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Rank conditions for finite group actions on 4-manifolds

Ian Hambleton and Semra Pamuk

Abstract. Let M be a closed, connected, orientable topological 4-manifold, and G be a finite group acting topologically and locally linearly on M. In this paper, we investigate the spectral sequence for the Borel cohomology $H^*_G(M)$ and establish new bounds on the rank of G for homologically trivial actions with discrete singular set.

1 Introduction

In this paper, we provide some new information about the existence of finite group actions on closed, connected, orientable 4-manifolds. In this dimension, the comparison between smooth and topological group actions is particularly interesting. Our focus will be on *locally linear topological actions* as background for future work on smooth actions.

For free actions on simply connected 4-manifolds, or equivalently for closed topological 4-manifolds with finite fundamental group, there are a number of classification results in the literature (for example, see [6–9]). One challenging open problem is to compute the Hausmann–Weinberger invariant [11], namely to determine the minimal Euler characteristic of a 4-manifold with a given fundamental group. The answer is only known at present in special cases (see [12]).

We will extend the scope of previous work by including *nonsimply connected* manifolds and concentrate on nonfree actions. We often assume that the actions are *homologically trivial*, meaning that the group of symmetries acts as the identity on the homology groups of the manifold.

A useful measure of the complexity of a finite group G is its *p-rank*, defined as the maximum rank r of an elementary abelian p-group $(\mathbb{Z}_p)^r \leq G$. We let $\operatorname{rank}_p(G)$ denote the p-rank of G for each prime p and let $\operatorname{rank}(G)$ denote the maximum over all primes of the p-ranks.

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Question *Given a closed orientable 4-manifold M, what is the maximum value of* $\mathrm{rank}(G)$ *over all the finite groups G which act effectively, locally linearly, and homologically trivially on M?*

We note that a $\mathbb{Z}_p$-action, for p a prime, will be homologically trivial if M has torsion-free homology and $(p-1)$ is larger than each of the Betti numbers of M. If M has Euler characteristic $\chi(M) \neq 0$, then any homologically trivial action of a finite group must be nonfree (by the Lefschetz fixed-point theorem).

Beyond the rank restrictions, we would like to know which finite groups G can act. For example, if M is the connected sum of two or more complex projective planes, then G is abelian and $\mathrm{rank}(G) \leq 2$ (see [10]). This was proved for smooth actions using techniques from gauge theory. Then, McCooey [14], building on earlier work by Edmonds [5], used methods from equivariant algebraic topology to prove a much stronger result:

Theorem (McCooey [14, Theorem 16]) *Let G be a (possibly finite) compact Lie group, and suppose M is a closed* 4*-manifold with* $H_1(M;\mathbb{Z}) = 0$ *and* $b_2(M) \geq 2$*, equipped with an effective, locally linear, homologically trivial G-action.*

(i) *If* $b_2(M) = 2$ *and* $\mathrm{Fix}(M) \neq \emptyset$*, then G is isomorphic to a subgroup of* $S^1 \times S^1$.

(ii) *If* $b_2(M) \geq 3$*, then G is isomorphic to a subgroup of* $S^1 \times S^1$*, and the fixed set* $\mathrm{Fix}(M)$ *is necessarily nonempty.*

What should we expect for actions on arbitrary *nonsimply connected* 4-manifolds? Here is a possible uniform answer to the rank question (compare [5, Conjecture 9.1]).

Conjecture *If a finite group G acts effectively, locally linearly, and homologically trivially on a closed orientable* 4*-manifold M with Euler characteristic* $\chi(M) \neq 0$*, then* $\mathrm{rank}_p(G) \leq 2$ *for p odd.*

The condition $\chi(M) \neq 0$ rules out actions on $M = T^4$ (for example), but the group $G = (\mathbb{Z}_2)^4$ acts linearly on S^4, so additional conditions must be found for $p = 2$.

Remark 1.1 Mann and Su [13, Theorem 2.2]) showed that $\mathrm{rank}_p(G) \leq 2$, for p an odd prime, provided that the fixed set $\mathrm{Fix}(M) \neq \emptyset$, without assuming that the action was locally linear or homologically trivial. However, the existence of a global fixed point is a strong assumption: in the locally linear case, the result follows easily from a result of P. A. Smith [19, Section 4] applied to the boundary S^3 of a G-invariant 4-ball at a point $x \in \mathrm{Fix}(M)$.

The main tool from equivariant algebraic topology used for the study of nonfree group actions is the Borel spectral sequence. Let BG denote the classifying space for principal G-bundles and EG the universal free, contractible G-space. Then, the Borel cohomology $H^*_G(M) := H^*(M \times_G EG)$ is "computable" in principle from the Serre spectral sequence of the fibration $M \to M \times_G EG \to BG$. We will use integral coefficients or $\mathbb{F}_p$-coefficients for $H^*(M)$, but note that, in general, this is a local coefficient system for the group cohomology of G. For homologically trivial actions, we have ordinary coefficients.

Theorem A *Suppose that* $G = \mathbb{Z}_p$ *acts locally linearly on a closed, connected, oriented* 4*-manifold M, preserving the orientation, with fixed-point set* $F = \mathrm{Fix}(M) \neq \emptyset$.

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(i) *If the map* $H_1(F;\mathbb{Z}) \twoheadrightarrow H_1(M;\mathbb{Z})$ *is surjective, then the Borel spectral sequence for* $H^*_G(M)$ *collapses with integral and* $\mathbb{F}_p$*-coefficients.*
(ii) *If* $\ker(H^1(M;\mathbb{Z}) \to H^1(F;\mathbb{Z}))$ *is nontrivial, but has trivial G-action, then the Borel spectral sequence with integral coefficients does not collapse.*

Edmonds [4, Proposition 2.3] showed that the Borel spectral sequence with integral or $\mathbb{F}_p$-coefficients collapses for orientation preserving $\mathbb{Z}_p$-actions with $F \neq \emptyset$ on closed simply connected 4-manifolds. We generalize this result to nonsimply connected 4-manifolds.

The two-dimensional components of $\mathrm{Fix}(M)$ are always orientable if $p > 2$, and for $p = 2$, this would follow, for example, by assuming that G preserves a $Spin^c$ structure in a suitable sense (see [4, Proposition 3.2] and Ono [17, Section 4]). However, the complex conjugation involution on $\mathbb{CP}^2$ with fixed set $\mathbb{RP}^2$ shows that orientability of the fixed set is not necessary, in general, for the collapse of the Borel spectral sequence with integral coefficients (see [4, Proposition 2.3]).

In his arXiv paper [16, Proposition 3.1], McCooey proposed a "collapse" result for homologically trivial actions under the assumption that $H_1(M)$ is torsion-free, but without our condition on $H_1(F)$ (see Example 7.2 for a counterexample).

Remark 1.2 Note that if $H_1(F) \twoheadrightarrow H_1(M)$ is surjective, then $H^1(M) \rightarrowtail H^1(F)$ is injective, but not conversely if $H_1(M)$ has torsion.

Recall that an action is called *pseudofree* if the singular set $\Sigma := \Sigma(M, G) \subset M$ consists of isolated points. For such actions, we can estimate the rank.

Theorem B *Let M be a closed, orientable* 4*-manifold with* $\chi(M) \neq 0$. *If a finite group G acts locally linearly, pseudofreely, and homologically trivially on M, then* $\mathrm{rank}_p(G) \leq 1$ *for* $p \geq 5$ *and* $\mathrm{rank}_p(G) \leq 2$ *for* $p = 2, 3$.

Remark 1.3 Note that the actions of $G = (\mathbb{Z}_2)^4$ on $M = S^4$ are not pseudofree (see Breton [3]). In addition, $M = \mathbb{CP}^2$ admits a pseudofree action of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ (see Example 6.2), and $S^2 \times S^2$ admits pseudofree actions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see [15]).

Here is a short outline of the paper. Throughout the paper, M denotes a closed, connected, oriented topological 4-manifold.

For orientation-preserving actions, the assumptions in Theorem A imply that the fixed-point set must be two-dimensional whenever $H_1(M) \neq 0$. In Section 7, we give some examples of group actions on a closed, connected oriented 4-manifold to illustrate various features. For example, there is an action with zero-dimensional fixed-point set, where the Borel spectral sequence does not collapse, and another with a two-dimensional fixed-point component where the Borel spectral sequence does not collapse. This motivates our assumption that $H_1(F) \twoheadrightarrow H_1(M)$ is surjective.

In Section 2, we give some general facts about the main tool we use in the proof; the Leray–Serre spectral sequence for the fibration $M \to M \times_G EG \to BG$ is also called the Borel Spectral Sequence. The details can be found in the books [2] and [20].

In Section 3, we prove the first part of Theorem A and complete the proof in Section 4. In Section 5, we give some applications under the extra assumptions of homological triviality and $H_1(M) = 0$. In Section 6, we prove Theorem B.

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2 The Borel spectral sequence

In this section, we recall some of the standard facts about $H_G^*(M)$, where G is a finite group acting on a finite-dimensional G– CW complex M. In particular, these results apply to G-manifolds and singular cohomology with coefficients in $R = \mathbb{Z}$ (or $R = \mathbb{F}_p$ when indicated). The details about this construction and the spectral sequence can be found in Borel [2] and tom Dieck [20].

The Leray–Serre spectral sequence for the fibration $M \to M \times_G EG \to BG$ is known as the Borel spectral sequence. The total space of this fibration is known as the Borel construction and is denoted by $M_G = M \times_G EG$. The E_2-page of this spectral sequence is

$$E_2^{k,l}(M) = H^k(G; H^l(M)),$$

which converges to the cohomology $H^*(M_G)$ of the total space M_G. These are denoted by $H_G^*(M)$ and are known as the Borel equivariant cohomology groups. This construction is natural with respect to G-maps of G-spaces.

In the examples in Section 7, we use Proposition 2.4 given below to decide whether the Borel spectral sequence collapses. Before we state this proposition, we recall some basic definitions for the convenience of the reader. In this section, we will denote the fixed set by M^G.

Since EG is path-connected, any fiber inclusion $j_b: M \to EG \times_G M$, with $j_b(m) = (b, m)$ for $b \in EG$ and $m \in M$, induces a well-defined map $j^*: H_G^*(M) \to H^*(M)$.

