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Derivation of the nonlinear bending–torsion model for a junction of elastic rods

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We derive the one-dimensional bending–torsion equilibrium model for the junction of straight rods. The starting point is a three-dimensional nonlinear elasticity equilibrium problem written as a minimization problem for a union of thin, rod-like bodies. By taking the limit as the thickness of the three-dimensional rods tends to zero, and by using ideas from the theory of Γ-convergence, we find that the resulting model consists of the union of the usual one-dimensional nonlinear bending–torsion rod models which satisfy the following transmission conditions at the junction point: continuity of displacement and rotation of the cross-sections; balance of contact forces and contact couples.

1. Introduction

In many real-life structures, such as, for example, certain types of bridges or buildings, two (or several) elastic rods are connected at one point. Such points where several rods meet are called junctions. Such multiple-rod systems may be as small as two rods joining in a non-smooth way, or as complex as several hundreds of interconnected rods forming a massive network. In either case, the basic principles of analysis are the same (although the complexity of the computation depends on the complexity of the system). Therefore, in the present paper, we limit our study to the case of one junction point.

We consider the equilibrium problem of a three-dimensional elastic body which consists of n straight, thin, rod-like bodies connected at a single point. Since the rods are thin, the behaviour of each rod should be well approximated by the one-dimensional rod model. In order to obtain a well-defined problem one needs to prescribe the conditions at the junction point. These conditions can also be seen as transmission conditions. Since we are interested in the bending–torsion behaviour of rods, such a rod is expected to be governed by the fourth-order equation [4]. Since this equation can be written as a first-order system in terms of the contact force, the contact couple, the rotation of the cross-section and the deformation (displacement), we expect the following four junction conditions (based on the continuity of the deformation and equilibrium laws) to hold:

1. the sum of all contact forces at the junction is zero;

2. the sum of all contact couples at the junction is zero;

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3. continuity of the rotation of the cross-section (the angles at the junction point are preserved);

4. continuity of the displacement (deformation/position) at the junction point.

These conditions follow physical intuition and are already used in modelling networks of elastic rods (see, for example, [13, 31]; in the case of strings see [11]).

In [15] these junction conditions have been mathematically justified for the case when the starting configuration is that of three-dimensional *linearized* elasticity. We justify these junction conditions starting from a three-dimensional nonlinearly hyperelastic material whose equilibrium problem is given in the energy minimization formulation. Since the rods are thin, a small parameter h describes their thickness. The mechanical response of rods strongly depends on the relative magnitude of the applied load with respect to the rod thickness h. In [24], a bending–torsion model of a single nonlinearly elastic inextensible rod was derived by the theory of Γ-convergence and the geometric rigidity theorem from [12]. In order to obtain the bending–torsion model, the main assumption is that the energy of the rod is of the order h^4. For other models see [1, 25]. In the present paper, we would like to obtain junction conditions at the junction of rods for the case when the total energy functional is of order h^4. However, unlike in the case of the single rod studied in [24], in the case when two or more rods meet at a junction, we cannot rescale our problem in such a way that the entire problem is defined on a canonical domain independent of h, at least in a simple way. To deal with the complications related to the geometry at the junction, we assume that the junction region of the rods forms a domain that scales with h (say a sphere, in which all the rods are connected). Then, as $h \to 0$, the junction region converges to a point. This leads to a problem with no obvious simple canonical domain, and so the results from [24] cannot be applied directly to this problem. To get around this difficulty we adapt the ideas from [24] to this new scenario and express the asymptotic behaviour of minimizers in norms depending on the thickness h.

Following [24], we first prove a compactness result (theorem 3.1) for the sequence of energy minimizers $\boldsymbol{y}^{(h)}$ deriving the asymptotic behaviour of $\nabla \boldsymbol{y}^{(h)}$. Moreover, we prove that the rotations of the cross-sections need to be continuous at the junction point in the limit as $h \to 0$. Since we are considering a pure traction problem for rods joining at a point, we still need to control the displacement of the entire structure. Under the assumption that the translation of the whole structure is controlled at the end of one rod, in lemma 4.1 and corollary 4.2 we derive the asymptotic behaviour of the minimizing sequence $\boldsymbol{y}^{(h)}$ and we obtain that, in the limit, the displacement (deformation) of the rods at the junction point is continuous. Finally, in theorem 5.1 we derive the model for the junction of rods.

The junction of elastic rods has been studied by several authors. However, most results in the literature are restricted to linearized elasticity. The first study of a junction of two rods was given by Le Dret in [19] (see also [22, 30]). For systems of rods see [27, 28] and references therein. The junction of two plates is studied in [14, 20, 21], while [29] deals with the junction of beams and plates. The case of the junction of a three-dimensional domain and a two-dimensional one is explored in [10] (see also [9] and references therein). For the asymptotic analysis of the junction between three-dimensional and one-dimensional structures see [5, 18].

Two efforts in the study of junction problems within nonlinear elasticity are made in [6] and [16] using the asymptotic expansion method. In [6] the model of a plate inserted in a three-dimensional elastic body is derived, while in [16] a model of junction of a rod and a plate is derived. See also [7, 23] for the asymptotic analysis of the problem of junctions of thin pipes filled with a fluid using the asymptotic expansion method.

2. Setting up the problem

The domain of the junction of rods we define as a union of cylinders and the 'junction' part. Let $n \in \mathbb{N}$ denote the number of rods meeting in junction and let $h > 0$. Let ith rod be of length L_i with the cross-section hS_i, where $S_i \subset \mathbb{R}^2$ (open, bounded, connected). Let the junction part be of the form $T^h = hT$, for $T \subset \mathbb{R}^3$ open, bounded, connected set. Let $\mathsf{Q}_i \in \mathrm{SO}(3)$, $i = 1, \dots, n$. The vector $\boldsymbol{t}_i = \mathsf{Q}_i \boldsymbol{e}_1$ denotes the tangential direction of the ith rod. Then the domain of the junction of rods is given as

$$\Omega^h = T^h \cup \bigcup_{i=1}^{n} C_i^h, \qquad C_i^h = \mathsf{Q}_i((h, L_i) \times hS_i).$$

We assume that the domain Ω^h is open, bounded, connected and with the Lipschitz boundary. We also assume, as in [24], that, for each i,

$$\int_{S_i} x_2 x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_3 = \int_{S_i} x_2 \,\mathrm{d}x_2 \,\mathrm{d}x_3 = \int_{S_i} x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_3 = 0.$$

We naturally interpret every function $\boldsymbol{y} \in W^{1,2}((a, b); \mathbb{R}^3)$ as an element of $W^{1,2}((a, b) \times \mathbb{R}^2; \mathbb{R}^3)$. We also define the mapping $\boldsymbol{P}^{(h)} : (a, b) \times S_i \to (a, b) \times hS_i$ by $\boldsymbol{P}^{(h)}(x_1, x_2, x_3) = (x_1, hx_2, hx_3)$ and use it to change between thin and thick domains.

The starting point of our analysis is the equilibrium problem of the junction of rods, i.e. the elastic body $\overline{\Omega^h}$. The internal energy of the junction of rods is given by

$$E^{(h)}(\boldsymbol{y}) = \int_{\Omega^h} W(\nabla \boldsymbol{y}(x)) \,\mathrm{d}x,$$

for a deformation $\boldsymbol{y} \in W^{1,2}(\Omega^h, \mathbb{R}^3)$, where $W : \mathbb{M}^{3\times 3} \to [0, +\infty]$ is an internal energy density function. As in [24], W is supposed to satisfy

- $W \in C^0(\mathbb{M}^{3\times 3})$, W is of class C^2 in a neighbourhood of $\mathrm{SO}(3)$,
- W is frame-indifferent, i.e. $W(\mathsf{F}) = W(\mathsf{RF})$ for every $\mathsf{F} \in \mathbb{M}^{3\times 3}$ and $\mathsf{R} \in \mathrm{SO}(3)$,
- $W(\mathsf{F}) \geqslant C_W \operatorname{dist}^2(\mathsf{F}, \mathrm{SO}(3)), W(\mathsf{F}) = 0$ if $\mathsf{F} \in \mathrm{SO}(3)$.

We are looking for the one-dimensional bending–torsion model of junction of rods. Thus, motivated by [24], we assume that the energy $E^{(h)}$ behaves as h^4. Then we analyse the behaviour of $E^{(h)}(\boldsymbol{y})/h^4$ and derive the one-dimensional model. In [24] this is obtained by Γ-convergence, but in the junction problem there is no obvious and simple domain independent of the thickness h. However, using the ideas and techniques of Γ-convergence we are able to give the asymptotics (in the form (4.17)) of the infimizing sequence of the total energy functional and the total energy functional itself.

We shall need the following theorem, which can be found in [12].

THEOREM 2.1 (geometric rigidity). *Let $U \subset \mathbb{R}^m$ be a bounded Lipschitz domain, $m \geqslant 2$. Then there exists a constant $C(U)$ with the following property: for every $\boldsymbol{v} \in W^{1,2}(U;\mathbb{R}^m)$ there is an associated rotation $R \in \mathrm{SO}(m)$ such that*

$$\|\nabla \boldsymbol{v} - R\|_{L^2(U)} \leqslant C(U)\|\mathrm{dist}(\nabla \boldsymbol{v}, \mathrm{SO}(m))\|_{L^2(U)}. \tag{2.1}$$

We will apply this theorem in the next section, on subdomains of Ω^h which are of size h in each direction. This is possible since the constant $C(U)$ in the estimate is independent of the translation and dilatation of U. Let us consider the domain hU, for $h > 0$. Take $\boldsymbol{v} \in W^{1,2}(hU;\mathbb{R}^m)$. Then the function

$$\boldsymbol{v}^{(h)}(x) = \frac{1}{h}\boldsymbol{v}(hx)$$

belongs to $W^{1,2}(U;\mathbb{R}^m)$ and satisfies the estimate

$$\|\nabla \boldsymbol{v}^{(h)} - R\|_{L^2(U)} \leqslant C(U)\|\mathrm{dist}(\nabla \boldsymbol{v}^{(h)}, \mathrm{SO}(m))\|_{L^2(U)}.$$

Since $\nabla \boldsymbol{v}^{(h)} = \nabla \boldsymbol{v}(hx)$, after the change of variables in the norms we obtain that the estimate (2.1) holds for $\boldsymbol{v}$ with the same constant $C(U)$. (See also [12].)

Throughout the paper we use the following function space:

$$W^{1,p}(\Omega;\mathrm{SO}(3)) = \{R \in W^{1,p}(\Omega;\mathbb{R}^{3\times 3}) \mid R(x) \in \mathrm{SO}(3) \text{ for a.e. } x \in \Omega\}.$$

Moreover, by $\|\cdot\|$ (without subscript) we denote the Frobenius matrix norm.

3. Compactness

In this section, following [24], we prove the compactness result (theorem 3.1). Namely, for $\boldsymbol{y}^{(h)}$ that satisfy (3.1) (this will be shown for infimizers $\boldsymbol{y}^{(h)}$ of the energy of order h^4) we obtain asymptotics of $\nabla \boldsymbol{y}^{(h)}$. Moreover, it turns out that rotations of the cross-sections in the limit, when $h \to 0$, need to be continuous at the junction point.

THEOREM 3.1. *Let $(\boldsymbol{y}^{(h)}) \subset W^{1,2}(\Omega^h;\mathbb{R}^3)$ be such that*

$$\limsup_{h\to 0} \frac{1}{h^4}\int_{\Omega^h} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3))\,\mathrm{d}x < +\infty. \tag{3.1}$$

Then there exist a subsequence (not relabelled) and $\bar{R}_i \in W^{1,2}((0,L_i),\mathrm{SO}(3))$, $i = 1,\dots,n$, such that $\bar{R}_1(0) = \bar{R}_2(0) = \cdots = \bar{R}_n(0)$ in the sense of traces and

$$\lim_{h\to 0}\frac{1}{h^2}\sum_{i=1}^{n}\int_{C_i^h}\|\nabla \boldsymbol{y}^{(h)}(x) - \bar{R}_i(x\cdot \boldsymbol{t}_i)\|^2\,\mathrm{d}x = 0. \tag{3.2}$$

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Proof. We follow the proof of [24, theorem 2.2].

Now we cover Ω^h with subdomains of size h in each direction and apply theorem 2.1 to each of them. For every $h > 0$ and $i = 1, \dots, n$ let $k_i^h \in \mathbb{N}$ be such that $h \leqslant L_i/k_i^h < 2h$ and let

$$I^i_{a,k_i^h} := \left(a, a + \frac{L_i}{k_i^h}\right), \quad a \in [0, L_i) \cap \frac{L_i}{k_i^h}\mathbb{N}.$$

We apply theorem 2.1 to domains $\mathsf{Q}_i((a, a+2h) \times hS_i)$ (when $a = L_i - L_i/k_i^h$ we take $(L_i - 2h, L_i))$ and $T^h \cup \bigcup_{i=1}^n \mathsf{Q}_i((h, 2h) \times hS_i)$. Note that $\mathsf{Q}_i(I^i_{a,k_i^h} \times hS_i) \subset \mathsf{Q}_i((a, a+2h) \times hS_i)$. Then there exist a constant C (independent of i, as there is a finite number of domains, and h, by the remark after theorem 2.1) and a piecewise constant map

$$\mathsf{R}^{(h)}\colon \bigcup_{i=1}^n \mathsf{Q}_i([0, L_i] \times \{0\} \times \{0\}) \to \mathrm{SO}(3),$$

constant on each $[a, a + L_i/k_i^h)$ for $a \in [0, L_i) \cap (L_i/k_i^h)\mathbb{N}$ and on

$$T^h \cup \bigcup_{i=1}^n \mathsf{Q}_i\left(\left[h, \frac{L_i}{k_i^h}\right) \times hS_i\right),$$

such that for every $i \in \{1, \dots, n\}$ we have, for every $a \in [0, L_i) \cap (L_i/k_i^h)\mathbb{N}$,

$$\int_{\mathsf{Q}_i(I^i_{a,k_i^h} \times hS_i)} \|\nabla \boldsymbol{y}^{(h)} - \mathsf{R}^{(h)}\|^2 \,\mathrm{d}x \leqslant C \int_{\mathsf{Q}_i((a,a+2h)\times hS_i)} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3)) \,\mathrm{d}x$$

and

$$\int_{T^h \cup \bigcup_{i=1}^n \mathsf{Q}_i((h, L_i/k_i^h)\times hS_i)} \|\nabla \boldsymbol{y}^{(h)} - \mathsf{R}^{(h)}\|^2 \,\mathrm{d}x \leqslant C \int_{T^h \cup \bigcup_{i=1}^n \mathsf{Q}_i((h,2h)\times hS_i)} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3)) \,\mathrm{d}x.$$

By summing all these estimates, since only neighbouring subdomains overlap, we obtain the inequality

$$\frac{1}{h^2} \int_{\Omega^h} \|\nabla \boldsymbol{y}^{(h)} - \mathsf{R}^{(h)}\|^2 \,\mathrm{d}x \leqslant \frac{2C}{h^2} \int_{\Omega^h} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3)) \,\mathrm{d}x \leqslant C_1 h^2, \tag{3.3}$$

where the last inequality holds for sufficiently small h by (3.1).

In the following we show that on a subsequence $\mathsf{R}^{(h)}$ converges to a $W^{1,2}$ function. In order to do that, we first estimate the difference of $\mathsf{R}^{(h)}$ on neighbouring subdomains.

