}\left(\frac{\partial\omega_k}{\partial u_l} - \frac{\partial\omega_l}{\partial u_k}\right)\right)^2$$

and proceed as in the proof of (3.9). The proof of (3.11) is very similar. □

We start proving proposition 3.4 in a special case.

Lemma 3.10. *Let* $\Omega\subset\mathbb{R}^n$ *be a bounded open Lipschitz set and let* $\lambda\in\mathbb{R}^n$ *be a nonzero constant vector. Then the equality*

$$\int_\Omega |\nabla\omega|^2 = \int_\Omega (|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2)$$

holds true for all $\omega\in \mathrm{C}^1(\overline{\Omega})^n$ *which satisfy* $\lambda\times\omega = 0 \quad$ *on* $\partial\Omega$.

Remark 3.11. We will only use this lemma for $\lambda = e_1$, and will, therefore, only prove that case. The result for general λ follows easily from this particular case, lemma 3.8 and (3.8).

Proof. As remarked, we only prove the lemma in the case when $\lambda = e_1 = (1,0,\ldots,0)$. In this case, the boundary condition $\lambda\times\omega = 0$ is equivalent with $\omega_2 = \cdots = \omega_n = 0$ on $\partial\Omega$. Recall that, see (3.2),

$$\int_\Omega \left(|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 - |\nabla\omega|^2\right) = -\sum_{i<j}\int_{\partial\Omega}\omega_i\partial_{ij}[\omega_j] + \sum_{i<j}\int_{\partial\Omega}\omega_j\partial_{ij}[\omega_i].$$

The right-hand side of the previous equality cancels since for any $i\neq j$, $\omega_i\partial_{ij}[\omega_j]$ is pointwise zero on $\partial\Omega$: indeed, either $\omega_i = 0$ on $\partial\Omega$ or

$$\partial_{ij}[\omega_j] = \frac{\partial\omega_j}{\partial x_i}\nu_j - \frac{\partial\omega_j}{\partial x_j}\nu_i = 0 \quad \text{on } \partial\Omega$$

if $\omega_j = 0$ on $\partial\Omega$ recalling that $\partial_{ij}[\omega_j]$ is a tangential derivative. □

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The main statement of the next lemma (Part (ii)), states that the change of the L^2 norms of $\nabla\omega$, $\operatorname{curl}\omega$ and $\operatorname{div}\omega$ under the pushforward of Φ can be estimated appropriately, if $\nabla\phi \in SO(n)$ at some point and if a neighbourhood is taken small enough near that point.

Lemma 3.12. *Let $x_0 \in \mathbb{R}^n$ and λ be a $C^{1,1}$ vector field defined in a neighbourhood of x_0, such that $|\lambda(x_0)| = 1$.*

(i) *Then there exist open sets $O, W \subset \mathbb{R}^n$, $x_0 \in O$, $0 \in W$, and a diffeomorphism $\Phi \in \operatorname{Diff}^{1,1}(\overline{O};\overline{W})$ such that $\Phi(x_0) = 0$,*

$$\Phi_*(\lambda) = e_1 \quad \text{in } W \quad \text{and} \quad \nabla\Phi(x_0) \in SO(n).$$

(ii) *Moreover, for any $0 < \epsilon \leqslant 1$, up to taking O and W smaller, there exists a constant $C = C(\Phi)$ satisfying the following three inequalities:*

$$\left|\int_O |\nabla\omega|^2 - \int_W |\nabla(\Phi_*(\omega))|^2\right| \leqslant \epsilon \int_O |\nabla\omega|^2 + \frac{C}{\epsilon}\int_O |\omega|^2 \tag{3.13}$$

$$\left|\int_O |\operatorname{curl}\omega|^2 - \int_W |\operatorname{curl}(\Phi_*(\omega))|^2\right| \leqslant \epsilon \int_O |\nabla\omega|^2 + \frac{C}{\epsilon}\int_O |\omega|^2 \tag{3.14}$$

$$\left|\int_O |\operatorname{div}\omega|^2 - \int_W |\operatorname{div}(\Phi_*(\omega))|^2\right| \leqslant \epsilon \int_O |\nabla\omega|^2 + \frac{C}{\epsilon}\int_O |\omega|^2 \tag{3.15}$$

for all $\omega \in C^{0,1}(\overline{O})^n$.

