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A new proof of the characterization of the weighted Hardy inequality

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Maz'ja and Sinnamon proved a characterization of the boundedness of the Hardy operator from $L^p(v)$ into $L^q(w)$ in the case $0 < q < p$, $1 < p < \infty$. We present here a new simple proof of the sufficiency part of that result.

Maz'ja [2] and Sinnamon [3] characterized the weights such that

$$\left(\int_0^\infty \left(\int_0^x f\right)^q w(x)\,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{1/p} \tag{1}$$

for all measurable $f \geqslant 0$ with a constant independent of f, where $0 < q < p$ and $1 < p < \infty$. The characterization reduces to the condition that the function

$$\Psi(x) = \left(\int_x^\infty w\right)^{1/p}\left(\int_0^x v^{1-p'}\right)^{1/p'}$$

belongs to $L^r(w)$, where $1/r = 1/q - 1/p$. The proof of the necessity in [2] is not direct. However a direct and easy proof can be found in [4]. The proof of the sufficiency in [2] uses the Hölder inequality for three exponents and the characterization of Hardy's inequality for the case $p = q$. The proof of the sufficiency in [4] is simpler but uses Hardy's inequality in the case $p = q$.

The aim of this paper is to present a new proof of the implication $\Psi \in L^r(w) \Rightarrow$ (1) that we believe is simple, elementary and standard because it is more similar to the proofs of the characterization in the easy case $p \leqslant q$ and it does not use the characterization in the case $p = q$. The key point is to show that $\Psi \in L^r(w)$ implies that the function

$$\Phi(x) = \sup_{0<a<x} \Psi(a) = \sup_{0<a<x}\left(\int_a^\infty w\right)^{1/p}\left(\int_0^a v^{1-p'}\right)^{1/p'}$$

belongs also to $L^r(w)$. Once we obtain this result, (1) follows by partitioning $(0,\infty)$ as is usual and by using the Hölder inequality in a natural way. In this way, we obtain a new characterization of (1), namely $\Phi \in L^r(w)$. We also believe that the ideas in the proof could be of some help in overcoming some difficulties appearing in the study of the Hardy's inequalities with $q < p$. In fact, the ideas of this paper are used in [1] to study the generalized Hardy–Steklov operator in weighted spaces.

Next we establish the theorem (due to Maz'ja and Sinnamon) and we give a complete simple proof of the theorem. We remark that the proof of the necessity, (a) $\Rightarrow$ (b), is taken from [4], while the proofs of (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are our contribution. We include the proof of (a) $\Rightarrow$ (b) to present a complete proof of the theorem.

THEOREM 1. *Let w and v be nonnegative measurable functions. Let q, p, p' and r be such that $0 < q < p < \infty$, $1 < p < \infty$, $p + p' = pp'$ and $1/r = 1/q - 1/p$. The following statements are equivalent.*

(a) *There exists a positive constant C such that*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q w(x)\,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{1/p}$$

for all measurable $f \geqslant 0$.

(b) *The function*

$$\Psi(x) = \left(\int_x^\infty w\right)^{1/p}\left(\int_0^x v^{1-p'}\right)^{1/p'}$$

belongs to $L^r(w)$.

(c) *The function*

$$\Phi(x) = \sup_{0<a<x}\left(\int_a^\infty w\right)^{1/p}\left(\int_0^a v^{1-p'}\right)^{1/p'}$$

belongs to $L^r(w)$.

Proof. (b) $\Rightarrow$ (c). We may assume that w and $v^{1-p'}$ are integrable functions (the general case follows by approximating these weights by integrable functions). Observe that if

$$\Phi_1(x) = \sup_{0<a<x}\left(\int_a^x w\right)^{1/p}\left(\int_0^a v^{1-p'}\right)^{1/p'},$$

then

$$\Phi(x) \leqslant \Psi(x) + \Phi_1(x).$$

Therefore, it will suffice to prove that $\Phi_1 \in L^r(w)$. This follows from the inequality

$$w(\{x : \Phi_1(x) > \lambda\}) \leqslant 2w(\{x : \Psi(x) > \lambda\}),$$

which we will now prove. Since

$$w(\{x : \Phi_1(x) > \lambda\}) \leqslant w(\{x : \Psi(x) > \lambda\}) + w(\{x : \Psi(x) \leqslant \lambda < \Phi_1(x)\}),$$

we have only to show that, if $E = \{x : \Psi(x) \leqslant \lambda < \Phi_1(x)\}$, then

$$w(E) \leqslant w(\{x : \Psi(x) > \lambda\}).$$

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To prove this last inequality we have only to establish that

$$\int_x^\infty w \leqslant w(\{y : \Psi(y) > \lambda\}) \tag{2}$$

for all $x \in E$. Let us fix a point $x \in E$. There then exists a, $0 < a < x$, such that

$$\left(\int_x^\infty w\right)^{1/p}\left(\int_0^x v^{1-p'}\right)^{1/p'} \leqslant \lambda < \left(\int_a^x w\right)^{1/p}\left(\int_0^a v^{1-p'}\right)^{1/p'}.$$

These inequalities imply that

$$\int_x^\infty w < \int_a^x w.$$

If $(a, x) \subset \{y : \Psi(y) > \lambda\}$, then (2) follows immediately. Assume that the set $F = \{y \in (a, x) : \Psi(y) \leqslant \lambda\}$ is non-empty and let $y \in F$. Then

$$\left(\int_y^\infty w\right)^{1/p}\left(\int_0^y v^{1-p'}\right)^{1/p'} \leqslant \lambda < \left(\int_a^x w\right)^{1/p}\left(\int_0^a v^{1-p'}\right)^{1/p'}.$$

Since $a < y$, it follows that

$$\int_y^\infty w \leqslant \int_a^x w = \int_a^y w + \int_y^x w.$$

Thus,

$$\int_x^\infty w \leqslant \int_a^y w.$$

for all $y \in F$. If β is the infimum of F, the last inequality implies that

$$\int_x^\infty w \leqslant \int_a^\beta w.$$

This inequality implies that (2) holds, since $(a, \beta) \subset \{y : \Psi(y) > \lambda\}$.