Let $a_i \in (0, L_i - 4h] \cap (L_i/k_i^h)\mathbb{N}$, $b_i = a_i + L_i/k_i^h$. Now we apply theorem 2.1 on the set $\mathsf{Q}_i((a_i, a_i + 4h) \times hS_i)$. We obtain that there exists $\bar{\mathsf{R}} \in \mathrm{SO}(3)$ such that

$$\int_{\mathsf{Q}_i((a_i,a_i+4h)\times hS_i)} \|\nabla \boldsymbol{y}^{(h)} - \bar{\mathsf{R}}\|^2 \,\mathrm{d}x \leqslant C_2 \int_{\mathsf{Q}_i((a_i,a_i+4h)\times hS_i)} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3)) \,\mathrm{d}x.$$

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Then using the fact that $I^i_{a_i,k_1^h}$ and $I^i_{b_i,k_1^h}$ are contained in $(a_i,a_i+4h)\times hS_i$, we have, for every i,

$$
\begin{aligned}
\frac{L_i}{k_i^h}\|R^{(h)}(a_i\boldsymbol{t}_i)-R^{(h)}(b_i\boldsymbol{t}_i)\|^2
&\leqslant 2\frac{L_i}{k_i^h}(\|R^{(h)}(a_i\boldsymbol{t}_i)-\bar{R}\|^2+\|R-\bar{R}^{(h)}(b_i\boldsymbol{t}_i)\|^2)\\
&\leqslant \frac{2}{h^2}\int_{Q_i(I^i_{a_i,k_i^h}\times hS_i)}\|R^{(h)}(a_i\boldsymbol{t}_i)-\bar{R}\|^2\\
&\quad+\frac{2}{h^2}\int_{Q_i(I^i_{b_i,k_i^h}\times hS_i)}\|\bar{R}-R^{(h)}(b_i\boldsymbol{t}_i)\|^2\\
&\leqslant \frac{4}{h^2}\int_{Q_i(I^i_{a_i,k_i^h}\times hS_i)}\|R^{(h)}(a_i\boldsymbol{t}_i)-\nabla\boldsymbol{y}^{(h)}\|^2+\|\nabla\boldsymbol{y}^{(h)}-\bar{R}\|^2\\
&\quad+\frac{4}{h^2}\int_{Q_i(I^i_{b_i,k_i^h}\times hS_i)}\|\bar{R}-\nabla\boldsymbol{y}^{(h)}\|^2+\|\nabla\boldsymbol{y}^{(h)}-R^{(h)}(b_i\boldsymbol{t}_i)\|^2\\
&\leqslant \frac{4}{h^2}\int_{Q_i(I^i_{a_i,k_i^h}\times hS_i)}\|R^{(h)}(a_i\boldsymbol{t}_i)-\nabla\boldsymbol{y}^{(h)}\|^2\\
&\quad+\frac{4}{h^2}\int_{Q_i((a_i,a_i+4h)\times hS_i)}\|\nabla\boldsymbol{y}^{(h)}-\bar{R}\|^2\\
&\quad+\frac{4}{h^2}\int_{Q_i(I^i_{b_i,k_i^h}\times hS_i)}\|\nabla\boldsymbol{y}^{(h)}-R^{(h)}(b_i\boldsymbol{t}_i)\|^2.
\end{aligned}
$$

All the terms on the right-hand side of the estimate can be estimated by theorem 2.1, so we obtain

$$\frac{L_i}{k_i^h}\|R^{(h)}(a_i\boldsymbol{t}_i)-R^{(h)}(b_i\boldsymbol{t}_i)\|^2\leqslant\frac{C_3}{h^2}\int_{Q_i((a_i,a_i+4h)\times hS_i)}\operatorname{dist}^2(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\,\mathrm{d}x,\quad (3.4)$$

and similarly, as $I^i_{L_i/k_i^h,k_i^h}$ and T_h are contained in $T^h\cup Q_i((h,4h)\times hS_i)$, we obtain

$$\frac{L_i}{k_i^h}\left\|R^{(h)}(0)-R^{(h)}\left(\frac{L_i}{k_i^h}\boldsymbol{t}_i\right)\right\|^2\leqslant\frac{C_3}{h^2}\int_{T^h\cup Q_i((h,4h)\times hS_i)}\operatorname{dist}^2(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\,\mathrm{d}x.\quad (3.5)$$

Thus, we have (since $R^{(h)}$ is piecewise constant) for every $0\leqslant\xi\leqslant L_i/k_i^h$ and every i and for every $a\in(0,L_i)\cap(L_i/k_i^h)\mathbb{N}$ such that $(a,a+4h)\subset(0,L_i)$:

$$
\begin{aligned}
&\int_{I^i_{a,k_i^h}}\|R^{(h)}((x_1+\xi)\boldsymbol{t}_i)-R^{(h)}(x_1\boldsymbol{t}_i)\|^2\,\mathrm{d}x_1\\
&\qquad\leqslant\frac{C_3}{h^2}\int_{Q_i((a,a+4h)\times hS_i)}\operatorname{dist}^2(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\,\mathrm{d}x,\qquad (3.6)
\end{aligned}
$$

since $x_1+\xi$ and x_1 belong to the same or neighbouring subdomains and we can apply estimate (3.4). In the same way we can show that, for every i and a such

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that $(a-2h,a+2h)\subset(0,L_i)$ and every $-L_i/k_i^h\leqslant\xi\leqslant 0$,

$$\int_{I^i_{a,k_i^h}}\|R^{(h)}((x_1+\xi)\boldsymbol{t}_i)-R^{(h)}(x_1\boldsymbol{t}_i)\|^2\,\mathrm{d}x_1 \leqslant \frac{C_3}{h^2}\int_{Q_i((a-2h,a+2h)\times hS_i)}\mathrm{dist}^2(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\,\mathrm{d}x. \tag{3.7}$$

Let us now look at cylinders C_1^h and C_2^h. By summing estimates (3.4)–(3.7), for every open interval I' compactly contained in $(-L_1,L_2)$ and $\xi\in\mathbb{R}$ which satisfies

$$|\xi|\leqslant\mathrm{dist}(I',\{-L_1,L_2\}),\qquad |\xi|\leqslant\frac{L_i}{k_i^h}\quad\text{for all } i,$$

we have that

$$\int_{I'}\|R_m^{(h)}(x_1+\xi)-R_m^{(h)}(x_1)\|^2\,\mathrm{d}x_1\leqslant\frac{C}{h^2}\int_{\Omega^h}\mathrm{dist}^2(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\,\mathrm{d}x, \tag{3.8}$$

where $R_m^{(h)}\colon(-L_1,L_2)\to\mathrm{SO}(3)$ is defined by

$$R_m^{(h)}(x_1)=\begin{cases}R^{(h)}(-x_1\boldsymbol{t}_1) & \text{if } x_1\in(-L_1,0],\\ R^{(h)}(x_1\boldsymbol{t}_2) & \text{if } x_1\in(0,L_2).\end{cases} \tag{3.9}$$

By the iterative application of (3.8) and use of the inequality $(x_1+\cdots+x_n)^2\leqslant n(x_1^2+\cdots+x_n^2)$ and the assumption (3.1) for every open interval I' compactly contained in $(-L_1,L_2)$ and $\xi\in\mathbb{R}$, which satisfies $|\xi|\leqslant\mathrm{dist}(I',\{-L_1,L_2\})$, we have

$$\begin{aligned}\int_{I'}\|R_m^{(h)}(x_1+\xi)-R_m^{(h)}(x_1)\|^2\,\mathrm{d}x_1&\leqslant C_4\left(\frac{|\xi|}{h}+1\right)^2\frac{1}{h^2}\int_{\Omega^h}\mathrm{dist}^2(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\,\mathrm{d}x\\&\leqslant C_5(|\xi|+h)^2.\end{aligned} \tag{3.10}$$

Note here that the factor $(|\xi|/h+1)^2$ is the upper estimate of the number of terms by which the left-hand side of (3.10) has to be estimated. Using the Fréchet–Kolmogorov criterion [2, theorems 2.21, 2.22], one can deduce from this that, for any sequence $h_j\to 0$, there exists a subsequence $(R_m^{(h_{j_{1,2}})})$ strongly converging in $L^2(-L_1,L_2)$ to some $\bar{R}\in L^2(-L_1,L_2)$ with $\bar{R}(x_1)\in\mathrm{SO}(3)$ for a.e. $x_1\in(-L_1,L_2)$. We define $\bar{R}_1\colon(0,L_1)\to\mathrm{SO}(3)$, $\bar{R}_2\colon(0,L_2)\to\mathrm{SO}(3)$ as

$$\begin{aligned}\bar{R}_1(x_1)&=\bar{R}(-x_1)\quad\text{if } x_1\in(-L_1,0),\\ \bar{R}_2(x_1)&=\bar{R}(x_1)\qquad\text{if } x_1\in(0,L_2).\end{aligned}$$

We shall prove that $\bar{R}\in W^{1,2}((-L_1,L_2);\mathbb{R}^{3\times3})$. Using the estimate (3.10) and letting $h\to 0$, we obtain that, for every I' compactly contained in $(-L_1,L_2)$ and every ξ which satisfies $|\xi|\leqslant\mathrm{dist}(I',\{-L_1,L_2\})$, there exists a constant C independent of I' and ξ such that

$$\int_{I'}\frac{\|\bar{R}(x_1+\xi)-\bar{R}(x_1)\|^2}{|\xi|^2}\,\mathrm{d}x_1\leqslant C. \tag{3.11}$$

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From standard theorems we obtain that $\bar{R} \in W^{1,2}((-L_1, L_2); \mathbb{R}^{3\times 3})$. This is equivalent to the fact that $\bar{R}_1 \in W^{1,2}((0, L_1); \mathbb{R}^{3\times 3})$, $\bar{R}_2 \in W^{1,2}((0, L_2); \mathbb{R}^{3\times 3})$ and $\bar{R}_1(0) = \bar{R}_2(0)$ in the sense of traces. In the same way, one can take cylinders C_1^h and C_i^h for $i = 3, \dots, n$ (by choosing every time a subsequence $\bar{R}^{(h_{j_1,\dots,i})}$ of the previously chosen sequence $\bar{R}^{(h_{j_1,\dots,i-1})}$), so we obtain existence of $\bar{R}_i \in W^{1,2}((0, L_i); \mathbb{R}^{3\times 3})$. Moreover, the definition of $\bar{R}_1$ is not ambiguous and $\bar{R}_1(0) = \bar{R}_2(0) = \cdots = \bar{R}_n(0)$. Now,

$$\frac{1}{h^2}\sum_{i=1}^n \int_{C_i^h} \|\nabla \boldsymbol{y}^{(h)}(x) - \bar{R}_i(x\cdot \boldsymbol{t}_i)\|^2 \,\mathrm{d}x \leqslant \frac{2}{h^2}\int_{\Omega^h} \|\nabla \boldsymbol{y}^{(h)} - R^{(h)}\|^2 \,\mathrm{d}x + \frac{2}{h^2}\sum_{i=1}^n \int_{C_i^h} \|\bar{R}_i(x\cdot \boldsymbol{t}_i) - R^{(h)}(x)\|^2 \,\mathrm{d}x.$$

Using the estimate (3.3) and $R^{(h)} \to \bar{R}_i$ in $L^2(0, L_i)$ we obtain that

$$\lim_{h\to 0}\frac{1}{h^2}\sum_{i=1}^n \int_{C_i^h} \|\nabla \boldsymbol{y}^{(h)}(x) - \bar{R}_i(x\cdot \boldsymbol{t}_i)\|^2 \,\mathrm{d}x = 0.$$

□

4. *Γ*-convergence

In the proof of theorem 3.1 we obtained the asymptotics of $\nabla \boldsymbol{y}^{(h)}$. However, as we are considering the pure traction case, in order to obtain the asymptotics of $\boldsymbol{y}^{(h)}$ one needs to control the constant. Thus, we additionally assume that the mean value at the end of the first rod behaves nicely. Then we obtain that, in the limit at the junction point, displacements from different rods must be equal.

LEMMA 4.1. *Let (h^j) be a sequence that converges to 0 and $(\boldsymbol{y}^{(h_j)}) \subset W^{1,2}(\Omega^{h_j}; \mathbb{R}^3)$ such that*

$$\limsup_{j\to\infty} \frac{1}{h_j^2}\int_{\Omega^{h_j}} \|\nabla \boldsymbol{y}^{(h_j)}\|^2 \,\mathrm{d}x < \infty. \tag{4.1}$$

Let there exist $\boldsymbol{y}_i^0 \in W^{1,2}((0, L_i); \mathbb{R}^3)$ such that $\boldsymbol{y}_i^0(0) = 0$ in the sense of traces and let us suppose that, for every i,

$$\lim_{j\to\infty}\int_{(h_j, L_i)\times S_i} \|\nabla(\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)}) - ((\boldsymbol{y}_i^0)'|0|0)\|^2 \,\mathrm{d}x = 0. \tag{4.2}$$

Let us also suppose that there exists

$$\lim_{j\to\infty} \fint_{\{L_1\}\times S_1} \boldsymbol{y}^{(h_j)} \circ Q_1 \circ \boldsymbol{P}^{(h_j)} \,\mathrm{d}x := C_{L_1} \in \mathbb{R}^3. \tag{4.3}$$

Then for $C_0 := C_{L_1} - \boldsymbol{y}_1^0(L_1)$ we have

$$\lim_{j\to\infty} \|\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)} - C_0\|_{L^2(\{h_j\}\times S_i)} = 0, \quad i = 1, \dots, n, \tag{4.4}$$

and

$$\lim_{j\to\infty}\sum_{i=1}^n \|\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)} - \boldsymbol{y}_i^c\|_{W^{1,2}((h_j, L_i)\times S_i)} = 0,$$

where $\boldsymbol{y}_i^c(x_1) = C_0 + \boldsymbol{y}_i^0(x_1)$.

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Proof. By applying the Poincaré inequality [8, Theorem 6.1-8(b)] to the cylinders $(0,1)\times S_i$ we have that there exists a constant K_1 such that, for every $i\in\{1,\dots,n\}$ and every $\boldsymbol{y}\in W^{1,2}((0,1)\times S_i;\mathbb{R}^3)$, one has

$$\left\|\boldsymbol{y}-⨍_{\{1\}\times S_i}\boldsymbol{y}\,\mathrm{d}x\right\|_{L^2((0,1)\times S_i;\mathbb{R}^3)}\leqslant K_1\|\nabla\boldsymbol{y}\|_{L^2((0,1)\times S_i;\mathbb{R}^3)}.$$

By applying this estimate to functions of the form $\tilde{\boldsymbol{y}}(x)=\boldsymbol{y}((L_i-h)x_1+h,x_2,x_3)$, we obtain that there is a constant $K_2=\max\{1,L_i-h\}K_1$ such that, for all $i\in\{1,\dots,n\}$, all $h>0$ (small enough) and all $\boldsymbol{y}\in W^{1,2}((h,L_i)\times S_i;\mathbb{R}^3)$, one has

$$\left\|\boldsymbol{y}-⨍_{\{L_i\}\times S_i}\boldsymbol{y}\,\mathrm{d}x\right\|_{L^2((h,L_i)\times S_i;\mathbb{R}^3)}\leqslant K_2\|\nabla\boldsymbol{y}\|_{L^2((h,L_i)\times S_i;\mathbb{R}^3)}. \tag{4.5}$$

In a similar way, we obtain

$$\left\|\boldsymbol{y}-⨍_{\{h\}\times S_i}\boldsymbol{y}\,\mathrm{d}x\right\|_{L^2((h,L_i)\times S_i;\mathbb{R}^3)}\leqslant K_2'\|\nabla\boldsymbol{y}\|_{L^2((h,L_i)\times S_i;\mathbb{R}^3)}. \tag{4.6}$$

Moreover, by using the same rescaling of the domain, from continuity of traces we obtain that there is a constant K_3 such that, for all i, h and $\boldsymbol{y}\in W^{1,2}((h,L_i)\times S_i;\mathbb{R}^3)$, one has

$$\|\boldsymbol{y}\|_{L^2(\{h\}\times S_i;\mathbb{R}^3)}+\|\boldsymbol{y}\|_{L^2(\{L_i\}\times S_i;\mathbb{R}^3)}\leqslant K_3\|\boldsymbol{y}\|_{W^{1,2}((h,L_i)\times S_i;\mathbb{R}^3)}. \tag{4.7}$$

By applying the Poincaré inequality (of the same form as before) to the domain T on functions given by $\tilde{\boldsymbol{y}}(x)=\boldsymbol{y}(hx)$, we have that there exists a constant K_4 such that, for all i,h and $\boldsymbol{y}\in W^{1,2}(T^h;\mathbb{R}^3)$, one has

$$\left\|\boldsymbol{y}-⨍_{Q_i(\{h\}\times hS_i)}\boldsymbol{y}\,\mathrm{d}x\right\|_{L^2(T^h;\mathbb{R}^3)}\leqslant hK_4\|\nabla\boldsymbol{y}\|_{L^2(T^h;\mathbb{R}^3)}.$$

