Canad. Math. Bull. Vol. 63 (2), 2020 pp. 262–268
http://dx.doi.org/10.4153/S0008439519000195

New Facts about the Vanishing Off Subgroup $V(G)$

Nabil Mlaiki

Abstract. In this manuscript, we generalize Lewis's result about a central series associated with the vanishing off subgroup. We write $V_1 = V(G)$ for the vanishing off subgroup of G, and $V_i = [V_{i-1}, G]$ for the terms in this central series. Lewis proved that there exists a positive integer n such that if $V_3 < G_3$, then $|G : V_1| = |G' : V_2|^2 = p^{2n}$. Let $D_3/V_3 = C_{G/V_3}(G'/V_3)$. He also showed that if $V_3 < G_3$, then either $|G : D_3| = p^n$ or $D_3 = V_1$. We show that if $V_i < G_i$ for $i \geqslant 4$, where G_i is the i-th term in the lower central series of G, then $|G_{i-1} : V_{i-1}| = |G : D_3|$.

1 Introduction

Throughout this paper, G is a finite group. We write $\mathrm{Irr}(G)$ for the set of irreducible characters of G and $\mathrm{nl}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) \neq 1\}$. Define the vanishing off subgroup of G, denoted by $V(G)$, by $V(G) = \langle g \in G \mid \text{there exists } \chi \in \mathrm{nl}(G) \text{ such that } \chi(g) \neq 0\rangle$. This subgroup was first introduced by Lewis in [4]. Note that $V(G)$ is the smallest subgroup of G such that all nonlinear irreducible characters vanish on $G \setminus V(G)$. Moreover, $V(G)$ is a proper subgroup only if G is solvable (and of course nonabelian). Let G_i be the i-th term in the lower central series, which is defined by $G_1 = G$, $G_2 = G' = [G, G]$, and $G_i = [G_{i-1}, G]$ for $i \geq 3$. We are going to study a central series associated with the vanishing off subgroup, defined inductively by $V_1 = V(G)$ and $V_i = [V_{i-1}, G]$ for $i \geq 2$. Lewis proved in [4] that $G_{i+1} \leq V_i \leq G_i$. In [4], Lewis showed that when $V_i < G_i$, we have $V_j < G_j$ for all j such that $1 \leq j \leq i$. Also, in [4], Lewis proved that if $V_2 < G_2$, then there exists a prime p such that G_i/V_i is an elementary abelian p-group for all $i \geq 1$. In addition, as shown in Figure 1 he proved that there exists a positive integer n such that if $V_3 < G_3$, then $|G : V_1| = |G' : V_2|^2 = p^{2n}$.

We define some subgroups that are useful to prove our results. First, set $D_3/V_3 = C_{G/V_3}(G'/V_3)$. Lewis proved in [4] that if $V_3 < G_3$, then either $|G : D_3| = \sqrt{|G : V_1|}$ or $D_3 = V_1$. We are able to generalize the results in [4] to the case where $V_i < G_i$ for $i > 3$. Also, we prove that the index of V_{i-1} in G_{i-1} is the same as the index of D_3 in G. To study the case when $i > 3$, we define some more subgroups. For each integer $i \geq 3$, set $Y_i/V_i = Z(G/V_i)$ and $D_i/V_i = C_{G/V_i}(G_{i-1}/V_i)$.

We say G_k is H_1, if for every normal subgroup N of G where $V_k \leq N < G_k$ we have $V_{k-1}/N = G_{k-1}/N \cap Y_k(G/N)$.

Received by the editors March 25, 2019; revised March 28, 2019.
Published online on Cambridge Core August 23, 2019.
AMS subject classification: **20D10**, 20D15.
Keywords: vanishing off subgroup $V(G)$, lower and upper central series.

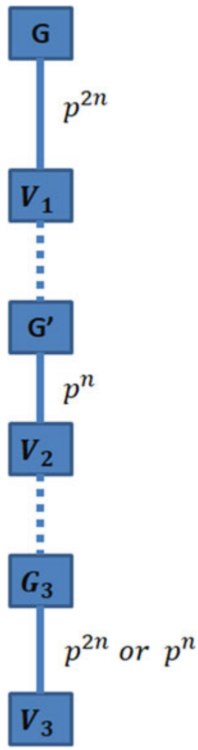

Figure 1: The index of V_1 in G and the index of V_2 in G', n the case where $V_3 < G_3$.