A *cohomology extension of the fiber* is an R-module homomorphism of degree zero $t: H^*(M) \to H_G^*(M)$ such that $j^* \circ t$ is the identity. M is called *totally nonhomologous to zero* in M_G with respect to $H^*(-)$ if j^* is surjective.

Since a surjective map onto a free R-module splits, if M is totally nonhomologous to zero and $H^*(M)$ is a free R-module, then a cohomology extension of the fiber exists. Also, if M is totally nonhomologous to zero in M_G, then G acts trivially on $H^*(M)$.

One can show that [20, Chapter III, Proposition 1.17] M is totally nonhomologous to zero in M_G if and only if G acts trivially on $H^*(M)$ and $E_2^{0,*}$ consists of permanent cocyles (i.e., $E_2^{0,p} = E_\infty^{0,p}$). Also, if we have $H^*(M)$ is finitely generated free R-module, then [20, Chapter III, Proposition 1.18] M is totally nonhomologous to zero in M_G if and only if G acts trivially on $H^*(M)$ and the Borel spectral sequence collapses. In this case, $H_G^*(M)$ is a free $H^*(BG)$-module.

The [20, Chapter III, Proposition 4.16] comes as an application of Localization Theorem, so let us recall it briefly. Let S be a multiplicatively closed subset of homogeneous elements in $H^*(BG)$ and $\mathcal{F}(S) = \{H \leqslant G \mid S \cap \ker(H_G^*(G/G) \to H_G^*(G/H)) \neq \emptyset\}$, then [20, Chapter III, Theorem 3.8]:

Theorem 2.1 *Let (M, A) be a finite-dimensional relative G-complex. Suppose $M \backslash A$ has finite orbit types with orbits isomorphic to G/H for $H \in \mathcal{F}(S)$. Then, the inclusion $A \subset M$ induces the isomorphism $S^{-1}H_G^*(M) \cong S^{-1}H_G^*(A)$.*

Assumption In the remainder of this section, we list some results about the Borel cohomology $H_G^*(M)$ *for finite p-group actions, with coefficients in* $\mathbf{k} := \mathbb{F}_p$ *understood.*

In this setting, the Localization Theorem has a stronger conclusion.

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Theorem 2.2 [20, Chapter III, Theorem 3.13] *Let $G = (\mathbb{Z}_p)^n$ be a p-torus and M a finite-dimensional G– CW complex. Then, $S^{-1}H_G^*(M) \cong S^{-1}H_G^*(M^G)$.*

Let $j: M \to \{pt\}$ denote the map of M to a point.

Corollary 2.3 *Let $G = (\mathbb{Z}_p)^n$ be a p-torus and M a finite-dimensional G– CW complex. Then, $M^G \neq \emptyset$ if and only if $j^*: H_G^*(pt) \to H_G^*(M)$ is injective.*

Here are some useful criteria for the collapse of the Borel spectral sequence: we are combining statements from Borel [2, Chapter XII, Theorem 3.4] and tom Dieck [20, Chapter III, Proposition 4.16].

Proposition 2.4 (Borel) *Let $G = (\mathbb{Z}_p)^n$ be a p-torus and $\mathbf{k} = \mathbb{F}_p$. Suppose the total dimension $\sum_r \dim_k H^r(M)$ is finite and $H^q(M) = 0$ for $q > \dim M = n$. Then,*

$$\sum_r \dim_k H^r(M^G) \leq \sum_r \dim_k H^r(M).$$

Moreover, the following are equivalent:

(i) $\sum_r \dim_k H^r(M^G) = \sum_r \dim_k H^r(M)$.
(ii) *M is totally nonhomologous to zero in M_G with respect to $H^*(-)$.*
(iii) $\dim_k H_G^q(M) = \sum_r \dim_k H^r(M)$ *for $q > n$.*
(iv) *G acts trivially on $H^*(M)$, and the Borel spectral sequence collapses.*

With some extra assumptions, the following statement can be proved:

Corollary 2.5 [2, Chapter XII, Corollary 3.5] *Let G be an elementary abelian p-group and M be a compact G-space for which $\dim_k M$, $\dim_k H^*(M)$, and the number of orbit types are all finite. Assume that*

(i) *G acts homologically trivially, and*
(ii) *$H^*(M)$ is generated by elements which are transgressive in the Borel spectral sequence.*

Then, the fixed-point set M^G is nonempty if and only if the Borel spectral sequence collapses.

3 Collapse of the spectral sequence

Under some conditions, including the strong assumption that $H_1(F) \twoheadrightarrow H_1(M)$ is surjective, we prove the first part of Theorem A, namely a "collapse" result for the Borel spectral sequence.

Theorem 3.1 *Let $G = \mathbb{Z}_p$ act locally linearly on a closed, connected, oriented 4-manifold M, preserving the orientation, with fixed-point set $F \neq \emptyset$. If $H_1(F;\mathbb{Z}) \twoheadrightarrow H_1(M;\mathbb{Z})$ is surjective, then the Borel spectral sequence for $H_G^*(M)$ collapses with integral and $\mathbb{F}_p$-coefficients.*

Remark 3.2 Since all the arguments in the proof of this result are cohomological, the conclusion should hold (at least for integral coefficients) if $\operatorname{coker}\{H_1(F) \to H_1(M)\} \neq 0$ is a cohomologically trivial $\mathbb{Z}G$-module and $H_1(M)$ is torsion-free. We have not checked the details. If $H_1(M)$ has p-primary torsion, then the situation in this extra generality appears much more complicated.

At various points, we will need to use some properties of the group cohomology of $G = \mathbb{Z}_p$. Recall that the integral cohomology is a polynomial algebra $H^*(G;\mathbb{Z}) = \mathbb{Z}[\theta]$, where $|\theta| = 2$ is a class of degree 2. For p odd, we have

$$H^*(G;\mathbb{F}_p) = \mathbb{F}_p[u] \otimes \Lambda(x),$$

where $|u| = 2$ and $|x| = 1$, with $x^2 = 0$. For $p = 2$, $H^*(G;\mathbf{F}_2) = \mathbf{F}_2[x]$, where $|x| = 1$. The cup products are natural with respect to the change of coefficients $\mathbb{Z} \to \mathbb{F}_p$, and the induced maps $H^{2k}(G;\mathbb{Z}) \to H^{2k}(G;\mathbb{F}_p)$ are isomorphisms for $k > 0$ and surjective for $k = 0$. The differentials in the E_r terms of the Borel spectral sequence for $H^*_G(M)$ are multiplicative with respect to cup products in the cohomology of M and G.

Before starting the proof of Theorem 3.1, we will collect some useful remarks:

(i) Since G preserves the orientation on M (automatic for p odd) and $H_1(F) \to H_1(M)$ is surjective, G acts trivially on the homology and cohomology of M, except possibly for $H_2(M) \cong H^2(M)$.

(ii) Let $A \subset F$ denote a nonempty one-dimensional subset of the fixed-point set, such that the map $H_1(A) \twoheadrightarrow H_1(M)$ is surjective. For example, take A to be a 1-skeleton of F.

(iii) The induced map $H^1(M) \to H^1(A)$ is injective.

(iv) By applying duality to a neighborhood of A in M, we have $H^*(M\text{-}A) \cong H_{4-*}(M, A)$, and similarly $H^*(M, A) \cong H_{4-*}(M\text{-}A)$.

(v) The statements so far also hold for homology and cohomology with $\mathbb{F}_p$-coefficients.

(vi) $H_1(M, A) = \ker\{H_0(A) \to H_0(M)\}$ is $\mathbb{Z}$-torsion-free, with trivial G-action.

(vii) $H^2(M, A)$ is $\mathbb{Z}$-torsion-free: its torsion subgroup is $\mathrm{Ext}(H_1(M, A), \mathbb{Z}) = 0$.

Proof We first consider the E_2-page of the Borel spectral sequence $E_2^{k,l}(M) = H^k(G; H^l(M))$ and show that d_2 differentials are zero. Integral coefficients are understood unless $\mathbb{F}_p$-coefficients are stated explicitly.

3A The maps $d_2^{k,4}: E_2^{k,4}(M) \to E_2^{k+2,3}(M)$

For any fixed point $x \in F$, the inclusion map $i: M\text{-}\{x\} \hookrightarrow M$ induces a homomorphism $i^*: H^n(M) \to H^n(M\text{-}\{x\})$, which is zero for $n \geq 4$ and isomorphism for other dimensions. The corresponding map of spectral sequences $E_r^{k,l}(M) \to E_r^{k,l}(M\text{-}\{x\})$ is trivial when $l = 4$ and an isomorphism otherwise. By naturality, we have the commutative diagrams of differentials:

$$\begin{tikzcd}
H^k(G; H^4(M)) \arrow[r, "d_2^{k,4}"] \arrow[d, "i^*"'] & H^{k+2}(G; H^3(M)) \arrow[d, "\cong"] \\
H^k(G; H^4(M\text{-}\{x\})) \arrow[r, "d_2^{k,4}"] & H^{k+2}(G; H^3(M\text{-}\{x\}))
\end{tikzcd}$$

Since $H^4(M\text{-}\{x\}) = 0$, we have $i^* = 0$ and $H^3(M\text{-}\{x\}) \cong H^3(M)$, so $d_2^{k,4} = 0$. The same argument works for $\mathbb{F}_p$-coefficients.

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3B The maps $d_2^{k,1}\colon E_2^{k,1}(M) \to E_2^{k+2,0}(M)$

Similarly, for any $x \in F$, consider the map $j^*\colon H^n(M,\{x\}) \to H^n(M)$ induced by $j\colon (M,\emptyset) \to (M,\{x\})$. From the long exact sequence in relative cohomology, j^* is isomorphism for all $n \geq 1$. The corresponding map of spectral sequences $E_r^{k,l}(M,\{x\}) \to E_r^{k,l}(M)$ is also isomorphism for $l \geq 1$. By naturality, we again have the commutative diagrams of differentials:

$$\begin{tikzcd}
H^k(G;H^1(M,\{x\})) \arrow[r, "d_2^{k,1}"] \arrow[d, "j^*"'] & H^{k+2}(G;H^0(M,\{x\})) \arrow[d] \\
H^k(G;H^1(M)) \arrow[r, "d_2^{k,1}"] & H^{k+2}(G;H^0(M))
\end{tikzcd}$$

Since $H^0(M,\{x\}) = 0$ and j^* is isomorphism, then $d_2^{k,1} = 0$. The same argument works for $\mathbb{F}_p$-coefficients.

