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ON TALAGRAND'S EXHAUSTIVE PATHOLOGICAL SUBMEASURE

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Abstract. We investigate Talagrand's construction of an exhaustive pathological submeasure. We consider the forcing associated with this submeasure and we also begin an effort to explicitly describe this construction.

§1. Introduction. Given a Boolean algebra $\mathfrak{A}$, a function $\mu : \mathfrak{A} \to \mathbb{R}$ is called a *submeasure* if the following conditions are satisfied:

- $\mu(0) = 0$;
- $(\forall a, b \in \mathfrak{A})(a \leq b \to \mu(a) \leq \mu(b))$;
- $(\forall a, b \in \mathfrak{A})(\mu(a \cup b) \leq \mu(a) + \mu(b))$.

A submeasure is *additive* if, given disjoint $a, b \in \mathfrak{A}$, we have $\mu(a \cup b) = \mu(a) + \mu(b)$. Additive submeasures are called *measures*. Two submeasures μ and λ are *equivalent* if, for any sequence $(a_n)_n$ from $\mathfrak{A}$, we have

$$\lim_n \mu(a_n) = 0 \leftrightarrow \lim_n \lambda(a_n) = 0.$$

A submeasure μ on $\mathfrak{A}$ is called *exhaustive* if, given a pairwise disjoint sequence $(a_n)_n$ from $\mathfrak{A}$, we have $\lim_n \mu(a_n) = 0$. Maharam's problem, also known as the control measure problem, asks if every exhaustive submeasure is equivalent to a measure. This problem first appeared in [16] and in [19], building on [17] and [4], M. Talagrand constructs an exhaustive submeasure on the clopen (closed and open) sets of the Cantor space that is not equivalent to a measure. Maharam's problem has many equivalent formulations and these are, by now, well documented. A detailed treatment of this topic is given in [6, Chapter 39], and a very accessible survey, which also discusses a related problem of von Neumann, is given in [1].

A frustrating aspect of Maharam's problem is the complexity of its solution. It seems, at least in the literature, that very little progress has been made (or perhaps, attempted) in trying to analyse Talagrand's construction. The only discussion other than [19] on Talagrand's solution that we are aware of is [5]. Consequently it is still not clear how much more insight we have into Maharam's problem now that it has been settled, and in particular why it was so difficult. This article considers

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Talagrand's construction and in Section 2 we present the background material that concerns what follows.

In Section 3 we investigate the forcing associated to Talagrand's construction. We show that in any such forcing extension, the collection of random reals will have ν-measure 0, where ν is Talagrand's submeasure (Theorem 3.1). We also give a proof that, in any such extension, the collection of ground model reals will be Lebesgue null and meagre (Theorem 3.4). A motivating and still open problem for this section is whether or not the nonmeasurable Maharam algebra associated to Talagrand's submeasure adds a random real (see for example [5, Problem 3A]).

In Section 4 we consider the first pathological submeasure ψ constructed by Talagrand in [19]. We give explicit values that this submeasure assumes on the entire Cantor space and also explicit values for relative atoms (Theorem 4.1). The motivation here is that the values of the Lebesgue measure on the Cantor space are indeed easily calculable, and it would be very helpful if the same could be said for Talagrand's submeasure.

§**2. Preliminaries.** We have tried to make our notation and terminology as standard as possible. Unless otherwise stated, our set theory follows [14] and in particular "$p \leq q$" is taken to mean that "p is stronger than q". Everything concerning Boolean algebras follows [12]. We let $\mathbb{N} = \{1, 2, \dots\}$ and $\omega = \{0, 1, 2, \dots\}$. If $n \in \mathbb{N}$ then we let $[n] = \{1, 2, \dots, n\}$. Given sets $(X_n)_{n \in \mathbb{N}}$, $I \subseteq \mathbb{N}$ and $s \in \prod_{n \in I} X_n$, we let

$$[s] = \{f \in \prod_{n \in \mathbb{N}} X_n : (\forall n \in I)(f(n) = s(n))\}.$$

Given a topological space K, we let $\operatorname{Clopen}(K)$ and $\operatorname{Borel}(K)$ be the collections of clopen sets and Borel sets of K, respectively.

Unless otherwise stated, everything concerning submeasures may be found in [6]. Given a submeasure μ on a Boolean algebra $\mathfrak{A}$, we say that μ is *strictly positive* if $(\forall a \in \mathfrak{A})(\mu(a) = 0 \to a = 0)$. The submeasure μ is called *uniformly exhaustive* if, for every $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that for any antichain $a_1, \dots, a_N$ from $\mathfrak{A}$ we have $\min_n \mu(a_n) < \varepsilon$. The submeasure μ is called *pathological* if there does not exist a nonzero measure λ on $\mathfrak{A}$ such that $\lambda \leq \mu$, where by $\lambda \leq \mu$ we mean $(\forall a \in \mathfrak{A})(\lambda(a) \leq \mu(a))$. The well-known Kalton–Roberts theorem reads as follows.

THEOREM 2.1 ([13]). *A submeasure is uniformly exhaustive if and only if it is equivalent to a measure.*

If $\mathfrak{A}$ is σ-complete then μ is called *continuous* if, for each sequence $a_1 \geq a_2 \geq \cdots$ from $\mathfrak{A}$ such that $\prod_n a_n = 0$, we have $\lim_n \mu(a_n) = 0$. It follows that if μ is continuous and $(a_n)_n$ is a sequence from $\mathfrak{A}$ such that

$$a := \limsup_n a_n = \liminf_n a_n$$

then $\lim_n \mu(a_n) = \mu(a)$. An atomless σ-complete Boolean algebra that carries a strictly positive continuous submeasure is called a *Maharam algebra*. Every Maharam algebra satisfies the countable chain condition (ccc) and no such algebra can add a Cohen real (see [1, Theorem 5.9]). If $\mathfrak{A}$ is σ-complete then μ is called *σ-additive* if for every antichain $(a_n)_n$ from $\mathfrak{A}$ we have

$$\mu\left(\sum_n a_n\right) = \sum_n \mu(a_n).$$

An atomless σ-complete Boolean algebra that carries a strictly positive σ-additive measure is called a *measure algebra*.

Since σ-additivity implies continuity, every measure algebra is a Maharam algebra. The more prevalent formulations of Maharam's problem are as follows.