Under the additional hypothesis that G'/V_i is abelian, we are able to show that G_i is H_1 for all $i > 3$. We are also interested in computing the index of V_i in G_i. We will see that this index depends on the size of D_3. In other words, it depends on the size of the centralizer of G' modulo V_i. The following theorem is very useful to prove other results of the paper.

Theorem 1 *Assume that $V_k < G_k$, G'/V_k is abelian, and G_i is H_1 for all $i = 4, \ldots, k$. Then $D_k = D_3$.*

Our second theorem should be considered to be the main result of this paper. We are able to prove that $|G_{i-1} : V_{i-1}| = |G : D_3|$ for every $i \geq 4$, where $V_i < G_i$ and G'/V_i is abelian. Hence, for a nilpotent group of class c, if $V_c < G_c$, and G'/V_c is abelian, then we have $|G_{i-1} : V_{i-1}| = |G : D_3|$ for all $4 \leq i \leq c$ and $|G_c : V_c| \leq |G : D_3|$.

Theorem 2 *Assume that $V_k < G_k$, G'/V_k is abelian, for some $k \geq 3$.*

(a) $|G_{k-1} : V_{k-1}| = |G : D_3|$ *for* $k \geq 4$.
(b) $D_k = D_3$.
(c) G_k *is* H_1.
(d) $|G_k : V_k| \leq |G : D_3|$.

https://doi.org/10.4153/S0008439519000195 Published online by Cambridge University Press

Let G be a finite group. We say that G is a Camina group if the conjugacy class $\mathrm{cl}(x) = xG'$ for every $x \in G \setminus G'$. If $3 \leq i \leq k-1$, then $D_i = D_3$, G_i is H_1, and when $i \geq 4$, $|G_{i-1} : V_{i-1}| = |G : D_3|$. Note that the above result was motivated from the bound of subgroups by MacDonald in [3], where he proved that $|G_3| \leq |G : G'|$ for a Camina group G. Our motivation for adding the hypothesis that G/V_k is abelian is that the results in [3] were under the hypothesis that G is metabelian (*i.e.*, G' is abelian.) Hence, proving this conclusion under a similar metabelian hypothesis seems like a reasonable first step. In the Camina group case, removing the metabelian hypothesis required totally different techniques.

2 General Lemmas

In this section, we prove some lemmas that are useful for the proofs of our theorems. Also, some of these facts give us a good idea about the relation between the lower central series and the central series associated with the vanishing off subgroup that we defined in the introduction. Lewis showed in [4] that both series are related by proving that $V_i \leq G_i \leq V_{i-1}$. We now show that if G_k is H_1, then $V_{k-1} = G_{k-1} \cap Y_k$.

Lemma 2.1 *Assume that $V_k < G_k$. If there exists N such that $V_k \leq N < G_k$ with $V_{k-1}/N = (G_{k-1}/N) \cap Z(G/N)$, then $V_{k-1} = G_{k-1} \cap Y_k$.*

Proof Observe that $Y_k/N \leq Z(G/N)$. We have

$$\begin{aligned} V_{k-1}/N &\leq (Y_k \cap G_{k-1})/N = (Y_k/N) \cap (G_{k-1}/N) \\ &\leq Z(G/N) \cap (G_{k-1}/N) = V_{k-1}/N. \end{aligned}$$

Thus, we obtain equality throughout, and $V_{k-1} = G_{k-1} \cap Y_k$ as desired. ∎

As an immediate consequence, note that if G_k is H_1, then $V_{k-1} = G_{k-1} \cap Y_k$. This next lemma is well known.