Remark 3.13. The proof will actually show that (3.13)–(3.15) remain valid with the same constant C replacing O by any of its own open subsets V and replacing W with $U = \Phi(V)$.

Proof. Without loss of generality, we can assume that $x_0 = 0$.

Step 1. We first prove (i). Let $\tilde{\Psi}(t, x)$ be the solution of

$$\frac{\partial\tilde{\Psi}}{\partial t} = \lambda(\tilde{\Psi}) \quad \text{and} \quad \tilde{\Psi}(0, x) = Ax,$$

where $A \in SO(n)$ is such that its first column is equal to $\lambda(x_0)$. Then define $\Psi(x) = \tilde{\Psi}(x_1, 0, x_2, \ldots, x_n)$. It can be easily verified that $\Phi = \Psi^{-1}$ has all the desired properties.

Step 2. We now prove (ii). We will only do the proof for (3.13). The proof for (3.14) and (3.15) is very similar. Let $\Phi \in \operatorname{Diff}^{1,1}\left(\overline{O}, \overline{W}\right)$ be as in (i) and $\Psi = \Phi^{-1}$. Throughout the proof C_1, C_2, C_3 and C_4 will denote

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constants depending only on Φ. Let us write

$$\begin{aligned}\nabla(\Phi_*(\omega))(y) &= \nabla((\nabla\Phi\circ\Psi)(\omega\circ\Psi))(y)\\ &= \sum_{k=1}^{n} S^k(\Phi,y)\omega_k(\Psi(y)) + \nabla\Phi(\Psi(y))\,\nabla\omega(\Psi(y))\,\nabla\Psi(y),\end{aligned}$$

where $S^k(\Phi,y)$, $k=1,\ldots,n$, are matrix valued functions depending only on derivatives of at most second order of Φ. Its entries shall be donoted by $S^k_{ij}(\Phi,y)$. So we have

$$|\nabla(\Phi_*(\omega))|^2 = D+E+F, \tag{3.16}$$

where

$$\begin{aligned}D(y) &= \sum_{i,j=1}^{n}(\nabla\Phi(\Psi(y))\,\nabla\omega(\Psi(y))\,\nabla\Psi(y))^2_{ij},\\ F(y) &= \sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} S^k_{ij}(\Phi,y)\omega_k(\Psi(y))\right)^2\\ E(y) &= 2\sum_{i,j,k=1}^{n} S^k_{ij}(\Phi,y)\omega_k(\Psi(y))(\nabla\Phi(\Psi(y))\,\nabla\omega(\Psi(y))\,\nabla\Psi(y))_{ij}\end{aligned}$$

Fix $0<\epsilon\leqslant 1$. Using the inequality $2ab\leqslant a^2/\epsilon + b^2\epsilon$ and the fact that Φ is $C^{1,1}$, one immediately obtains

$$\begin{gathered}E(y)\leqslant C_1\epsilon|\nabla\omega|^2(\Psi(y)) + \frac{C_1}{\epsilon}|\omega|^2(\Psi(y)) \quad \text{and}\\ F(y)\leqslant C_2|\omega|^2(\Psi(y)) \quad \text{for all } y\in\overline{O}.\end{gathered}$$