3C The maps $d_2^{k,3}\colon E_2^{k,3}(M) \to E_2^{k+2,2}(M)$

From the long exact sequence of relative homology, and $H_2(A) = 0$, we get injectivity of $H_2(M) \rightarrowtail H_2(M,A)$. Since $H_1(A) \twoheadrightarrow H_1(M)$ is surjective, we conclude that the map $H_1(M) \to H_1(M,A)$ is zero. By duality, the map $H^3(M) \to H^3(M\text{-}A)$ is zero. This also holds for $\mathbb{F}_p$-coefficients. We obtain the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^2(M) \arrow[r] \arrow[d, "\cong"'] & H^2(M\text{-}A) \arrow[r] \arrow[d, "\cong"'] & K \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & H_2(M) \arrow[r, tail] & H_2(M,A) \arrow[r] & H_1(A) \arrow[r, two heads] & H_1(M) \arrow[r] & 0
\end{tikzcd} \tag{3.3}$$

where $K := \ker\{H_1(A) \to H_1(M)\}$. For $k \geq 0$ even, $H^{k+1}(G;K) = 0$, since K is $\mathbb{Z}$-torsion-free with trivial G-action. When we apply group cohomology to the upper short exact sequence in (3.3), we get the long exact sequence:

$$\cdots \longrightarrow H^{k+1}(G;K) \longrightarrow H^{k+2}(G;H^2(M)) \longrightarrow H^{k+2}(G;H^2(M\text{-}A)) \longrightarrow \cdots$$

It follows that the map $H^{k+2}(G;H^2(M)) \rightarrowtail H^{k+2}(G;H^2(M\text{-}A))$ is injective for k even.

Since the map $H^3(M) \to H^3(M\text{-}A)$ is zero, the induced map in group cohomology $H^k(G;H^3(M)) \to H^k(G;H^3(M\text{-}A))$ is also zero. By naturality of spectral sequences with respect to the inclusion $M\text{-}A \hookrightarrow M$, we have the following commutative diagram:

$$\begin{tikzcd}
H^k(G;H^3(M)) \arrow[r, "d_2^{k,3}"] \arrow[d, "0"'] & H^{k+2}(G;H^2(M)) \arrow[d, tail] \\
H^k(G;H^3(M\text{-}A)) \arrow[r, "d_2^{k,3}"] & H^{k+2}(G;H^2(M\text{-}A))
\end{tikzcd}$$

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implying $d_2^{k,3} = 0$ for k even. For $\mathbb{F}_p$, we are missing the injectivity of the right-hand vertical map. However, the isomorphism $H^3(M) \otimes \mathbb{F}_p \cong H^3(M; \mathbb{F}_p)$ implies that $H^0(G; H^3(M)) \to H^0(G; H^3(M; \mathbb{F}_p))$ is surjective, since both coefficients have trivial G-action, so reduces to the surjection $H^3(M) \to H^3(M; \mathbb{F}_p)$. Then, naturality gives $d_2^{2i,3} = 0$ for integral or $\mathbb{F}_p$-coefficients.

The differentials $d_2^{k,3}$ for k odd (-coefficients). If $H_1(M)$ has no p-torsion, then $H^3(M; \mathbb{F}_p) = 0$ and $d_2^{k,3} = 0$ with $\mathbb{F}_p$-coefficients for k odd. To handle the case where $H^3(M)$ has p-primary torsion and k is odd, we compare with the $\mathbb{F}_p$-coefficient spectral sequence via the change of coefficients $\mathbb{Z} \to \mathbb{F}_p$. Note that since $H^0(G; H^3(M)) \to H^0(G; H^3(M; \mathbb{F}_p))$ is surjective, and $d_2^{0,3} = 0$, we see that

$$d_2^{0,3} \colon H^0(G; H^3(M; \mathbb{F}_p)) \to H^2(G; H^2(M; \mathbb{F}_p))$$

is also zero. Now we use the multiplicativity of the $\mathbb{F}_p$-coefficient spectral sequence, and the fact that

$$\cup x \colon H^0(G; H^3(M; \mathbb{F}_p)) \to H^1(G; H^3(M; \mathbb{F}_p))$$

is surjective (since the coefficients have trivial G-action), where $0 \neq x \in H^1(G; \mathbb{F}_p)$, to conclude that

$$d_2^{1,3} \colon H^1(G; H^3(M; \mathbb{F}_p)) \to H^3(G; H^2(M; \mathbb{F}_p))$$

is zero for $\mathbb{F}_p$-coefficients, and hence for all odd k by naturality and periodicity. We have now shown that $d_2^{k,3} = 0$ for all k in the spectral sequence with $\mathbb{F}_p$-coefficients .

Remark 3.4 If $H_1(M) \cong H^3(M)$ is p-primary torsion-free, then $H^k(G, H^3(M)) = 0$ for k odd, since $H_1(M) \cong H^3(M)$ is a trivial G-module, due to the assumption that $H_1(F) \twoheadrightarrow H_1(M)$ is surjective. Hence, the differentials $d_2^{k,3} = 0$ with integral coefficients for all odd k, if the order of $H_1(M)$ is not divisible by p.

We will return to the remaining differentials $d_2^{k,3}$, for k odd and integral coefficients, after showing that the spectral sequence collapses for $\mathbb{F}_p$-coefficients.

3D The maps $d_2^{k,2} \colon E_2^{k,2}(M) \to E_2^{k+2,1}(M)$

Let $T = \ker\{H^2(M, A) \to H^2(M)\} = \operatorname{coker}\{H^1(M) \to H^1(A)\}$. Since T is a quotient of $H^1(A)$, it has trivial G-action. Since $T \subseteq H^2(M, A)$, it is $\mathbb{Z}$-torsion-free. Since $H^2(A) = 0$, we have a short exact sequence:

$$0 \longrightarrow T \longrightarrow H^2(M, A) \xrightarrow{\alpha} H^2(M) \longrightarrow 0$$

which induces a long exact sequence in group cohomology:

$$\cdots \longrightarrow H^k(G; T) \longrightarrow H^k(G; H^2(M, A)) \longrightarrow H^k(G; H^2(M)) \longrightarrow H^{k+1}(G; T) \longrightarrow \cdots$$

Therefore, the map $H^k(G; H^2(M, A)) \to H^k(G; H^2(M))$ is surjective for k even, since $H^{k+1}(G; T) = 0$ in this case.

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By naturality with respect to the map of pairs $(M,\emptyset)\to(M,A)$, we have the commutative diagram:

$$\begin{tikzcd} H^k(G;H^2(M,A)) \arrow[r, "d_2^{k,2}"] \arrow[d] & H^{k+2}(G;H^1(M,A)) \arrow[d, "0"'] \\ H^k(G;H^2(M)) \arrow[r, "d_2^{k,2}"] & H^{k+2}(G;H^1(M)) \end{tikzcd}$$

We note that the map $H^1(M,A)\to H^1(M)$ is zero, since $H^1(M)\to H^1(A)$ is injective. Hence, the map $H^{k+2}(G;H^1(M,A))\to H^{k+2}(G;H^1(M))$ is zero, for k even, and $d_2^{k,2}=0$, for k even. For odd k, we have $H^{k+2}(G;H^1(M))=0$, since $H^1(M)$ is torsion-free with trivial G-action, and $d_2^{k,2}=0$ also for k odd (with integral coefficients).

To understand the $d_2^{k,2}$ differentials with $\mathbb{F}_p$-coefficients, we use the multiplicative structure in the spectral sequence. Suppose that $0\neq d_2^{0,2}(z)\in E_2^{2,1}=H^2(G;H^1(M;\mathbb{F}_p))$, for some $z\in E_2^{0,2}$. Since the cup product pairing

$$H^2(G;H^1(M;\mathbb{F}_p))\times H^2(G;H^3(M;\mathbb{F}_p))\to H^4(G;H^4(M;\mathbb{F}_p))=\mathbb{F}_p$$

is nonsingular, there exists $w\in H^2(G;H^3(M;\mathbb{F}_p))=E_2^{2,3}$ such that $d_2^{0,2}(z)\cdot w\neq 0$. But

$$0=d_2^{2,5}(z\cdot w)=z\cdot d_2^{2,3}(w)-d_2^{0,2}(z)\cdot w,$$

since $z\cdot w\in E_2^{2,5}=0$ and $d_2^{2,3}(w)=0$, as shown above. This is a contradiction, and hence $d_2^{0,2}=0$. Since $\cup x\colon H^2(G;H^1(M;\mathbb{F}_p))\cong H^3(G;H^1(M;\mathbb{F}_p))$, we have $d_2^{1,2}=0$. This completes the proof that all the d_2 differentials are zero for $\mathbb{F}_p$-coefficients.

3E Vanishing of differentials in the E_3-page

Obviously, $d_3^{k,1}=0$, and we again use the maps induced from $i\colon M\text{-}\{x\}\hookrightarrow M$ to show $d_3^{k,4}=0$, and $j\colon(M,\emptyset)\to(M,\{x\})$ to show $d_3^{k,2}=0$.

$$\begin{tikzcd} E_3^{k,4}(M)=H^k(G;H^4(M)) \arrow[r, "d_3^{k,4}"] \arrow[d, "i^*"'] & H^{k+3}(G;H^2(M))/\operatorname{Im} d_2^{k+1,3}(M) \arrow[d, "\cong"] \\ E_3^{k,4}(M\text{-}\{x\})=H^k(G;H^4(M\text{-}\{x\})) \arrow[r, "d_3^{k,4}"] & H^{k+3}(G;H^2(M\text{-}\{x\}))/\operatorname{Im} d_2^{k+1,3}(M\text{-}\{x\}) \end{tikzcd}$$

We have $H^2(M\text{-}\{x\})\cong H^2(M)$ and $H^3(M\text{-}\{x\})\cong H^3(M)$, which gives the right-hand vertical isomorphism. The map $i^*=0$, since $H^4(M\text{-}\{x\})=0$, so $d_3^{k,4}=0$.