Lemma 2.2 *If G is nilpotent and $|G_i| = p$, then for every $x \in G_{i-1} \setminus (G_{i-1} \cap Y_i)$, we have $\mathrm{cl}(x) = xG_i$.*

Proof Because G is nilpotent, we can write $G = P \times Q$, where P is a p-group and Q is a p'-group. Hence, $G_{i-1} = P_{i-1} \times Q_{i-1}$. As $|G_i| = p$, we have $G_i = P_i$. In particular, $Q_{i-1} \leq Z(G)$. Observe that G_{i-1}/G_i is central in G/G_i. Thus, it follows that $\mathrm{cl}(x) \subseteq xG_i$. We deduce that $|\mathrm{cl}(x)| \leq p$. Recall that $x \in G_{i-1} \setminus Y_i$, which implies that $Q \leq C_G(x)$. Now, $|\mathrm{cl}(x)| = |G : C_G(x)|$ divides $|G : Q| = |P|$. Therefore, $|\mathrm{cl}(x)|$ is either 1 or p. Since x is not central, we must have $|\mathrm{cl}(x)| = p = |xG_i|$. We conclude that $\mathrm{cl}(x) = xG_i$. ∎

Now we get a relationship between the central series associated with the vanishing off subgroup of the whole group and a quotient group of that group.

Lemma 2.3 *Assume that $V_k < G_k$, for some $k \geq 3$. Then for every normal subgroup $N < G_k$ we have $V_i(G/N) = V_i/N$ for every $2 \leq i \leq k$.*

https://doi.org/10.4153/S0008439519000195 Published online by Cambridge University Press

Proof We prove this by induction. In Lemma 2.2 in [4], we have $V_1(G/V_2)=V(G)/V_2$. Let $X/N = V(G/N)$. By Lemma 3.3 in [4], $X \leq V(G)$. On the other hand, V_2/N is normal in G/N. By Lemma 3.3 in [4] applied to G/N, we have $V(G)/V_2 = V_1(G/V_2) = V_1\big((G/N)/(V_2/N)\big) \leq V(G/N)/(V_2/N) = (X/N)/(V_2/N) \cong X/V_2$. So, $V(G) \leq X$. We deduce that $X = V(G)$, and $V_2(G/N) = V_2/N$. This is the initial case of the induction. Now suppose that $i > 2$ and assume that $V_{i-1}(G/N) = V_{i-1}/N$. Therefore, $V_i(G/N) = [V_{i-1}(G/N), G/N] = [V_{i-1}/N, G/N] = [V_{i-1}, G]N/N = V_i/N$ as desired. ∎

Now we see the importance of the H_1 hypothesis.

Lemma 2.4 *If $V_i = 1$ and G_i is H_1, then for every $x \in G_{i-1} \setminus V_{i-1}$ we have* $\mathrm{cl}(x) = xG_i$.

Proof Since $V_i = 1$, we have G_i is central in G. Thus, $[x, G]$ is central. This implies that $[x, G] = \{x^{-1}x^g \mid g \in G\}$. It follows that the map $a \mapsto x^{-1}a$ is a bijection from $\mathrm{cl}(x)$ to $[x, G]$. Hence, $\mathrm{cl}(x) = xG_i$ if and only if $[x, G] = G_i$. Since $x \in G_{i-1}$, it follows that $[x, G] \leq G_i$. Suppose that $[x, G] < G_i$, and we want to find a contradiction. We can find N such that $[x, G] \leq N < G_i$, where $|G_i : N| = p$. Since $x \notin Y_i$, $[x, G] \neq 1$. Thus, $N > 1$. Applying Lemma 2.3, it is not difficult to see that $V_{i-1}(G/N) = V_{i-1}/N$. Notice that $xN \in Y_i(G/N)$. On the other hand, we have $xN \in G_{i-1}/N = (G/N)_{i-1}$. Thus, since G_i is H_1, we have $xN \in Y_i(G/N) \cap (G_{i-1}/N) = V_{i-1}(G/N) \leq V_{i-1}/N$. Therefore, $x \in V_{i-1}$, which contradicts the choice of x. ∎

The following result is a nice consequence of Lemma 2.4 that gives us a good idea about the irreducible characters in $\mathrm{Irr}(G|G_k)$.