Changing the variables we, therefore, get

$$\begin{aligned}\int_W E &\leqslant \int_O \left(C_1\epsilon|\nabla\omega|^2(x) + \frac{C_1}{\epsilon}|\omega|^2(x)\right)\det\nabla\Phi(x)\mathrm{d}x\\ &\leqslant \int_O\left(C_3\epsilon|\nabla\omega|^2 + \frac{C_3}{\epsilon}|\omega|^2\right)\end{aligned} \tag{3.17}$$

and similarly

$$\int_W F \leqslant \int_O C_4|\omega|^2. \tag{3.18}$$

Combining (3.16), (3.17) and (3.18) it is enough to estimate

$$\left|\int_W D - \int_O |\nabla\omega|^2\right|$$

to prove (3.13). By the change of variables formula, we get

$$\int_W D = \int_O |\nabla\Phi(x)\,\nabla\omega(x)\,(\nabla\Phi(x))^{-1}|^2\det\nabla\Phi(x)\mathrm{d}x.$$

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Thus

$$\int_W D - \int_O |\nabla\omega|^2 = \int_O \Big(\big|\nabla\Phi(x)\,\nabla\omega(x)\,(\nabla\Phi(x))^{-1}\big|^2 \det\nabla\Phi(x) - \big|\nabla\Phi(0)\,\nabla\omega(x)\,(\nabla\Phi(0))^{-1}\big|^2\Big)\,\mathrm{d}x + \int_O \Big(\big|\nabla\Phi(0)\,\nabla\omega(x)\,(\nabla\Phi(0))^{-1}\big|^2 - |\nabla\omega(x)|^2\Big)\,\mathrm{d}x.$$

It follows from (3.9) that the integrand in the second integral of the right-hand side of the previous equation is pointwise 0 in O. To see this, fix $x \in O$, set $A = \nabla\Phi(0) \in SO(n)$ and apply the map $\psi(u) = Au$ to lemma 3.8: then (3.9) evaluated at $u = x$ gives

$$|\nabla\omega(x)|^2 = |\nabla(\psi_*(\omega))|^2(\psi(x)) = |A\,\nabla\omega(x)\,A^t|^2 = |\nabla\Phi(0)\,\nabla\omega(x)\,(\nabla\Phi(0))^{-1}|^2.$$

Hence, recalling that $\det\nabla\Phi(0) = 1$, it follows from continuity of $\nabla\Phi$ that, taking O smaller (and consequently W as well) if necessary, that

$$\left|\int_W D - \int_O |\nabla\omega|^2\right| \leqslant \epsilon \int_O |\nabla\omega|^2.$$

This concludes the proof of the lemma since the estimates on E and F remain valid for the new smaller open sets O and W with the same constants C_1, C_2, C_3 and C_4. □

We now prove proposition 3.4 in the special case when the vector fields ω have compact support in a sufficiently small neighbourhood of a boundary point $x_0 \in \partial\Omega$.

Lemma 3.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set and $\lambda \in C^{1,1}(\partial\Omega)^n$ be such that $\lambda \neq 0$ on $\partial\Omega$ and assume that $x_0 \in \partial\Omega$. Then there exists an open set $O \in \mathbb{R}^n$, $x_0 \in O$ and a constant $C = C(\Omega, O, \lambda)$ such that*

$$\int_V |\nabla\omega|^2 \leqslant C \int_V (|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 + |\omega|^2),$$

where $V = \Omega \cap O$, for all $\omega \in \mathrm{C}^1(\overline{O})$ which satisfy

$$\lambda \times \omega = 0 \quad \text{on } \partial\Omega \qquad \text{and} \qquad \operatorname{supp}(\omega) \subset O.$$

Proof. The proof follows from lemmas 3.10 and 3.12. With no loss of generality, we can assume that $|\lambda(x_0)| = 1$. We claim that O given by lemma 3.12 will have the desired property and we shall use the notation of that lemma. If $\lambda \times \omega = 0$ on $\partial\Omega$,

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we get that, using (3.8),

$$e_1 \times \Phi_*(\omega) = 0 \quad \text{on } \Phi(\partial\Omega \cap O)$$

Also, since ω has compact support in O, then $\Phi^*(\omega)$ has compact support in $\Phi(O)$. We conclude that setting $U = \Phi(V)$,

$$e_1 \times \Phi_*(\omega) = 0 \quad \text{all over } \partial U$$

for any ω satisfying the assumptions of the lemma. We thus conclude from lemma 3.10 that

$$\int_U |\nabla(\Phi_*(\omega))|^2 = \int_U (|\operatorname{curl}(\Phi_*(\omega))|^2 + |\operatorname{div}(\Phi_*(\omega))|^2).$$