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Next, consider the diagram:

$$\begin{tikzcd} H^k(G;H^2(M,\{x\}))/\operatorname{Im} d_2^{k-2,3}(M,\{x\}) \arrow[r, "d_3^{k,2}"] \arrow[d, "j^*"'] & H^{k+3}(G;H^0(M,\{x\})) \arrow[d] \\ H^k(G;H^2(M))/\operatorname{Im} d_2^{k-2,3}(M) \arrow[r, "d_3^{k,2}"] & H^{k+3}(G;H^0(M)) \end{tikzcd}$$

Since $H^0(M,\{x\}) = 0$ and j^* is an isomorphism, we have $d_3^{k,2} = 0$. The same arguments work for $\mathbb{F}_p$-coefficients.

For $d_3^{k,3}$, we again use naturality and the following commutative diagram:

$$\begin{tikzcd} H^k(G;H^3(M)) \supseteq \ker d_2^{k,3}(M) \arrow[r, "d_3^{k,3}"] \arrow[d, "i^*"'] & H^{k+3}(G;H^1(M)) \arrow[d, "i^*"', "\cong"] \\ H^k(G;H^3(M\text{-}A)) \supseteq \ker d_2^{k,3}(M\text{-}A) \arrow[r, "d_3^{k,3}"] & H^{k+3}(G;H^1(M\text{-}A)) \end{tikzcd}$$

By duality, $H^1(M) \to H^1(M\text{-}A)$ is an isomorphism, and so is the map

$$i^*\colon H^{k+3}(G;H^1(M)) \to H^{k+3}(G;H^1(M\text{-}A)).$$

Since the map $H^3(M) \to H^3(M\text{-}A)$ is zero (as noted above), we have $d_3^{k,3} = 0$. The same arguments work for $\mathbb{F}_p$-coefficients. For integral coefficients (where we have not yet shown $\ker d_2^{k,3} = 0$ if k is odd), we are using the vanishing of $d_2^{k-2,4}$ to see that the domain of $d_3^{k,3}$ is $\ker d_2^{k,3} \subseteq H^k(G;H^3(M))$.

3F Vanishing of differentials in the E_4-page

Obviously, $d_4^{k,1} = 0$ and $d_4^{k,2} = 0$, and again we use the induced maps i^* to show $d_4^{k,4} = 0$ and j^* to show $d_4^{k,3} = 0$.

$$\begin{tikzcd} H^k(G;H^4(M)) \arrow[r, "d_4^{k,4}"] \arrow[d, "i^*"'] & H^{k+4}(G;H^1(M)) \arrow[d, "\cong"] \\ H^k(G;H^4(M\text{-}\{x\})) \arrow[r, "d_4^{k,4}"] & H^{k+4}(G;H^1(M\text{-}\{x\})) \end{tikzcd}$$

Since $H^4(M\text{-}\{x\}) = 0$, we have $i^* = 0$ and $H^1(M\text{-}\{x\}) \cong H^1(M)$, so $d_4^{k,4} = 0$.

$$\begin{tikzcd} H^k(G;H^3(M,\{x\})) \supseteq \ker d_2^{k,3}(M,\{x\}) \arrow[r, "d_4^{k,3}"] \arrow[d, "j^*"'] & H^{k+4}(G;H^0(M,\{x\})) \arrow[d] \\ H^k(G;H^3(M)) \supseteq \ker d_2^{k,3}(M) \arrow[r, "d_4^{k,3}"] & H^{k+4}(G;H^0(M)) \end{tikzcd}$$

Since $H^0(M,\{x\}) = 0$ and j^* is an isomorphism, it follows that $d_4^{k,3} = 0$. The same arguments work for $\mathbb{F}_p$-coefficients.

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3G Vanishing of differentials in the E_5-page

There is only one differential to consider $d_5^{k,4}\colon E_5^{k,4}(M) \to E_5^{k+5,0}(M)$, which can easily shown to be zero by again using j^*:

$$\begin{tikzcd} H^k(G; H^4(M,\{x\})) \arrow[r, "d_5^{k,4}"] \arrow[d, "j^*"'] & H^{k+5}(G; H^0(M,\{x\})) \arrow[d] \\ H^k(G; H^4(M)) \arrow[r, "d_5^{k,4}"] & H^{k+5}(G; H^0(M)) \end{tikzcd}$$

Since $H^0(M,\{x\}) = 0$ and j^* is isomorphism, then $d_5^{k,4} = 0$. The same arguments work for $\mathbb{F}_p$-coefficients.

Remark 3.5 For the vanishing of the differentials $d_r^{k,r-1}$ hitting the $(*,0)$ line, we could just have cited Corollary 2.3, since $F \neq \emptyset$.

We have now shown that the Borel spectral sequence with $\mathbb{F}_p$-coefficients collapses and that $E_3 = E_\infty$ with integral coefficients (independently of the vanishing of $d_2^{2i+1,3}$).

3H The maps $d_2^{2i+1,3}\colon E_2^{2i+1,3}(M;\mathbb{Z}) \to E_2^{2i+3,2}(M;\mathbb{Z})$

We will show that the differentials $d_2^{2i+1,3} = 0$ by comparing the integral calculations with the mod p calculations.

Note that the groups $H^q_G(M)$ with integral coefficients are all $\mathbf{k}$-vector spaces for $q > 4$ (with notation $\mathbf{k} := \mathbb{F}_p$ as before). This follows from the isomorphism $H^q_G(M) \cong H^q_G(F) \cong H^q(F \times BG)$ (see [5, Proposition 2.1]).

There is a short exact sequence of $\mathbf{k}$-vector spaces

$$0 \to H^5_G(M) \to H^5_G(M;\mathbb{F}_p) \to H^6_G(M) \to 0, \tag{3.6}$$

and we will compute both sides of the resulting equality

$$\dim_{\mathbf{k}} H^5_G(M) + \dim_{\mathbf{k}} H^6_G(M) = \dim_{\mathbf{k}} H^5_G(M;\mathbb{F}_p) \tag{3.7}$$

via separate calculations.

Suppose that $d_2^{1,3} \neq 0$, and we let $b = \dim_{\mathbf{k}}(\operatorname{Im} d_2^{1,3}) = \dim_{\mathbf{k}}(\operatorname{Im} d_2^{3,3})$ (by periodicity). Let $t = \dim_{\mathbf{k}} H^{odd}(G; H^3(M))$ and note that $H^{odd}(G; H^1(M)) = 0$. Since $H^3(M) \cong H_1(M)$ has trivial G-action, we have $\dim_{\mathbf{k}} H^2(G; H^3(M)) = b_1(M) + t$.

Lemma 3.8 (i) $\dim_k H^5_G(M) = 2b_1(M) + (t-b) + \dim_k H^3(G; H^2(M))$;
(ii) $\dim_k H^6_G(M) = 2 + (t-b) + \dim_k H^4(G; H^2(M))$;
(iii) $\dim_k H^5_G(M;\mathbb{F}_p) = 2 + 2b_1(M) + 2t + \dim_k H^3(G; H^2(M;\mathbb{F}_p))$.

Proof We compute

$$\dim_{\mathbf{k}} H^5_G(M) = \sum_{i=0}^{4} \dim_{\mathbf{k}} H^{5-i}(G; H^i(M)) - b$$

and note that

$$\dim_{\mathbf{k}} E_\infty^{3,2} = \dim_{\mathbf{k}} H^3(G; H^2(M)) - b.$$

We then obtain the first formula after taking into account the vanishing of all the other differentials. Similarly,

$$\dim_{\mathbf{k}} H^6_G(M) = \sum_{i=0}^{4} \dim_{\mathbf{k}} H^{6-i}(G; H^i(M)) - b$$

after substituting the value

$$\dim_{\mathbf{k}} E_\infty^{3,3} = \dim_{\mathbf{k}} H^3(G; H^3(M)) - b = t - b,$$

and we obtain the second formula. To compute the third formula, we note that

$$\dim_{\mathbf{k}} H^1(G; H^4(M; \mathbb{F}_p)) = \dim_{\mathbf{k}} H^5(G; \mathbb{F}_p) = 1$$

and

$$\dim_{\mathbf{k}} H^2(G; H^3(M; \mathbb{F}_p)) = \dim_{\mathbf{k}} H^4(G; H^1(M; \mathbb{F}_p)) = t + b_1(M). \qquad \blacksquare$$

In order to compare the integral and mod p formulas, we need some information about the structure of $H^2(M; \mathbb{F}_p)$ as a G-module. We can decompose

$$H^2(M)/\operatorname{Tors} \cong \mathbb{Z}^{r_0(M)} \oplus \mathbb{Z}[\zeta_p]^{r_1(M)} \oplus \Lambda^{r_2(M)}$$

as a G-module (this uses the classification of $\mathbb{Z}$-torsion-free $\mathbb{Z}G$-modules and an argument with $G_0(\mathbb{Z}G)$ due to Swan). This module supports a nonsingular G-invariant symmetric bilinear form arising from the intersection form on M.

Let $T = \operatorname{Tors}(H^2(M))$ and note that $T^* = \operatorname{Ext}^1(T, \mathbb{Z}) = \operatorname{Tors}(H^3(M)) \cong \operatorname{Tors}(H_1(M))$. In our case, $T^* \cong T$ as G-modules with trivial G-action. We introduce the notation $V := T \otimes \mathbb{F}_p$ and $V^* := {}_pT \cong \operatorname{Hom}_{\mathbf{k}}(V, \mathbb{F}_p)$ for the elements of exponent p in T.

Definition 3.9 Let $V := \operatorname{Tors}(H_1(M)) \otimes \mathbb{F}_p$ and $V^* = \operatorname{Hom}_{\mathbf{k}}(V, \mathbb{F}_p)$. We say that the G-representation $H^2(M; \mathbb{F}_p)$ has *split type* if the short exact sequences

$$0 \to H^2(M) \otimes \mathbb{F}_p \to H^2(M; \mathbb{F}_p) \to V^* \to 0$$

and

$$0 \to V \to H^2(M) \otimes \mathbb{F}_p \to (H^2(M)/\operatorname{Tors}) \otimes \mathbb{F}_p \to 0$$

of G-representations are *split exact over G*.

We will show that this condition is always satisfied in the setting of Theorem A.