FACT 2.2 ([6, §393]). *The following statements are equivalent.*

- *Every Maharam algebra is a measure algebra.*
- *Every exhaustive submeasure is equivalent to a measure.*
- *Every exhaustive submeasure is uniformly exhaustive.*
- *Every exhaustive submeasure is not pathological.*

Finally, although our notation follows quite closely to that of [19], for completeness we present Talagrand's example of an exhaustive pathological submeasure that is not uniformly exhaustive. For the remainder of this section everything is taken from [19]. Let

$$\mathcal{T} = \prod_{n\in\mathbb{N}}[2^n].$$

We also fix

$$\mathbb{T} = \text{Clopen}(\mathcal{T}).$$

For each $n \in \mathbb{N}$, let $\mathcal{A}_n = \{[f \upharpoonright [n]] : f \in \mathcal{T}\}$ and $\mathcal{B}_n$ be the subalgebra of $\mathbb{T}$ generated by $\mathcal{A}_n$. Members of $\mathcal{B}_n$ will be finite unions of sets of the form $[s]$, for $s \in \prod_{k\in[n]}[2^k]$.

Let

$$\mathcal{M} = \mathbb{T} \times [\mathbb{N}]^{<\omega} \times \mathbb{R}_{\geq 0}.$$

For finite $X \subseteq \mathcal{M}$, where $X = \{(X_1, I_1, w_1), \ldots, (X_n, I_n, w_n)\}$, let

$$w(\emptyset) = 0, \quad w(X) = \sum_{i=1}^{n} w_i, \quad \bigcup X = \bigcup_{i=1}^{n} X_i.$$

The value $w(X)$ is called the *weight* of X.

We have the following general construction.

DEFINITION 2.3. If $Y \subseteq \mathcal{M}$ and is such that there exists a finite $Y' \subseteq Y$ such that $\mathcal{T} = \bigcup Y'$ then Y defines a submeasure ϕ_Y given by

$$\phi_Y(B) = \inf\{w(Y') : Y' \subseteq Y \wedge Y' \text{ is finite} \wedge B \subseteq \bigcup Y'\}.$$

For $k \in \mathbb{N}$ and $\tau \in [2^n]$ let

$$S_{n,\tau} = \{f \in \mathcal{T} : f(n) \neq \tau\}.$$

For $k \in \mathbb{N}$ let

$$\eta(k) = 2^{2k+10} 2^{(k+5)^4}(2^3 + 2^{k+5}2^{(k+4)^4}), \ \alpha(k) = (k+5)^{-3}$$

and set

$$\mathcal{D}_k = \{(X, I, w) \in \mathcal{M} : |I| \in [\eta(k)] \wedge w = 2^{-k}\left(\frac{\eta(k)}{|I|}\right)^{\alpha(k)} \wedge (\exists \tau \in \prod_{n\in I}[2^n])(X = \bigcap_{n\in I} S_{n,\tau(n)})\}.$$

Let $\mathcal{D} = \bigcup_{k\in\mathbb{N}} \mathcal{D}_k$ and

$$\psi = \phi_{\mathcal{D}}.$$

An important property of ψ is the following.

Proposition 2.4. *Any nontrivial submeasure μ such that $\mu \leq \psi$ must be pathological and cannot be uniformly exhaustive.*

Thus it is enough to now construct a nontrivial exhaustive submeasure that lies below ψ.

Definition 2.5. Let $\mu : \mathbb{T} \to \mathbb{R}$ be a submeasure and let $m, n \in \mathbb{N}$.

- For each $s \in \prod_{i\in[m]}[2^i]$ we define the map

  $$\pi_{[s]} : \mathcal{T} \to [s]$$

  by

  $$(\pi_{[s]}(x))(i) = \begin{cases} s(i), & \text{if } i \in [m]; \\ x(i), & \text{otherwise.} \end{cases}$$

- For $m < n$ we say a set $X \subseteq \mathcal{T}$ is (m, n, μ)-***thin*** if and only if

  $$(\forall A \in \mathcal{A}_m)(\exists B \in \mathcal{B}_n)(B \subseteq A \wedge B \cap X = \emptyset \wedge \mu(\pi_A^{-1}[B]) \geq 1).$$

  For $I \subseteq \mathbb{N}$, we say that X is (I, μ)-***thin*** if it is (m, n, μ)-thin for each $m, n \in I$ with $m < n$.

The rest of the construction proceeds by a downward induction. For $p \in \mathbb{N}$ let $\mathcal{E}_{p,p} = \mathcal{C}_{p,p} = \mathcal{D}$ and $\psi_{p,p} = \phi_{\mathcal{C}_{p,p}}$. Now for $k < p$, given $\mathcal{E}_{k+1,p}$, $\mathcal{C}_{k+1,p}$, and $\psi_{k+1,p}$, we let

$$\mathcal{E}_{k,p} = \{(X, I, w) \in \mathcal{M} : X \text{ is } (I, \psi_{k+1,p})\text{-thin}, |I| \in [\eta(k)] \text{ and } w = 2^{-k}\left(\tfrac{\eta(k)}{|I|}\right)^{\alpha(k)}\},$$

$\mathcal{C}_{k,p} = \mathcal{C}_{k+1,p} \cup \mathcal{E}_{k,p}$ and $\psi_{k,p} = \phi_{\mathcal{C}_{k,p}}$.

Next let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. For each $k \in \mathbb{N}$ let $\mathcal{E}_k$ and $\mathcal{C}_k$ be subsets of $\mathcal{M}$ defined by

$$x \in \mathcal{E}_k \leftrightarrow \{p : x \in \mathcal{C}_{k,p}\} \in \mathcal{U},$$

and $\mathcal{C}_k = \mathcal{D} \cup \bigcup_{l\geq k} \mathcal{E}_l$.

Finally, let $\nu_k = \phi_{\mathcal{C}_k}$. It is clear from Definition 2.3 that we have

$$\nu_1 \leq \nu_2 \leq \nu_3 \cdots \leq \psi.$$

Now the submeasure ν_1, which we shall denote by ν from here on, is the desired counter example to Maharam's problem. The fact that ν is nontrivial and exhaustive requires two separate arguments. Exhaustivity follows by showing that for each k and antichain $(a_n)_n$ from $\mathbb{T}$ we have

$$\limsup_n \nu_k(a_n) \leq 2^{-k}.$$

This last property is known as 2^{-k}-*exhaustivity*.