Finally using (3.13)–(3.15) (and remark 3.13) with $\epsilon = 1/6$ and the previous equality, we obtain that

$$\begin{aligned}\int_V |\nabla\omega|^2 &\leqslant \epsilon \int_V |\nabla\omega|^2 + \frac{C}{\epsilon}\int_V |\omega|^2 + \int_U |\nabla(\Phi^*(\omega))|^2 \\ &\leqslant 3\epsilon \int_V |\nabla\omega|^2 + 3\frac{C}{\epsilon}\int_V |\omega|^2 + \int_V (|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2) \\ &= \frac{1}{2}\int_V |\nabla\omega|^2 + 18C\int_V |\omega|^2 + \int_V (|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2).\end{aligned}$$

which proves the lemma. □

We give the second proof of the main proposition under the more restrictive hypothesis that $\lambda \in C^{1,1}(\partial\Omega)^n$.

Second Proof (Proposition 3.4*).* Since $\partial\Omega$ is compact, we can cover it by open neighbourhoods $O_i \subset \mathbb{R}^n$, $i = 1, \ldots, M$ which satisfy the conclusion of lemma 3.14. Moreover, let us choose a further open set $O_0 \subset \overline{O_0} \subset \Omega$ such that $\Omega \subset \cup_{i=0}^M O_i$. Let $\{\xi_i\}_{i=0}^M$ be a partition of unity subordinate to the O_i:

$$0 \leqslant \xi_i \leqslant 1, \quad \operatorname{supp}(\xi_i) \subset O_i \quad \text{and} \quad \sum_{i=0}^M \xi_i = 1 \quad \text{in } \overline{\Omega}.$$

Let now $\omega \in C^1(\overline{\Omega})^n$ be a vector field such that $\lambda \times \omega = 0$ on $\partial\Omega$. Then using lemma 3.10 for $i = 0$, respectively lemma 3.14 for $i = 1, \ldots, M$, we obtain that

$$\int_{V_i} |\nabla(\xi_i\omega)|^2 \leqslant C_i \int_{V_i} \left(|\operatorname{curl}(\xi_i\omega)|^2 + |\operatorname{div}(\xi_i\omega)|^2 + |\xi_i\omega|^2\right), \tag{3.19}$$

for some constants $C_i = C_i(\Omega, \lambda)$, where $V_i = \Omega \cap O_i$. Note that

$$\int_\Omega |\nabla\omega|^2 = \int_\Omega \left|\nabla\left(\sum_{i=0}^M \xi_i\omega\right)\right|^2 \leqslant M\sum_{i=0}^M \int_\Omega |\nabla(\xi_i\omega)|^2 = M\sum_{i=0}^M \int_{V_i} |\nabla(\xi_i\omega)|^2. \tag{3.20}$$

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Thus combining (3.20) and (3.19), one gets

$$\int_\Omega |\nabla\omega|^2 \leqslant C_1 \sum_{i=0}^{M} \int_{V_i} (|\operatorname{curl}\xi_i\omega|^2 + |\operatorname{div}\xi_i\omega|^2 + |\xi_i\omega|^2)$$
$$\leqslant C_2 \int_\Omega (|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 + |\omega|^2),$$

for some constants C_1 and C_2 depending only on Ω and λ. □

To extend proposition 3.4 to H^1 vector fields, we need to show that a vector field $\omega \in H^1(\Omega)^n$ which satisfies $\lambda \times \omega = 0$ on the boundary can be approximated by C^1 vector fields also satisfying the same boundary condition. This is possible according to the next proposition.

