THE EFFECT OF A FIXED VERTICAL BARRIER ON SURFACE WAVES IN DEEP WATER

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Introduction. In this paper the two-dimensional reflection of surface waves from a vertical barrier in deep water is studied theoretically.

It can be shown that when the normal velocity is prescribed at each point of an infinite vertical plane extending from the surface, the motion on each side of the plane is completely determined, apart from a motion consisting of simple standing waves. In the cases considered here the normal velocity is prescribed on a part of the vertical plane and is taken to be unknown elsewhere. From the condition of continuity of the motion above and below the barrier an integral equation for the normal velocity can be derived, which is of a simple type, in the case of deep water. We begin by considering in detail the reflection from a fixed vertical barrier extending from depth a to some point above the mean surface.

We consider a two-dimensional motion, in which a vertical plane occupies the line $x = 0$, $0 \leqslant y \leqslant a$, where the axis of y is taken vertically downwards and $y = 0$ is the mean surface. A regular train of waves advancing from negative infinity will be partially reflected and partially transmitted, a steady state being finally set up, whose frequency $n/2\pi$ is equal to the frequency of the wave train.

It is assumed that the fluid is incompressible and inviscid, and that the motion is irrotational and simple harmonic, so that a velocity potential $\phi(x, y, t)$ exists, where

$$\frac{\partial^2\phi}{\partial t^2} + n^2\phi = 0. \tag{1}$$

Further

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{2}$$

These two conditions hold throughout the fluid. To find the analytic condition at the free surface, we assume that the inclination of the waves to the surface is small, which will be the case if the waves at a great distance are small. It is easy to see that near the barrier the surface elevation remains small, by considering the limiting case when the barrier is infinite. Let $\eta(x, t)$ be the surface elevation at the point $(x, 0)$. Since the gradient is small, the downward velocity

$$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}. \tag{3}$$

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From Bernoulli's equation $$\frac{p}{\rho} = g\eta - \frac{\partial\phi}{\partial t} \tag{4}$$

is constant on the surface, whence $$\frac{\partial^2\phi}{\partial t^2} = g\frac{\partial\phi}{\partial y} \tag{5}$$

$$= g\frac{\partial\eta}{\partial t}, \tag{6}$$

on the surface, whence, from (1) $$K\phi + \frac{\partial\phi}{\partial y} = 0, \tag{7}$$

where $$n^2 = gK. \tag{8}$$

On the barrier, the velocity is zero. Thus

$$\frac{\partial\phi}{\partial x} = 0 \quad (x = 0,\ 0 \leqslant y \leqslant a).$$

We have the further conditions

$$\frac{n}{g}\phi \to A_1 e^{-Ky}\cos(Kx - nt) + B_1 e^{-Ky}\sin(Kx - nt), \quad \text{as} \quad x \to +\infty,$$

$$\frac{n}{g}\phi \to A_2 e^{-Ky}\cos(Kx - nt) + B_2 e^{-Ky}\sin(Kx - nt)$$
$$+ A_3 e^{-Ky}\cos(Kx + nt) + B_3 e^{-Ky}\sin(Kx + nt), \quad \text{as} \quad x \to -\infty.$$

We begin by stating two lemmas:

LEMMA 1. $$e^{-kx}(k\cos ky - K\sin ky)\begin{matrix}\cos\\ \sin\end{matrix}\, nt \quad (x \geqslant 0)$$
satisfies equations (1), (2), (7) *for all* k.

We attempt to express the disturbance on the positive side of the barrier by a superposition of such expressions; clearly this tends to zero when x tends to $+\infty$. For this purpose we use the following expansion theorem given by Havelock (1).

LEMMA 2. *The solution of* (2) *satisfying conditions* (1) *and* (7), *which is defined for positive* x, *which satisfies*

$$\lim_{x\to 0} \frac{n}{g}\frac{\partial\phi}{\partial x} = f(y)\sin nt \quad (0 < y < \infty),$$

and which is of the form

$$\frac{n}{g}\phi = Ae^{-Ky}\cos(Kx - nt) + Be^{-Ky}\sin(Kx - nt)$$

for large x, *is given by*

$$\frac{n}{g}\phi = Ae^{-Ky}\cos(Kx - nt) + Be^{-Ky}\sin(Kx - nt)$$
$$+ \int_0^\infty [C(k)\sin nt + S(k)\cos nt]\, e^{-kx}(k\cos ky - K\sin ky)\, dk,$$

where $$A = 2\int_0^\infty f(y)\, e^{-Ky}\, dy, \quad B = 0,$$

$$C(k) = -\frac{2}{\pi}(K^2 + k^2)^{-1} k^{-1}\int_0^\infty f(y)(k\cos ky - K\sin ky)\, dy, \quad S(k) = 0.$$