Lemma 3.10 *If $G = \mathbb{Z}_p$, then $H^2(M; \mathbb{F}_p)$ has split type as a G-representation, and $H^2(M; \mathbb{F}_p) \cong (H^2(M)/\operatorname{Tors}) \otimes \mathbb{F}_p \oplus V \oplus V^*$ as a G-module.*

Proof We have a short exact sequence of $\mathbf{k}$-vector spaces with G-action:

$$0 \to H^2(M) \otimes \mathbb{F}_p \to H^2(M; \mathbb{F}_p) \to V^* \to 0,$$

where $V^* \cong {}_p(H^3(M))$ as a trivial G-representation. Let $\bar{L} := (H^2(M)/\operatorname{Tors}) \otimes \mathbb{F}_p$ and consider the short exact sequence

$$0 \to V \to H^2(M) \otimes \mathbb{F}_p \to \bar{L} \to 0.$$

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Since $\bar{L}$ supports a nondegenerate G-invariant symmetric bilinear form $\bar{L} \times \bar{L} \to \mathbb{F}_p$ (induced by the intersection from of M), it follows that this sequence splits over G and we have $H^2(M) \otimes \mathbb{F}_p \cong V \oplus \bar{L}$ as G-modules. Similarly, the submodule $\bar{L}$ of $H^2(M;\mathbb{F}_p)$ is a direct summand, and we have a splitting

$$H^2(M;\mathbb{F}_p) \cong \bar{L} \oplus H(V),$$

where $H(V)$ is determined by an extension $0 \to V \to H(V) \to V^* \to 0$. The G-module is an $\mathbb{F}_p$-vector space, with isometry t given by a generator of $G = \langle t \rangle$.

To show that the extension determining $H(V) \subseteq H^2(M;\mathbb{F}_p)$ is G-split, consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^2(M) \otimes \mathbb{F}_p \arrow[r] \arrow[d] & H^2(M;\mathbb{F}_p) \arrow[r] \arrow[d] & V^* \to 0 \\ & H^2(M\text{-}A) \otimes \mathbb{F}_p \arrow[r, "\cong"] & H^2(M\text{-}A;\mathbb{F}_p) & \end{tikzcd}$$

The lower isomorphism comes from the Bockstein sequence for $M\text{-}A$ and the fact that $H^3(M\text{-}A)$ is $\mathbb{Z}$-torsion-free. The short exact sequence in diagram (3.3)

$$0 \to H^2(M) \to H^2(M\text{-}A) \to K \to 0$$

shows that $\operatorname{Tors} H^2(M) \xrightarrow{\cong} \operatorname{Tors} H^2(M\text{-}A)$, since K is $\mathbb{Z}$-torsion-free. After tensoring with $\mathbb{F}_p$, we obtain a G-splitting of the submodule $H(V)$. ∎

Corollary 3.11 $\dim_k H^3(G; H^2(M;\mathbb{F}_p)) = 2t + r_0(M) + r_1(M)$.

Proof This follows from Lemma 3.10. ∎

We can put this information together with the formulas in Lemma 3.8. We have

$$\dim_{\mathrm{k}} H^3(G; H^2(M)) = t + r_1(M), \qquad \dim_{\mathrm{k}} H^4(G; H^2(M)) = t + r_0(M).$$

By substituting the values obtained into the dimension formula (3.7), we conclude that $b = \dim_{\mathrm{k}}(\operatorname{Im} d_2^{1,3}) = 0$. In other words, we have shown that the differential $d_2^{k,3} = 0$ for k odd in the Borel spectral sequence with integral coefficients. This completes the proof of Theorem 3.1 and establishes the first part of Theorem A. ∎

Remark 3.12 Since $H^r_G(M) \cong H^r_G(F) = H^r(F \times BG)$ for $r > 4$, we have $\dim_{\mathrm{k}} H^5_G(M) = \dim_{\mathrm{k}} H^5_G(F) = b_1(F)$ and $\dim_{\mathrm{k}} H^6_G(F) = b_0(F) + b_2(F)$. In addition, $\dim_{\mathrm{k}} H^5_G(F) = \dim_{\mathrm{k}} H^3_G(F)$, since F consists of surfaces and isolated points. By computing the trace of the action of a generator on $H^*(M)$, we obtain the relation

$$\chi(F) = b_2(F) - b_1(F) + b_0(F) = 2 - 2b_1(M) + r_0(M) - r_1(M).$$

However, since $H^q_G(M) = H^q_G(F)$ for $q > 4$, we can use the Herbrand quotient formula

$$\dim H^4((G; H^2(M)) - \dim H^3((G; H^2(M)) = r_0(M) - r_1(M)$$

and further calculations similar to those above for $H^q_G(M)$, to show directly that $\chi(F) = \dim H^6_G(M) - \dim H^5_G(M)$.

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4 A non-collapse result

We complete the proof of Theorem A by showing that our surjectivity condition for the map $H_1(F) \to H_1(M)$ is necessary in many cases (see Section 7 for some examples).

Proposition 4.1 *Suppose that $G = \mathbb{Z}_p$ acts locally linearly on a closed, connected, oriented 4-manifold M, preserving the orientation, with nonempty fixed-point set F. If*

$$\ker(H^1(M;\mathbb{Z}) \to H^1(F;\mathbb{Z}))$$

is nontrivial, but has trivial G-action, then the Borel spectral sequence with integral coefficients does not collapse.

Proof The proof uses the fact that $H^q_G(M,F) = 0$ for $q > 4$, implying that $E^{q,1}_\infty(M,F) = 0$ for $q > 4$ in the Borel spectral sequence. We will show that this leads to a contradiction.

If $H^1(M) \to H^1(F)$ is not injective, we let $0 \neq K = \ker(H^1(M) \to H^1(F))$, which by assumption has trivial G-action. Therefore, $H^{2r}(G;K) \neq 0$. We consider the relative long exact sequence for the pair (M,F), and we get short exact sequences

$$0 \to H^0(F)/H^0(M) \to H^1(M,F) \to K \to 0$$

and

$$0 \to K \to H^1(M) \to L \to 0,$$

where $L = \mathrm{Im}(H^1(M) \to H^1(F))$. Since both L and $H^0(F)/H^0(M)$ are $\mathbb{Z}$-torsion-free with trivial G-action, we have $H^{2r-1}(G;L) = 0$ and $H^{2r+1}(G;H^0(F)/H^0(M)) = 0$. By applying group cohomology to the sequences above, we obtain

$$\begin{tikzcd} H^{2r}(G;H^1(M,F)) \arrow[rr, "\alpha"] \arrow[dr] & & H^{2r}(G;H^1(M)) \\ & H^{2r}(G;K) \arrow[ur] & \end{tikzcd}$$

where $H^{2r}(G;H^1(M,F)) \twoheadrightarrow H^{2r}(G;K)$ is surjective and $H^{2r}(G;K) \rightarrowtail H^{2r}(G;H^1(M))$ is injective. Since $E^{2r,1}_\infty(M,F) = 0$ for $2r > 4$, some differential must hit a preimage of a nonzero element in $H^{2r}(G;K)$. By comparison, we see that the Borel spectral sequence for $H^*_G(M)$ has a nonzero differential and hence does not collapse. ∎

5 Homologically trivial actions

We will first consider the Borel spectral sequence for a cyclic p-group acting homologically trivially. We use $\mathbb{F}_p$-coefficients throughout this section.

Proposition 5.1 *Let $G = \mathbb{Z}_p$ act homologically trivially on M. Assume that $\chi(M) \neq 0$ and the fixed set F is discrete. Then, the differentials $d_r = 0$, for $r \geq 3$, in the Borel spectral sequence with $\mathbb{F}_p$ coefficients. Moreover, $b_2(M) \geq 2b_1(M)$, and the Borel spectral sequence does not collapse unless $b_1(M) = 0$. In particular, $d_2^{k,3}$ is injective for $k \geq 0$ and $d_2^{k,2}$ is surjective for $k \geq 0$.*

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Proof The difference in the multiplicative structure of the $\mathbb{F}_p$-cohomology algebras of $G = \mathbb{Z}_p$ for p odd and $p = 2$ does not affect the proof, so we consider both cases together. Since the action is homologically trivial and $\chi(M) \neq 0$, the fixed set $F \neq \emptyset$, and F consists of $\chi(M)$ isolated points.

We will prove the result by computing the dimension of $H^5_G(M)$, which must be equal to $\dim H^5_G(F) = \chi(M)$ by [5, Proposition 2.1].

The same arguments used in the proof of Theorem A show that the differentials $d_2^{0,4}$ and $d_2^{0,1}$ are both zero (these work the same way for $p = 2$ as for p odd). Moreover, since $F \neq \emptyset$, the inclusion induces an injection $H^*(G) \to H^*(M_G)$, so $E_2^{*,0} = E_\infty^{*,0}$.

The key to understanding the other d_2 differentials is the result of Sikora [18, Section 3.3], which shows that $E_3^{2,1} \cong E_3^{2,3}$ by recognizing a Poincaré duality structure on certain terms of the Borel spectral sequence. Since $E_2^{2,1} \cong E_2^{2,3}$, and $d_2^{2,1} = 0$, it follows that $\ker d_2^{2,3} \cong \operatorname{coker} d_2^{0,2}$. Therefore, if $R = \dim \ker d_2^{2,3}$, we have $\dim \operatorname{Im} d_2^{0,2} = b_1(M) - R$. Therefore,

$$\dim \operatorname{Im} d_2^{2,3} = \dim \operatorname{Im} d_2^{1,3} = \dim \operatorname{Im} d_2^{3,2} = b_1(M) - R,$$

so we have $\dim E_3^{2,3} = \dim E_3^{4,1} = R$, and $\dim E_3^{3,2} = b_2(M) - 2b_1(M) + 2R$. A detailed study of the possible d_3 differentials now shows that from the relation

$$\sum \dim E_3^{k,5-k} = 2 + b_2(M) - 2b_1(M) + 4R$$

and the convergence to $H^5_G(M) \cong H^5_G(F)$, we must have $R = 0$. The details are similar to those in Section 3. Since $F \neq \emptyset$, we conclude that $d_r^{*,4} = 0$ for $r = 3, 4, 5$. The only remaining differential to consider is $d_3^{*,3}$, but since Poincaré duality is preserved between $E_4^{4,1} \cong E_4^{4,3}$, we see that $d_3^{*,3} = 0$. Therefore, $d_2^{k,3}$ is injective for $k \geq 0$ and $d_2^{k,2}$ is surjective for $k \geq 0$. By the dimension count above, this shows that the higher differentials $d_r = 0$ for $r \geq 3$. ∎

With extra assumptions such as homological triviality and torsion-free $H_1(M)$, we can prove the converse of Theorem 3.1.