In the similar way as before we conclude that there exists a constant $K_5=2K_4$ such that, for all i, l, h and $\boldsymbol{y}\in W^{1,2}(T^h;\mathbb{R}^3)$, one has

$$\left\|⨍_{Q_i(\{h\}\times hS_i)}\boldsymbol{y}\,\mathrm{d}x-⨍_{Q_l(\{h\}\times hS_l)}\boldsymbol{y}\,\mathrm{d}x\right\|\leqslant\frac{K_5}{\sqrt{h}}\|\nabla\boldsymbol{y}\|_{L^2(T^h;\mathbb{R}^3)}. \tag{4.8}$$

We now apply inequality (4.5) to the sequence $\boldsymbol{y}^{(h_j)}\circ Q_1\circ\boldsymbol{P}^{(h_j)}-\boldsymbol{y}_1^c$ to obtain

$$\begin{aligned}\left\|\boldsymbol{y}^{(h_j)}\circ Q_1\circ\boldsymbol{P}^{(h_j)}-\boldsymbol{y}_1^c-\left(⨍_{\{L_1\}\times S_1}\boldsymbol{y}^{(h_j)}\circ Q_1\circ\boldsymbol{P}^{(h_j)}\,\mathrm{d}x-C_{L_1}\right)\right\|_{L^2((h_j,L_1)\times S_1;\mathbb{R}^3)}\\ \leqslant K_2\|\nabla(\boldsymbol{y}^{(h_j)}\circ Q_1\circ\boldsymbol{P}^{(h_j)})-((\boldsymbol{y}_1^0)'|0|0)\|_{L^2((h_j,L_1)\times S_1;\mathbb{R}^3)}.\end{aligned}$$

Now, using the assumptions (4.2) and (4.3), we obtain that

$$\|\boldsymbol{y}^{(h_j)}\circ Q_1\circ\boldsymbol{P}^{(h_j)}-\boldsymbol{y}_1^c\|_{W^{1,2}((h_j,L_1)\times S_1;\mathbb{R}^3)}\to 0.$$

The estimate (4.7) now implies

$$\lim_{j\to\infty}\|\boldsymbol{y}^{(h_j)}\circ Q_1\circ\boldsymbol{P}^{(h_j)}-C_0\|_{L^2(\{h_j\}\times S_1;\mathbb{R}^3)}=0. \tag{4.9}$$

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By applying (4.8) for $l=1$ and $i \neq 1$ to the sequence $\boldsymbol{y}^{(h_j)}$, we obtain

$$\left\| \fint_{Q_i(\{h_j\}\times h_j S_i)} \boldsymbol{y}^{(h_j)}\,\mathrm{d}x - \fint_{Q_1(\{h_j\}\times h_j S_1)} \boldsymbol{y}^{(h_j)}\,\mathrm{d}x \right\| \leqslant \frac{K_5}{\sqrt{h_j}} \|\nabla \boldsymbol{y}^{(h_j)}\|_{L^2(T^{h_j};\mathbb{R}^3)}.$$

Now we change the variables in the integrals on the left-hand side (also note that $T^{h_j} \subset \Omega^{h_j}$) to obtain

$$\left\| \fint_{\{h_j\}\times S_i} \boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)}\,\mathrm{d}x - \fint_{\{h_j\}\times S_1} \boldsymbol{y}^{(h_j)} \circ Q_1 \circ \boldsymbol{P}^{(h_j)}\,\mathrm{d}x \right\| \leqslant \frac{K_5}{\sqrt{h_j}} \|\nabla \boldsymbol{y}^{(h_j)}\|_{L^2(\Omega^{h_j};\mathbb{R}^3)}.$$

Therefore, (4.1) and (4.9) imply

$$\fint_{\{h_j\}\times S_i} \boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)}\,\mathrm{d}x \to C_0 \quad \text{for all } i.$$

By applying the inequality (4.6) to the sequence $\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)} - \boldsymbol{y}_i^c$ for $i \neq 1$ we obtain that

$$\|\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)} - \boldsymbol{y}_i^c\|_{W^{1,2}((h_j,L_i)\times S_i;\mathbb{R}^3)} \to 0.$$

Then (4.4) follows immediately from (4.7) for $i=1$ and using the fact that

$$\|\boldsymbol{y}_i^c - C_0\|_{L^2(\{h_j\}\times S_i;\mathbb{R}^3)} = |S_i|^{1/2} \|\boldsymbol{y}_i^c(h_j) - C_0\| \to 0.$$

□

In the following we use the notation

$$(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) = ((\boldsymbol{y}_1, \boldsymbol{d}_1^2, \boldsymbol{d}_1^3), \dots, (\boldsymbol{y}_n, \boldsymbol{d}_n^2, \boldsymbol{d}_n^3))$$

to collect deformations of all rods.

Combining the results of theorem 3.1 and lemma 4.1, we obtain the following result.

Corollary 4.2. *Let $(\boldsymbol{y}^{(h)}) \subset W^{1,2}(\Omega^h;\mathbb{R}^3)$ be such that*

$$\limsup_{h\to 0} \frac{1}{h^4} \int_{\Omega^h} \operatorname{dist}^2(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3))\,\mathrm{d}x < +\infty, \tag{4.10}$$

$$\lim_{j\to\infty} \fint_{\{L_1\}\times S_1} \boldsymbol{y}^{(h_j)} \circ Q_1 \circ \boldsymbol{P}^{(h_j)}\,\mathrm{d}x := C_{L_1} \in \mathbb{R}^3. \tag{4.11}$$

Then for every sequence in $\mathbb{R}_+$ converging to 0 there exist a subsequence (h_j) and $\boldsymbol{y}_i \in W^{2,2}((0,L_i);\mathbb{R}^3)$, $\boldsymbol{d}_i^2, \boldsymbol{d}^3 \in W^{1,2}((0,L_i);\mathbb{R}^3)$ such that for $R_i = (\boldsymbol{y}_i' | \boldsymbol{d}_i^2 | \boldsymbol{d}_i^3)$ one has $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \in \mathcal{A}$, where

$$\begin{aligned}\mathcal{A} := \{&((\boldsymbol{y}_1, \boldsymbol{d}_1^2, \boldsymbol{d}_1^3), \dots, (\boldsymbol{y}_n, \boldsymbol{d}_n^2, \boldsymbol{d}_n^3)) \\ &\in (W^{2,2}((0,L_1);\mathbb{R}^3) \times W^{1,2}((0,L_1);\mathbb{R}^3) \times W^{1,2}((0,L_1);\mathbb{R}^3)) \times \cdots \\ &\quad \times (W^{2,2}((0,L_n);\mathbb{R}^3) \times W^{1,2}((0,L_n);\mathbb{R}^3) \times W^{1,2}((0,L_n);\mathbb{R}^3)) : \\ &\quad R_i \in \mathrm{SO}(3) \text{ a.e. and } \boldsymbol{y}_1(0) = \cdots = \boldsymbol{y}_n(0),\ R_1(0)Q_1^{\mathrm{T}} = \cdots = R_n(0)Q_n^{\mathrm{T}}\}\end{aligned}$$

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and

$$\lim_{j\to\infty}\frac{1}{h_j^2}\sum_{i=1}^{n}\|\boldsymbol{y}^{(h_j)}\circ Q_i - D_i(\boldsymbol{y}_i,\boldsymbol{d}_i^2,\boldsymbol{d}_i^3)\|^2_{W^{1,2}((h_j,L_i)\times h_jS_i;\mathbb{R}^3)} = 0, \tag{4.12}$$

where

$$D_i(\boldsymbol{y}_i,\boldsymbol{d}_i^2,\boldsymbol{d}_i^3)(x_1,x_2,x_3) = \boldsymbol{y}_i(x_1) + x_2\boldsymbol{d}_i^2(x_1) + x_3\boldsymbol{d}_i^3(x_1) \quad \textit{for } x\in(h_j,L_i)\times h_jS_i.$$

Proof. From (4.10) it follows that the assumption of theorem 3.1 is satisfied. Therefore, there exist a subsequence (h_j) converging to 0 and $\bar{R}_i \in W^{1,2}((0,L_i),\mathrm{SO}(3))$, $i=1,\dots,n$, such that $\bar{R}_1(0)=\bar{R}_2(0)=\cdots=\bar{R}_n(0)$ in the sense of traces and

$$\lim_{j\to\infty}\frac{1}{h_j^2}\sum_{i=1}^{n}\int_{C_i^{h_j}}\|\nabla\boldsymbol{y}^{(h_j)}(x)-\bar{R}_i(x\cdot\boldsymbol{t}_i)\|^2\,\mathrm{d}x = 0.$$

We rewrite this convergence to obtain

$$\begin{aligned}
0 &= \lim_{j\to\infty}\frac{1}{h_j^2}\sum_{i=1}^{n}\int_{(h_j,L_i)\times h_jS_i}\|\nabla\boldsymbol{y}^{(h_j)}(Q_ix)-\bar{R}_i(Q_ix\cdot\boldsymbol{t}_i)\|^2\,\mathrm{d}x\\
&= \lim_{j\to\infty}\frac{1}{h_j^2}\sum_{i=1}^{n}\int_{(h_j,L_i)\times h_jS_i}\|\nabla(\boldsymbol{y}^{(h_j)}\circ Q_i)(x)Q_i^{\mathrm{T}}-\bar{R}_i(x\cdot\boldsymbol{e}_1)\|^2\,\mathrm{d}x\\
&= \lim_{j\to\infty}\frac{1}{h_j^2}\sum_{i=1}^{n}\int_{(h_j,L_i)\times h_jS_i}\|\nabla(\boldsymbol{y}^{(h_j)}\circ Q_i)(x)-\bar{R}_i(x_1)Q_i\|^2\,\mathrm{d}x.
\end{aligned} \tag{4.13}$$

Now we define

$$\begin{aligned}
(\boldsymbol{y}_i^0)' &= \bar{R}_i(x_1)Q_i\boldsymbol{e}_1,\\
\boldsymbol{y}_i^0(0) &= 0,\\
\boldsymbol{d}_i^2 &= \bar{R}_i(x_1)Q_i\boldsymbol{e}_2,\\
\boldsymbol{d}_i^3 &= \bar{R}_i(x_1)Q_i\boldsymbol{e}_3,\\
R_i &= \bar{R}_i(x_1)Q_i.
\end{aligned}$$

Since $\bar{R}_i\in W^{1,2}((0,L_i),\mathrm{SO}(3))$, it follows that $\boldsymbol{y}_i^0\in W^{2,2}((0,L_i);\mathbb{R}^3)$ and $R_i\in W^{1,2}((0,L_i),\mathrm{SO}(3))$. By the trace property of $\bar{R}_i$ we obtain

$$R_1(0)Q_1^{\mathrm{T}} = \cdots = R_n(0)Q_n^{\mathrm{T}}.$$

In the following we want to apply lemma 4.1. Therefore, we check its assumptions. First, we estimate the norm of a matrix by the distance of the matrix to SO(3) and the norm of an arbitrary rotation to obtain

$$\int_{\Omega^{h_j}}\|\nabla\boldsymbol{y}^{(h_j)}\|^2\,\mathrm{d}x \leqslant 2\int_{\Omega^{h_j}}\mathrm{dist}^2(\nabla\boldsymbol{y}^{(h_j)}(x),\mathrm{SO}(3))\,\mathrm{d}x + Ch_j^2.$$

Using (4.10) we obtain that (4.1) is satisfied.

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Changing the coordinates in (4.13), we obtain

$$0 = \lim_{j\to\infty} \sum_{i=1}^{n} \int_{(h_j,L_i)\times S_i} \|\nabla(\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)})(x)\nabla \boldsymbol{P}^{(h_j)} - \bar{R}_i(x_1)Q_i\|^2 \,\mathrm{d}x.$$

This implies that (4.2) is satisfied with $\boldsymbol{y}_i^0$ defined above. The assumption (4.3) is satisfied by (4.11). Therefore, we can apply lemma 4.1 to obtain that for $C_0 := C_{L_1} - \boldsymbol{y}_1^0(L_1)$ we have (4.4) and

$$\lim_{j\to\infty} \sum_{i=1}^{n} \|\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)} - \boldsymbol{y}_i\|_{W^{1,2}((h_j,L_i)\times S_i)} = 0, \tag{4.14}$$

where

$$\boldsymbol{y}_i(x_1) = C_0 + \boldsymbol{y}_i^0(x_1).$$

Since $\boldsymbol{y}_i' = (\boldsymbol{y}_i^0)'$ from (4.14) and (4.13), we obtain (4.12). From (4.12) and the estimate

$$\|\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)} - \boldsymbol{y}_i\|_{L^2(\{h_j\}\times S_i)} \leqslant C\|\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)} - \boldsymbol{y}_i\|_{W^{1,2}((h_j,L_i)\times S_i)}$$

(for details see the proof of lemma 4.1) we obtain

$$\lim_{j\to\infty} \|\boldsymbol{y}^{(h_j)} \circ Q_i \circ \boldsymbol{P}^{(h_j)} - \boldsymbol{y}_i\|_{L^2(\{h_j\}\times S_i)} = 0, \quad i = 1,\dots,n. \tag{4.15}$$

Now, (4.4) and (4.15) imply

$$|S_i|^{1/2}\|\boldsymbol{y}_i(h_j) - C_0\| = \|\boldsymbol{y}_i - C_0\|_{L^2(\{h_j\}\times S_i)} \to 0$$

for all $i = 1,\dots,n$. This implies that $\boldsymbol{y}_1(0) = \cdots = \boldsymbol{y}_n(0) = C_0$.

Thus, we obtain that $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \in \mathcal{A}$. □

Remark 4.3. The structure of the functions $D_i(\boldsymbol{y}_i, \boldsymbol{d}_i^2, \boldsymbol{d}_i^3)$ defined after (4.12) is essentially one dimensional. It stands as the limit deformation for the ith rod. The function $\boldsymbol{y}_i$ describes the deformation of the middle curve of the ith rod, while the vectors $\boldsymbol{d}_i^2$ and $\boldsymbol{d}_i^3$ span the normal plane of the deformed middle curve (since $R_i = (\boldsymbol{y}_i'|\boldsymbol{d}_i^2|\boldsymbol{d}_i^3) \in \mathrm{SO}(3)$). Since the rod is assumed to be thin, variables x_2 and x_3 (cross-sectional coordinates of hS_i) are of order h, so the terms involving these terms can be considered as first correctors to the leading-order approximation $\boldsymbol{y}_i$ of the ith rod. Note also that the convergence (4.12) will be the one which will be used to formulate the asymptotics of the infimizing sequence.

Proposition 4.4. *Let the functional I be defined by*

$$I(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) = \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{1}{2}\int_0^{L_i} q_2^i(R_i^{\mathrm{T}} R_i')\,\mathrm{d}x_1 & \text{if } (\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \in \mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $R_i := (\boldsymbol{y}_i'|\boldsymbol{d}_i^2|\boldsymbol{d}_i^3)$, while the class $\mathcal{A}$ is given in corollary 4.2. The quadratic forms $q_2^i\colon \mathbb{M}^{3\times 3}_{\mathrm{skew}} \to [0,\infty)$ are defined by

$$q_2^i(A) := \min_{\alpha\in W^{1,2}(S_i;\mathbb{R}^3)} \int_{S_i} q_3^i\left(A\begin{pmatrix}0\\x_2\\x_3\end{pmatrix} \Big| \partial_2\alpha \Big| \partial_3\alpha\right)\mathrm{d}x_2\,\mathrm{d}x_3, \tag{4.16}$$

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where

$$q_3^i(\mathsf{G}) = \frac{\partial^2 W}{\partial \mathsf{F}^2}(\mathsf{Q}_i^{\mathrm{T}})(\mathsf{G}\mathsf{Q}_i^{\mathrm{T}}, \mathsf{G}\mathsf{Q}_i^{\mathrm{T}}).$$

Then the following two statements hold.