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The lemma is equivalent to the following:

LEMMA 2A. *Suppose that $f(y)$ belongs to $L(0,\infty)$. Then*

$$\lim_{x\to 0}\int_0^\infty G(k)\,(k\cos ky - K\sin ky)\,e^{-kx}\,dk \equiv f(y) - 2K\int_0^\infty f(u)\,e^{-K(y+u)}\,du,$$

where
$$G(k) = \frac{2}{\pi}(K^2+k^2)^{-1}\int_0^\infty f(y)\,(k\cos ky - K\sin ky)\,dy.$$

To the solution of Lemma 2 may be added waves whose velocity across the imaginary axis is zero. The only waves of this type which have physical significance are standing waves of the type
$$\frac{n\phi}{g} = ae^{-Ky}\cos Kx\,\cos(nt+b).$$
These waves are excluded by the condition that the motion at infinity consists of waves of progressive type.

We assume that horizontal motions under the barrier are in phase
$$\frac{n}{g}\frac{\partial\phi}{\partial x} = f(y)\sin nt \quad (x = 0,\ a<y<\infty),$$
where $f(y)$ is such that the flow near the bottom edge of the barrier is of the same type as the steady flow past a similar barrier, i.e.
$$\sqrt{(y^2-a^2)}\,f(y) \quad \text{is bounded near } y = a,$$
and the total flow under the barrier in a half-period is bounded. i.e.
$$\int_0^\infty f(y)\,dy \quad \text{exists.}$$
We assume that the integral is absolutely convergent.

The assumption about phase will be justified if a solution satisfying all the boundary conditions can be found.

The solution proceeds on the following lines. Given the form of the motion for large (positive and negative) values of x and the velocity on $x = 0$, we can, from Lemma 2, find a corresponding potential ϕ_+ defined in the region $x \geqslant 0$ and a similar potential ϕ_- defined in the region $x \leqslant 0$. If we consider the potential function equal to ϕ_+ for positive x, and equal to ϕ_- for negative x, then by construction $\partial\phi/\partial x$ is continuous across $x = 0$, but in general there is a discontinuity in the vertical velocity. The condition for continuity in this component
$$\phi_+ = \phi_- \quad (x = 0,\ a<y<\infty)$$
provides an integral equation for $f(y)$. We put
$$\frac{n}{g}\phi_+ = A_1e^{-Ky}\cos(Kx-nt) + B_1e^{-Ky}\sin(Kx-nt)$$
$$+\int_0^\infty [C(k)\sin nt + S(k)\cos nt]\,e^{-kx}(k\cos ky - K\sin ky)\,dk \quad (x>0),$$
$$\frac{n}{g}\phi_- = A_2e^{-Ky}\cos(Kx-nt) + B_2e^{-Ky}\sin(Kx-nt)$$
$$+A_3e^{-Ky}\cos(Kx+nt) + B_3e^{-Ky}\sin(Kx+nt)$$
$$+\int_0^\infty [\overline{C(k)}\sin nt + \overline{S(k)}\cos nt]\,e^{kx}(k\cos ky - K\sin ky)\,dk \quad (x<0),$$

and $$\frac{n}{g}\frac{\partial\phi}{\partial x} = f(y)\sin nt \quad (0<y<\infty), \qquad f(y) = 0 \quad (0<y<a),$$

whence, from Lemma 2,

$$\left.\begin{aligned} A_2 - A_3 = A_1 &= 2\int_0^\infty f(y)\,e^{-Ky}\,dy, \\ B_2 + B_3 = B_1 &= 0, \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} C(k) &= -\overline{C(k)} = -\frac{2}{\pi}(K^2+k^2)^{-1}\int_0^\infty f(y)\,(k\cos ky - K\sin ky)\,dy, \\ S(k) &= -\overline{S(k)} = 0. \end{aligned}\right\} \tag{10}$$