§3. Talagrand's ideal. Recall that $\mathcal{T}$ is the product space $\prod_{i\in\mathbb{N}}[2^i]$, $\mathbb{T} = \mathrm{Clopen}(\mathcal{T})$ and $\nu : \mathbb{T} \to \mathbb{R}$ is Talagrand's submeasure. We may extend ν to a σ-subadditive submeasure on $\mathcal{P}(\mathcal{T})$ by

$$\nu(A) = \inf\left\{\sum_{i\in\mathbb{N}} \nu(A_i) : A_i \in \mathbb{T} \wedge A \subseteq \bigcup_{i\in\mathbb{N}} A_i\right\}, \tag{3.1}$$

where the restriction of ν to $\mathrm{Borel}(\mathcal{T})$ is a continuous submeasure (see for example, [17, Proposition 7.1]). Plainly this extension remains pathological. Let

$$\mathtt{path} = \{A \in \mathcal{P}(\mathcal{T}) : \nu(A) = 0\}.$$

Let $\mathtt{meagre}$ be the ideal of meagre subsets of $\mathcal{T}$ and $\mathtt{null}$ be the ideal of Lebesgue null sets. For the rest of this section fix a countable transitive model M of ZFC and $\mathcal{U} \in M$ such that, in M, $\mathcal{U}$ is a nonprincipal ultrafilter. By ν^M we mean Talagrand's submeasure as defined in M and with respect to $\mathcal{U}$. By $\mathtt{path}_M$ we mean the collection (in M) of ν^M-null sets. We will also denote the complete Boolean algebra $\mathrm{Borel}(\mathcal{T})/\mathtt{path}$, as computed in M, by $\mathtt{path}_M$. By N we mean either a countable transitive model of *ZFC* such that $M \subseteq N$ or V itself. By ν^N we mean ν as defined in N with respect to *any* nonprincipal ultrafilter $\mathcal{V}$ (in N) such that $\mathcal{U} \subseteq \mathcal{V}$. Such an ultrafilter exists since $\mathcal{U}$ will always have the finite intersection property and will not contain any finite sets, and so any nonprincipal extension will do. We do not know if different ultrafilters produce different ideals, nevertheless, the choice of the ultrafilter here will not matter. We let $\mathtt{path}_N$ denote the collection of ν^N-null sets. If $\mathcal{V}$ is any nonprincipal ultrafilter over $\mathbb{N}$ we let $\nu_{\mathcal{V}}$ be Talagrand's submeasure defined with respect to the ultrafilter $\mathcal{V}$.

By BC we mean the collection of Borel codes as described in [10, Chapter 25]. Given a subset A of BC let $R(A) = \{f \in \mathcal{T} : (\forall c \in A)(A_c \in \mathtt{null} \to f \notin A_c)\}$. If H is a countable transitive model of *ZFC* then $R(\mathrm{BC} \cap H)$ is just the collection of random reals over H. We prove the following.

THEOREM 3.1. *Let G be a $\mathtt{path}_M$-generic filter over M. Then in $M[G]$ we have*

$$(\forall \mathcal{V})((\mathcal{V} \textit{ is a nonprincipal ultrafilter on } \mathbb{N} \wedge \mathcal{U} \subseteq \mathcal{V}) \to \nu_{\mathcal{V}}(R(\mathrm{BC} \cap M)) = 0).$$

The two claims we will need are as follows, the first is due to Christensen.

FACT 3.2 ([2, Theorem 1]). *There exists a Borel set A such that $A \in \mathtt{path}$ and $\mathcal{T} \setminus A \in \mathtt{null}$.*

Fact 3.2 is saying that the ideals $\mathtt{path}$ and $\mathtt{null}$ are *dual*, according to [15].

PROPOSITION 3.3. *If $c \in \mathrm{BC} \cap M$ then $\nu^M(A_c^M) \geq \nu^N(A_c^N)$. In particular for any $c \in \mathrm{BC} \cap M$, if $A_c \cap M \in \mathtt{path}_M$ then $A_c \cap N \in \mathtt{path}_N$.*

Assuming this for now, we have the following.

PROOF OF THEOREM 3.1. By Fact 3.2, we can find $c, d \in \mathrm{BC} \cap M$ such that $A_c \cap M \in \mathtt{null}_M$ and $A_d \cap M = \mathcal{T} \setminus A_c \cap M \in \mathtt{path}_M$. Let G be a $\mathtt{path}$-generic filter over M. In $M[G]$, if $f \in R(\mathrm{BC} \cap M)$ then $f \notin A_c$ so that $R(\mathrm{BC} \cap M) \subseteq A_d$. But by Proposition 3.3 we know that, since $A_d \cap M \in \mathtt{path}_M$, for any appropriate $\mathcal{V}$ we have $\nu_{\mathcal{V}}^{M[G]}(A_d \cap M[G]) = 0$. ⊣

PROOF OF PROPOSITION 3.3. This proof follows the same induction argument as in [5, Proposition N]. Let $\mathcal{T}^*$ be the collection $\bigcup_{I\in[\mathbb{N}]^{<\omega}}\prod_{n\in I}[2^n]$. Let $\phi_1(f,\tau)$ be the formula

$$\tau\in\mathcal{T}^*\wedge f\in\mathcal{T}\wedge(\forall n\in\operatorname{dom}(\tau))(f(n)\neq\tau(n)))$$

Of course

$$f\in\bigcap_{n\in\operatorname{dom}(\tau)}S_{n,\tau(n)}\leftrightarrow\phi_1(f,\tau).$$

Since ${\mathcal{T}^*}^M=\mathcal{T}^*$ and $\mathcal{T}^M=\mathcal{T}\cap M$, we have

$$(\forall\tau)(\forall f\in M)\left(\phi_1(f,\tau)\leftrightarrow\phi_1^M(f,\tau)\right).$$

So if $\tau\in\mathcal{T}^*$

$$(\bigcap_{n\in I}S_{n,\tau(n)})^M=\{f:\phi_1(f,\tau)\}^M=\{f\in M:\phi_1^M(f,\tau)\}=\bigcap_{n\in I}S_{n,\tau(n)}\cap M. \quad (3.2)$$

Let $\phi_2(x)$ be the formula

$$x\text{ is a function}\wedge\operatorname{dom}(x)=3\wedge(\exists\tau\in\mathcal{T}^*)(\exists k)\left(x(0)=\bigcap_{n\in\operatorname{dom}(\tau)}S_{n,\tau(n)}\right.$$
$$\left.\wedge\,x(1)=|\tau|\wedge x(2)=\left(\frac{\eta(k)}{|\tau|}\right)^{\alpha(k)}\right).$$

Of course

$$\phi_2(X)\leftrightarrow X\in\mathcal{D}.$$

By this and (3.2) we see that

$$\mathcal{D}^M=\{(A\cap M,I,w):(A,I,w)\in\mathcal{D}\}. \quad (3.3)$$

Note that the sequences $(\eta(k))_{k\in\mathbb{N}}$ and $(\alpha(k))_{k\in\mathbb{N}}$ are in M.