Corollary 5.2 *Let $G = \mathbb{Z}_p$ for p odd act locally linearly and homologically trivially on a closed, connected, oriented 4-manifold M with the fixed-point set F nonempty and $H_1(M;\mathbb{Z})$-torsion-free. Then, the Borel spectral sequence with integral coefficients collapses if and only if $H_1(F) \twoheadrightarrow H_1(M)$ is surjective.*

Proof Since we are assuming that $H_1(M;\mathbb{Z})$-torsion-free, the condition that $H_1(F) \to H_1(M)$ is surjective is equivalent to the condition that $H^1(M;\mathbb{Z}) \to H^1(F;\mathbb{Z})$ is injective. The result now follows from Theorem 3.1 and Proposition 4.1. ∎

Corollary 5.3 *Let p be an odd prime. If $G = \mathbb{Z}_p$ acts homologically trivially and locally linearly on M with $\chi(M) \neq 0$, such that $H_1(F) \twoheadrightarrow H_1(M)$ is surjective, then the $\mathbb{F}_p$-Betti numbers satisfy $b_1(F) = 2b_1(M)$ and $b_0(F) + b_2(F) = 2 + b_2(M)$.*

Proof Since the action is homologically trivial, $\chi(F) = \chi(M) \neq 0$ by the Lefschetz fixed-point theorem and hence $F \neq \emptyset$. By Theorem 3.1, we know that Borel spectral sequence collapses, and by Proposition 2.4 (with $\mathbf{k} = \mathbb{F}_p$-coefficients), we have

$$\sum_r \dim_{\mathbf{k}} H^r(F) = \sum_r \dim_{\mathbf{k}} H^r(M).$$

It follows that $b_1(F) = 2b_1(M)$ and $b_0(F) + b_2(F) = 2 + b_2(M)$ for odd p. ∎

We can also apply our results to some actions of rank-two groups (compare [5, Proposition 6.1]).

Remark 5.4 If G acts homologically trivially and the Borel spectral sequence $E(M_K)$ does not collapse for the subgroup $K \leq G$ of a group G, then $E(M_G)$ does not collapse.

Proposition 5.5 *For an odd prime p, let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ act homologically trivially and locally linearly on M with nonempty fixed-point set. Suppose that $H_1(M;\mathbb{Z})$ is torsion-free. Then, the Borel spectral sequence with $\mathbb{F}_p$-coefficients collapses if and only if $H_1(M) = 0$.*

Proof Suppose that the fixed set F contains a two-dimensional component $F_1 \subseteq F$. Consider the action of G on the boundary of a G-equivariant normal 2-disk neighborhood of a point $x \in F_1$. Since $G = \mathbb{Z}_p \times \mathbb{Z}_p$ and p is odd, this gives a contradiction, since there is no such G-action on a circle. Hence, the fixed set F consists of a finite set of isolated points.

Next, we remark that in a small G-invariant neighborhood U of each fixed point $x \in F$ has $T_x U \cong V_1 \oplus V_2$, where $V_i = \mathrm{Fix}(T_x U, K_i)$, for two-order p subgroups $K_1 = \langle a \rangle$ and $K_2 = \langle b \rangle$ of G which have $K_1 \cap K_2 = \{1\}$.

Therefore, each G-fixed point $x \in F$ is contained in exactly two singular surfaces S_1 and S_2, where $S_1 \subseteq \mathrm{Fix}(K_1)$ and $S_2 \subseteq \mathrm{Fix}(K_2)$. Note that the action of G/K on a K-fixed surface S has an even number of fixed points, equal to $2 + \dim_{\mathbf{k}} H^1(G/K; H^1(S))$.

We now restrict the G-action to any index p subgroup $K \leq G$ and let $\mathrm{Fix}(K)$ denote its fixed set. The remarks above show that $\mathrm{Fix}(K)$ contains fixed orientable surfaces, each with an effective action of $G/K \cong \mathbb{Z}_p$. Since a $\mathbb{Z}_p$-action on an orientable surface $S \neq S^2$ induces an effective action on $H^1(S)$, we see that the map $H^1(M) \to H^1(\mathrm{Fix}(K))$ must be zero: either all the surfaces are 2-spheres, so that $H^1(\mathrm{Fix}(K)) = 0$ or the G/K-action on $H^1(M)$ would be nontrivial, contradicting our homologically trivial assumption.

Therefore, if $H_1(M) \neq 0$, the Borel spectral sequence for $E_K(M)$ does not collapse with $\mathbb{Z}$-coefficients (by Proposition 4.1). Since the homology of M is torsion-free, $H^r(M) \otimes \mathbb{F}_p \cong H^r(M;\mathbb{F}_p)$, and it follows from the Bockstein sequence that the maps $H^r(K; H^s(M)) \to H^r(K; H^s(M;\mathbb{F}_p))$ are injective for all $r > 0$. Therefore, the Borel spectral sequence for $E_K(M)$ does not collapse with $\mathbb{F}_p$-coefficients either. Hence, if $H_1(M) \neq 0$, the Borel spectral sequence for $E(M_G)$ does not collapse (see Remark 5.4).

If $H_1(M) = 0$, then our assumption that the fixed set $F \neq \emptyset$ and multiplicativity implies that the Borel spectral sequence for $E(M_G)$ does collapse (since no differentials can hit the line $E_2^{*,0}$). ∎

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6 The proof of Theorem B

Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, for p odd, and recall that the mod p cohomology algebra

$$H^*(G) = \mathbb{F}_p[u_1, u_2] \otimes \Lambda(x_1, x_2),$$

where $|u_i| = 2$ and $|x_i| = 1$, with $x_i^2 = 0$. We will use cohomology with $\mathbb{F}_p$-coefficients throughout this section.

The *essential cohomology*, denoted $\mathrm{Ess}^*(G) \subset H^*(G)$, is defined as the intersection of the kernels of the restriction maps induced by the $(p+1)$ nontrivial cyclic subgroups $K \leq G$. A nice description is given below:

Theorem 6.1 (Aksu and Green [1]) *For $G = \mathbb{Z}_p \times \mathbb{Z}_p$, the essential cohomology $\mathrm{Ess}(G)$ is the smallest ideal in $H^*(G)$ containing x_1x_2 and closed under the action of the Steenrod algebra. Moreover, as a module over $\mathbb{F}_p[u_1, u_2]$, the essential ideal $\mathrm{Ess}^*(G)$ is free on the set of Mùi generators.*

This statement is a special case of their general result. For the rank-two case, the Mùi generators are as follows:

$$\gamma_1 = x_1x_2,\ \gamma_2 = x_1u_2 - x_2u_1,\ \gamma_3 = x_1u_2^p - x_2u_1^p, \text{ and } \gamma_4 = u_1u_2^p - u_2u_1^p.$$

We note that the degrees are $2, 3, 2p+1$, and $2p+2$, respectively. For detailed calculations, it is useful to let $R := \mathbb{F}_p[u_1, u_2]$ and $\Lambda := \Lambda(x_1, x_2)$. These are graded rings with $\dim R^{2k} = k+1$, $\dim \Lambda^1 = 2$, and $\dim \Lambda^2 = 1$. In this notation, $H^{2k}(G) = R^{2k} \oplus (R^{2k-2} \otimes \Lambda^2)$ and $H^{2k+1}(G) = R^{2k} \otimes \Lambda^1$. Note that $H^*(G)$ is generated as an R-module by the cohomology groups $H^k(G)$, for $k \leq 2$.

The proof of Theorem B is based on a detailed study of the Borel spectral sequence. Here is an example for the case $p = 3$, which illustrates some of the features. As explained in that proof, the images of any differentials in the Borel spectral sequence for $H_G^*(M)$ with range $E_r^{k,0}$, for any $k \geq 0$, must belong to $\mathrm{Ess}^*(G)$.

Example 6.2 Let $M = \mathbb{CP}^2$ with the pseudofree action of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ given by $S(z_1, z_2, z_3) = (z_1, \omega z_2, \omega^2 z_3)$ and $T(z_1, z_2, z_3) = (z_2, z_3, z_1)$. The singular set consists of 12 points, arranged in 4 triangles each fixed by one of the 4 subgroups of order 3 in G. By [5, Proposition 2.1], we have an isomorphism $H_G^q(M) \to H_G^q(\Sigma)$, for $q > 4$, and hence $\dim H^q(G; H^0(\Sigma)) = 4$ for $q > 4$ (see the proof of Theorem B for details). It turns out that for this dimension bound to hold, the Múi generators γ_2, γ_3, and γ_4 (in degrees 3, 7, and 8, respectively) must be hit by differentials. We will use the dimension bound $\dim E_\infty^{q,0} \leq \dim H_G^q(M) = 4$, for $q > 4$.

Since $d_2 = 0$, $E_2 = E_3$ and the E_3-page has three lines, where the differentials are determined by the values of $d_3^{0,q}\colon E_2^{0,q} \to E_2^{3,q-2}$, for $q = 2, 4$, and the multiplicative structure of $H^*(G)$. Let $z \in H^2(\mathbb{CP}^2; \mathbb{Z})$ be the generator dual to the homology class of $\mathbb{CP}^1 \subset \mathbb{CP}^2$ and let $w = z^2 \in H^4(\mathbb{CP}^2; \mathbb{Z})$ be the orientation class. Then, $d_3(z) = \gamma_2$ and $d_3(w) = -\gamma_2 z$. Therefore, $d_3^{2,4}(\gamma_1 w) = -\gamma_1\gamma_2 = 0$ and $d_3^{3,4}(\gamma_2 w) = -(\gamma_2)^2 z = 0$, so these elements persist to the E_5-page, with $\dim E_5^{2,4} = \dim E_5^{3,4} = 1$.

For the dimension count of $H_G^6(M)$, we also need to compute $E_4^{4,2}$ and $E_4^{6,0}$ (and note that $E_4 = E_5$). It is not hard to check that $\mathrm{Im}\, d_3^{1,4} = \langle \gamma_1 u_1, \gamma_1 u_2 \rangle \subset E_3^{4,2} \cong H^4(G)$,

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and this equals the kernel of $d_3^{4,2}$. Therefore, $E_4^{4,2} = 0$. Next,

$$\operatorname{Im} d_3^{3,2} = \Lambda^1 \cdot R^2 \cdot \gamma_2 = \langle \gamma_1 u_1^2, \gamma_1 u_1 u_2, \gamma_1 u_2^2 \rangle \subset E_3^{6,0} \cong H^6(G).$$

Therefore, $\dim E_4^{6,0} = 4$, and the dimension count shows that there is one remaining nonzero differential $d_5^{2,4}\colon E_5^{2,4} \to E_5^{7,0}$ affecting the line $k + l = 6$.