- ***lim inf inequality.*** *Let* $\boldsymbol{y}_i \in W^{1,2}((0, L_i); \mathbb{R}^3)$, $\boldsymbol{d}_i^2, \boldsymbol{d}_i^3 \in L^2((0, L_i); \mathbb{R}^3)$. *Then for every sequence* $(h_j) \subset (0, \infty)$ *converging to* 0 *and every sequence* $(\boldsymbol{y}^{(h_j)}) \subset W^{1,2}(\Omega^{h_j}; \mathbb{R}^3)$ *such that*

$$\lim_{j\to\infty} \frac{1}{h_j^2} \sum_{i=1}^n \|\boldsymbol{y}^{(h_j)} \circ \mathsf{Q}_i - D_i(\boldsymbol{y}_i, \boldsymbol{d}_i^2, \boldsymbol{d}_i^3)\|^2_{W^{1,2}((h_j, L_i)\times h_j S_i; \mathbb{R}^3)} = 0, \tag{4.17}$$

  where D_i *are defined in (4.12), we have that*

$$I(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \leqslant \liminf_{j\to\infty} \frac{1}{h_j^4} E^{(h_j)}(\boldsymbol{y}^{(h_j)}).$$

- ***lim sup inequality.*** *For every sequence* $(h_j) \subset (0, \infty)$ *converging to* 0 *and for every* $\boldsymbol{y}_i \in W^{1,2}((0, L_i); \mathbb{R}^3)$, $\boldsymbol{d}_i^2, \boldsymbol{d}_i^3 \in L^2((0, L_i); \mathbb{R}^3)$ *there exists a sequence* $(\boldsymbol{y}^{(h_j)}) \subset W^{1,2}(\Omega^{h_j}; \mathbb{R}^3)$ *such that*

$$\lim_{j\to\infty} \frac{1}{h_j^2} \sum_{i=1}^n \|\boldsymbol{y}^{(h_j)} \circ \mathsf{Q}_i - D_i(\boldsymbol{y}_i, \boldsymbol{d}_i^2, \boldsymbol{d}_i^3)\|^2_{W^{1,2}((h_j, L_i)\times h_j S_i; \mathbb{R}^3)} = 0$$

  and

$$\lim_{j\to\infty} \frac{1}{h_j^4} E^{(h_j)}(\boldsymbol{y}^{(h_j)}) = I(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3).$$

REMARK 4.5. As noted in [24, remark 3.4.], each minimization problem in (4.16) has a solution and this can be equivalently computed on the class of functions

$$V_i = \left\{\boldsymbol{\alpha} \in W^{1,2}(S_i; \mathbb{R}^3) \colon \int_{S_i} \boldsymbol{\alpha}\, \mathrm{d}x_2\, \mathrm{d}x_3 = \int \nabla\boldsymbol{\alpha}\, \mathrm{d}x_2\, \mathrm{d}x_3 = 0\right\}.$$

It can also be shown that for every i the minimizer is unique in V_i and that the minimizer in V_i depends linearly on the entries (a_{ij}) of A. Hence, q_2^i is in fact a quadratic form of A. In the isotropic case ($W(\mathsf{F}) = W(\mathsf{F}\mathsf{R})$ for every $\mathsf{F} \in \mathbb{M}^{3\times 3}$ and $\mathsf{R} \in \mathrm{SO}(3)$) for every i we have

$$q_3^i(\mathsf{G}) = \frac{\partial^2 W}{\partial \mathsf{F}^2}(\boldsymbol{i})(\mathsf{G}, \mathsf{G}).$$

In this case there are also some explicit formulae for q_2^i [24, remarks 3.5 and 3.6].

Proof. Let us first prove the lim inf inequality.

Let $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \in \mathcal{A}$ and let $0 < h_j \to 0$ and $\boldsymbol{y}^{(h_j)} \subset W^{1,2}(\Omega^{h_j}; \mathbb{R}^3)$ satisfy (4.17). Let us also fix $\delta > 0$. Then, after rescaling each convergence in the sum (4.17) to

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the fixed domain $(\delta, L_i)\times S_i$, we obtain that for every $i\in\{1,\dots,n\}$ one has

$$\|\boldsymbol{y}^{(h_j)}\circ Q_i\circ \boldsymbol{P}^{(h_j)}-\boldsymbol{y}_i\|_{W^{1,2}((\delta,L_i)\times S_i;\mathbb{R}^3)}\to 0,$$

$$\left\|\frac{1}{h_j}(\partial_2(\boldsymbol{y}^{(h_j)}\circ Q_i\circ \boldsymbol{P}^{(h_j)}),\ \partial_3(\boldsymbol{y}^{(h_j)}\circ Q_i\circ \boldsymbol{P}^{(h_j)}))-(\boldsymbol{d}_i^2,\boldsymbol{d}_i^3)\right\|_{L^2((\delta,L_i)\times S_i;\mathbb{R}^3\times\mathbb{R}^3)}\to 0.$$

Now, by using [24, theorem 3.1] on each rod separately (applying it to the energy density functions $W^{Q_i^{\mathrm{T}}}(F):=W(FQ_i^{\mathrm{T}})$), we conclude that for every δ and for every i we have

$$\begin{aligned}\frac{1}{2}\int_\delta^{L_i} q_2^i(R_i^{\mathrm{T}}R_i')\,\mathrm{d}x_1 &\leqslant \liminf_{j\to\infty}\frac{1}{h_j^4}\int_{(\delta,L_i)\times S_i} W^{Q_i^{\mathrm{T}}}(\nabla_{h_j}(\boldsymbol{y}^{(h_j)}\circ Q_i\circ \boldsymbol{P}^{(h_j)}))\,\mathrm{d}x\\ &=\liminf_{j\to\infty}\frac{1}{h_j^4}\int_{(\delta,L_i)\times h_jS_i} W(\nabla \boldsymbol{y}_i^{(h_j)}(Q_ix)Q_iQ_i^{\mathrm{T}})\,\mathrm{d}x\\ &=\liminf_{j\to\infty}\frac{1}{h_j^4}\int_{Q_i((\delta,L_i)\times h_jS_i)} W(\nabla \boldsymbol{y}_i^{(h_j)}(x))\,\mathrm{d}x,\end{aligned}$$

where we have used the notation

$$\nabla_h=\left(\partial_1\left|\frac{1}{h}\partial_2\right|\frac{1}{h}\partial_3\right).$$

By summing all these inequalities, we obtain that for every $\delta>0$ one has

$$\sum_{i=1}^n\frac{1}{2}\int_\delta^{L_i} q_2^i(R_i^{\mathrm{T}}R_i')\,\mathrm{d}x_1\leqslant\liminf_{j\to\infty}\frac{1}{h_j^4}E^{(h_j)}(\boldsymbol{y}^{(h_j)}).$$

By letting $\delta\to 0$ we obtain

$$I(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)\leqslant\liminf_{j\to\infty}\frac{1}{h_j^4}E^{(h_j)}(\boldsymbol{y}^{(h_j)}).$$

Let us now suppose that $(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)\notin\mathcal{A}$. We must show that for every sequence $(\boldsymbol{y}^{(h_j)})\subset W^{1,2}(\Omega^{h_j};\mathbb{R}^3)$ such that (4.17) holds, one has

$$\liminf_{j\to\infty}\frac{1}{h_j^4}E^{(h_j)}(\boldsymbol{y}^{(h_j)})=+\infty.$$

Let us suppose to the contrary that

$$\liminf_{j\to\infty}\frac{1}{h_j^4}E^{(h_j)}(\boldsymbol{y}^{(h_j)})<\infty.$$

Using the property of the stored energy function W, we estimate

$$C_W\frac{1}{h_j^4}\int_{\Omega^{h_j}}\operatorname{dist}^2(\nabla\boldsymbol{y}^{(h_j)}(x),\mathrm{SO}(3))\,\mathrm{d}x\leqslant\frac{1}{h_j^4}E^{(h_j)}(\boldsymbol{y}^{(h_j)})<\infty. \tag{4.18}$$

From the convergence (4.17) one can easily conclude, using the continuity of the trace operator and the fact that we can control the change of the domain (analogously to lemma 4.1 and corollary 4.2), that

$$\lim_{j\to\infty} \fint_{\{L_1\}\times S_1} \boldsymbol{y}^{(h_j)} \circ Q_1 \circ \boldsymbol{P}^{(h_j)}\,\mathrm{d}x := \boldsymbol{y}_1(L_1).$$

Thus, the assumptions of corollary 4.2 are satisfied and we can conclude, by the uniqueness of the limit, that $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \in \mathcal{A}$, which is a contradiction.

To prove the lim sup inequality we have to construct the appropriate sequence. Let us take $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \in \mathcal{A}$. Let us in addition suppose that $\boldsymbol{y}_i \in C^2([0, L_i]; \mathbb{R}^3)$, $\boldsymbol{d}_i^2, \boldsymbol{d}_i^3 \in C^1([0, L_i]; \mathbb{R}^3)$ (note that $\boldsymbol{y}_i \in C^2([0, L_i]; \mathbb{R}^3)$ is an immediate consequence of $\boldsymbol{d}_i^2, \boldsymbol{d}_i^3 \in C^1([0, L_i]; \mathbb{R}^3)$ and $R_i \in \mathrm{SO}(3)$). Let us define $\boldsymbol{y}^{(h_j)}$ in the following way:

$$\boldsymbol{y}^{(h_j)}(x) = \boldsymbol{y}_i(0) + R_i(0)Q_i^{\mathrm{T}} x \quad \text{for } x \in T^{h_j}$$

(the definition is not ambiguous),

$$\begin{aligned}\boldsymbol{y}^{(h_j)}(Q_i\boldsymbol{P}^{(h_j)}(x)) = \boldsymbol{y}_i(x_1 - h_j) &+ h_j \boldsymbol{y}_i'(0) \\ &+ h_j x_2(\boldsymbol{d}_i^2(x_1 - h_j) - \alpha^{(h_j)}(x_1)(\boldsymbol{d}_i^2)'(0)) \\ &+ h_j x_3(\boldsymbol{d}_i^3(x_1 - h_j) - \alpha^{(h_j)}(x_1)(\boldsymbol{d}_i^3)'(0)) + h_j^2 \boldsymbol{\beta}_i^j(x),\end{aligned}$$

for $x \in (h_j, L_i) \times S_i$, where $\alpha^{(h_j)} \in C^1([h_j, L_i]; \mathbb{R}^3)$ are such that

$$\begin{gathered}\alpha^{(h_j)}(h_j) = 0, \qquad (\alpha^{(h_j)})'(h_j) = 1, \qquad (\alpha^{(h_j)})'(2h_j) = 0, \\ \alpha^{(h_j)}(x_1) = 0 \quad \text{for } x_1 \geqslant 2h_j, \qquad \sup_j \|\alpha^{(h_j)}\|_\infty < Ch_j, \qquad \sup_j \|(\alpha^{(h_j)})'\|_\infty < \infty\end{gathered}$$

(e.g. $\alpha^{(h_j)}(x_1) = (1/h_j^2)x_1^3 - (5/h_j)x_1^2 + 8x_1 - 4h_j$, for $x_1 \in [h_j, 2h_j]$ and 0 otherwise). The functions $\boldsymbol{\beta}_i^j \colon [0, L_i] \times S_i \to \mathbb{R}^3$ are chosen such that $\boldsymbol{\beta}_i^j(x) = \gamma_i^j(x_1)\boldsymbol{\beta}_i(x)$, where $\gamma_i^j \in C^1([0, L_i]; \mathbb{R})$ are such that

$$\begin{gathered}\gamma_i^j(x_1) = 0 \quad \text{for } x_1 \leqslant h_j, \qquad \gamma_i^j(x_1) = 1 \quad \text{for } x_1 \geqslant 2h_j, \\ \|\gamma_i^j\|_\infty < C, \qquad \|(\gamma_i^j)'\|_\infty < \frac{C}{h_j}\end{gathered}$$

(e.g. $\gamma_i^j(x) = -(2/h_j^3)x^3 + (9/h_j^2)x^2 - (12/h_j)x + 5$) and $\boldsymbol{\beta}_i \in C^1([0, L_i] \times S_i; \mathbb{R}^3)$. Then we have

$$\nabla \boldsymbol{y}^{(h_j)}(x) = R_i(0)Q_i^{\mathrm{T}} \quad \text{for } x \in T^{h_j},$$

and

$$\begin{aligned}&\nabla \boldsymbol{y}^{(h_j)}(Q_i\boldsymbol{P}^{(h_j)}(x))Q_i \\ &\quad = R_i(x_1 - h_j) - (0|\alpha^{(h_j)}(x_1)(\boldsymbol{d}_i^2)'(0)|\alpha^{(h_j)}(x_1)(\boldsymbol{d}_i^3)'(0))1_{h_j \leqslant x_1 \leqslant 2h_j} \\ &\qquad + h_j(x_2(\boldsymbol{d}_i^2)'(x_1 - h_j) + x_3(\boldsymbol{d}_i^3)'(x_1 - h_j)|\partial_2\boldsymbol{\beta}_i^j(x)|\partial_3\boldsymbol{\beta}_i^j(x)) \\ &\qquad - h_j(x_2(\alpha^{(h_j)})'(x_1)(\boldsymbol{d}_i^2)'(0) + x_3(\alpha^{(h_j)})'(x_1)(\boldsymbol{d}_i^3)'(0)|0|0)1_{h_j \leqslant x_1 \leqslant 2h_j} \\ &\qquad + h_j^2(\partial_1\boldsymbol{\beta}_i^j(x)|0|0) \quad \text{for } x \in (h_j, L_i) \times S_i.\end{aligned}$$

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Note that $\boldsymbol{y}^{(h_j)} \in C^1(\Omega^{h_j};\mathbb{R}^3)$. It can be easily seen, by the dominated convergence theorem, that for every i we have

$$\lim_{j\to\infty} \frac{1}{h_j} \|\boldsymbol{y}^{(h_j)} \circ Q_i - \boldsymbol{y}_i\|_{L^2((h_j,L_i)\times h_j S_i;\mathbb{R}^3)} = 0$$

and

$$\lim_{j\to\infty} \frac{1}{h_j} \|\nabla\boldsymbol{y}^{(h_j)} \circ Q_i Q_i - R_i\|_{L^2((h_j,L_i)\times h_j S_i;\mathbb{R}^3)} = 0,$$

which together imply that $\boldsymbol{y}^{(h_j)}$ satisfies (4.17). Now we have to prove the lim sup inequality for this sequence.

For $x \in (h_j, L_i) \times S_i$, let us define

$$B_i^{(h_j)}(x) = \frac{R_i(x_1-h_j)^{\mathrm{T}} \nabla \boldsymbol{y}^{(h_j)}(Q_i \boldsymbol{P}^{(h_j)}(x)) - Q_i^{\mathrm{T}}}{h_j}. \tag{4.19}$$

Then

$$\begin{aligned} B_i^{(h_j)}(x) Q_i = {} & R_i(x_1-h_j)^{\mathrm{T}} (x_2 (\boldsymbol{d}_i^2)'(x_1-h_j) + x_3(\boldsymbol{d}_i^3)'(x_1-h_j) | \partial_2 \boldsymbol{\beta}_i^j(x) | \partial_3 \boldsymbol{\beta}_i^j(x)) \\ & - R_i(x_1-h_j)^{\mathrm{T}} \left(0 \middle| \frac{\alpha^{(h_j)}(x_1)}{h_j} (\boldsymbol{d}_i^2)'(0) \middle| \frac{\alpha^{(h_j)}(x_1)}{h_j} (\boldsymbol{d}_i^3)'(0)\right) 1_{h_j \leqslant x_1 \leqslant 2h_j} \\ & - R_i(x_1-h_j)^{\mathrm{T}} (x_2 (\alpha^{(h_j)})'(x_1)(\boldsymbol{d}_i^2)'(0) \\ & \qquad + x_3 (\alpha^{(h_j)})'(x_1)(\boldsymbol{d}_i^3)'(0)|0|0) 1_{h_j \leqslant x_1 \leqslant 2h_j} \\ & + h_j R_i(x_1-h_j)^{\mathrm{T}} (\partial_1 \boldsymbol{\beta}_i^j(x)|0|0). \end{aligned}$$

Note that for every $\delta > 0$ one has

$$B_i^{(h_j)}(x) Q_i \to R_i(x_1)^{\mathrm{T}} (x_2 (\boldsymbol{d}_i^2)'(x_1) + x_3 (\boldsymbol{d}_i^3)'(x_1) | \partial_2 \boldsymbol{\beta}_i(x) | \partial_3 \boldsymbol{\beta}_i(x)) \quad \text{a.e. } x \in (\delta, L_i).$$