Now we proceed by downwards induction. Let $[k,p]$ be the statement that

$$(\mathcal{C}_{k,p}^M=\{(A^M,I,w):(A,I,w)\in\mathcal{C}_{k,p}\})\wedge(\forall A\in\mathbb{T})(\psi^M_{\mathcal{C}_{k,p}}(A^M)=\psi^N_{\mathcal{C}_{k,p}}(A^N)).$$

We show that for each $k\leq p$ the statement $[k,p]$ holds. First we show that $\psi^M_{\mathcal{D}}(A^M)=\psi^N_{\mathcal{D}}(A^N)$, this along with (3.3) will prove $[p,p]$. Suppose $\psi^M_{\mathcal{D}}(A^M)<\eta$, for some $\eta\in\mathbb{Q}_{>0}$. Then we can find $\{(X_i\cap M,I_i,w_i):i\in I\}\subseteq\mathcal{D}^M$ such that $A^M=A\cap M\subseteq\bigcup_{i\in I}X_i\cap M=(\bigcup_{i\in I}X_i)^M$ and $\sum_{i\in I}w_i<\eta$. Thus $\{(X_i\cap N,I_i,w_i):i\in I\}\subseteq\mathcal{D}^N$ witnesses $\psi^N_{\mathcal{D}}(A)<\eta$. The other direction is the same but just using the fact that if $\{(X_i\cap N,I_i,w_i):i\in I\}\subseteq\mathcal{D}^N$ then $\{(X_i\cap M,I_i,w_i):i\in I\}\subseteq\mathcal{D}^M$.

Suppose now that for some $k<p$ we have $[k+1,p]$ holds. By $[k+1,p]$, for every $s\in\mathcal{T}^*$ and $B\in\mathbb{T}$, we have

$$\psi^M_{k+1,p}\left(\left(\pi^{-1}_{[s]}(B)\right)^M\right)=\psi^N_{k+1,p}\left(\left(\pi^{-1}_{[s]}(B)\right)^N\right),$$

from which it follows that

$$(\forall X\in\mathbb{T})(\ X\cap M\text{ is }(I,\psi^M_{k+1,p})\text{-thin if and only if }X\cap N\text{ is }(I,\psi^N_{k+1,p})\text{-thin}).$$

From this, arguing as in the case for $[p,p]$, we obtain $[k,p]$.

Finally, since $\mathcal{U} \subseteq \mathcal{V}$, we have for each $k \in \mathbb{N}$:

$$\textit{If } (X \cap M, I, w) \in \mathcal{E}_k^M \textit{ then } (X \cap N, I, w) \in \mathcal{E}_k^N,$$

where of course $\mathcal{E}_k^M = \{(X, I, w) : \{p : (X, I, p) \in \mathcal{C}_{k,p}^M\} \in \mathcal{U}\}$ and $\mathcal{E}_k^N = \{(X, I, w) : \{p : (X, I, p) \in \mathcal{C}_{k,p}^N\} \in \mathcal{V}\}$. This completes the proof. ⊣

An ideal $\mathcal{I}$ on Borel$(\mathcal{T})$ is called *analytic on* G_δ if, for every G_δ set $A \subseteq \mathcal{T} \times \mathcal{T}$, the set $\{x : A_x \in \mathcal{I}\}$ is analytic. By Proposition 3.5, path is analytic on G_δ. The following now follows from [7].

Theorem 3.4. path $\Vdash \check{\mathcal{T}} \in \dot{\text{meagre}} \cap \dot{\text{null}}$.

The following should be compared to [11, Theorem 17.25].

Proposition 3.5. *Let μ : Borel$(\mathcal{T}) \to \mathbb{R}$ be a Maharam submeasure. Then for each Borel set $A \subseteq \mathcal{T} \times \mathcal{T}$, the map*

$$x \mapsto \mu(A_x)$$

is Borel. In particular, by considering the preimage of $\{0\}$, the ideal Null(μ) *is analytic on G_δ.*

Proof. Fix a Maharam submeasure μ on Borel$(\mathcal{T})$. Given $A \in$ Borel$(\mathcal{T} \times \mathcal{T})$, let $[A]$ be the statement:

$$\text{The map } \mathcal{T} \to \mathcal{T} : x \mapsto \mu(A_x) \text{ is Borel.}$$

We show that the collection of all $A \in \mathcal{P}(\mathcal{T} \times \mathcal{T})$ such that $[A]$ holds is closed under countable intersections of decreasing sequences and countable unions of increasing sequences and contains all open sets. By the Monotone Class Theorem (see [9, Theorem 6B]) it follows that $[A]$ holds for each $A \in$ Borel$(\mathcal{T}\times\mathcal{T})$. Indeed, let $(A_i)_{i\in\mathbb{N}}$ be a decreasing sequence such that $[A_i]$ holds for each i and let $A = \bigcap_i A_i$. Let $f : \mathcal{T} \to \mathbb{R}$ be the map $x \mapsto \mu(A_x)$ and, for each $n \in \mathbb{N}$, let $f_n : \mathcal{T} \to \mathbb{R}$ be the map

$$x \mapsto \mu\left((A_n)_x\right).$$

By the monotonicity of μ, we have that $f_1(x) \geq f_2(x) \geq \cdots$, and since μ is Maharam we have

$$f(x) = \lim_n f_n(x).$$

Since each f_n is Borel the map f remains Borel (see [3, Theorem 4.2.2]), and so we must have $[A]$. The same argument shows that, if $(A_i)_{i\in\mathbb{N}}$ is an increasing sequence such that $[A_i]$ holds for each i, then $[\bigcup_i A_i]$ also holds. Let us now show that $[A]$ holds for each open set A of $\mathcal{T} \times \mathcal{T}$. If $A = \bigcup_{i\in[n]}[s_i] \times [t_i] \subseteq \mathcal{T} \times \mathcal{T}$, for some finite sequences s_i and t_i, then for each $x \in \mathcal{T}$ and function μ : Borel$(\mathcal{T}) \to \mathbb{R}$ we have

$$\mu(A_x) = \mu\left(\bigcup\{[t_i] : i \in [n] \wedge x \in [s_i]\}\right).$$

From this it is straightforward to see that the map $x \mapsto \mu(A_x)$ is continuous (and so Borel). Now suppose A is an open set in $\mathcal{T} \times \mathcal{T}$. Then we can find finite sequences $(s_i)_{i\in\mathbb{N}}$ and $(t_i)_{i\in\mathbb{N}}$ such that $A = \bigcup_{i\in\mathbb{N}}[s_i]\times[t_i]$. For each n, let $A_n = \bigcup_{i\in[n]}[s_i]\times[t_i]$. Then, by the above, we see that $[A]$ holds. ⊣

Finally for this section let us remark that [7] may be avoided in justifying Theorem 3.4. By [5, Proposition N], the ideal path satisfies the following condition:

$$(\forall f \in \mathcal{T})(\forall A \in \text{path})(A + f \in \text{path}).$$

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This is called 0-1-*invariance* in [15]. Now we have the following.