Lemma 2.5 *If $V_k = 1$ and G_k is H_1, then all the characters in* $\mathrm{Irr}(G|G_k)$ *vanish on $G_{k-1} \setminus V_{k-1}$.*

Proof Consider $x \in G_{k-1} \setminus V_{k-1}$. By Lemma 2.4 we have $\mathrm{cl}(x) = xG_k$. Applying the second orthogonality relation, which is Theorem 2.18 in [1], we obtain

$$\begin{aligned}|G|/|G_k| &= |G|/|\mathrm{cl}(x)| = |C_G(x)| \\ &= \sum_{\chi\in\mathrm{Irr}(G)} |\chi(x)|^2 = \sum_{\chi\in\mathrm{Irr}(G/G_k)} |\chi(x)|^2 + \sum_{\chi\in\mathrm{Irr}(G|G_k)} |\chi(x)|^2.\end{aligned}$$

Since G_{k-1}/G_k is central in G/G_k, we can use the second orthogonality relation in G/N to see that

$$|G:G_k| = \sum_{\chi\in\mathrm{Irr}(G/G_k)} |\chi(xG_k)|^2 = \sum_{\chi\in\mathrm{Irr}(G/G_k)} |\chi(x)|^2.$$

Hence,

$$\sum_{\chi\in\mathrm{Irr}(G|G_k)} |\chi(x)|^2 = 0.$$

Since $|\chi(x)|^2 \geq 0$ for each $\chi \in \mathrm{Irr}(G \mid G_k)$, this implies that all characters in $\mathrm{Irr}(G \mid G_k)$ vanish on $G_{k-1} \setminus V_{k-1}$ as desired. ∎

https://doi.org/10.4153/S0008439519000195 Published online by Cambridge University Press

Define $E_i/(G_{i-1} \cap Y_i) = C_{G/(G_{i-1} \cap Y_i)}\big(G_{i-2}/(G_{i-1} \cap Y_i)\big)$. We know that $V_{i-1} \leq G_{i-1}$. Since $V_i = [V_{i-1}, G]$, we have $V_{i-1} \leq Y_i$, and hence, $V_{i-1} \leq G_{i-1} \cap Y_i$. Because $[G_{i-1}, D_{i-1}] \leq V_{i-1} \leq G_{i-1} \cap Y_i$, it follows that $D_{i-1} \leq E_i$.

Recall, as a consequence of Lemma 2.1, that if G_i is H_1, then $V_{i-1} = G_{i-1} \cap Y_i$. Hence, $D_{i-1}/V_{i-1} = C_{G/V_{i-1}}(G_{i-2}/V_{i-1}) = C_{G/(G_{i-1} \cap Y_i)}\big(G_{i-2}/(G_{i-1} \cap Y_i)\big) = E_i/(G_{i-1} \cap Y_i)$. In particular, $D_{i-1} = E_i$.

Notice that our next lemma is the only time we use the hypothesis that G'/V_i is abelian.

Lemma 2.6 *Let $V_i < G_i$, suppose that $i \geq 4$, and assume that G'/V_i is abelian. Then $D_i \leq E_i$.*

Proof We may assume that $V_i = 1$. Hence, $D_i = C_G(G_{i-1})$, G' is abelian, and $Y_i = Z(G)$. Since G' is abelian, we obtain $[G, D_i, G_{i-2}] \leq [G', G'] = 1$. On the other hand, we have $[G_{i-2}, G, D_i] = [G_{i-1}, D_i] = 1$. By the Three Subgroups Lemma, which is Lemma 8.27 in [2], we get $[D_i, G_{i-2}, G] = 1$. Therefore, $[D_i, G_{i-2}] \leq Y_i$. Now, we know that $[D_i, G_{i-2}] = [G_{i-2}, D_i] \leq G_{i-1}$, and $[D_i, G_{i-2}] \leq G_{i-1} \cap Y_i$. We conclude that $D_i \leq E_i$, as desired. ∎

In the next lemma, we get an upper bound for the index of D_i in G.

Lemma 2.7 *Assume that $V_i = 1$. If $|G_i| = p$, then $|G : D_i| \leq |G_{i-1} : G_{i-1} \cap Y_i|$.*

Proof By Theorem 1 in [4], we know that G_{i-1}/V_{i-1} is an elementary abelian p-group. Hence, we can find $x_1, \ldots, x_t \in G_{i-1} \setminus Y_i$, such that $G_{i-1} = \langle x_1, \ldots, x_t, G_{i-1} \cap Y_i \rangle$, where $|G_{i-1} : G_{i-1} \cap Y_i| = p^t$. Since $|G_i| = p$, we know by Lemma 2.2 that $|G : C_G(x_j)| = p$ for all $j = 1, \ldots, t$. Thus,

$$|G : D_i| = \Big|G : \bigcap_{j=1}^{t} C_G(x_j)\Big| \leq \prod_{j=1}^{t} |G : C_G(x_j)| = p^t = |G_{i-1} : G_{i-1} \cap Y_i|. \qquad ∎$$

In our next lemma, we prove a very interesting isomorphism that will a be a key to getting the index of V_i in G_i.