For every i, we look at the sequence $(f_j^i)_j$ of functions $f_j^i\colon (0,L_i)\times S_i \to [0,+\infty)$ defined by

$$\begin{aligned} f_j^i(x) &= 0 \quad \text{for } x \in (0,h_j)\times S_i, \\ f_j^i(x) &= \frac{1}{h_j^2} W(\nabla \boldsymbol{y}^{(h_j)}(Q_i \boldsymbol{P}^{(h_j)}(x))) \\ &= \frac{1}{h_j^2} W(Q_i^{\mathrm{T}} + h_j B_i^{(h_j)}(x)) \quad \text{for } x \in (h_j, L_i)\times S_i, \end{aligned}$$

where the equality in the second line holds by the objectivity of W. Since W is C^2 in the neighbourhood of SO(3) and has extreme on SO(3) and $B_i^{(h_j)}$ is bounded, by the Taylor theorem, for every i, one has

$$f_j^i(x) \to \tfrac{1}{2} q_3^i (R_i^{\mathrm{T}} (x_2 (\boldsymbol{d}_i^2)' + x_3 (\boldsymbol{d}_i^3)' | \partial_2 \boldsymbol{\beta}_i | \partial_3 \boldsymbol{\beta}_i)) \quad \text{a.e. } x \in (0,L_i)\times S_i.$$

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Since $B_i^{(h_j)}$ is bounded, the sequence f_j^i is bounded in $L^\infty((0,L_i)\times S_i;\mathbb{R}^3)$, also by the Taylor theorem. Thus, by the dominated convergence theorem, we have

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{h_j^4}\int_{C_i^{h_j}} W(\nabla \boldsymbol{y}^{(h_j)})\,\mathrm{d}x &= \lim_{j\to\infty}\int_{(h_j,L_i)\times S_i}\frac{1}{h_j^2}W(\nabla \boldsymbol{y}^{(h_j)}(Q_i\boldsymbol{P}^{(h_j)}(x)))\,\mathrm{d}x\\ &= \lim_{j\to\infty}\int_{(0,L_i)\times S_i} f_j^i(x)\,\mathrm{d}x\\ &= \frac{1}{2}\int_{(0,L_i)\times S_i} q_3^i(R_i^{\mathrm{T}}(x_2(\boldsymbol{d}_i^2)'+x_3(\boldsymbol{d}_i^3)'|\partial_2\boldsymbol{\beta}_i|\partial_3\boldsymbol{\beta}_i)).\end{aligned}$$

Also note that, for the chosen sequence $\boldsymbol{y}^{(h_j)}$, for every j, one has

$$W(\nabla \boldsymbol{y}^{(h_j)}|_{T^{h_j}})=0,$$

and thus

$$\begin{aligned}&\lim_{j\to\infty}\frac{1}{h_j^4}\int_{\Omega^{h_j}} W(\nabla \boldsymbol{y}^{(h_j)})\,\mathrm{d}x\\ &\quad=\sum_{i=1}^n\lim_{j\to\infty}\frac{1}{h_j^4}\int_{C_i^{h_j}} W(\nabla \boldsymbol{y}^{(h_j)})\,\mathrm{d}x\\ &\quad=\sum_{i=1}^n\frac{1}{2}\int_{(0,L_i)\times S_i} q_3^i(R_i^{\mathrm{T}}(x_2(\boldsymbol{d}_i^2)'+x_3(\boldsymbol{d}_i^3)'|\partial_2\boldsymbol{\beta}_i|\partial_3\boldsymbol{\beta}_i)).\end{aligned}$$

Thus, for $(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)\in\mathcal{A}$. such that $\boldsymbol{y}_i\in C^2([0,L_i];\mathbb{R}^3)$, $\boldsymbol{d}_i^2,\boldsymbol{d}_i^3\in C^1([0,L_i];\mathbb{R}^3)$ and arbitrary $\boldsymbol{\beta}_i\in C^1([0,L_i]\times S_i;\mathbb{R}^3)$, we have that there exists a sequence $(\boldsymbol{y}^{(h_j)})\subset W^{1,2}(\Omega^{h_j};\mathbb{R}^3)$ such that

$$\lim_{j\to\infty}\frac{1}{h_j^2}\sum_{i=1}^n\|\boldsymbol{y}^{(h_j)}\circ Q_i - D_i(\boldsymbol{y}_i,\boldsymbol{d}_i^2,\boldsymbol{d}_i^3)\|^2_{W^{1,2}((h_j,L_i)\times h_jS_i;\mathbb{R}^3)}=0 \tag{4.20}$$

and

$$\lim_{j\to\infty}\frac{1}{h_j^4}E^{(h_j)}(\boldsymbol{y}^{(h_j)})=\frac{1}{2}\sum_{i=1}^n\int_{(0,L_i)\times S_i} q_3^i(R_i^{\mathrm{T}}(x_2(\boldsymbol{d}_i^2)'+x_3(\boldsymbol{d}_i^3)'|\partial_2\boldsymbol{\beta}_i|\partial_3\boldsymbol{\beta}_i)). \tag{4.21}$$

Let us now consider the general case and take an arbitrary $(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)\in\mathcal{A}$. For every i we choose a sequence $(\tilde{R}_i^{(j)})\subset C^1([0,L_i];\mathbb{M}^{3\times3})$ such that $\tilde{R}_i^{(j)}\to(\boldsymbol{y}_i'|\boldsymbol{d}_i^2|\boldsymbol{d}_i^3)=R_i$ in $W^{1,2}((0,L_i);\mathbb{M}^{3\times3})$. By making a slight correction, namely taking

$$\hat{R}_i^{(j)}=R_i(0)\tilde{R}_i^{(j)}(0)^{-1}\tilde{R}_i^{(j)}$$

(this can be done for j large enough, due to the Sobolev embedding theorem), we also have

$$(\hat{R}_i^{(j)})\subset C^1([0,L_i];\mathbb{M}^{3\times3}),\qquad \hat{R}_i^{(j)}\to R_i=(\boldsymbol{y}_i'|\boldsymbol{d}_i^2|\boldsymbol{d}_i^3)\quad\text{in } W^{1,2}((0,L_i);\mathbb{M}^{3\times3})$$

(this follows from the trace theorem) and $\hat{R}_i^{(j)}(0)=R_i(0)$. Now take $R_i^{(j)}=\Pi\hat{R}_i^{(j)}$ where $\Pi\colon\mathbb{M}^{3\times3}\to\mathbb{M}^{3\times3}$ is a smooth function in the neighbourhood of SO(3)

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defining projection from the neighbourhood of SO(3) to SO(3). We define

$$\boldsymbol{y}_i^{(j)}(x_1) := \boldsymbol{y}_i(0) + \int_0^{x_1} R^{(j)}(s)\boldsymbol{e}_1\,\mathrm{d}s, \qquad \boldsymbol{d}_k^{i,(j)} = R^{(j)}(x_1)\boldsymbol{e}_k \quad \text{for } k = 2, 3.$$

Then

$$((\boldsymbol{y}_1^{(j)}, \boldsymbol{d}_1^{2,(j)}, \boldsymbol{d}_1^{3,(j)}), \dots, (\boldsymbol{y}_n^{(j)}, \boldsymbol{d}_n^{2,(j)}, \boldsymbol{d}_n^{3,(j)})) \in \mathcal{A}$$

and $\boldsymbol{y}_i^{(j)}$ in $C^2([0, L_i]; \mathbb{R}^3)$, $\boldsymbol{d}_i^{2,(j)}, \boldsymbol{d}_i^{3,(j)} \in C^1([0, L_i]; \mathbb{R}^3)$ and we also have that

$$((\boldsymbol{y}_i^{(j)})', \boldsymbol{d}_i^{2,(j)}, \boldsymbol{d}_i^{3,(j)}) = R_i^{(j)}$$

is converging to $R_i = (\boldsymbol{y}_i' | \boldsymbol{d}_i^2 | \boldsymbol{d}_i^3)$ in $W^{1,2}((0, L_i); \mathbb{M}^{3\times 3})$. The functions $\boldsymbol{\beta}_i$ are chosen in the following way. We choose $\boldsymbol{\alpha}_i(x_1, \cdot) \in V_i$ (see remark 4.5) to be the solution of the minimum problem defining $q_i^2(R_i^{\mathrm{T}}(x_1)R_i'(x_1))$ (the affine function of $R_i^{\mathrm{T}}(x_1)R_i'(x_1)$). Now take $\boldsymbol{\beta}_i = R_i\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i^{(j)} \in C^1([0, L_i] \times S_i; \mathbb{R}^3)$ defined by convolution (first by the first variable and then by the last two variables) such that $\boldsymbol{\beta}_i^{(j)} \to \boldsymbol{\beta}_i$ and $\partial_k\boldsymbol{\beta}_i^{(j)} \to \partial_k\boldsymbol{\beta}_i$ (for $k = 2, 3$) in $L^2(\Omega; \mathbb{R}^3)$. By an application of the Nemytskii operator theory [3, p. 15] we have that, for every i,

$$\int_{[0,L_i]} q_i^3((R_i^{(j)})^{\mathrm{T}}(x_2(\boldsymbol{d}_i^2)' + x_3(\boldsymbol{d}_i^3)' | \partial_2\boldsymbol{\beta}_i^{(j)} | \partial_3\boldsymbol{\beta}_i^{(j)}))\,\mathrm{d}x \\ \to \int_{[0,L_i]} q_i^3((x_2 R_i^{\mathrm{T}}(\boldsymbol{d}_i^2)' + x_3 R_i^{\mathrm{T}}(\boldsymbol{d}_i^3)' | \partial_2\boldsymbol{\alpha}_i | \partial_3\boldsymbol{\alpha}_i))\,\mathrm{d}x.$$

Therefore, we can assume (by taking a subsequence) that

$$\bigg| I(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \\ - \frac{1}{2}\sum_{i=1}^n \int_{[0,L_i]} q_i^3((R_i^{(j)})^{\mathrm{T}}(x_2(\boldsymbol{d}_i^{2,(j)})' + x_3(\boldsymbol{d}_i^{3,(j)})' | \partial_2\boldsymbol{\beta}_i^{(j)} | \partial_3\boldsymbol{\beta}_i^{(j)}))\,\mathrm{d}x \bigg| < \frac{1}{j}.$$

From (4.20) and (4.21) for a given j, we can find $\boldsymbol{y}^{(h_j)} \in W^{1,2}(\Omega^{h_j}; \mathbb{R}^3)$ such that

$$\frac{1}{h_j^2}\sum_{i=1}^n \|\boldsymbol{y}^{(h_j)} \circ Q_i - D_i(\boldsymbol{y}_i^{(j)}, \boldsymbol{d}_i^{2,(j)}, \boldsymbol{d}_i^{3,(j)})\|^2_{W^{1,2}((h_j, L_i)\times h_j S_i; \mathbb{R}^3)} < \frac{1}{j}$$

and

$$\bigg| \frac{1}{h_j^4} E^{(h_j)}(\boldsymbol{y}^{(h_j)}) \\ - \frac{1}{2}\sum_{i=1}^n \int_{(0,L_i)\times S_i} q_3^i((R_i^{(j)})^{\mathrm{T}}(x_2(\boldsymbol{d}_i^{2,(j)})' + x_3(\boldsymbol{d}_i^{3,(j)})' | \partial_2\boldsymbol{\beta}_i^{(j)} | \partial_3\boldsymbol{\beta}_i^{(j)})) \bigg| < \frac{1}{j}.$$

By the triangle inequality we have that $\boldsymbol{y}^{(h_j)}$ satisfies (4.17) and

$$\left| \frac{1}{h_j^4} E^{(h_j)}(\boldsymbol{y}^{(h_j)}) - I(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \right| \to 0.$$

The case $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \notin \mathcal{A}$ is obvious. □

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5. Minimization

Since we cannot formulate the junction problem on a canonical domain in a simple way, we have to adapt techniques of Γ-convergence and use the asymptotics of the infimizing sequence in the form (4.17).

We suppose that the external body force is given by the density $\boldsymbol{f}_{\mathrm{r}}^{(h)} \in L^2(\Omega^h;\mathbb{R}^3)$ and that the external surface force is given by the density $\boldsymbol{g}_{\mathrm{r}}^{(h)} \in L^2(\partial\Omega^h;\mathbb{R}^3)$ (we assume both are dead loads). As is usual in lower-dimensional modelling, the scaling of the surface force densities is different at the rod ends and the lateral boundary. Therefore, we introduce the notation

$$\boldsymbol{g}_{\mathrm{rl}}^{(h)} = \boldsymbol{g}_{\mathrm{r}}^{(h)}|_{\partial\Omega^h\setminus\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i)}, \qquad \boldsymbol{g}_{\mathrm{re}}^{(h)} = \boldsymbol{g}_{\mathrm{r}}^{(h)}|_{\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i)}.$$

We give the result for the Neumann boundary condition on the whole domain, i.e. for the pure traction problem. Therefore, we suppose that the resultant of all forces is zero, i.e.

$$\int_{\Omega^h} \boldsymbol{f}_{\mathrm{r}}^{(h)}(x)\,\mathrm{d}x + \int_{\partial\Omega^h} \boldsymbol{g}_{\mathrm{r}}^{(h)}(x)\,\mathrm{d}x = 0,$$

and look for the minimum that satisfies

$$\fint_{Q_1(\{L_1\}\times hS_1)} \boldsymbol{y}^{(h)}(x)\,\mathrm{d}x = 0.$$

THEOREM 5.1. *For every h we define the functional*

$$J^{(h)}(\boldsymbol{v}) = \int_{\Omega^h} W(\nabla\boldsymbol{v}(x))\,\mathrm{d}x - \int_{\Omega^h} \boldsymbol{f}_{\mathrm{r}}^{(h)}(x)\cdot\boldsymbol{v}(x)\,\mathrm{d}x - \int_{\partial\Omega^h} \boldsymbol{g}_{\mathrm{r}}^{(h)}(x)\cdot\boldsymbol{v}(x)\,\mathrm{d}x$$

in the space

$$V^h = \left\{\boldsymbol{v}\in W^{1,2}(\Omega^h;\mathbb{R}^3)\;\middle|\;\fint_{Q_1(\{L_1\}\times hS_1)} \boldsymbol{v}(x)\,\mathrm{d}x = 0\right\}.$$

Let the scaling of loads be as follows:

$$\boldsymbol{f}^{(h)} = \frac{\boldsymbol{f}_{\mathrm{r}}^{(h)}}{h^2}, \qquad \boldsymbol{g}_{\mathrm{l}}^{(h)} = \frac{\boldsymbol{g}_{\mathrm{rl}}^{(h)}}{h^3}, \qquad \boldsymbol{g}_{\mathrm{e}}^{(h)} = \frac{\boldsymbol{g}_{\mathrm{re}}^{(h)}}{h^2},$$

where

$$\frac{1}{h}\|\boldsymbol{f}^{(h)}\|_{L^2(\Omega^h;\mathbb{R}^3)}, \quad \frac{1}{h}\|\boldsymbol{g}_{\mathrm{e}}^{(h)}\|_{L^2(\bigcup_{i=1}^n\{L_i\}\times hS_i;\mathbb{R}^3)}, \quad \frac{1}{h}\|\boldsymbol{g}_{\mathrm{l}}^{(h)}\|^2_{L^2(\partial\Omega^h\setminus\bigcup_{i=1}^n\{L_i\}\times hS_i;\mathbb{R}^3)}$$

are bounded. Moreover, let us suppose that

$$\int_{\Omega^h} \boldsymbol{f}_{\mathrm{r}}^{(h)}(x)\,\mathrm{d}x + \int_{\partial\Omega^h} \boldsymbol{g}_{\mathrm{r}}^{(h)}(x)\,\mathrm{d}x = 0, \tag{5.1}$$

$$\lim_{h\to 0}\sum_{i=1}^n \frac{1}{h^2}\int_{C_i^h} \|\boldsymbol{f}^{(h)}(x) - \boldsymbol{f}_i((\boldsymbol{P}^{(h)})^{-1}(Q_i^{\mathrm{T}}x))\|^2\,\mathrm{d}x = 0, \tag{5.2}$$

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$$\lim_{h\to 0}\sum_{i=1}^{n}\frac{1}{h}\int_{Q_i((h,L_i)\times h\partial S_i)}\|\boldsymbol{g}_{\mathrm{l}}^{(h)}(x)-\boldsymbol{g}_{\mathrm{l}i}((\boldsymbol{P}^{(h)})^{-1}(Q_i^{\mathrm{T}}x))\|^2\,\mathrm{d}x=0, \tag{5.3}$$

$$\lim_{h\to 0}\sum_{i=1}^{n}\frac{1}{h^2}\int_{Q_i(\{L_i\}\times hS_i)}\|\boldsymbol{g}_{\mathrm{e}}^{(h)}(x)-\boldsymbol{g}_{\mathrm{e}i}((\boldsymbol{P}^{(h)})^{-1}(Q_i^{\mathrm{T}}x))\|^2\,\mathrm{d}x=0, \tag{5.4}$$

where $\boldsymbol{f}_i\in L^2((0,L_i)\times S_i;\mathbb{R}^3)$, $\boldsymbol{g}_{\mathrm{l}i}\in L^2((0,L_i)\times\partial S_i;\mathbb{R}^3)$, $\boldsymbol{g}_{\mathrm{e}i}\in L^2(\{L_i\}\times S_i;\mathbb{R}^3)$ *for* $i=1,\dots,n$.