Proposition 3.15. *Let $n \geqslant 2$, $r \geqslant 0$ be an integer and $0 \leqslant \alpha \leqslant 1$, with $r + \alpha \geqslant 1$. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open Lipschitz set and $\lambda \in C^{r,\alpha}(\partial\Omega)^n$ be such that*

$$\lambda \neq 0 \quad on\ \partial\Omega.$$

Suppose $\omega \in H^1(\Omega)^n$ is such that $\lambda \times \omega = 0$ on $\partial\Omega$. Then there exists a sequence $\{\omega^k\}_{k\in\mathbb{N}} \subset C^{r,\alpha}(\overline{\Omega})^n$ such that for $k \to \infty$

$$\omega^k \to \omega \in H^1(\Omega)^n \quad and \quad \lambda \times \omega^k = 0 \quad on\ \partial\Omega \quad for\ all\ k.$$

Proof.

Step 1. We first prove the following claim: For every $x_0 \in \partial\Omega$, there exists a neighbourhood $W \subset \mathbb{R}^n$ of x_0 such that for all $\omega \in H^1(\Omega)$ satisfying

$$\operatorname{supp}(\omega) \subset W \quad \text{and} \quad \lambda \times \omega = 0 \quad \text{on } \partial\Omega, \tag{3.21}$$

there exists a sequence $\{\omega^k\}_{k\in\mathbb{N}} \subset C^{r,\alpha}\left(\overline{\Omega \cap W}\right)^n$ such that

$$\omega^k \to \omega \quad \text{in } H^1(\Omega \cap W)^n \quad \text{and} \quad \lambda \times \omega^k = 0 \quad \text{on } \partial\Omega \cap W \text{ for all } k.$$

We extend λ to a $C^{r,\alpha}$ vector field in $\mathbb{R}^n$, see definition 3.1. Since λ does not vanish on the boundary, we can assume with no loss of generality that $\lambda_1 \neq 0$ in $\overline{W}$ where W is a small enough neighbourhood of x_0. Let us define

$$\alpha_i = \lambda_1\omega_i - \lambda_i\omega_1 = (\lambda \times \omega)_{1i}\,.$$

Note that by the additional assumptions (3.21) the support of ω is contained in W and, in particular, vanishes on ∂W. Therefore, $\alpha_i \in$

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$H_0^1(\Omega\cap W)$ and hence there exists a sequence α_i^k with the properties

$$\{\alpha_i^k\}_{k\in\mathbb{N}}\in C_c^\infty(\Omega\cap W),\qquad \alpha_i^k\to\alpha_i\quad\text{in } H^1(\Omega\cap W).$$

Moreover, we choose a sequence $\{\beta^k\}\in C^\infty(\overline{\Omega\cap W})$ such that $\beta^k\to\omega_1$ in $H^1(\Omega\cap W)$. We finally define $\omega^k=(\omega_1^k,\dots,\omega_n^k)$ by

$$\begin{aligned}\omega_1^k&=\beta^k\\ \omega_i^k&=\frac{\alpha_i^k+\lambda_i\beta^k}{\lambda_1}\qquad\text{for } i=2,\dots,n.\end{aligned}$$

Using that $\alpha_i^k=0$ on $\partial\Omega\cap W$, we obtain that for any $i,j\in\{1,\dots,n\}$

$$\lambda_j\omega_i^k-\lambda_i\omega_j^k=\frac{\lambda_j}{\lambda_1}\lambda_i\beta^k-\frac{\lambda_i}{\lambda_1}\lambda_j\beta^k=0\qquad\text{on } \partial\Omega\cap W\,.$$

and thus ω^k has all the desired properties claimed in Step 1.