Lemma 2.8 *Assume that G_i is H_1. Let $a \in G_{i-1} \setminus V_{i-1}$ and set $K/V_i = C_{G/V_i}(aV_i)$. Then $G/K \cong G_i/V_i$.*

Proof Without loss of generality, we may assume that $V_i = 1$. Consider the map from G to G_i defined by $g \mapsto [g, a]$. Since $a \in G_{i-1}$, we have $[g, a] \in G_i$ for every $g \in G$. Hence, this map is well defined. Also, we know that G_i is central in G. Thus, this map is a homomorphism with kernel K. By Lemma 2.4, this map is onto. Therefore, by the First Isomorphism Theorem, we conclude that $G/K \cong G_i$. ∎

Now we prove the following result.

Corollary 2.9 *Assume that G_i is H_1. Then $|G_i : V_i| \leq |G : D_i|$.*

Proof Let a and K be as in Lemma 2.8. We know since $a \in G_{i-1}$ and $D_i/V_i = C_{G/V_i}(G_{i-1}/V_i)$ that $D_i \leq K$. Hence, $|G_i : V_i| = |G : K| \leq |G : D_i|$. ∎

https://doi.org/10.4153/S0008439519000195 Published online by Cambridge University Press

The following result is very useful to prove our main theorem.

Lemma 2.10 *Assume that $V_i < G_i$, G'/V_i is abelian, and G_{i-1} is H_1, for $i \geq 4$. Let $a \in G_{i-2} \setminus V_{i-2}$ and set $K/V_{i-1} = C_{G/V_{i-1}}(aV_{i-1})$. Then $K \leq D_i$.*

Proof We may assume that $V_i = 1$. Hence, V_{i-1} is central in G, G' is abelian, $Y_i = Z(G)$, and $D_i = C_G(G_{i-1})$. Fix $x \in K$, and let $w \in G$ be arbitrary. Notice that $[a, x] \in V_{i-1} \leq Y_i$. Thus, $[a, x, w] = 1$. Also, $[x, w] \in G'$. Because $i \geq 4$, $G_{i-2} \leq G'$ so $a \in G'$. Since G' is abelian, $[x, w, a] \leq [G', G'] = 1$. Therefore, by Hall's Identity, which is Lemma 8.26 in [2], we obtain $[w, a, x] = 1$. This implies that x centralizes $[w, a]$. Since $a \notin V_{i-2}$ and G_{i-1} is H_1, we deduce by Lemma 2.4 that as w runs through all of G, $[w, a]$ runs through all of G_{i-1}. Hence, x centralizes G_{i-1}. Thus, $x \leq D_i$. Therefore, $K \leq D_i$. ∎

As a consequence of the previous lemma, we get the following corollary.

Corollary 2.11 *Assume that $V_i < G_i$, G'/V_i is abelian, and G_{i-1} is H_1, for $i \geq 4$. Then $D_{i-1} \leq D_i$.*

Proof Let $a \in G_{i-2} \setminus V_{i-2}$ and set $K/V_{i-1} = C_{G/V_{i-1}}(aV_{i-1})$. Then by Lemma 2.10 we have $K \leq D_i$. Also, we know that $D_{i-1} \leq K$. Thus, $D_{i-1} \leq D_i$. ∎

We now get an upper bound for $|G_{i-1} : G_{i-1} \cap Y_i|$.

Lemma 2.12 *Assume that $V_i < G_i$ and G_{i-1} is H_1. Then $|G : E_i| \geq |G_{i-1} : G_{i-1} \cap Y_i|$.*

Proof Fix $a \in G_{i-2} \setminus V_{i-2}$, and consider the map f from G to G_{i-1}/V_{i-1} defined by $f(g) = [a, g]V_{i-1}$. As in the proof of Lemma 2.8, we know that f is an onto homomorphism. It follows that f maps G/E_i onto $G_{i-1}/f(E_i)$. Thus, $|G_{i-1} : f(E_i)| \leq |G : E_i|$. Since $a \in G_{i-2}$, $[E_i, a] \leq G_{i-1} \cap Y_i$, and thus $f(E_i) \leq G_{i-1} \cap Y_i$. Then $|G_{i-1} : G_{i-1} \cap Y_i| \leq |G_{i-1} : f(E_i)|$. Hence, $|G : E_i| \geq |G_{i-1} : G_{i-1} \cap Y_i|$ as required. ∎

3 Proofs of Theorems 1 and 2

In this section, we prove our three theorems using the general lemmas that we proved in the previous section.