Substituting in the equation

$$\phi_+ = \phi_- \quad (x=0,\ a<y<\infty),$$

we have $$A_3 = 0,$$

$$\begin{aligned} B_3 e^{-Ky} &= \int_0^\infty C(k)\,(k\cos ky - K\sin ky)\,dk \\ &= -\frac{2}{\pi}\int_0^\infty f(u)\,du\int_0^\infty \frac{(k\cos ky - K\sin ky)\,(k\cos ku - K\sin ku)}{k(K^2+k^2)}\,dk \\ &= -\frac{1}{\pi}\int_0^\infty f(u)\,du\left[\log\left|\frac{y+u}{y-u}\right| - 2e^{-K(y+u)}\int_{-\infty}^{K(y+u)}\frac{e^v}{v}\,dv\right] \quad (a<y<\infty). \end{aligned} \tag{11}$$

Formally differentiating the last equation with respect to y, we obtain

$$-KB_3e^{-Ky} = -\frac{1}{\pi}\int_0^\infty f(u)\,du\left[-\frac{1}{y+u} - \frac{1}{y-u} + 2Ke^{-K(y+u)}\int_{-\infty}^{K(y+u)}\frac{e^v}{v}\,dv\right],$$

whence, by addition,

$$\int_0^\infty f(u)\,du\left[K\log\left|\frac{y+u}{y-u}\right| - \frac{1}{y+u} - \frac{1}{y-u}\right] = 0 \quad (a<y<\infty).$$

Now $$f(y) = 0 \quad (0<y<a);$$

by hypothesis $$F(y) = \int_a^y f(u)\,du$$

exists and $F(\infty)$ is finite. By integration by parts

$$\int_a^\infty f(u)\,du\log\left|\frac{y+u}{y-u}\right| = -\int_a^\infty F(u)\,du\left(\frac{1}{y+u} + \frac{1}{y-u}\right),$$

and the integral equation becomes

$$\int_a^\infty [KF(u) + f(u)]\frac{du}{y^2-u^2} = 0 \quad (a<y<\infty). \tag{12}$$

(The integrals in the last equations are to be understood as Cauchy principal values.)

We now solve integral equations of this type in another lemma. An allied type occurs in the theory of aerofoils.

LEMMA 3. *The equation*

$$P\int_a^\infty \frac{\mu(u)\,du}{y^2-u^2} = \lambda(y) \quad (a<y<\infty)$$

has for suitable $\lambda(y)$ *a solution* $\mu(u)$ *such that* $\sqrt{(u^2-a^2)}\,\mu(u)$ *is bounded near* $u=a$, *this solution being*

$$\mu(u) = \frac{Cu}{\sqrt{(u^2-a^2)}} + \frac{4}{\pi^2}\frac{u^3}{\sqrt{(u^2-a^2)}}P\int_a^\infty \frac{\lambda(y)\sqrt{(y^2-a^2)}\,dy}{y(y^2-u^2)},$$

where C *is an arbitrary constant.*

Put $\quad u = a\sec\theta, \quad y = a\sec\alpha \quad (0\leqslant\theta<\tfrac{1}{2}\pi,\ 0\leqslant\alpha\leqslant\tfrac{1}{2}\pi).$

By hypothesis, $\mu(a\sec\theta)\sin\theta$ is bounded near $\theta=0$, and we assume that

$$\mu(a\sec\theta)\sin\theta = \tfrac{1}{2}a_0 + \sum_1^\infty a_{2r}\cos 2r\theta \quad (0<\theta<\tfrac{1}{2}\pi),$$

where the infinite series is uniformly convergent.

$$\begin{aligned} a\sec^2\alpha\,\lambda(a\sec\alpha) &= \int_0^{\frac{1}{2}\pi}\frac{\mu(a\sec\theta)\sin\theta\,d\theta}{\cos^2\theta-\cos^2\alpha} \\ &= \tfrac{1}{2}a\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\cos^2\theta-\cos^2\alpha} + \sum_1^\infty a_{2r}\int_0^{\frac{1}{2}\pi}\frac{\cos 2r\theta\,d\theta}{\cos^2\theta-\cos^2\alpha} \\ &= \frac{\pi}{\sin 2\alpha}\sum_1^\infty a_{2r}\sin 2r\alpha, \end{aligned}$$

whence, by uniqueness of trigonometric series,

$$a_{2r} = \frac{8a}{\pi^2}\int_0^{\frac{1}{2}\pi}\lambda(a\sec\alpha)\tan\alpha\,\sin 2r\alpha\,d\alpha.$$