For the dimension count of $H_G^7(M)$, we have $\dim E_4^{3,4} = 1$ and we make similar calculations to determine $E_4^{5,2}$ and $E_4^{7,0}$. We see that $\operatorname{Im} d_3^{2,4} = \langle \gamma_2 u_1, \gamma_2 u_2 \rangle = \ker d_3^{5,2}$, so $E_4^{5,2} = 0$. We compute

$$\operatorname{Im} d_3^{4,2} = \gamma_2 \cdot R^4 = \langle x_1 u_1^2 u_2 - x_2 u_1^3, x_1 u_1 u_2^2 - x_2 u_1^2 u_2, x_1 u_2^3 - x_2 u_1 u_2^2 \rangle \subset H^7(G).$$

Therefore, $\dim E_4^{7,0} = 5$, and the dimension count confirms that $d_5^{2,4}$ is nonzero with one-dimensional image. More precisely, $\operatorname{Im} d_5^{2,4} = \langle \gamma_3 \rangle$, since

$$0 \neq \operatorname{Im} d_5^{2,4} \subseteq \operatorname{Ess}^7(G)/\operatorname{Im} d_3^{4,2} = \langle \gamma_3, \gamma_2 \cdot R^4 \rangle / \operatorname{Im} d_3^{4,2} \cong \langle \gamma_3 \rangle.$$

By a similar calculation, $\operatorname{Im} d_3^{5,2} = \operatorname{Im} d_3^{4,2} \cdot \{x_1, x_2\} = \Lambda^2 \cdot R^6$ has dimension 4 and $\dim E_4^{8,0} = 5$, so that $d_5^{3,4}$ must have one-dimensional image. Since $\operatorname{Ess}^8(G) = \langle \gamma_4, \Lambda^2 \cdot R^6, \gamma_4 \rangle$, it follows that $d_5^{3,4}$ hits γ_4, and hence the differentials surject onto $\operatorname{Ess}^q(G)$, for $q > 2$.

Remark 6.3 To rule out higher rank actions as asserted in Theorem B, we will show that the Mùi generators γ_{2p+1} and γ_{2p+2} for $p \geq 5$ cannot be hit by differentials in the Borel spectral sequence for $G = \mathbb{Z}_p \times \mathbb{Z}_p$. This would imply that the groups $H_G^q(M)$ for large values of q would have dimensions contradicting the bound (6.4) from the singular set and hence rule out the existence of these actions.

In order to prove this claim, the key point is that the differentials are determined through multiplicativity by their values on $E_r^{k,l}$ for $k \leq 3$. This is a consequence of the structure of the cohomology ring $H^*(G)$, which is generated by classes in degrees ≤ 2 (as explained in Example 6.2).

Proof [The proof of Theorem B] Suppose that G is acting homologically trivially on M with $\chi(M) \neq 0$. In addition, we are assuming that the action is pseudofree, meaning that the singular set Σ is a discrete set of points. Note that $M^G = \emptyset$ since G cannot act freely on S^3. Each subgroup $K \cong \mathbb{Z}_p$ has $\chi(M) > 0$ fixed points, which are then permuted in $\chi(M)/p$ orbits of size p by G/K, so that $H^0(\operatorname{Fix}(K))$ is the direct sum of $\chi(M)/p$ copies of the permutation G-module $\mathbb{F}_p[G/K]$.

By [5, Proposition 2.1], we have an isomorphism $H_G^q(M) \xrightarrow{\approx} H_G^q(\Sigma)$, for $q > 4$, and this provides a dimension count as above. In this case, we have $p \mid \chi(M)$ and there are $p + 1$ distinct subgroups of order p in G, so that

$$\dim H^q(G; H^0(\Sigma)) = \sum_i \dim H^q(G; \mathbb{F}_p[G/K_i])^{\chi(M)/p} = \frac{\chi(M)}{p} \cdot (p+1) \tag{6.4}$$

by Shapiro's Lemma. The main observation is that the images of any differentials in the Borel spectral sequence for $H_G^*(M)$ with range $E_r^{k,0}$, for any $k \geq 0$, must belong to $\operatorname{Ess}^*(G)$. This follows immediately by comparing the spectral sequences for $H_G^*(M)$ and $H_G^*(\Sigma)$. Similarly, by Proposition 5.1, the images of the higher differentials d_r, for $r \geq 3$, must lie in $\operatorname{Ess}^*(G)$ modulo indeterminacy from the earlier differentials.

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Moreover, since the $\mathrm{Res}_K\colon H^r(G;\mathbb{F}_p[G/K]) \to H^r(K;\mathbb{F}_p[G/K])$ is an injection, the sum of the restriction maps

$$\bigoplus_K \mathrm{Res}_K\colon H^q_G(M) \to \bigoplus_K \{H^q_K(M) \,|\, 1 \neq K \neq G\}$$

is also an injection for $q > 4$.

We have commutative diagram (for $q > 4$):

$$\begin{tikzcd}
H^q(G) \arrow[r, "\cong"] \arrow[d, "\mathrm{Res}_K"] & E_2^{q,0}(M_G) \arrow[r, two heads] \arrow[d, "\mathrm{Res}_K"] & E_\infty^{q,0}(M_G) \arrow[r, tail] \arrow[d, "\mathrm{Res}_K"] & H^q_G(M) \arrow[d, "\mathrm{Res}_K"] \\
\oplus H^q(K) \arrow[r, "\cong"] & \oplus E_2^{q,0}(M_K) \arrow[r, "\cong"] & \oplus E_\infty^{q,0}(M_K) \arrow[r, "\cong"] & \oplus H^q_K(M)
\end{tikzcd}$$

It follows from this diagram, and the fact that the images of differentials with range in $E_r^{k,0}$ are contained in $\mathrm{Ess}^*(G)$ and that $\mathrm{Ess}^q(G) = \ker\{E_2^{q,0}(M_G) \twoheadrightarrow E_\infty^{q,0}(M_G)\}$, for $q > 4$, is a necessary condition for the G-action to exist.

For $p = 3$, the Mùi generators have dimensions $2, 3, 7$, and 8, and these are all within the range of the differentials $d_r^{k,l}$, for $r \le 5$ and $k \le 3$ (as in Example 6.2). However, for $p > 5$, only the first two Mùi generators $\gamma_1 = x_1x_2$ and $\gamma_2 = x_1u_2 - x_2u_1$ can be hit by a nonzero differential $d_r^{k,r-1}$ if $k = 2p+1-r$ or $k = 2p+2-r$, with $r \le 5$. $2p+1 \ge 11$ Since, this implies $k \ge 6$.

Consider the differentials d_r with range in the line $E_r^{*,0}$. These are $d_2^{k,1}$, $d_3^{k,2}$, $d_4^{k,3}$, and $d_5^{k,4}$. At each page, if the differential $d_r^{k,r-1}$ is nonzero, its image must lie in $\mathrm{Ess}^r(G)$. We claim that the images of the differentials $d_r^{k,r-1}$, for all $k \ge 0$, will be contained in the module generated by the first two Mùi generators γ_1 and γ_2 under the action of the polynomial algebra $\mathbb{F}_p[u_1, u_2]$. Since $\mathrm{Ess}^*(G)$ is a free module on all the Mùi generators (by [1, Theorem 1.2]), we will have a contradiction to the dimension bound on $H^q_G(M)$ for large q, and the assumed G-action does not exist.

To verify this, we tabulate the generators of $\mathrm{Ess}^k(G)$ for $2 \le k \le 6$ as follows:

$$\mathrm{Ess}^k(G) = \{\langle\gamma_1\rangle, \langle\gamma_2\rangle, \langle\gamma_1u_1, \gamma_1u_2\rangle, \langle\gamma_2u_1, \gamma_2u_2\rangle, \langle\gamma_1u_1^2, \gamma_1u_1u_2, \gamma_1u_2^2\rangle\}.$$

For use in our arguments below, we also note that $\mathrm{Ess}^k(G)$ is generated by γ_1 and γ_2 over R in degrees $k \le 10$ (for all primes $p \ge 5$).

We first fix some notation for an $\mathbb{F}_p$-basis of the cohomology of M: let us denote them by $w \in H^4(M)$, $\langle\beta_1, \dots, \beta_t\rangle \subset H^3(M)$, $\langle z_1, \dots, z_s\rangle \subset H^2(M)$, and $\langle\alpha_1, \dots, \alpha_t\rangle \subset H^1(M)$. We will check the images of the differentials $d_r^{k,r-1}$ in each case.

The image of $d_2^{k,1}\colon E_2^{k,1} \to E_2^{k+2,0}$. Since $\operatorname{Im} d_2^{0,1} \subseteq \mathrm{Ess}^2(G) = \langle\gamma_1\rangle$, either $d_2^{k,1} = 0$, for $k \ge 0$, or $d_2^{0,1}(\alpha_1) = \gamma_1$, and we may assume that $d_2^{k,1}(\alpha_k) = 0$, for $k \ge 2$. In the second case, $\operatorname{Im} d_2^{k,1} \subseteq \gamma_1 \cdot R$ and $\ker d_2^{k,1} = \langle\alpha_1 \cdot (\Lambda^1 \otimes R), \alpha_2, \dots, \alpha_t\rangle$. In particular, the image of $d_2^{k,1}\colon E_2^{k,1} \to E_2^{k+2,0}$ does not contain γ_2, γ_3, or γ_4.

The image of $d_3^{k,2}\colon E_3^{k,2} \to E_2^{k+3,0}$. The image of $d_2^{k,2}$ restricted to any order p subgroup of G must be surjective, by Proposition 5.1. It follows that either $d_2^{0,2}(z_i) \neq 0$ and projects nontrivially to $\alpha_j \cdot H^2(G)/\langle\gamma_1\rangle$, for some α_j, or $d_2^{0,2}(z_i) = 0$, and $\operatorname{Im} d_3^{k,2}(z_i) \subseteq \gamma_2 \cdot R$. Therefore, $\operatorname{Im} d_3^{k,2}$ does not contain γ_3 or γ_4.