Step 2. Using that $\partial\Omega$ is compact, we can cover it by a finite number of open sets $W_1,\dots,W_L$ with the properties given by Step 1. Clearly, we can add W_0 such that W_0 is also open, $\overline{\Omega}\subset\bigcup_{l=0}^L W_l$ and any $\omega^0\in H^1(W_0)$ with compact support in W_0 can be approximated by smooth vector fields $\omega^{0,k}$ with compact support in W_0. In particular, $\lambda\times\omega^{0,k}=0$ on $\partial\Omega$ for all k. Let η_l be a smooth partition of unity subordinate to this covering such that

$$\sum_{l=0}^L\eta_l^2=1\quad\text{in } \overline{\Omega}.$$

Define $\omega^l=\eta_l\omega$. Using Step 1 there exists for each $l=1,\dots,L$ sequences $\{\omega^{l,k}\}_{k\in\mathbb{N}}$ of $C^{r,\alpha}$ vector fields such that for $k\to\infty$

$$\omega^{l,k}\to\omega^l\quad\text{in } H^1(\Omega\cap W)\qquad\text{and}\qquad\lambda\times\omega^{l,k}=0\quad\text{on } \partial\Omega\cap W\quad\text{for all } k.$$

Then $\eta_l\omega^{l,k}\in C^{r,\alpha}(\overline{\Omega})^n$ is well defined and $\omega^k=\sum_{l=0}^L\eta_l\omega^{l,k}$ has all the desired properties. □

4. Formulation in $\mathbb{R}^2$ for discontinuous $\boldsymbol{\lambda}$

In two dimensions, we improve theorem 3.2: we no longer require λ to be continuous on the whole boundary, but still Lipschitz on different pieces of $\partial\Omega$. More precisely, we make the following assumption.

Assumption 4.1. Assume that $\Omega\subset\mathbb{R}^2$ is a bounded open Lipschitz set, such that for some integer N

$$\partial\Omega=\bigcup_{i=1}^N\overline{\Gamma}_i\qquad\text{and}\qquad\overline{\Gamma}_i\cap\overline{\Gamma}_{i+1}=\{S_i\}\qquad\text{for } i=1,\dots,N,$$

where Γ_i are disjoint open sets in $\partial\Omega$ (with the convention that $\Gamma_{N+1}=\Gamma_1$) and the S_i are N different points on the boundary, called vertices. Let $\lambda_i\in C^{0,1}(\overline{\Gamma}_i)^2$

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for $i = 1, \dots, N$ and define

$$\lambda : \bigcup_{i=1}^{N} \Gamma_i \to \mathbb{R}^2,$$

by $\lambda^i = \lambda$ on Γ_i. We also assume that

$$\lambda_i \neq 0 \quad \text{on } \overline{\Gamma}_i\,.$$

Note that we allow that at a vertex S_i the segments Γ_i and Γ_{i+1} can meet at an angle π. In this setting, we have the following theorem.

THEOREM 4.2. *Let Ω and λ be as in assumption 4.1. Then there exists a constant $C = C(\Omega, \lambda)$ such that*

$$\|\nabla\omega\|^2_{L^2(\Omega)} \leqslant C \left(\|\operatorname{curl}\omega\|^2_{L^2(\Omega)} + \|\operatorname{div}\omega\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)} \right), \tag{4.1}$$

for all $\omega \in H^1(\Omega)^2$ which satisfy

$$\lambda \times \omega = 0 \quad \text{on } \partial\Omega,$$

where the last equality is understood as $\lambda_i \times \omega = 0$ on Γ_i for each $i = 1, \dots, N$.

EXAMPLE 4.3. As a special case, we obtain Gaffney inequality with the classical boundary conditions in polygonial domains.

The proof of Theorem 4.2 is essentially the same as the corresponding result for globally Lipschitz λ: only the approximation result, that is, the analogy to proposition 3.15 has to be adapted. This is done in the next proposition.

PROPOSITION 4.4. *Let Ω and λ be as in assumption 4.1. Suppose $\omega \in H^1(\Omega)^2$ is such that $\lambda \times \omega = 0$ on $\partial\Omega$. Then there exists a sequence $\{\omega^k\}_{k\in\mathbb{N}} \subset C^{0,1}(\overline{\Omega})^2$ such that for $k \to \infty$*

$$\omega^k \to \omega \in H^1(\Omega)^2 \quad \text{and} \quad \lambda \times \omega^k = 0 \quad \text{on } \partial\Omega \quad \text{for all } k.$$

Proof.