(c) $\Rightarrow$ (a). It will suffice to prove this for integrable functions f. Let us choose a decreasing sequence x_k such that $x_0 = +\infty$ and $\int_0^{x_k} f = 2^{-k}\int_0^\infty f$. By the Hölder inequality with exponents p and p', we see that

$$\begin{aligned}\int_0^\infty\left(\int_0^x f\right)^q w(x)\,\mathrm{d}x &\leqslant \sum_k \int_{x_{k+1}}^{x_k} w(x)\,\mathrm{d}x\left(\int_0^{x_k} f\right)^q\\ &\leqslant 4^q \sum_k \int_{x_{k+1}}^{x_k} w(x)\,\mathrm{d}x\left(\int_{x_{k+2}}^{x_{k+1}} f\right)^q\\ &\leqslant 4^q \sum_k \left(\int_{x_{k+2}}^{x_{k+1}} f^p v\right)^{q/p}\int_{x_{k+1}}^{x_k} w(x)\,\mathrm{d}x\left(\int_{x_{k+2}}^{x_{k+1}} v^{1-p'}\right)^{q/p'}.\end{aligned}$$

Applying the Hölder inequality with exponents p/q and r/q and the definition of Φ, we find that the last term is dominated by

$$
\begin{aligned}
4^q\biggl(\sum_k \int_{x_{k+2}}^{x_{k+1}} f^p v\biggr)^{q/p}\biggl(\sum_k \int_{x_{k+1}}^{x_k} w(x)\,\mathrm{d}x\biggl(\int_{x_{k+1}}^{x_k} w\biggr)^{r/p}\biggl(\int_{x_{k+2}}^{x_{k+1}} v^{1-p'}\biggr)^{r/p'}\biggr)^{q/r}& \\
\leqslant 4^q\biggl(\int_0^\infty f^p v\biggr)^{q/p}\biggl(\sum_k \int_{x_{k+1}}^{x_k} \Phi^r w\biggr)^{q/r}& \\
\leqslant 4^q\biggl(\int_0^\infty f^p v\biggr)^{q/p}\biggl(\int_0^\infty \Phi^r w\biggr)^{q/r}.&
\end{aligned}
$$

(a) $\Rightarrow$ (b). Let w_0 and v_0 be nonnegative integrable functions such that $w_0 \leqslant w$ and $v_0 \leqslant v^{1-p'}$ and let

$$
f(t) = \biggl(\int_t^\infty w_0\biggr)^{r/(pq)}\biggl(\int_0^t v_0\biggr)^{r/(p'q)-1} v_0(t).
$$

Then

$$
\begin{aligned}
\int_0^x f(t)\,\mathrm{d}t &\geqslant \biggl(\int_x^\infty w_0\biggr)^{r/(pq)}\int_0^x\biggl(\int_0^t v_0\biggr)^{r/(p'q)-1} v_0(t)\,\mathrm{d}t \\
&= \frac{p'q}{r}\biggl(\int_x^\infty w_0\biggr)^{r/(pq)}\biggl(\int_0^x v_0\biggr)^{r/(p'q)}.
\end{aligned}
$$

Then, by (a),

$$
\begin{aligned}
\int_0^\infty \biggl(\frac{p'q}{r}\biggr)^q\biggl(\int_x^\infty w_0\biggr)^{r/p}\biggl(\int_0^x v_0\biggr)^{r/p'} w_0(x)\,\mathrm{d}x& \\
\leqslant \int_0^\infty\biggl(\int_0^x f\biggr)^q w(x)\,\mathrm{d}x& \\
\leqslant C^q\biggl(\int_0^\infty f^p v\biggr)^{q/p}& \\
= C^q\biggl(\int_0^\infty\biggl(\int_t^\infty w_0\biggr)^{r/q}\biggl(\int_0^t v_0\biggr)^{r/q'} v_0^p(t)v(t)\,\mathrm{d}t\biggr)^{q/p}& \\
\leqslant C^q\biggl(\int_0^\infty\biggl(\int_t^\infty w_0\biggr)^{r/q}\biggl(\int_0^t v_0\biggr)^{r/q'} v_0(t)\,\mathrm{d}t\biggr)^{q/p}& \\
= C^q\biggl(\frac{p'}{q}\biggr)^{q/p}\biggl(\int_0^\infty\biggl(\int_x^\infty w_0\biggr)^{r/p}\biggl(\int_0^x v_0\biggr)^{r/p'} w_0(x)\,\mathrm{d}x\biggr)^{q/p}&,
\end{aligned}
$$

where in the last equality we have used integration by parts. Since w_0 and v_0 are integrable functions, we find that

$$
\int_0^\infty\biggl(\int_x^\infty w_0\biggr)^{r/p}\biggl(\int_0^x v_0\biggr)^{r/p'} w_0(x)\,\mathrm{d}x \leqslant C^r\biggl(\frac{r}{p'q}\biggr)^r\biggl(\frac{p'}{q}\biggr)^{r/p}.
$$

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Approximating w and $v^{1-p'}$ by increasing sequences of integrable functions we obtain (b). □

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