FACT 3.6 ([15]). *If $\mathcal{I}$ is a* 0-1-*invariant ideal on $\mathcal{P}$ that is dual to* $\mathtt{null}$ *then*

$$\mathrm{Borel}(\mathcal{T})/\mathcal{I} \Vdash \check{\mathcal{T}} \in \dot{\mathtt{null}}.$$

The above is still true if we replace $\mathtt{null}$ *by* $\mathtt{meagre}$.[1]

Theorem 3.4 now follows by Fact 3.2 and the following.

LEMMA 3.7. *For every $A \in \mathrm{Borel}(\mathcal{T}) \setminus \mathtt{path}$ there exists $B \in (\mathrm{Borel}(\mathcal{T}) \cap \mathtt{meagre}) \setminus \mathtt{path}$ such that $B \subseteq A$. In particular,* $\mathtt{meagre}$ *and* $\mathtt{path}$ *are dual.*

PROOF. Suppose that for some $A \in \mathrm{Borel}(\mathcal{T}) \setminus \mathtt{path}$ we have $\mathrm{Borel}(\mathcal{T}) \cap \mathtt{meagre} \cap \mathcal{P}(A) \subseteq \mathtt{path}$. Let $\dot{r}$ be a name such that

$$\mathtt{path} \Vdash (\forall c \in \check{\mathrm{BC}})(A_c \cap \check{\mathcal{T}} \in \dot{\mathtt{path}} \rightarrow \dot{r} \notin A_c).^2 \tag{3.4}$$

We claim that

$$A \Vdash \text{“}\dot{r} \text{ is a Cohen real”}. \tag{3.5}$$

If not then for some $B \subseteq A$ and some $c \in \mathrm{BC}$ with $A_c \in \mathtt{meagre}$ we have $B \Vdash \dot{r} \in A_c$. If $d \in \mathrm{BC}$ is such that $B = A_d$ then $B \Vdash \dot{r} \in A_d \cap A_c$. Let $e \in \mathrm{BC}$ be such that $A_e = A_c \cap A_d$. But then

$$A_c \cap A_d \in \mathrm{Borel}(\mathcal{T}) \cap \mathtt{meagre} \cap \mathcal{P}(A) \subseteq \mathtt{path}.$$

In particular $B \Vdash A_e \cap \check{T} \in \dot{\mathtt{path}} \wedge \dot{r} \in A_e$, which contradicts (3.4). Thus (3.5) holds which contradicts the fact that a Maharam algebra cannot add a Cohen real.

Now use the above to find, for each $A \in \mathrm{Borel}(\mathcal{T}) \setminus \mathtt{path}$, a meagre Borel set $\Gamma(A) \notin \mathtt{path}$ such that $\Gamma(A) \subseteq A$. Let $B_1 = \Gamma(\mathcal{T})$. If B_β for $\beta < \alpha < \omega_1$ has been constructed let

$$B_\alpha = \begin{cases} \Gamma(\mathcal{T} \setminus (\bigcup_{\beta<\alpha} B_\beta)), & \text{if } \mathcal{T} \setminus (\bigcup_{\beta<\alpha} B_\beta) \notin \mathtt{path}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since $\mathrm{Borel}(\mathcal{T})/\mathtt{path}$ is ccc, we know that $B := \{B_\alpha : \alpha < \omega_1 \wedge B_\alpha \notin \mathtt{path}\}$ is countable. Thus $\mathcal{T} \setminus \bigcup B \in \mathtt{path}$ and $\bigcup B \in \mathtt{meagre}$, since each $B_\alpha \in \mathtt{meagre}$. ⊣

§4. Talagrand's ψ.

Let λ be the Lebesgue measure on $\mathrm{Clopen}(2^\omega)$. If $A \in \mathrm{Clopen}(2^\omega)$ then we know that for some $n \in \omega$ we have

$$\lambda(A) = |\{s \in {}^n2 : [s] \subseteq A\}| \cdot 2^{-n}.$$

A similarly explicit description for Talagrand's submeasure would be very useful. To this end, we investigate the first (pathological) submeasure constructed in [19], this is the submeasure denoted by ψ in Section 2 (and in [19]). We remark that in [19] the value $\eta(k)$ was set to $2^{2k+10}2^{(k+5)^4}(2^3 + 2^{k+5}2^{(k+4)^4})$. As pointed out in [19] anything larger will do, so for simplicity we take the value

$$\eta(k) = 2^{2500k^4}.$$

[1] Note that the assumptions in this claim do not include the *absoluteness* of $\mathcal{I}$, as in [15], but the proof still works.

[2] See [20, Proposition 2.1.2].

In Subsection 4.1 we prove the following:

THEOREM 4.1. *We have* $\psi(\mathcal{T}) = \eta(1)^{\alpha(1)} = 2^{\frac{2500}{216}}$ *and*

$$\psi([s]) = \min\left\{2^{-\delta(|I|)+1}, 2^{-\delta(|I|)}\left(\frac{\eta(\delta(|I|))}{|I|}\right)^{\alpha(\delta(|I|))}\right\},$$

where $s \in \prod_{i\in I}[2^i]$, *for some finite* $I \subseteq \mathbb{N}$, *and* $\delta(m) = \min\{n \in \mathbb{N} : \eta(n) \geq m\}$.

In Subsection 4.2 we list the numerical inequalities that we shall need for Subsection 4.1.

4.1. Main calculations. We begin with the following two definitions.

DEFINITION 4.2. For $X \in \mathbb{T}$ we say that X is a $\mathcal{D}$***-set*** if and only if for some (nonempty) finite set $I \subseteq \mathbb{N}$ and some $\tau \in \prod_{n\in I}[2^n]$ we have

$$X = \bigcap_{n\in I}\{y \in \mathcal{T} : (\forall n \in I)\left(y(n) \neq \tau(n)\}\right) = \bigcap_{n\in I} S_{n,\tau(n)}.$$

Since we can recover I and τ from X we allow ourselves to denote I by X^{Ind} and $\tau(n)$ by $X(n)$.