The image of $d_4^{k,3}\colon E_4^{k,3} \to E_2^{k+4,0}$. Since $d_2^{k,3}$ is injective when restricted to any order p subgroup, by Proposition 5.1, it follows that $\operatorname{Im} d_2^{k,3}(\beta_i)$ projects nontrivially to $H^k(G;H^2(M))/\langle \mathrm{Ess}^k(G)\cdot H^2(M)\rangle$. Therefore, $d_2^{k,3}$ is injective, and $E_r^{k,3}=0$ for $r\geq 3$ implies $d_4^{k,3}=0$.

The image of $d_5^{k,4}\colon E_4^{k,4} \to E_4^{k+5,0}$. Since $d_2^{k,3}$ is injective, we have $d_2^{k,4}=0$, for $k\geq 0$. Suppose first that $0\neq d_3^{0,4}(w)\in \gamma_2\cdot H^2(M)$. Then, $\operatorname{Im} d_3^{k,4}\subseteq (\langle \gamma_1,\gamma_2\rangle\cdot R)\cdot H^2(M)$. Therefore, $\ker d_3^{k,4}\subseteq \langle \overline{\gamma_1 w},\overline{\gamma_2 w}\rangle\cdot R\subseteq E_4^{k,4}$, and $\operatorname{Im} d_4^{k,4}$ is generated by the images $d_4^{2,4}(\overline{\gamma_1 w})\in \mathrm{Ess}^6(G)\cdot E_4^{6,1}$ and $d_4^{3,4}(\overline{\gamma_2 w})\in \mathrm{Ess}^7(G)\cdot E_4^{7,1}$ under the action of R.

If both these images under $d_4^{k,4}$ are nonzero, then $\ker d_4^{k,4}=0$, since multiplication by elements of R is injective on $\operatorname{Im} d_4^{k,4}$. Therefore, $d_5^{k,4}=0$; hence , γ_3 or γ_4 cannot be hit.

If either of these images under $d_4^{k,4}$ is zero, then their corresponding images under $d_5^{k,4}$ will be contained in $\mathrm{Ess}^q(G)$ for $q\leq 7$, and again d_5 cannot hit γ_3 or γ_4.

For $p=3$, we rule out actions of $G=\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$ by similar arguments. In the rank-three case, there are eight Mùi generators, starting with $\gamma_1=x_1x_2x_3$ and $\gamma_2=\beta(\gamma_1)$ in degrees 3 and 4, and continuing in degrees $8,9,20,21,25$, and 26 (see [1, Section 3]). The higher Mùi generators are outside the range of differentials hitting the line $E_r^{*,0}$. Hence, such an action does not exist.

For $p=2$ and $G=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, the cohomology ring is now $H^*(G)=\mathbb{F}_2[x_1,x_2,x_3]$ and there is just one Mùi generator

$$\gamma = x_1x_2x_3(x_1+x_2)(x_1+x_3)(x_2+x_3)(x_1+x_2+x_3)$$

in degree 7, which is the product of the distinct linear forms. The ideal $\mathrm{Ess}^*(G)=\langle\gamma\rangle$ is a free module over $\mathbb{F}_p[x_1,x_2,x_3]$, and $\mathrm{Ess}^*(G)$ is the Steenrod closure of γ in $H^*(G)$ (see [1, Lemma 2.2]). This means that the rank-two actions cannot be ruled out by the method above (in fact, such actions exist on $S^2\times S^2$).

However, we can use the information contained in the proof of Proposition 5.1 to see that the images of the differential $d_2^{0,2}$ in $E_2^{2,1}(K)$ must be nonzero in each summand of $H^2(K)\otimes H^1(M)$, and for each subgroup $K\cong\mathbb{Z}_2$. Therefore, there must be a class $\alpha\in H^2(G)$ such that $\mathrm{Res}_K(\alpha)\neq 0$ for each $K<G$ of order 2. We claim that no such class exists. To see this, let $H\cong\mathbb{Z}_2\times\mathbb{Z}_2$ be an index-two subgroup. The only possibility for $\mathrm{Res}_H(\alpha)$ is the class $\delta=\bar{x}_1^2+\bar{x}_1\bar{x}_2+\bar{x}_2^2$, where $\bar{x}_i$ denote the degree- 1 generators of the cohomology of H. We look at the restriction of a general element

$$\alpha = \sum_{1\leq i\leq 3} a_i x_i^2 + \sum_{i<j} b_{ij}x_ix_j \in H^2(G)$$

to each of the index-two subgroups H obtained by imposing one of the seven linear relations in the formula for γ. First, to get $\mathrm{Res}_H(\alpha)=\delta$ by setting $x_i=0$ for each $1\leq i\leq 3$ separately, we find that all the coefficients a_i and b_{ij} must be nonzero. But then, setting $x_1+x_2=0$ gives $\mathrm{Res}_H(\alpha)=\bar{x}_1^2+\bar{x}_3^2\neq\delta$. Hence, α does not exist, and such a rank-three pseudofree G-action is ruled out. ∎

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7 Some examples

In this section, we give some illustrative examples of group actions on a closed, connected oriented 4-manifold. These indicate the necessity of the conditions in Theorem 3.1 for the Borel spectral sequence to collapse. We let $\mathbf{k} = \mathbb{F}_p$ with the prime p under consideration understood.

Example 7.1 Consider (i) $S^1 \times S^3$ with $\mathbb{Z}_3$ acting trivially on S^1 and by rotation on S^3, so that the fixed-point set $S^1 \times S^1$ and (ii) $\mathbb{C}P^2$ with a $\mathbb{Z}_3$-action fixing $\mathbb{C}P^1$ and a point. Taking the equivariant connected sum along the two-dimensional fixed set, we get $M = S^1 \times S^3 \# \mathbb{C}P^2$ with the fixed-point set $F = S^1 \times S^1 \# \mathbb{C}P^1 \cup \{pt\}$.

By Theorem 3.1, since $H_1(F) = \mathbb{Z} \oplus \mathbb{Z}$ surjects onto $H_1(M) = \mathbb{Z}$, the Borel spectral sequence with integral coefficients collapses for this example. Since the action is homologically trivial, and the total dimensions satisfy

$$\sum_r \dim_{\mathbf{k}} H^r(F) = 5 = \sum_r \dim_{\mathbf{k}} H^r(M),$$

the Borel spectral sequence with $\mathbb{F}_3$-coefficients collapses by Proposition 2.4.

Next, we have a case where the fixed-point set consists of isolated points and $H_1(M)$ is torsion-free.

Example 7.2 Consider the diagonal action of $\mathbb{Z}_p$ on $S^2 \times S^2$ with four fixed points. Now, take two copies of $S^2 \times S^2$ with this action and take the equivariant connected sum along two pairs of fixed points where the representations of the tangent bundles are equivalent. We obtain a 4-manifold M which has a $\mathbb{Z}_p$-action with four fixed points. M has $H_i(M) = \mathbb{Z}$ for $i = 0, 1, 3, 4$ and $H_2(M) = (\mathbb{Z})^4$ as homology groups. Since the action is homologically trivial, we can also again use Proposition 2.4

$$\sum_r \dim_{\mathbf{k}} H^r(M^G) = 4 \neq 8 = \sum_r \dim_{\mathbf{k}} H^r(M),$$

showing that the Borel spectral sequence with $\mathbb{F}_p$-coefficients does not collapse.

There are also examples where the fixed-point set is two-dimensional, but the Borel spectral sequence does not collapse.

Example 7.3 Consider again a $\mathbb{Z}_3$-action on $\mathbb{C}P^2$ fixing a $\mathbb{C}P^1$ and a point. Take two copies of this and take the equivariant connected sum along the two-dimensional fixed sets and the fixed points. The manifold we obtain is a 4-manifold having $\mathbb{Z}_3$-action with a connected two-dimensional fixed set which has the homology of the two sphere. Again, the action is homologically trivial and by Proposition 2.4

$$\sum_r \dim_{\mathbf{k}} H^r(M^G) = 2 \neq 6 = \sum_r \dim_{\mathbf{k}} H^r(M),$$

showing that the Borel spectral sequence with $\mathbb{F}_3$-coefficients does not collapse. Here the map $H_1(F) \to H_1(M)$ is not surjective.

Here is an example with p-torsion in $H_1(M)$.

Example 7.4 Let $M = L^3(\mathbb{Z}_p, 1) \times S^1$, with the action of $G = \mathbb{Z}_p$ given by

$$\zeta \cdot ([z_1 : z_2], z_3) = ([\zeta \cdot z_1 : z_2], z_3).$$

Note that $[\zeta \cdot z_1 : z_2] = [z_1 : \zeta^{-1} \cdot z_2]$ because of the equivalence relation used to define $L^3(\mathbb{Z}_p, 1)$. The fixed set $F = S^1 \times S^1 \bigsqcup S^1 \times S^1$, and $H_1(F) \to H_1(M)$ is surjective; hence, the Borel spectral sequence collapses.

Here is an example for which $H_1(M)$ has nontrivial G-action.

Example 7.5 For $G = \mathbb{Z}_2$, consider the diagonal reflection on $M = S^1 \times S^3$ which reverses the orientation on each factor. The fixed-point set $F = S^2 \bigsqcup S^2$ and $H^1(M) = \mathbb{Z}_-$. Since the total dimensions satisfy

$$\sum_r \dim_{\mathbf{k}} H^r(M^G) = 4 = \sum_r \dim_{\mathbf{k}} H^r(M),$$

the Borel spectral sequence collapses.

Finally, we will give an example with $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acting homologically trivially.

Example 7.6 Consider the $\mathbb{Z}_p \times \mathbb{Z}_p$-action on $S^2 \times S^2$ given by the product of two rotation actions of $\mathbb{Z}_p$ on S^2. This action has four fixed points and a singular set consisting of four 2-spheres. Let M be obtained by taking the equivariant connected sum of two copies of $S^2 \times S^2$ along two of the fixed points. Then, M admits a $\mathbb{Z}_p \times \mathbb{Z}_p$-action with four global fixed points and which is homologically trivial and locally linear (in fact, smooth). The Borel spectral sequence with $\mathbb{F}_p$-coefficients does not collapse (by Remark 5.4 and Corollary 5.2). This is a counterexample to [16, Corollary 3.2].

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McMaster University, Hamilton, ON, Canada
e-mail: hambleton@mcmaster.ca

Middle East Technical University, Ankara, Turkey
e-mail: pasemra@metu.edu.tr

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