Then we have that $|\inf_{\boldsymbol{v}\in V^h}J^{(h)}(\boldsymbol{v})|\leqslant Ch^4$. *Let us take the sequence* $\boldsymbol{y}^{(h)}\in V^h$ *that satisfies*

$$J^{(h)}(\boldsymbol{y}^{(h)})\leqslant\inf_{\boldsymbol{v}\in V^h}J^{(h)}(\boldsymbol{v})+o(h^4), \tag{5.5}$$

($o(h^4)$ *means that* $\lim_{h\to 0}o(h^4)/h^4=0$). *Let the sequence* (h_j) *converge to* 0. *Then there exist a subsequence of* (h_j) *(still denoted by* h_j*) and* $(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)\in\mathcal{A}$ *such that*

$$\lim_{j\to\infty}\frac{1}{h_j^2}\sum_{i=1}^{n}\|\boldsymbol{y}^{(h_j)}\circ Q_i-D_i(\boldsymbol{y}_i,\boldsymbol{d}_i^2,\boldsymbol{d}_i^3)\|^2_{W^{1,2}((h_j,L_i)\times h_jS_i;\mathbb{R}^3)}=0. \tag{5.6}$$

The limit $(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)$ *minimizes the functional*

$$\begin{aligned}J(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)=I(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)&-\sum_{i=1}^{n}\int_0^{L_i}\int_{S_i}\boldsymbol{f}_i(x)\,\mathrm{d}x_2\,\mathrm{d}x_3\cdot\boldsymbol{y}_i(x_1)\,\mathrm{d}x_1\\&-\sum_{i=1}^{n}\int_0^{L_i}\int_{\partial S_i}\boldsymbol{g}_{\mathrm{l}i}(x)\,\mathrm{d}s\cdot\boldsymbol{y}_i(x_1)\,\mathrm{d}x_1\\&-\sum_{i=1}^{n}\int_{\{L_i\}\times S_i}\boldsymbol{g}_{\mathrm{e}i}(L_i,x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3\cdot\boldsymbol{y}_i(L_i)\end{aligned}$$

in the space $V_{\mathrm{l}}=\{(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3)\in\mathcal{A}\colon\boldsymbol{y}_1(L_1)=0\}$. *Moreover, the energies converge to the energy of the limit*

$$\lim_{h\to 0}\frac{1}{h^4}J^{(h)}(\boldsymbol{y}^{(h)})=J(\boldsymbol{y},\boldsymbol{d}^2,\boldsymbol{d}^3).$$

Proof.

STEP 1 (*a priori* estimate for the total energy and $\boldsymbol{y}^{(h)}$). Let us estimate

$$|J^{(h)}(\boldsymbol{i}+\boldsymbol{a}^{(h)})|,$$

where $\boldsymbol{i}$ is the identity mapping and $\boldsymbol{a}^{(h)}\in\mathbb{R}^3$ is chosen such that $\boldsymbol{i}+\boldsymbol{a}^{(h)}\in V^h$ (such an $\boldsymbol{a}^{(h)}$ exists and is unique). Using (5.1) and $W(I)=0$, we obtain

$$\begin{aligned}|J^{(h)}(\boldsymbol{i}+\boldsymbol{a}^{(h)})|&=\left|\int_{\Omega^h}\boldsymbol{f}_{\mathrm{r}}^{(h)}(x)\cdot\boldsymbol{i}(x)\,\mathrm{d}x+\int_{\partial\Omega^h}\boldsymbol{g}_{\mathrm{r}}^{(h)}(x)\cdot\boldsymbol{i}(x)\,\mathrm{d}x\right|\\&\leqslant Ch^3\|\boldsymbol{f}^{(h)}\|_{L^2(\Omega^h)}+Ch^{7/2}\|\boldsymbol{g}_{\mathrm{l}}^{(h)}\|_{L^2(\partial\Omega^h\setminus(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i)))}\\&\quad+Ch^3\|\boldsymbol{g}_{\mathrm{e}}^{(h)}\|_{L^2(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i))}\\&\leqslant Ch^4.\end{aligned}$$

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Then from (5.5) we conclude that $(1/h^4)J^{(h)}(\boldsymbol{y}^{(h)}) \leqslant C$. From this we want to conclude that

$$\frac{1}{h^4}\int_{\Omega^h} \operatorname{dist}^2(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3))^2\,\mathrm{d}x < \infty,$$

so we have to estimate the energy from below:

$$\begin{aligned}\frac{1}{h^4}J^{(h)}(\boldsymbol{y}^{(h)}) &\geqslant C_W\frac{1}{h^4}\int_{\Omega^h}\operatorname{dist}^2(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3)) - \frac{1}{h^2}\|\boldsymbol{f}^{(h)}\|_{L^2(\Omega^h)}\|\boldsymbol{y}^{(h)}\|_{L^2(\Omega^h)}\\ &\quad - \frac{1}{h}\|\boldsymbol{g}_1^{(h)}\|_{L^2(\partial\Omega^h\setminus(\bigcup_{i=1}^n\{L_i\}\times hS_i))}\|\boldsymbol{y}^{(h)}\|_{L^2(\partial\Omega^h\setminus(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i)))}\\ &\quad - \frac{1}{h^2}\|\boldsymbol{g}_\mathrm{e}^{(h)}\|_{L^2(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i))}\|\boldsymbol{y}^{(h)}\|_{L^2(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i))}\\ &\geqslant C_W\frac{1}{h^4}\int_{\Omega^h}\operatorname{dist}^2(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\\ &\quad - C\biggl(\frac{1}{h}\|\boldsymbol{y}^{(h)}\|_{L^2(\Omega^h)} + \frac{1}{h^{1/2}}\|\boldsymbol{y}^{(h)}\|_{L^2(\partial\Omega^h\setminus(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i)))}\\ &\qquad\qquad + \frac{1}{h}\|\boldsymbol{y}^{(h)}\|_{L^2(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i))}\biggr).\end{aligned} \tag{5.7}$$

In the same way as in lemma 4.1, we conclude that there exists a constant C independent of h (using rescaling $\alpha^{(h)}(x_1,x_2,x_3) = (x_1, hx_2, hx_3)$) such that for every i and every $\boldsymbol{y} \in W^{1,2}(\Omega^h;\mathbb{R}^3)$ we have

$$\biggl\|\boldsymbol{y} - ⨍_{Q_i(\{L_i\}\times hS_i)}\boldsymbol{y}\,\mathrm{d}x\biggr\|_{L^2(C_i^h;\mathbb{R}^3)} \leqslant C\|\nabla\boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}, \tag{5.8}$$

$$\biggl\|\boldsymbol{y} - ⨍_{Q_i(\{h\}\times hS_i)}\boldsymbol{y}\,\mathrm{d}x\biggr\|_{L^2(C_i^h;\mathbb{R}^3)} \leqslant C\|\nabla\boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}. \tag{5.9}$$

From this we conclude that there exists a constant C independent of h such that

$$\biggl\|⨍_{Q_i(\{h\}\times hS_i)}\boldsymbol{y}\,\mathrm{d}x - ⨍_{Q_i(\{L_i\}\times hS_i)}\boldsymbol{y}\,\mathrm{d}x\biggr\| \leqslant \frac{C}{h}\|\nabla\boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}. \tag{5.10}$$

By using scaling $\alpha^{(h)}(x_1,x_2,x_3) = (hx_1, hx_2, hx_3)$ we conclude that there exists C independent of h such that, for every i, l,

$$\biggl\|\boldsymbol{y} - ⨍_{Q_i(\{h\}\times hS_i)}\boldsymbol{y}\,\mathrm{d}x\biggr\|_{L^2(T^h;\mathbb{R}^3)} \leqslant hC\|\nabla\boldsymbol{y}\|_{L^2(T^h;\mathbb{R}^3)}, \tag{5.11}$$

$$\biggl\|⨍_{Q_i(\{h\}\times hS_i)}\boldsymbol{y}\,\mathrm{d}x - ⨍_{Q_l(\{h\}\times hS_l)}\boldsymbol{y}\,\mathrm{d}x\biggr\| \leqslant \frac{C}{\sqrt{h}}\|\nabla\boldsymbol{y}\|_{L^2(T^h;\mathbb{R}^3)}. \tag{5.12}$$

Using estimate (5.8) for $i = 1$ and the fact that $\boldsymbol{y}^{(h)} \in V^h$, we conclude that

$$\|\boldsymbol{y}^{(h)}\|_{L^2(C_1^h;\mathbb{R}^3)} \leqslant C\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(C_1^h;\mathbb{R}^3)} \leqslant C(\|\operatorname{dist}(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h).$$

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Using estimate (5.10) we conclude that

$$\left\| \fint_{Q_1(\{h\}\times hS_1)} \boldsymbol{y}^{(h)} \right\| \leqslant \frac{C}{h} \|\nabla \boldsymbol{y}^{(h)}\|_{L^2(C_1^h;\mathbb{R}^3)} \\ \leqslant C\left(\frac{1}{h}\left\|\mathrm{dist}(\nabla \boldsymbol{y}^{(h)},\mathrm{SO}(3))\right\|_{L^2(C_1^h;\mathbb{R}^3)} + 1\right).$$

Using estimate (5.11) we conclude that

$$\|\boldsymbol{y}^{(h)}\|_{L^2(T^h;\mathbb{R}^3)} \leqslant h^{3/2}\left\| \fint_{Q_1(\{h\}\times hS_1)} \boldsymbol{y}^{(h)} \right\| + Ch\|\nabla \boldsymbol{y}^{(h)}\|_{L^2(T^h;\mathbb{R}^3)}.$$

Since

$$\|\nabla \boldsymbol{y}^{(h)}\|_{L^2(T^h;\mathbb{R}^3)} \leqslant C(\|\mathrm{dist}(\nabla \boldsymbol{y}^{(h)},\mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h^{3/2}), \tag{5.13}$$

we conclude that

$$\|\boldsymbol{y}^{(h)}\|_{L^2(T^h;\mathbb{R}^3)} \leqslant C(h^{1/2}\|\,\mathrm{dist}(\nabla \boldsymbol{y}^{(h)},\mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h^{3/2}). \tag{5.14}$$

Using estimates (5.9) and (5.12) for $l = 1$ we conclude that, for every i,

$$\|\boldsymbol{y}^{(h)}\|_{L^2(C_i^h;\mathbb{R}^3)} \leqslant C(\|\mathrm{dist}(\nabla \boldsymbol{y}^{(h)},\mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h).$$

Thus, we have

$$\|\boldsymbol{y}^{(h)}\|_{L^2(\Omega^h;\mathbb{R}^3)} \leqslant C(\|\mathrm{dist}(\nabla \boldsymbol{y}^{(h)},\mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h). \tag{5.15}$$

In the same way, one can analyse traces. First we start from the trace inequality on the cylinder $C_i = (0,1)\times S_i$. For every $\boldsymbol{y} \in W^{1,2}(C_i;\mathbb{R}^3)$ we have that there exists a constant C such that

$$\left\|\boldsymbol{y} - \fint_{\{1\}\times S_i} \boldsymbol{y}\right\|_{L^2(\partial C_i)} \leqslant C\left\|\boldsymbol{y} - \fint_{\{1\}\times S_i} \boldsymbol{y}\right\|_{W^{1,2}(C_i;\mathbb{R}^3)} \leqslant C\|\nabla \boldsymbol{y}\|_{L^2(C_i;\mathbb{R}^3)}, \tag{5.16}$$

$$\left\|\boldsymbol{y} - \fint_{\{0\}\times S_i} \boldsymbol{y}\right\|_{L^2(\partial C_i)} \leqslant C\left\|\boldsymbol{y} - \fint_{\{0\}\times S_i} \boldsymbol{y}\right\|_{W^{1,2}(C_i;\mathbb{R}^3)} \leqslant C\|\nabla \boldsymbol{y}\|_{L^2(C_i;\mathbb{R}^3)}. \tag{5.17}$$

By using appropriate scaling and rotation, we have that there exists a constant C such that for every C_i^h and $\boldsymbol{y} \in W^{1,2}(C_i^h;\mathbb{R}^3)$ we have

$$\left\|\boldsymbol{y} - \fint_{Q_i(\{h\}\times hS_i)} \boldsymbol{y}\right\|_{L^2(Q_i(\{L_i\}\times hS_i);\mathbb{R}^3)} \leqslant C\|\nabla \boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}, \tag{5.18}$$

$$\left\|\boldsymbol{y} - \fint_{Q_i(\{L_i\}\times hS_i)} \boldsymbol{y}\right\|_{L^2(Q_i(\{L_i\}\times hS_i);\mathbb{R}^3)} \leqslant C\|\nabla \boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}, \tag{5.19}$$

$$\left\|\boldsymbol{y} - \fint_{Q_i(\{h\}\times hS_i)} \boldsymbol{y}\right\|_{L^2(Q_i(\{h\}\times hS_i);\mathbb{R}^3)} \leqslant C\|\nabla \boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}, \tag{5.20}$$

$$\left\|\boldsymbol{y} - \fint_{Q_i(\{L_i\}\times hS_i)} \boldsymbol{y}\right\|_{L^2(Q_i(\{h\}\times hS_i);\mathbb{R}^3)} \leqslant C\|\nabla \boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)} \tag{5.21}$$

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and

$$\left\| \boldsymbol{y} - \fint_{Q_i(\{h\}\times hS_i)} \boldsymbol{y} \right\|_{L^2(Q_i((h,L_i)\times\partial hS_i);\mathbb{R}^3)} \leqslant C\frac{1}{h^{1/2}}\|\nabla\boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}, \tag{5.22}$$

$$\left\| \boldsymbol{y} - \fint_{Q_i(\{L_i\}\times hS_i)} \boldsymbol{y} \right\|_{L^2(Q_i((h,L_i)\times\partial hS_i);\mathbb{R}^3)} \leqslant C\frac{1}{h^{1/2}}\|\nabla\boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}, \tag{5.23}$$

$$\left\| \boldsymbol{y} - \fint_{Q_i(\{h\}\times hS_i)} \boldsymbol{y} \right\|_{L^2(Q_i((h,L_i)\times\partial hS_i);\mathbb{R}^3)} \leqslant C\frac{1}{h^{1/2}}\|\nabla\boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}, \tag{5.24}$$

$$\left\| \boldsymbol{y} - \fint_{Q_i(\{L_i\}\times hS_i)} \boldsymbol{y} \right\|_{L^2(Q_i((h,L_i)\times\partial hS_i);\mathbb{R}^3)} \leqslant C\frac{1}{h^{1/2}}\|\nabla\boldsymbol{y}\|_{L^2(C_i^h;\mathbb{R}^3)}. \tag{5.25}$$

In the same way, we conclude that

$$\left\| \boldsymbol{y} - \fint_{Q_i(\{h\}\times hS_i)} \boldsymbol{y} \right\|_{L^2(\partial T^h;\mathbb{R}^3)} \leqslant Ch^{1/2}\|\nabla\boldsymbol{y}\|_{L^2(T^h;\mathbb{R}^3)}. \tag{5.26}$$

Now, by using $\boldsymbol{y}^{(h)} \in V^h$, we have from (5.19) and (5.23) that

$$\begin{aligned} \|\boldsymbol{y}^{(h)}\|_{L^2(Q_1(\{L_1\}\times hS_1);\mathbb{R}^3)} &\leqslant C\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(C_1^h;\mathbb{R}^3)} \\ &\leqslant C(\|\operatorname{dist}(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h), \\ \|\boldsymbol{y}^{(h)}\|_{L^2(Q_1((h,L_1)\times\partial hS_1);\mathbb{R}^3)} &\leqslant C\frac{1}{h^{1/2}}\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(C_1^h;\mathbb{R}^3)} \\ &\leqslant C\left(\frac{1}{h^{1/2}}\|\operatorname{dist}(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h^{1/2}\right). \end{aligned}$$