DEFINITION 4.3. Let $A \subseteq \mathcal{T}$, X a collection of $\mathcal{D}$-sets and $Y \in [\mathcal{D}]^{<\omega}$. We say that X (resp. Y) is a *cover of* A if and only if $A \subseteq \bigcup X$ (resp. $A \subseteq \bigcup Y$). We say that X (resp. Y) is a *proper cover of* A if and only if it is a cover of A and for any $X' \subsetneq X$ (resp. $Y' \subsetneq Y$)

$$A \not\subseteq \bigcup X' (\text{resp. } A \not\subseteq \bigcup Y').$$

Clearly then given $A \subseteq \mathcal{T}$ we have

$$\psi(A) = \inf\{w(X) : X \subseteq \mathcal{D} \text{ and } X \text{ properly covers } A\}. \tag{4.1}$$

Let us now measure $\mathcal{T}$. The idea here is as follows. For each proper cover X of $\mathcal{T}$ we find another cover Y of $\mathcal{T}$ of lower weight, where the Y here will have a very regular structure and so will have an easily calculable weight. Of course it will be sufficient to consider the infimum over all such regular structures.

DEFINITION 4.4. For any $n \in \mathbb{N}$ let

$$w(n) = 2^{-\delta(n)}\left(\frac{\eta(\delta(n))}{n}\right)^{\alpha(\delta(n))}.$$

If X is a finite collection of $\mathcal{D}$-sets then we will denote the ***weight*** of X by

$$w(X) = \sum_{Y\in X} w\left(|Y^{\mathrm{Ind}}|\right).^{3}$$

By Inequality 1 from Subsection 4.2, we see that if X is a $\mathcal{D}$-set then $w(|I(X)|)$ will be the least weight that we can possibly attach to it. Specifically, we will always have $(X, I(X), w(|I(X)|)) \in \mathcal{D}$ and, if $(X, I(X), w) \in \mathcal{D}$ then $w \geq w(|I(X)|)$.

Here is the regular structure we mentioned above.

[3]We now are using the term *weight* for $\mathcal{D}$-sets and members of $\mathcal{D}$, but with a slight variation in meaning.

DEFINITION 4.5. Let $X = \{X_i : i \in I\}$ be a collection of $\mathcal{D}$-sets. We call X an N-***rectangle*** for some integer $N \geq 2$ if and only if the following hold:

- $|I| = N$;
- $X_i^{\text{Ind}} = X_j^{\text{Ind}}$ for all $i, j \in I$;
- $X_i(m) \neq X_j(m)$, whenever $i \neq j$ and $m \in X_i^{\text{Ind}}$;
- $|X_i^{\text{Ind}}| = N - 1$ for all (any) $i \in I$.

Notice that the weight of an N-rectangle is given by

$$N \cdot w(N-1). \tag{4.2}$$

Rectangles give rise to proper covers of $\mathcal{T}$:

LEMMA 4.6. *If $X := \{X_i : i \in I\}$ is an N-rectangle then X is a proper cover of $\mathcal{T}$.*

PROOF. Assume that $x \in \mathcal{T} \setminus \bigcup_i X_i$. Then for each i we can find an $m_i \in X_i^{\text{Ind}}$ such that $x(m_i) = X_i(m_i)$. These m_i must be distinct for if $i \neq j$ and $m := m_i = m_j$, then $X_i(m) = x(m) = X_j(m)$, for some i, contradicting the third item from Definition 4.5. But then $\{m_1, \ldots, m_N\} \subseteq X_i^{\text{Ind}}$ a (cardinality) contradiction. To see that this cover is proper let J be a nonempty strict subset of $\{1, 2, \ldots, N\}$. Then $|J| \leq N - 1 = |X_i^{\text{Ind}}|$, for each $i \in J$. Enumerate

$$J = \{a_1, a_2, \ldots, a_k\}.$$

Inductively, choose $b_1 \in X_{a_1}^{\text{Ind}}$, $b_2 \in X_{a_2}^{\text{Ind}} \setminus \{b_2\}$, $b_3 \in X_{a_3}^{\text{Ind}} \setminus \{b_1, b_2\}, \ldots, b_k \in X_{a_k}^{\text{Ind}} \setminus \{b_1, \ldots, b_{k-1}\}$. Now define $y \in \prod_{i \in J}[2^{b_i}]$ by

$$y_i = \begin{cases} X_{a_i}(b_i), & \text{if } i \in \{b_1, \ldots, b_k\}; \\ 1, & \text{if } i \notin J. \end{cases}$$

and note that $y \in \mathcal{T} \setminus \bigcup_{i \in J} X_i$. ⊣

Given a proper cover of $\mathcal{T}$ we claim that we can find an N-rectangle of lower weight. Before we can demonstrate this we need one more claim.

LEMMA 4.7. *Let $X = \{X_i : i \in I\}$ be a collection of $\mathcal{D}$-sets that properly covers $\mathcal{T}$. Then*

$$|\bigcup_{i \in I} X_i^{\text{Ind}}| \leq |I| - 1.$$

PROOF. For each $i \in I$ let $I_i = X_i^{\text{Ind}}$. Recall that a *complete system of distinct representatives* for $\{I_i : i \in I\}$ (a CDR) is an injective function $F : I \to \bigcup_{i \in I} I_i$ such that $(\forall i \in I)(F(i) \in I_i)$, and that by Hall's marriage theorem a CDR exists if and only if

$$(\forall J \subseteq I)(|J| \leq |\bigcup_{i \in J} I_i|).^4$$

Clearly if a CDR existed for $\{I_i : i \in I\}$ then $\bigcup_{i \in I} X_i$ would not cover $\mathcal{T}$ (just argue as in the proof of Lemma 4.6). So for some $J \subseteq I$ we have $|\bigcup_{i \in J} I_i| \leq |J| - 1$. Assume that $|J|$ is as large as possible so that

$$(J' \subseteq I \wedge |J'| > |J|) \to (|J'| \leq |\bigcup_{i \in J'} I_i|). \tag{4.3}$$

[4]See [8].