Step 1. We first prove the following claim: For every $x_0 \in \partial\Omega$, there exists a neighbourhood $W \subset \mathbb{R}^2$ of x_0 such that for all $\omega \in H^1(\Omega)$ satisfying

$$\operatorname{supp}(\omega) \subset W \quad \text{and} \quad \lambda \times \omega = 0 \quad \text{on } \partial\Omega, \tag{4.2}$$

there exists a sequence $\{\omega^k\}_{k\in\mathbb{N}} \subset C^{0,1}(\overline{\Omega \cap W})^2$ such that

$$\omega^k \to \omega \quad \text{in } H^1(\Omega \cap W)^2 \quad \text{and} \quad \lambda \times \omega^k = 0 \quad \text{on } \partial\Omega \cap W \quad \text{for all } k.$$

The proof of this claim is the same as the proof of Proposition 3.15 if x_0 is not a vertex and so we can assume that $x_0 = \overline{\Gamma}_i \cap \overline{\Gamma}_{i+1}$ is a vertex. Then we distinguish two cases.

Case 1. We assume that $\lambda^i(x_0)$ and $\lambda^{i+1}(x_0)$ are linearly dependent. Since the boundary condition $\lambda \times \omega = 0$ is invariant under scaling or sign change of λ, and neither λ^i nor λ^{i+1} vanish, we can assume that $\lambda^i(x_0) = \lambda^{i+1}(x_0)$. But then λ is Lipschitz in $\overline{\Gamma}_i \cup \overline{\Gamma}_{i+1}$ and thus we can again proceed as in proposition 3.15.

Case 2. We assume that $\det(\lambda^i(x_0)|\lambda^{i+1}(x_0)) \neq 0$. In this case, we extend both λ^i and λ^{i+1} separately to $C^{0,1}$ vector fields defined in $\mathbb{R}^2$. By continuity, there exists a neighbourhood W of x_0 such that $\det(\lambda^i|\lambda^{i+1}) \neq 0$ in $\overline{W}$. Let ω be a vector field satisfying (4.2). Define the two functions

$$p = \lambda^i \times \omega \in H^1(\Omega), \quad \text{and} \quad q = \lambda^{i+1} \times \omega \in H^1(\Omega),$$

which can also be written in the matrix form

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\lambda_2^i & \lambda_1^i \\ -\lambda_2^{i+1} & \lambda_1^{i+1} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = M \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

By an extension theorem (see Bernard [**3**] or theorem 1.6.1 of Grisvard [**22**] for polygonial domains) there exists sequences $\{p^k\}_{k\in\mathbb{N}}, \{q^k\}_{k\in\mathbb{N}} \in C^1(\overline{\Omega})$ such that both p^k (respectively q^k) converges to p (respectively q) in $H^1(\Omega)$ and

$$p^k = 0 \quad \text{on } \Gamma_i \qquad \text{and} \qquad q^k = 0 \quad \text{on } \Gamma_{i+1} \quad \text{for all } k \in \mathbb{N}.$$

Since $\det(\lambda^i|\lambda^{i+1}) \neq 0$ on $\overline{W}$, we can define $\omega^k \in C^{0,1}(\overline{W \cap \Omega})$ by

$$\omega^k = M^{-1} \begin{pmatrix} p^k \\ q^k \end{pmatrix}.$$

Note that $\lambda^i \times \omega^k = p^k$, respectively, $\lambda^{i+1} \times \omega^k = q^k$. It straightforward to check that ω^k has all the desired properties claimed by Step 1.

Step 2. We finally conclude exactly as in Step 2 of the proof of Proposition 3.15.

□

We now prove the main theorem of this section.