Substituting in the cosine series, the sum being defined as

$$\lim_{R\to 1-0}\ \sum_1^\infty a_{2r}R^{2r}\cos 2r\theta,$$

we obtain $$\mu(a\sec\theta)\sin\theta = \frac{4a}{\pi^2}\int_0^{\frac{1}{2}\pi}\frac{\lambda(a\sec\alpha)\,(1-\cos^2\alpha)}{\cos^2\theta-\cos^2\alpha}\,d\alpha + \text{const.},$$

i.e. $$\mu(u) = \frac{Cu}{\sqrt{(u^2-a^2)}} + \frac{4}{\pi^2}\frac{u^3}{\sqrt{(u^2-a^2)}}P\int_a^\infty\frac{\lambda(y)\sqrt{(y^2-a^2)}}{y(y^2-u^2)}\,dy.$$

In the integral equation (12) considered here $\lambda(y)=0$, so that

$$K\int_a^u f(v)\,dv + f(u) = \frac{Cu}{\sqrt{(u^2-a^2)}}.$$

We normalize $f(u)$ so that $C=1$. Then

$$f(u) = \frac{d}{du}\left[e^{-Ku}\int_a^u e^{Kv}\frac{v}{\sqrt{(v^2-a^2)}}\,dv\right], \tag{13}$$

on solving the differential equation and noting that

$$\lim_{u\to a}\left[f(u)-\frac{u}{\sqrt{(u^2-a^2)}}\right] = 0.$$

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We can now easily complete the solution, substituting in equation (10):

$$-\tfrac{1}{2}\pi(K^2+k^2)\,kC(k) = \int_a^\infty f(y)\,(k\cos ky - K\sin ky)\,dy$$

$$= \int_a^\infty (k\cos ky - K\sin ky)\frac{d}{dy}\left[e^{-Ky}\int_a^y e^{Kv}\frac{v}{\sqrt{(v^2-a^2)}}\,dv\right]dy,$$

$$-\tfrac{1}{2}\pi(K^2+k^2)\,C(k) = \int_a^\infty \left(\frac{y}{\sqrt{(y^2-a^2)}}-1\right)\cos ky\,dy - \frac{\sin ka}{k}. \tag{14}$$

It can be shown (cf. Watson, *Bessel Functions*, § 13·42) that

$$\int_0^\infty J_1(ak)\cos ku\,dk = \frac{1}{a} \quad (u<a)$$

$$= -\frac{1}{a}\left(\frac{u}{\sqrt{(u^2-a^2)}}-1\right) \quad (u>a),$$

whence, by Fourier transforms,

$$\tfrac{1}{2}\pi J_1(ak) = \int_0^a \frac{1}{a}\cos ku\,du - \frac{1}{a}\int_a^\infty \left(\frac{u}{\sqrt{(u^2-a^2)}}-1\right)\cos ku\,du,$$

so that, from (14), $$C(k) = \frac{aJ_1(ak)}{K^2+k^2}.$$

B_3 can now be evaluated from (11) and A_1 from (9).

Let L be the contour consisting of the real axis and a large semicircle in the upper half-plane, the centre of the circle being at the origin. Then

$$\int_0^\infty C(k)\,(k\cos ky - K\sin ky)\,dk = \frac{1}{2}\int_L \frac{aJ_1(az)}{z-iK}e^{izy}\,dz \quad (y>a)$$

$$= -\pi a I_1(aK)\,e^{-Ky} = B_z e^{-Ky},$$

from (11), so that $$B_3 = -\pi a I_1(aK).$$

From equation (9), by integration by parts,

$$Ai = 2\int_a^\infty f(y)\,e^{-Ky}\,dy = \int_a^\infty e^{-Ky}\frac{y}{\sqrt{(y^2-a^2)}}\,dy = aK_1(aK),$$

so that finally

$$\frac{n}{g}\phi_+ = aK_1(aK)\,e^{-Ky}\cos(Kx-nt)$$

$$+\int_0^\infty \frac{aJ_1(ak)}{K^2+k^2}e^{-kx}(k\cos ky - K\sin ky)\,dk\sin nt \quad (x\geqslant 0),$$

$$\frac{n}{g}\phi_- = aK_1(aK)\,e^{-Ky}\cos(Kx-nt)$$

$$+\pi a I_1(aK)\,e^{-Ky}\sin(Kx-nt) - \pi a I_1(aK)\,e^{-Ky}\sin(Kx+nt)$$

$$-\int_0^\infty \frac{aJ_1(ak)}{K^2+k^2}e^{kx}(k\cos ky - K\sin ky)\,dk\sin nt \quad (x\leqslant 0).$$