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If $J = I$ then we are done. So we may assume that $J \subsetneq I$. Since X is a proper cover of $\mathcal{T}$ there exists $t \in \mathcal{T}$ such that $t \notin \bigcup_{i\in J} X_i$. For $i \in I \setminus J$ let $I_i' = I_i \setminus \bigcup_{j\in J} I_j$. Suppose that $\{I_i' : i \in I \setminus J\}$ has a CDR $F : I \setminus J \to \bigcup_{i\in I\setminus J} I_i'$. Let $s \in \prod_{k\in \operatorname{ran}(F)}[2^k]$ be defined by $s(k) = X_{F^{-1}(k)}(k)$. Then the function $(t \setminus \{(k, t(k)) : k \in \operatorname{ran}(F)\}) \cup s \notin \bigcup_{i\in I} X_i$, which is a contradiction. Thus no such CDR can exist and so by Hall's theorem again, we may find a $J' \subseteq I \setminus J$ such that $|\bigcup_{i\in J'} I_i'| \leq |J'| - 1$. But then

$$|\bigcup_{i\in J\cup J'} I_i| = |\bigcup_{i\in J} I_i \cup \bigcup_{i\in J'} I_i'| \leq |J| - 1 + |J'| - 1 \leq |J| + |J'| - 1 = |J \cup J'| - 1.$$

But $|J \cup J'| > |J|$, contradicting (4.3). ⊣

Proposition 4.8. *For every proper cover of $\mathcal{T}$ there exists an N-rectangle of lower weight.*

Proof. Let $X = \{(X, I_i, w_i) : i \in [M]\}$ is a proper cover of $\mathcal{T}$ and assume that I_1 is such that $(\forall i \in [M])(w(|I_1|) \leq w(|I_i|))$. By Lemma 4.7 we have

$$(\forall i)(|I_i| + 1 \leq |\bigcup_{i\in[N]} I_i| + 1 \leq M). \tag{4.4}$$

So if Y is an $|I_1| + 1$-rectangle we get:

$$w(X) \geq \sum_{i\in[M]} w(|I_i|) \geq Mw(|I_1|) \geq (|I_1| + 1)w(|I_1|) \overset{(4.2)}{=} w(Y). \qquad ⊣$$

Thus we have

$$\psi(\mathcal{T}) = \inf\{w(X) : X \text{ is an } N\text{-rectangle, for some } N\}. \tag{4.5}$$

But by Inequality 3 we see that $\psi(\mathcal{T})$ is just the weight of a 2-rectangle, that is to say,

$$\psi(\mathcal{T}) = \eta(1)^{\alpha(1)}. \tag{4.6}$$

This gives the first half of Theorem 4.1.

Now let us try to measure sets of the form $[s]$. Fix a nonempty finite subset $\mathcal{I}$ of $\mathbb{N}$ and an $\tau \in \prod_{i\in\mathcal{I}}[2^i]$ and lets measure $A := [\tau]$.

Note that as before

$$\psi(A) = \inf\{w(X) : X \subseteq \mathcal{D} \text{ is a proper cover of } A\}.$$

The idea here is the same as before but instead of rectangles we use the following analogue of Definition 4.5 and also Definition 4.12, below.

Definition 4.9. Let $X := \{X_i : i \in I\}$ be a collection of $\mathcal{D}$-sets. We call X a (J, S, N)*-rectangle* for some nonempty finite subset J of $\mathbb{N}$, $S \subseteq \prod_{j\in J}[2^j]$ and integer $N \geq 2$ if and only if the following hold:

- $J \subsetneq X_i^{\text{Ind}}$, always;
- $\{\bigcap_{l\in X_i^{\text{Ind}}\setminus J} S_{l,X_i(l)} : i \in I\}$ is an N-rectangle;
- $(\forall s \in S)(\forall i \in I)(\forall j \in J)(X_i(j) \neq s(j))$.

In the case that $S = \{s\}$, we shall call X a (J, s, N)*-rectangle*.

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For example, in the case that $J = [m]$, for some $m \in \mathbb{N}$, this new type of rectangle is just an old rectangle with m rows attached to the bottom (most likely with a gap) where the values of the determining sequences along these rows miss the corresponding values of s.

Of course the weight of a (J, s, N)-rectangle is given by

$$N \cdot w(|J| + N - 1)$$

LEMMA 4.10. *Every $(\mathcal{I}, \tau, N)$-rectangle covers A.*

PROOF. Let $X = \{X_i : i \in I\}$ be an $(\mathcal{I}, \tau, N)$-rectangle, as in the above statement. Assume that we can find a $y \in A \setminus \bigcup X$. The assumption that $y \notin \bigcup X$ cannot be witnessed by $y(i)$ for some $i \in \mathcal{I}$ since for each such i, we have $y(i) = \tau(i) \neq X_j(i)$, for each $j \in I$. In particular y witnesses that $Y = \{\bigcap_{l \in X_i^{\text{Ind}} \setminus \mathcal{I}} S_{l, X_i(l)} : i \in I\}$ does not cover $\mathcal{T}$, which contradicts Lemma 4.6 and the fact that Y is an N-rectangle. ⊣

Next we see how to use Lemma 4.7 in this new situation and adapt what we have already done with $\psi(\mathcal{T})$ to $\psi(A)$ (compare (4.4) above, and (4.10) below).

LEMMA 4.11. *If $X = \{X_i : i \in I\}$ is a proper cover of A such that $(\forall i \in \mathcal{I})$ $(X_i^{\text{Ind}} \setminus \mathcal{I} \neq 0)$ then $\{\bigcap_{l \in X_i^{\text{Ind}} \setminus \mathcal{I}} S_{l, X_i(l)} : i \in I\}$ is a proper cover of $\mathcal{T}$*

PROOF. For each $i \in I$, let $I_i = X_i^{\text{Ind}}$. Let $X_i' = \bigcap_{l \in I_i \setminus \mathcal{I}} S_{l, X_i(l)}$ and lets show that $Y := \{X_i' : i \in I\}$ is a cover of $\mathcal{T}$. Suppose not and let $x \in \mathcal{T} \setminus \bigcup Y$. Thus for every $i \in I$ there exists an $m_i \in X_i^{\text{Ind}} \setminus \mathcal{I}$ such that $x(m_i) = X_i(m_i)$. Let $y \in A$ be such that $y(j) = x(j)$, for each $j \in \{m_i : i \in I\}$. Then it is straightforward to see that $y \notin \bigcup X$, which contradicts the assumption that X is a cover of A. Suppose now that Y is not proper. Then there exists $I' \subsetneq I$ such that $\{X_i' : i \in I'\}$ is a cover of $\mathcal{T}$. But then $\{X_i : i \in I'\}$ is a cover of A, contradicting the properness of X. ⊣

DEFINITION 4.12. A $\mathcal{D}$-set X is a (I, S, J)-*spike* for some nonempty finite subset I of $\mathbb{N}$, $S \subseteq \prod_{j \in I} [2^j]$ and $J \subseteq I$ if and only if X is of the form

$$X = \bigcap_{j \in J} S_{j, t(j)} \tag{4.7}$$

such that $t \in \prod_{j \in J} [2^j]$ and $(\forall s \in S)(\forall j \in J)(t(j) \neq s(j))$. In the case that $S = \{s\}$, we shall call X an (I, s, J)-*spike*.