From (5.18) and (5.19) we conclude that

$$\left\| \fint_{Q_1(\{h\}\times hS_1)} \boldsymbol{y}^{(h)} \right\| \leqslant \frac{C}{h}\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(C_1^h;\mathbb{R}^3)}. \tag{5.27}$$

From this and (5.12) we conclude that, for every i,

$$\begin{aligned} \left\| \fint_{Q_i(\{h\}\times hS_i)} \boldsymbol{y}^{(h)} \right\| &\leqslant \frac{C}{h}\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(C_1^h;\mathbb{R}^3)} + \frac{C}{h^{1/2}}\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(T^h;\mathbb{R}^3)} \\ &\leqslant \frac{C}{h}\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(\Omega^h;\mathbb{R}^3)}. \end{aligned} \tag{5.28}$$

From (5.22) and (5.15) we conclude that

$$\begin{aligned} &\|\boldsymbol{y}^{(h)}\|_{L^2(Q_i((h,L_i)\times\partial hS_i);\mathbb{R}^3)} \\ &\qquad \leqslant C\left(\frac{1}{h^{1/2}}\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(\Omega^h;\mathbb{R}^3)} + h^{1/2}\left\| \fint_{Q_i(\{h\}\times hS_i)} \boldsymbol{y}^{(h)} \right\|\right) \\ &\qquad \leqslant \frac{C}{h^{1/2}}\|\nabla\boldsymbol{y}^{(h)}\|_{L^2(\Omega^h;\mathbb{R}^3)} \\ &\qquad \leqslant C\left(\frac{1}{h^{1/2}}\|\operatorname{dist}(\nabla\boldsymbol{y}^{(h)},\mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h^{1/2}\right). \end{aligned} \tag{5.29}$$

From (5.26) for $i = 1$, (5.27) and (5.13) we conclude that

$$\|\boldsymbol{y}^{(h)}\|_{L^2(\partial T^h;\mathbb{R}^3)} \leqslant C\|\nabla \boldsymbol{y}^{(h)}\|_{L^2(T^h;\mathbb{R}^3)} \leqslant C(\|\mathrm{dist}(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h^{3/2}). \tag{5.30}$$

From (5.18) and (5.28) we conclude that, for every i,

$$\begin{aligned} \|\boldsymbol{y}^{(h)}\|_{L^2(Q_i(\{L_i\}\times hS_i);\mathbb{R}^3)} &\leqslant C\|\nabla \boldsymbol{y}^{(h)}\|_{L^2(C_i^h;\mathbb{R}^3)} \\ &\leqslant C(\|\mathrm{dist}(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h). \end{aligned} \tag{5.31}$$

From (5.29)–(5.31) we conclude that

$$\begin{aligned} \|\boldsymbol{y}^{(h)}\|_{L^2(\partial\Omega^h\setminus(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i)))} \\ \leqslant C\bigg(\frac{1}{h^{1/2}}\|\mathrm{dist}(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h^{1/2}\bigg), \end{aligned} \tag{5.32}$$

$$\|\boldsymbol{y}^{(h)}\|_{L^2(\bigcup_{i=1}^n Q_i(\{L_i\}\times hS_i))} \leqslant C(\|\mathrm{dist}(\nabla \boldsymbol{y}^{(h)}, \mathrm{SO}(3))\|_{L^2(\Omega^h;\mathbb{R}^3)} + h). \tag{5.33}$$

By using (5.7), (5.15), (5.32) and (5.33), we conclude that there exist C_2 and C_3 such that

$$\begin{aligned} C_W\frac{1}{h^4}\int_{\Omega^h} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}(x), \mathrm{SO}(3))\,\mathrm{d}x \\ - C_2\bigg(\frac{1}{h}\bigg(\int_{\Omega^h} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}(x), \mathrm{SO}(3))\,\mathrm{d}x\bigg)^{1/2} + 1\bigg) \\ \leqslant \frac{1}{h^4}J^{(h)}(\boldsymbol{y}^{(h)}) \\ \leqslant C_3. \end{aligned} \tag{5.34}$$

Using the fact that, for $h \leqslant 1$,

$$\begin{aligned} \frac{1}{h}\bigg(\int_{\Omega^h} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}(x), \mathrm{SO}(3))\,\mathrm{d}x\bigg)^{1/2} &\leqslant \frac{1}{h^2}\bigg(\int_{\Omega^h} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}(x), \mathrm{SO}(3))\,\mathrm{d}x\bigg)^{1/2} \\ &=: \alpha, \end{aligned}$$

we conclude from (5.34) that

$$C_W\alpha^2 - C_2\alpha \leqslant C_3,$$

which implies that α^2 is bounded, i.e, there exists $C > 0$ such that

$$\frac{1}{h^4}\int_{\Omega^h} \mathrm{dist}^2(\nabla \boldsymbol{y}^{(h)}(x), \mathrm{SO}(3))\,\mathrm{d}x \leqslant C, \tag{5.35}$$

which implies that the left-hand side of (5.34) is bounded as well. This implies

$$\Big|\inf_{\boldsymbol{v}\in V^h} J^{(h)}(\boldsymbol{v})\Big| \leqslant Ch^4.$$

STEP 2 (the convergence proof for $\boldsymbol{y}^{(h)}$ and the scaled total energy). The estimate (5.35) implies that the assumptions of theorem 3.1 (the compactness theorem)

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are satisfied. Therefore, the assumptions of corollary 4.2 are satisfied as well (with $C_{L_1} = 0$). Therefore, we conclude that for every sequence h_j there exists a subsequence (still denoted by h_j) and $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) = ((\boldsymbol{y}_1, \boldsymbol{d}_1^2, \boldsymbol{d}_1^3), \dots, (\boldsymbol{y}_n, \boldsymbol{d}_n^2, \boldsymbol{d}_n^3)) \in \mathcal{A}$ such that

$$\lim_{j\to\infty} \frac{1}{h_j^2} \sum_{i=1}^{n} \|\boldsymbol{y}^{(h_j)} \circ Q_i - D_i(\boldsymbol{y}_i, \boldsymbol{d}_i^2, \boldsymbol{d}_i^3)\|^2_{W^{1,2}((h_j,L_i)\times h_j S_i;\mathbb{R}^3)} = 0. \tag{5.36}$$

From this convergence it is obvious that

$$\boldsymbol{y}_1(L_1) = \lim_{j\to\infty} \fint_{Q_1(\{L_1\}\times h_j S_1)} \boldsymbol{y}^{(h_j)}(x)\,\mathrm{d}x = 0.$$

Thus, we have proved that $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \in V_{\mathrm{l}}$. What is left to be proven is that it minimizes the functional J in V_{l}. We can use a standard argument from the Γ-convergence, although we have variable domains (and cannot apply the Γ-convergence directly). Let $(\boldsymbol{y}_a, \boldsymbol{d}_a^2, \boldsymbol{d}_a^3) \in V_{\mathrm{l}}$ and $(\boldsymbol{y}_a, \boldsymbol{d}_a^2, \boldsymbol{d}_a^3) \neq (\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3)$. We have to prove that $J(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \leqslant J(\boldsymbol{y}_a, \boldsymbol{d}_a^2, \boldsymbol{d}_a^3)$. From the lim inf inequality from proposition 4.4 we conclude that

$$I(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \leqslant \liminf_{j\to\infty} \frac{1}{h_j^4} E^{(h_j)}(\boldsymbol{y}^{(h_j)}). \tag{5.37}$$

By using (5.14), (5.30) and (5.35) we have

$$\begin{aligned}
\frac{1}{h_j^4} \int_{T^{h_j}} |\boldsymbol{f}_{\mathrm{r}}^{(h_j)}(x) \cdot \boldsymbol{y}^{(h_j)}(x)|\,\mathrm{d}x &\leqslant \frac{1}{h_j^2} \|\boldsymbol{f}^{(h_j)}\|_{L^2(T^{h_j})} \|\boldsymbol{y}^{(h_j)}\|_{L^2(T^{h_j})} \\
&\leqslant \frac{1}{h_j^2} \|\boldsymbol{f}^{(h_j)}\|_{L^2(\Omega^{h_j})} C(h_j^2 + h_j^{3/2}) \to 0,
\end{aligned} \tag{5.38}$$

$$\begin{aligned}
\frac{1}{h_j^4} \int_{\partial T^{h_j} \setminus \bigcup_{i=1}^n \{h_j\}\times h_j S_i} &|\boldsymbol{g}_{\mathrm{r}}^{(h_j)}(x) \cdot \boldsymbol{y}^{(h_j)}(x)|\,\mathrm{d}x \\
&\leqslant \frac{1}{h_j} \|\boldsymbol{g}_{\mathrm{l}}^{(h_j)}\|_{L^2(\partial T^{h_j} \setminus \bigcup_{i=1}^n \{h_j\}\times h_j S_i)} \\
&\qquad \times \|\boldsymbol{y}^{(h_j)}\|_{L^2(\partial T^{h_j} \setminus \bigcup_{i=1}^n \{h_j\}\times h_j S_i)} \\
&\leqslant C(h_j^{3/2} + h_j) \to 0.
\end{aligned} \tag{5.39}$$

From this and (5.2)–(5.4), (5.15), (5.32), (5.33), (5.36) and (5.37) we conclude that

$$J(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \leqslant \liminf_{j\to\infty} \frac{1}{h_j^4} J^{(h_j)}(\boldsymbol{y}^{(h_j)}). \tag{5.40}$$

Let us, by the lim sup inequality from proposition 4.4, choose $\boldsymbol{y}_a^{(h_j)}$ such that

$$\lim_{j\to\infty} \frac{1}{h_j^2} \sum_{i=1}^{n} \|\boldsymbol{y}_a^{(h_j)} \circ Q_i - D_i(\boldsymbol{y}_{i,a}, \boldsymbol{d}_{i,a}^2, \boldsymbol{d}_{i,a}^3)\|^2_{W^{1,2}((h_j,L_i)\times h_j S_i;\mathbb{R}^3)} = 0 \tag{5.41}$$

and

$$\lim_{j\to\infty} \frac{1}{h_j^4} E^{(h_j)}(\boldsymbol{y}_a^{(h_j)}) = I(\boldsymbol{y}_a, \boldsymbol{d}_a^2, \boldsymbol{d}_a^3). \tag{5.42}$$

Let us choose $\boldsymbol{c}^{(h_j)} \in \mathbb{R}^3$ such that $\boldsymbol{z}_a^{(h_j)} = \boldsymbol{y}_a^{(h_j)} + \boldsymbol{c}^{(h_j)} \in V^{h_j}$. From the convergence (5.41) we conclude that

$$\lim_{j\to\infty} \boldsymbol{c}^{(h_j)} = 0.$$

Thus, we have that (5.41) is also satisfied for the sequence $\boldsymbol{z}_a^{(h_j)}$. We also see that (5.42) is also satisfied for $\boldsymbol{z}_a^{(h_j)}$. Therefore, using the lower bound on W, it follows that there exists a constant C such that

$$\sup_j \frac{1}{h_j^4} \int_{\Omega^{h_j}} \operatorname{dist}^2(\nabla \boldsymbol{z}_a^{(h_j)}(x), \mathrm{SO}(3))\,\mathrm{d}x < C.$$

In the same way as before we conclude

$$J(\boldsymbol{y}_a, \boldsymbol{d}_a^2, \boldsymbol{d}_a^3) = \lim_{j\to\infty} \frac{1}{h_j^4} J^{(h_j)}(\boldsymbol{y}_a^{(h_j)}). \tag{5.43}$$

Finally, from (5.5), (5.40) and (5.43) we have

$$J(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \leqslant \liminf_{j\to\infty} \frac{1}{h_j^4} J^{(h_j)}(\boldsymbol{y}^{(h_j)}) \leqslant \liminf_{j\to\infty} \frac{1}{h_j^4} J^{(h_j)}(\boldsymbol{y}_a^{(h_j)}) = J(\boldsymbol{y}_a, \boldsymbol{d}_a^2, \boldsymbol{d}_a^3).$$

That the energies converge can be easily seen by a standard argument in the Γ-convergence. (We first take the sequence $\boldsymbol{l}^{(h_j)}$ such that

$$\frac{1}{h_j^4} J^{(h_j)}(\boldsymbol{l}^{(h_j)}) \to J(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3)$$

and then, by using (5.5), conclude that

$$\lim_{j\to\infty} \frac{1}{h_j^4} J^{(h_j)}(\boldsymbol{l}^{(h_j)}) = \lim_{j\to\infty} \frac{1}{h_j^4} J^{(h_j)}(\boldsymbol{y}^{(h_j)}).$$

Since this can be done for any arbitrary sequence, we have the claim.) □

Remark 5.2. In the proof of the strong convergence of the deformations (5.36) we have only used the assumed boundedness of the external loads. Therefore, we can weaken the assumption of strong convergence of the loads given in (5.2)–(5.4) in the following way:

$$\boldsymbol{f}^{(h)} \circ Q_i \circ \boldsymbol{P}^{(h)} \chi_{(h,L_i)\times S_i} \rightharpoonup \boldsymbol{f}_i \quad \text{weakly in } L^2((0,L_i)\times S_i), \tag{5.44}$$

$$\boldsymbol{g}_{\mathrm{l}}^{(h)} \circ Q_i \circ \boldsymbol{P}^{(h)} \chi_{(h,L_i)\times \partial S_i} \rightharpoonup \boldsymbol{g}_{\mathrm{l}i} \quad \text{weakly in } L^2((0,L_i)\times \partial S_i), \tag{5.45}$$

$$\boldsymbol{g}_{\mathrm{e}}^{(h)} \circ Q_i \circ \boldsymbol{P}^{(h)} \chi_{\{L_i\}\times S_i} \rightharpoonup \boldsymbol{g}_{\mathrm{e}i} \quad \text{weakly in } L^2(\{L_i\}\times S_i), \tag{5.46}$$

where by χ we have denoted the characteristic function of the appropriate set.

Remark 5.3. Using (5.38) and (5.39), by a straightforward calculation from (5.1), we obtain

$$\sum_{i=1}^n \left(\int_0^{L_i} \left(\int_{S_i} \boldsymbol{f}_i(x)\,\mathrm{d}x_2\,\mathrm{d}x_3 + \int_{\partial S_i} \boldsymbol{g}_{\mathrm{l}i}(x)\,\mathrm{d}s \right) + \int_{S_i} \boldsymbol{g}_{\mathrm{e}i}(L_i, x_2, x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 \right) = 0.$$

This means that in the limit model the total force is zero as well. This can also be concluded under the assumptions of remark 5.2.

REMARK 5.4. Adding a constant to the solution of a pure traction problem gives a solution again, i.e. the set of solutions is closed under translations. Therefore, we had to control behaviour of this constant in the three-dimensional problem in order to obtain the limit. We did this by ensuring that the mean deformation at the end of the first rod (indexed by 1) vanishes. As expected, a consequence of this constraint in the limit model is the constraint that the end of the first rod is fixed at the origin ($\boldsymbol{y}_1(L_1) = 0$). In the limit model we can also consider this constraint as the one which just fixes the translation, since again the set of solutions of the pure traction problem is closed under translations.

6. Differential formulation of the model

In this section we formulate the weak and the differential formulations of the model. This enables us to interpret the limit model as a model of one-dimensional rods with the transmission conditions at the junction point (see (6.6)–(6.12)).

Let us define

$$\tilde{\boldsymbol{f}}_i(x_1) = \int_{S_i} \boldsymbol{f}_i(x)\,\mathrm{d}x_2\,\mathrm{d}x_3 + \int_{\partial S_i} \boldsymbol{g}_{\mathrm{l}i}(x)\,\mathrm{d}s,$$
$$\tilde{\boldsymbol{F}}_i = \int_{S_i} \boldsymbol{g}_{\mathrm{e}i}(L_i, x_2, x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3.$$

Then the total energy of the limit model is given by

$$J(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) = \sum_{i=1}^{n} \left(\frac{1}{2} \int_0^{L_i} q_2^i(R_i^{\mathrm{T}} R_i')\,\mathrm{d}x_1 - \int_0^{L_i} \tilde{\boldsymbol{f}}_i \cdot \boldsymbol{y}_i\,\mathrm{d}x_1 - \tilde{\boldsymbol{F}}_i \cdot \boldsymbol{y}(L_i) \right),$$

for $(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) \in \mathcal{A}$ and $+\infty$ otherwise.