The transmission and reflection coefficients are given by

$$\mathfrak{T} = \frac{K_1(n^2a/g)}{\sqrt{\{\pi^2 I_1^2(n^2a/g) + K_1^2(n^2a/g)\}}},$$

$$\mathfrak{R} = \frac{\pi I_1(n^2a/g)}{\sqrt{\{\pi^2 I_1^2(n^2a/g) + K_1^2(n^2a/g)\}}}.$$

These are shown in Fig. 1.

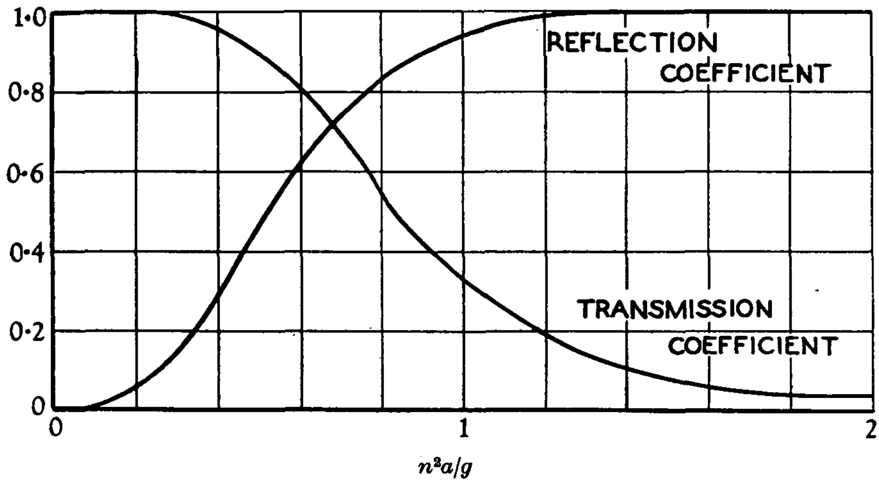

Fig. 1. Reflection and transmission coefficients for a vertical barrier extending upwards from depth a to the surface. $2\pi/n$ is the period of the motion.

The same method can be used to obtain the potential for an infinite vertical submerged barrier whose upper edge is at depth b.

$$\frac{n}{g}\phi_+ = \pi b I_0(bK)\, e^{-Ky} \cos(Kx - nt)$$
$$- \int_0^\infty \frac{bJ_0(bk)}{K^2+k^2} e^{-kx}(k\cos ky - K \sin ky)\, dk \sin nt \quad (x \geqslant 0),$$

$$\frac{n}{g}\phi_- = \pi b I_0(bK)\, e^{-Ky} \cos(Kx - nt)$$
$$+ bK_0(bK)\, e^{-Ky} \sin(Kx - nt) - bK_0(bK)\, e^{-Ky} \sin(Kx + nt)$$
$$+ \int_0^\infty \frac{bJ_0(bk)}{K^2+k^2} e^{kx}(k\cos ky - K \sin ky)\, dk \sin nt \quad (x \leqslant 0),$$

$$\mathfrak{T} = \frac{\pi I_0(n^2b/g)}{\sqrt{\{\pi^2 I_0^2(n^2b/g) + K_0^2(n^2b/g)\}}},$$

$$\mathfrak{R} = \frac{K_0(n^2b/g)}{\sqrt{\{\pi^2 I_0^2(n^2b/g) + K_0^2(n^2b/g)\}}}.$$

This problem has been solved by a different and simpler method by W. R. Dean (2).