Of course, every $(\mathcal{I}, \tau, J)$-spike covers A.

PROPOSITION 4.13. *For every proper cover of A there exists an $(\mathcal{I}, \tau, J)$-spike of lower weight.*

Assuming this for now we obtain

$$\begin{aligned} \psi(A) &= \min\{w(X) : X \text{ is an } (\mathcal{I}, \tau, J)\text{-spike for some } J \subseteq \mathcal{I}\} \\ &= \min\{2^{-\delta(|\mathcal{I}|)+1}, w(|\mathcal{I}|)\}. \end{aligned} \tag{4.8}$$

which completes the proof of Theorem 4.1.

PROOF OF LEMMA 4.13. Let $X = \{(X_i, I_i, w_i) : i \in [N]\}$ be a proper cover of A and let $m = |\mathcal{I}|$. If there exists $i \in [N]$ such that $|I_i| \leq m$ then any $(\mathcal{I}, \tau, J)$-spike such that $|J| = |I_i|$ will have a lower weight than X and will cover A and we will be done. So we may assume that

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$$(\forall i \in [N])(|I_i| > m). \tag{4.9}$$

By Lemma 4.11 and Lemma 4.7 we get

$$(\forall i \in [N])(|I_i| \leq N + m - 1). \tag{4.10}$$

We now divide the proof into the following cases.

- $\delta(N + m - 1) = 1$. Then

$$w(X) = \sum_{i \in [N]} 2^{-1} \left(\frac{\eta(1)}{|I_i|}\right)^{\alpha(1)} \overset{(4.10)}{\geq} N2^{-1} \left(\frac{\eta(1)}{N + m - 1}\right)^{\alpha(1)},$$

  and this lower bound can be achieved by any $(\mathcal{I}, \tau, N)$-rectangle.
- $\delta(N + m - 1) > 1$. Let $\delta_1 = \delta(N + m - 1) - 1$, $\delta_2 = \delta(N + m - 1)$, $J_1 = \{i \in [N] : \delta(|I_i|) \leq \delta_1\}$ and $J_2 = [N] \setminus J_1$. Of course

$$\eta(\delta_1) < N + m - 1 \leq \eta(\delta_2). \tag{4.11}$$

  Notice that if $2 > \eta(\delta_1) - m + 1$ then

$$(\forall i \in [N])(\eta(\delta_1) \overset{(4.11)}{\leq} m \overset{(4.9)}{<} |I_i| \leq N + m - 1 \leq \eta(\delta_2)),$$

  and so

$$w(X) = \sum_{i \in [N]} 2^{-\delta_2} \left(\frac{\eta(\delta_2)}{|I_i|}\right)^{\alpha(\delta_2)} \overset{(4.10)}{\geq} N2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N + m - 1}\right)^{\alpha(\delta_2)},$$

  which can be achieved by any $(\mathcal{I}, \tau, N)$-rectangle. So we may assume that

$$2 \leq \eta(\delta_1) - m + 1.$$

  By Inequality 2 we have

$$w(X) = \sum_{i \in J_1} w_{|I_i|} + \sum_{i \in J_2} w_{|I_i|} \geq |J_1| 2^{-\delta_1} + |J_2| 2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N + m - 1}\right)^{\alpha(\delta_2)}.$$

  - $2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N+m-1}\right)^{\alpha(\delta_2)} \leq 2^{-\delta_1}$. Then

$$w(X) \geq N2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N + m - 1}\right)^{\alpha(\delta_2)},$$

    which can be achieved by any $(\mathcal{I}, \tau, N)$-rectangle..
  - $2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N+m-1}\right)^{\alpha(\delta_2)} > 2^{-\delta_1}$. Then

$$w(X) \geq N2^{-\delta_1} \overset{(4.11)}{>} (\eta(\delta_1) - m + 1)2^{-\delta_1}.$$

    But this can be achieved by any $(\mathcal{I}, \tau, \eta(\delta_1) - m + 1)$-rectangle since

$$w(\eta(\delta_1) - m + 1) = 2^{-\delta(\eta(\delta_1))} \left(\frac{\eta(\delta(\eta(\delta_1)))}{\eta(\delta_1)}\right)^{-\alpha(\delta(\eta(\delta_1)))}$$
$$= 2^{-\delta_1} \left(\frac{\eta(\delta_1)}{\eta(\delta_1)}\right)^{-\alpha(\delta_1)} = 2^{-\delta_1}.$$

Now, by Inequality 4, any $(\mathcal{I}, \tau, \mathcal{I})$-spike has a lower weight than any $(\mathcal{I}, \tau, k)$-rectangle, and this completes the proof. ⊣

4.2. Inequalities. Here we list the inequalities that are needed for Subsection 4.1.

INEQUALITY 1. *For each* $k \in \mathbb{N}$ *and* $n \in [\eta(k)]$

$$2^{-k} \left(\frac{\eta(k)}{n} \right)^{\alpha(k)} < 2^{-(k+1)} \left(\frac{\eta(k+1)}{n} \right)^{\alpha(k+1)}.$$

INEQUALITY 2. *For* $\delta_1, \delta_2, k \in \mathbb{N}$ *such that* $\delta_1 \leq \delta_2$ *and* $k \in [\eta(\delta_1)]$ *we have*

$$2^{-\delta_1} \left(\frac{\eta(\delta_1)}{k} \right)^{\alpha(\delta_1)} \geq 2^{-\delta_2}.$$

INEQUALITY 3. *Let* $N, M, \delta_1, \delta_2 \in \mathbb{N}$ *be such that* $2 \leq N \leq M$ *and* $\delta_1 \leq \delta_2$. *Then*

$$N 2^{-\delta_1} \left(\frac{\eta(\delta_1)}{N-1} \right)^{\alpha(\delta_1)} \leq M 2^{-\delta_2} \left(\frac{\eta(\delta_2)}{M-1} \right)^{\alpha(\delta_2)}.$$

INEQUALITY 4. *Let* $k, N, \delta_1, \delta_2 \in \mathbb{N}$ *be such that* $\delta_1 \leq \delta_2$. *Then*

$$2^{-\delta_1} \left(\frac{\eta(\delta_1)}{N} \right)^{\alpha(\delta_1)} \leq k 2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N+k-1} \right)^{\alpha(\delta_2)}.$$

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