First, performing partial integration in the force terms in the total energy functional, in a similar way to that in [26], we remove appearance of $\boldsymbol{y}_i$ from the energy functional. In order to do that let us define

$$\tilde{\boldsymbol{p}}_i(x_1) = \int_{x_1}^{L_i} \tilde{\boldsymbol{f}}_i(z) dz + \tilde{\boldsymbol{F}}_i, \quad i = 1, \dots, n, \tag{6.1}$$

and note that the force equilibrium, according to remark 5.3, can be expressed by

$$\sum_{i=1}^{n} \tilde{\boldsymbol{p}}_i(0) = 0. \tag{6.2}$$

Then

$$\begin{aligned}
\sum_{i=1}^{n} \int_0^{L_i} \tilde{\boldsymbol{p}}_i \cdot \boldsymbol{y}_i'\,\mathrm{d}x_1 &= \sum_{i=1}^{n} \tilde{\boldsymbol{p}}_i(L_i) \cdot \boldsymbol{y}_i(L_i) - \sum_{i=1}^{n} \tilde{\boldsymbol{p}}_i(0) \cdot \boldsymbol{y}_i(0) - \sum_{i=1}^{n} \int_0^{L_i} \tilde{\boldsymbol{p}}_i' \cdot \boldsymbol{y}_i\,\mathrm{d}x_1 \\
&= \sum_{i=1}^{n} \tilde{\boldsymbol{F}}_i \cdot \boldsymbol{y}_i(L_i) - \sum_{i=1}^{n} \tilde{\boldsymbol{p}}_i(0) \cdot \boldsymbol{y}_i(0) + \sum_{i=1}^{n} \int_0^{L_i} \tilde{\boldsymbol{f}}_i \cdot \boldsymbol{y}_i\,\mathrm{d}x_1 \\
&= \sum_{i=1}^{n} \tilde{\boldsymbol{F}}_i \cdot \boldsymbol{y}_i(L_i) + \sum_{i=1}^{n} \int_0^{L_i} \tilde{\boldsymbol{f}}_i \cdot \boldsymbol{y}_i\,\mathrm{d}x_1,
\end{aligned}$$

since, in $\mathcal{A}$, deformations satisfy $\boldsymbol{y}_1(0) = \cdots = \boldsymbol{y}_n(0)$. Thus, the total energy functional can be expressed in terms of R_i, $i = 1, \dots, n$, only, by

$$\tilde{J}(R) := J(\boldsymbol{y}, \boldsymbol{d}^2, \boldsymbol{d}^3) = \sum_{i=1}^{n} \left(\frac{1}{2} \int_0^{L_i} q_2^i(R_i^{\mathrm{T}} R_i') \, \mathrm{d}x_1 - \int_0^{L_i} \tilde{\boldsymbol{p}}_i \cdot R_i \boldsymbol{e}_1 \, \mathrm{d}x_1 \right),$$

where we have used the notation $R = (R_1, \dots, R_n)$. Thus, we can split the minimization of J into two steps. In the first step we minimize $\tilde{J}$ in the space

$$\begin{aligned}\mathcal{R} := \{R = (R_1, \dots, R_n) \in W^{1,2}((0, L_1); \mathrm{SO}(3)) \times \cdots \\ \times W^{1,2}((0, L_n); \mathrm{SO}(3)) \colon R_1(0) Q_1^{\mathrm{T}} = \cdots = R_n(0) Q_n^{\mathrm{T}}\}\end{aligned}$$

and in the second step we determine deformations $\boldsymbol{y}_i$, $i = 1, \dots, n$, from the equations

$$\boldsymbol{y}_i' = R_i \boldsymbol{e}_1, \quad \boldsymbol{y}_i(0) = \boldsymbol{y}_0, \quad i = 1, \dots, n, \tag{6.3}$$

where the constant vector $\boldsymbol{y}_0 \in \mathbb{R}$ is freely determined from an additional constraint (e.g. $\boldsymbol{y}_1(L_1) = 0$; see remark 5.4).

Now we want to find the weak formulation of the problem

$$\min_{R \in \mathcal{R}} \tilde{J}(R).$$

First note that $R_i^{\mathrm{T}} R_i'$ are a.e. antisymmetric matrices. Therefore, they possess axial vectors $\boldsymbol{s}_i = \boldsymbol{s}_i(R_i) \in L^2((0, L_i); \mathbb{R}^3)$, i.e.

$$R_i^{\mathrm{T}} R_i' = A_{\boldsymbol{s}_i}, \quad i = 1, \dots, n,$$

where the notation $A_{\boldsymbol{s}}$ stands for the matrix such that $A_{\boldsymbol{s}} x = \boldsymbol{s} \times x$. Since q_2^i are quadratic forms of the elements of $R_i^{\mathrm{T}} R_i'$, there are positive definite matrices H_i (positive definiteness of the matrices H_i follows from the fact that the second derivative of W is greater or equal to 0 and equal to 0 exactly on antisymmetric matrices) such that

$$q_2^i(R_i^{\mathrm{T}} R_i') = H_i \boldsymbol{s}_i \cdot \boldsymbol{s}_i.$$

Thus, the total energy functional can be written as

$$\tilde{J}(R) = \sum_{i=1}^{n} \left(\frac{1}{2} \int_0^{L_i} H_i \boldsymbol{s}_i(R_i) \cdot \boldsymbol{s}_i(R_i) \, \mathrm{d}x_1 - \int_0^{L_i} \tilde{\boldsymbol{p}}_i \cdot R_i \boldsymbol{e}_1 \, \mathrm{d}x_1 \right).$$

In order to obtain the weak and differential formulation of the model we need to find the Gâteaux derivative of the functional $\tilde{J}$ over $\mathcal{R}$. Let $R \in \mathcal{R}$, $\varepsilon > 0$ and $\boldsymbol{v}_i \in C^\infty([0, L_i]; \mathbb{R}^3)$, $i = 1, \dots, n$. Let us choose a perturbation $R^\varepsilon = (R_1^\varepsilon, \dots, R_n^\varepsilon) \in \mathcal{R}$ of R in the following form:

$$R_i^\varepsilon = \mathrm{e}^{\varepsilon A_{\boldsymbol{v}_i}} R_i, \quad i = 1, \dots, n.$$

In order for R^ε to be in $\mathcal{R}$, one only needs to satisfy the condition that

$$R_i^\varepsilon(0) Q_i^{\mathrm{T}} = \mathrm{e}^{\varepsilon A_{\boldsymbol{v}_i(0)}} R_i(0) Q_i^{\mathrm{T}}$$

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is independent of i. Since $R_i(0)Q_i^{\mathrm{T}}$ is independent of i, $R_i^\varepsilon(0)Q_i^{\mathrm{T}}$ is independent of i if and only if

$$\boldsymbol{v}_1(0) = \cdots = \boldsymbol{v}_n(0).$$

Thus, in the following, we take $\boldsymbol{v} \in \mathcal{R}_t$, where

$$\mathcal{R}_t = \{(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) \in C^\infty([0, L_1]; \mathbb{R}^3) \times \cdots \times C^\infty([0, L_n]; \mathbb{R}^3) \colon \boldsymbol{v}_1(0) = \cdots = \boldsymbol{v}_n(0)\}.$$

Next we need to compute the axial vectors $\boldsymbol{s}_i^\varepsilon$ of $(R_i^\varepsilon)^{\mathrm{T}}(R_i^\varepsilon)'$:

$$\begin{aligned}(R_i^\varepsilon)^{\mathrm{T}}(R_i^\varepsilon)' &= R_i^{\mathrm{T}}\mathrm{e}^{-\varepsilon A_{\boldsymbol{v}_i}}(\mathrm{e}^{\varepsilon A_{\boldsymbol{v}_i'}} R_i + \mathrm{e}^{\varepsilon A_{\boldsymbol{v}_i}} R_i') \\ &= R_i^{\mathrm{T}}(I - \varepsilon A_{\boldsymbol{v}_i} + O(\varepsilon^2))(\varepsilon A_{\boldsymbol{v}_i'} + O(\varepsilon^2))R_i + R_i^{\mathrm{T}} R_i' \\ &= R_i^{\mathrm{T}} R_i' + \varepsilon R_i^{\mathrm{T}} A_{\boldsymbol{v}_i'} R_i + O(\varepsilon^2).\end{aligned}$$

Since $R_i^{\mathrm{T}} A_{\boldsymbol{v}_i'} R_i x = R_i^{\mathrm{T}} \boldsymbol{v}_i \times x$ (as $R_i(x_1) \in \mathrm{SO}(3)$ a.e.), we obtain

$$\boldsymbol{s}_i^\varepsilon = \boldsymbol{s}_i + \varepsilon R_i^{\mathrm{T}} \boldsymbol{v}_i' + O(\varepsilon^2), \quad i = 1, \dots, n.$$

Now, we plug this perturbation into the functional $\tilde{J}$:

$$\tilde{J}(R_i^\varepsilon) = \sum_{i=1}^n \biggl(\frac{1}{2}\int_0^{L_i} H_i(\boldsymbol{s}_i + \varepsilon R_i^{\mathrm{T}}\boldsymbol{v}_i') \cdot (\boldsymbol{s}_i + \varepsilon R_i^{\mathrm{T}}\boldsymbol{v}_i')\,\mathrm{d}x_1 \\ - \int_0^{L_i} \tilde{\boldsymbol{p}}_i \cdot (I + \varepsilon A_{\boldsymbol{v}_i}) R_i \boldsymbol{e}_1\,\mathrm{d}x_1\biggr) + O(\varepsilon^2).$$

Thus, the stationary point of the functional $\tilde{J}$ satisfies

$$\sum_{i=1}^n \biggl(\int_0^{L_i} H_i \boldsymbol{s}_i \cdot R_i^{\mathrm{T}} \boldsymbol{v}_i'\,\mathrm{d}x_1 - \int_0^{L_i} \tilde{\boldsymbol{p}}_i \cdot A_{\boldsymbol{v}_i} R_i \boldsymbol{e}_1\,\mathrm{d}x_1\biggr) = 0, \quad \boldsymbol{v} \in \mathcal{R}_t,$$

i.e.

$$\sum_{i=1}^n \biggl(\int_0^{L_i} R_i H_i \boldsymbol{s}_i \cdot \boldsymbol{v}_i'\,\mathrm{d}x_1 - \int_0^{L_i} \boldsymbol{v}_i \cdot R_i \boldsymbol{e}_1 \times \tilde{\boldsymbol{p}}_i\,\mathrm{d}x_1\biggr) = 0, \quad \boldsymbol{v} \in \mathcal{R}_t.$$

Thus, by the partial integration on every rod, we obtain differential equations and the boundary condition

$$(R_i H_i \boldsymbol{s}_i)' + R_i \boldsymbol{e}_1 \times \tilde{\boldsymbol{p}}_i = 0, \qquad R_i(L_i) H_i \boldsymbol{s}_i(L_i) = 0. \tag{6.4}$$

Moreover, since $\boldsymbol{v} \in \mathcal{R}_t$, we obtain just one condition in the junction point:

$$\sum_{i=1}^n R_i(0) H_i \boldsymbol{s}_i(0) = 0. \tag{6.5}$$

Let us now define

$$\tilde{\boldsymbol{s}}_i = R_i \boldsymbol{s}_i, \qquad \tilde{\boldsymbol{q}}_i = R_i H_i R_i^{\mathrm{T}} \tilde{\boldsymbol{s}}_i.$$

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Then the problem given by (6.1)–(6.5) can be formulated as

$$\tilde{\boldsymbol{p}}_i' + \tilde{\boldsymbol{f}}_i = 0, \qquad \tilde{\boldsymbol{p}}_i(L_i) = \tilde{\boldsymbol{F}}_i, \quad i = 1,\dots,n, \tag{6.6}$$

$$\tilde{\boldsymbol{q}}_i' + R_i\boldsymbol{e}_1 \times \tilde{\boldsymbol{p}}_i = 0, \qquad \tilde{\boldsymbol{q}}_i(L_i) = 0, \quad i = 1,\dots,n, \tag{6.7}$$

$$\tilde{\boldsymbol{q}}_i = R_i H_i R_i^{\mathrm{T}} \tilde{\boldsymbol{s}}_i, \qquad i = 1,\dots,n, \tag{6.8}$$

$$R_i' = A_{\tilde{\boldsymbol{s}}_i} R_i, \qquad i = 1,\dots,n, \tag{6.9}$$

$$\boldsymbol{y}_i' = R_i\boldsymbol{e}_1, \qquad i = 1,\dots,n, \tag{6.10}$$

and

$$\sum_{i=1}^n \tilde{\boldsymbol{p}}_i(0) = 0, \qquad \sum_{i=1}^n \tilde{\boldsymbol{q}}_i(0) = 0, \tag{6.11}$$

$$R_1(0)Q_1^{\mathrm{T}} = \cdots = R_n(0)Q_n^{\mathrm{T}}, \qquad \boldsymbol{y}_1(0) = \cdots = \boldsymbol{y}_n(0). \tag{6.12}$$

Equations (6.6)–(6.10) are the equilibrium equations of the nonlinear inextensible rod model (see [24] for the derivation of the model from the three-dimensional nonlinear elasticity and [4] for the direct foundation of the theory of nonlinear rods; see also [17, 30] for the rod model obtained by linearization of the present one). The model is written as a first-order system of ordinary differential equations. The introduced unknowns $\tilde{\boldsymbol{p}}_i$ and $\tilde{\boldsymbol{q}}_i$ are the contact force and contact couple, respectively, corresponding to the ith rod. Equations (6.6) and (6.7) are the equilibrium equations together with the boundary conditions; (6.8) is the constitutional law; (6.9) and (6.10) are material restrictions of unshearability and inextensibility. The conditions (6.11), (6.12) are conditions at the junction. The two conditions in (6.11) are the equilibrium conditions and say that the sum of all contact forces and the sum of all contact couples in the junction are both zero. The conditions in (6.12) are continuity conditions. The first says that the rotation of the cross-section in the junction is the same looking from all rods. Note here the difference between R_i and $R_i Q_i^{\mathrm{T}}$. The matrix $R_i(0)$ gives the actual position of the tangent vector $R_i(0)\boldsymbol{e}_1$ and the cross-section (spanned by $R_i(0)\boldsymbol{e}_2, R_i(0)\boldsymbol{e}_3$); $R_i(0)Q_i^{\mathrm{T}}$ is the rotation of the cross-section 'in the junction' of the rod for the ith rod (the 'difference' between the undeformed Q_i and deformed $R_i(0)$ configuration). The second equation in (6.12) says that the deformation at the junction point is the same for all rods.

Thus, we conclude that junction (transmission) conditions for the junction of rods are given by the equilibrium of contact forces and couples as well as by continuity of the deformations and rotations at the junction.

Remark 6.1. The minimization problem for the total energy J on $\mathcal{A}$ has at least one solution by theorem 5.1. Thus,

$$\boldsymbol{y}_i \in W^{2,2}((0, L_i); \mathbb{R}^3), \qquad R_i \in W^{1,2}((0, L_i); \mathrm{SO}(3)).$$

From the differential formulation for each rod we can conclude a certain regularity result. For $\tilde{\boldsymbol{f}}_i \in L^2((0, L_i); \mathbb{R}^3)$ one has that $\tilde{\boldsymbol{p}}_i \in W^{1,2}((0, L_i); \mathbb{R}^3)$. Therefore,

$$R_i\boldsymbol{e}_1 \times \tilde{\boldsymbol{p}}_i \in W^{1,2}((0, L_i); \mathbb{R}^3)$$

as well, so $\tilde{\boldsymbol{q}}_i \in W^{2,2}((0, L_i); \mathbb{R}^3)$. Using (6.8) this implies $\tilde{\boldsymbol{s}}_i \in W^{1,2}((0, L_i); \mathbb{R}^3)$. Now, using (6.9), we obtain that $R_i \in W^{2,2}((0, L_i); \mathrm{SO}(3))$. Returning to (6.8), we obtain that $\tilde{\boldsymbol{s}}_i \in W^{2,2}((0, L_i); \mathbb{R}^3)$, which, again using (6.9), implies that $R_i \in W^{3,2}((0, L_i); \mathrm{SO}(3))$ and $\boldsymbol{y}_i \in W^{4,2}((0, L_i); \mathbb{R}^3)$. This is the most that can be concluded for L^2 loads in this fashion.

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