Phase change at a barrier in the surface. The motion of the surface near the barrier is easily found. It is of interest to compare the relative phase of the motions on the two sides of the barrier, using the integral

$$\int_0^\infty \frac{kJ_1(ak)\,dk}{K^2+k^2} = \tfrac{1}{2}\pi[\mathbf{L}_{-1}(aK) - I_1(aK)],$$

where $\mathbf{L}_{-1}(x)$ is an associated Struve function defined by

$$\mathbf{L}_{-1}(x) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{2m}}{\Gamma(m+\frac{1}{2})\,\Gamma(m+\frac{3}{2})}.$$

The relative phase θ is given by

$$\tan\theta = \frac{\pi K_1(aK)\,[I_1(aK)+\mathbf{L}_{-1}(aK)]}{K_1^2(aK)+\pi^2 I_1^2(aK) - [\frac{1}{2}\pi\{I_1(aK)+\mathbf{L}_{-1}(aK)\}]^2}.$$

For very long waves there is no phase change; for very short waves the change of phase is two right angles.

Discussion

In the two problems discussed here the velocity at the edge of the barrier becomes infinite. An infinity of this type can be removed by rounding the edge of the barrier. In experimental work it is found that when the barrier extends from the surface downwards, the steady state corresponding to the foregoing solution is rapidly established outside a thin layer provided that the barrier extends far enough near the rounded edge to prevent breaking of waves over it. When the barrier extends upwards from the bottom, the reflection is small, unless the edge of the barrier is very near the surface, as has been shown by Dean. As the edge of the barrier approaches the surface, the character of the wave motion near the top of the barrier is no longer oscillatory; there is a rapid flow over the barrier when a crest arrives. It can be shown by computation that the part of the potential defined by the infinite integral is negligible at a distance of more than two wave-lengths from the barrier.

Other positions of the barrier. When the barrier occupies the whole of the imaginary axis except for a gap between ai and bi, the problem is solved by a trivial extension of Lemma 3, but the resulting solution is complicated. A simple solution can be obtained for the allied problem in which a barrier occupies the part of the imaginary axis between ai and bi. Following Dean, we shall attempt to find a complex potential

$$w = \phi + i\psi$$

of the form
$$\frac{nw}{g} = \left\{Ae^{iKz} + e^{iKz}\int_{ia}^{z} R(z)\,e^{-iKz}\,dz\right\}\cos nt + Be^{iKz}\sin nt.$$

The pressure condition on the surface requires that $\Re(z)$ should be real on the real axis; on the barrier, the imaginary part of w is zero, i.e.

$$\int_a^v R(iy)\,e^{Kv}\,dy$$

must be pure imaginary, when y lies between a and b. A suitable function is therefore

$$R(z) = \frac{z^2+c^2}{\sqrt{\{(z^2+a^2)\,(z^2+b^2)\}}};$$

c^2 is determined from the condition that dw/dz is one-valued in the half-plane $y > 0$ cut from ai to bi, whence

$$\int_a^b \frac{(c^2 - y^2)\, e^{Ky}\, dy}{\sqrt{\{(y^2 - a^2)(b^2 - y^2)\}}} = 0.$$

For large x the potential represents a progressive wave travelling away from the origin; this condition determines A and B. The problem will not be considered in detail here.

Extension of the method. When the normal velocity on the barrier is prescribed, the resulting wave motion can be determined from the general solution given in Lemma 3. If the motion in shallow water is to be determined, Lemma 2 must be replaced by the corresponding series expansion, which is of the form

$$\frac{n}{g}\phi = A_0 \cosh k_0(k - y) \cos(k_0 x - nt) + \sum_{s=1}^{\infty} A_s e^{-k_s x} \cos k_s(k - y) \sin nt,$$

where
$$K = k_0 \tanh k_0 h,$$

and $k_1, k_2, \ldots, k_s, \ldots$ are the real roots of

$$k \sin kh + K \cos kh = 0$$

in increasing order of magnitude, and

$$A_0 = \frac{2\int_0^h f(y) \cosh k_0(h - y)\, dy}{k_0 h + \sinh k_0 h \cosh k_0 h},$$

$$A_s = \frac{-2\int_0^h f(y) \cos k_s(h - y)\, dy}{k_s h + \sin k_s h \cos k_s h} \quad (s \geqslant 1).$$

The amplitude at infinity is $A_0 \cosh k_0 h$. An integral equation can then be derived, which may be solved by numerical methods. Lemma 2 gives the limiting form of the series expansion when the depth h tends to infinity.

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