Now we prove Theorem 1.

Proof of Theorem 1 We have $D_3 = D_3$. This is the initial case of induction. Assume that the theorem is true for $k - 1$. We are going to prove it for k. By hypothesis, we know that G_k is H_1, and by Lemma 2.1, we have $V_{k-1} = G_{k-1} \cap Y_k$. This implies $E_k = D_{k-1}$. By the inductive hypothesis we know that $D_{k-1} = D_3$, and so, $E_k = D_3$. By Lemma 2.6, we obtain $D_k \leq E_k$. Applying Corollary 2.11, we conclude that $D_{k-1} \leq D_k$. Thus, $D_{k-1} \leq D_k \leq E_k = D_{k-1}$. Therefore, we deduce that $D_k = E_k = D_{k-1} = D_3$. ∎

Now we are ready to prove our second theorem.

Proof of Theorem 2 We are going to prove this theorem by induction. Notice that the initial case of induction ($i = 3$) is done by Lewis in [4]. Now assume that the

https://doi.org/10.4153/S0008439519000195 Published online by Cambridge University Press

theorem is true for $k = i - 1$. We are going to prove it for $k = i$. Also in this proof, without loss of generality, we may assume that $V_i = 1$. We also know by the inductive hypothesis that $D_{i-1} = D_3$ and G_{i-1} is H_1. Now, by Lemma 2.6 we have that $D_i \leq E_i$. By Corollary 2.11, we have $D_i \leq D_{i-1}$. First we assume that $|G_i| = p$. Thus, we obtain

$$|G : D_i| \geq |G : D_{i-1}| \geq |G_{i-1} : V_{i-1}| \geq |G_{i-1} : G_{i-1} \cap Y_i|.$$

But by Lemma 2.7, we have $|G : D_i| \leq |G_{i-1} : G_{i-1} \cap Y_i|$. Hence, we have equality throughout the above inequality. Therefore, $V_{i-1} = G_{i-1} \cap Y_i$, and $|G_{i-1} : V_{i-1}| = |G : D_{i-1}|$.

Now assume that $|G_i| > p$. Consider a normal subgroup N, such that $V_i \leq N < G_i$ and $|G_i : N| = p$. The above argument shows that $V_{i-1}(G/N) = Y_i(G/N) \cap (G_{i-1}/N)$. Thus, G_i satisfies H_1. By strong induction we have $G_4, \ldots, G_{i-1}$ satisfy H_1. Thus, we may apply Theorem 1 to see that $D_i = D_3$. First define $D_{iN}/N = C_{G/N}(G_{i-1}/N)$. Note that $D_i \leq D_{iN}$, and so $D_{iN} = D_3$. The above argument yields $|G : D_3| = |G : D_{i-1}| = |G_{i-1} : V_{i-1}|$. To prove part (d), since G_i is H_1, by Corollary 2.9 we obtain $|G_i : V_i| \leq |G : D_i|$, as desired. ∎

Acknowledgements The author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17. ∎

References

[1] I. M. Isaacs, *Character Theory of Finite Groups*. Academic Press, San Diego, California, 1976.
[2] I. M. Isaacs, *Algebra: A Graduate Course*. Academic Press, Pacific Grove, California, 1993.
[3] I. D. MacDonald, *Some p-Groups of Frobenius and Extra-Special Type*. Israel J. Math. 40(1981), 350–364.
[4] M. L. Lewis, *The vanishing-off subgroup*. J. Algebra 321(2009), 1313–1325.
[5] R. Dark and C. M. Scoppola, *On Camina Groups of Prime Power Order*. J. Algebra 181(1996), 787–802.

Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia
e-mail: nmlaiki@psu.edu.sa nmlaiki2012@gmail.com