Proof of Theorem 4.2. Since Ω is a Lipschitz domain, we can use partial integration and obtain

$$\begin{aligned}
&\int_\Omega (|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 - |\nabla\omega|^2) \\
&= \int_{\partial\Omega} \omega_1 \left(\nu_1 \frac{\partial\omega_2}{\partial x_2} - \nu_2 \frac{\partial\omega_2}{\partial x_1}\right) - \int_{\partial\Omega} \omega_2 \left(\nu_1 \frac{\partial\omega_1}{\partial x_2} - \nu_2 \frac{\partial\omega_1}{\partial x_1}\right) \\
&= \sum_{i=1}^N \left[\int_{\Gamma_i} \omega_1 (\nu_1 \frac{\partial\omega_2}{\partial x_2} - \nu_2 \frac{\partial\omega_2}{\partial x_1}) - \int_{\Gamma_i} \omega_2 \left(\nu_1 \frac{\partial\omega_1}{\partial x_2} - \nu_2 \frac{\partial\omega_1}{\partial x_1}\right)\right]
\end{aligned}$$

that holds for any $\omega \in C^{0,1}(\overline{\Omega})^2$, where the first equality is exactly as in Step 1 of the first proof of Proposition 3.4). We now proceed as in Step 2 of the first proof

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of Proposition 3.4, working on each Γ_i separately: Using that each Γ_i is a $C^{0,1}$ curve and that λ^i does not vanish on Γ_i, one obtains that there exists a constant $C_1 = C_1(\Omega, \lambda) > 0$ such that

$$\int_\Omega \left(|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 - |\nabla\omega|^2\right) \geqslant -C_1 \int_{\partial\Omega} |\omega|^2$$

for all $\omega \in C^{0,1}(\overline{\Omega})^2$ satisfying $\lambda \times \omega = 0$ on $\partial\Omega$. This proves the theorem for $C^{0,1}$ vector fields ω. The general case follows from proposition 4.4 □

5. Counterexamples

In view of theorems 3.2, 4.2 and the classical boundary condition $\langle \nu; \omega \rangle = 0$, one could expect that if $n \geqslant 3$, we also have a Gaffney inequality under the boundary condition

$$\langle \lambda; \omega \rangle = 0 \quad \text{on } \partial\Omega,$$

if λ does not vanish on $\partial\Omega$. This is, however, not true as shown by the following simple example.

Example 5.1. Let $\Omega \subset \mathbb{R}^3$ be any bounded open smooth set and $\lambda = (0, 0, 1)$. Then there exists no constant $C = C(\Omega, \lambda)$ such that

$$\int_\Omega |\nabla\omega|^2 \leqslant C \int_\Omega \left(|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 + |\omega|^2\right)$$

for all $\omega \in C^2(\overline{\Omega}; \mathbb{R}^n)$ satisfying $\langle \lambda; \omega \rangle = 0$ on $\partial\Omega$. To see this, take

$$\omega(x) = (e^{nx_1}\cos(nx_2), -e^{nx_1}\sin(nx_2), 0).$$

Then one easily verifies that $\operatorname{div}\omega = 0$, $\operatorname{curl}\omega = 0$, $|\nabla\omega(x)|^2 = 2n^2 e^{2nx_1}$ and $|\omega(x)|^2 = e^{2nx_1}$. Hence, as in (3), Gaffney inequality cannot hold.

The question also arises whether theorem 3.2 generalizes to differential forms of a higher order (identifying vector fields with 1-forms). This is also not true. More precisely, we have the following counterexample for 2-forms.

Example 5.2. Let $n \geqslant 3$, $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Then there exists no constant $C = C(\Omega)$ such that

$$\int_\Omega |\nabla\omega|^2 \leqslant C \int_\Omega \left(|\mathrm{d}\omega|^2 + |\delta\omega|^2 + |\omega|^2\right)$$

for all $\omega \in C^2(\overline{\Omega}; \Lambda^2)$ such that $dx^3 \wedge \omega = 0$ on $\partial\Omega$. To see this, take

$$\omega = e^{nx_1}\cos(nx_2)dx^1 \wedge dx^3 + e^{nx_1}\sin(nx_2)dx^2 \wedge dx^3.$$

One can verify that $d\omega = 0$ and $\delta\omega = 0$. Thus one concludes exactly as in example 5.1.

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