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GROUND STATES OF SOME FRACTIONAL SCHRÖDINGER EQUATIONS ON $\mathbb{R}^N$

XIAOJUN CHANG

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, Jilin, People's Republic of China
and
College of Mathematics, Jilin University, Changchun 130012, Jilin, People's Republic of China (changxj1982@hotmail.com)

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Abstract In this paper, we study a time-independent fractional Schrödinger equation of the form $(-\Delta)^s u + V(x)u = g(u)$ in $\mathbb{R}^N$, where $N \geqslant 2$, $s \in (0,1)$ and $(-\Delta)^s$ is the fractional Laplacian. By variational methods, we prove the existence of ground state solutions when V is unbounded and the nonlinearity g is subcritical and satisfies the following geometry condition:

$$\limsup_{t\to 0^+} \frac{2\int_0^t g(\tau)\,\mathrm{d}\tau}{t^2} < \inf \sigma((-\Delta)^s + V(x)) < \liminf_{t\to +\infty} \frac{2\int_0^t g(\tau)\,\mathrm{d}\tau}{t^2}.$$

Keywords: ground states; fractional Schrödinger equation; variational methods

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1. Introduction and main results

We consider the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $(-\Delta)^s$ stands for the fractional Laplacian, $g \in C^1(\mathbb{R},\mathbb{R})$ and $N \geqslant 2$. Here the fractional Laplacian $(-\Delta)^s$, with $s \in (0,1)$, of a function $\phi\colon \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^s\phi)(\xi) = |\xi|^{2s}\mathcal{F}(\phi)(\xi) \quad \forall s \in (0,1),$$

where $\mathcal{F}$ is the Fourier transform, i.e.

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \mathrm{e}^{-2\pi \mathrm{i}\xi\cdot x}\phi(x)\,\mathrm{d}x.$$

If u is a solution of (1.1), then the function $\psi(t,x) = \exp(-\mathrm{i}ct)u(x)$, with c a constant, is the so-called standing wave solution of the following time-dependent fractional Schrödinger equation:

$$\mathrm{i}\frac{\partial \psi}{\partial t} + (-\Delta)^s\psi + (V(x) - c)\psi = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where i is the imaginary unit. The fractional Schrödinger equation is an important model in the study of fractional quantum mechanics, which was discovered by Laskin [**28**, **29**] as a result of expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. In Laskin's studies, the Feynman path integral leads to the classical Schrödinger equation and the path integral over Lévy trajectories leads to the fractional Schrödinger equation.

Recently, Guo and Huo [**26**] studied the global well-posedness of a fractional Schrödinger equation of the form (1.2) in $\mathbb{R}$. Some blow-up phenomena of the fractional Schrödinger equation in high-dimensional space was discussed in [**17**]. The investigation of stationary solutions of the fractional Schrödinger equation has also attracted the attention of many mathematicians. In [**18**, **19**], Coti Zelati and Nolasco studied the existence of positive ground states of some fractional Schrödinger equations involving the operator $(-\Delta + m^2)^{1/2}$ with $m > 0$. Frank and Lenzmann [**24**] studied the uniqueness and non-degeneracy of the ground state of $(-\Delta)^s u + u = u^{\alpha+1}$ in $\mathbb{R}$ for general $s \in (0,1)$, where $0 < \alpha < 4s/(1-2s)$ for $s \in (0, \frac{1}{2})$ and $0 < \alpha < \infty$ for $s \in [\frac{1}{2}, 1)$. Recently, the result of [**24**] has been extended in any dimension when s is sufficiently close to 1 by Fall and Valdinoci [**22**] and later for general $s \in (0,1)$ by Frank *et al*. [**25**]. In [**23**], Felmer *et al*. studied the existence of positive solutions of (1.1) with $V(x) \equiv 1$ for general $s \in (0,1)$ when g has subcritical growth and satisfies the Ambrosetti–Rabinowitz condition (see [**1**]), which implies that the nonlinearity g satisfies superlinear growth at ∞. For the case in which the potential V is non-constant, Secchi [**34**] obtained the existence of ground state solutions of (1.1) for general $s \in (0,1)$ when $V(x) \to +\infty$ as $|x| \to +\infty$ and the Ambrosetti–Rabinowitz condition holds. When the nonlinearity g satisfies the asymptotically linear growth, one can see [**14**]. For other results related to the operator $(-\Delta)^s$, one can also see [**4**, **7**, **9**, **10**, **12**, **13**, **15**, **16**, **20**, **37**] and the references therein.

In this paper, we study the existence of ground states of (1.1) when V is unbounded and the nonlinearity g is subcritical and satisfies the following geometry condition:

$$\limsup_{t\to 0^+} \frac{2G(t)}{t^2} < \inf \sigma((-\Delta)^s + V(x)) < \liminf_{t\to+\infty} \frac{2G(t)}{t^2},$$

where $G(t) = \int_0^t g(\tau)\,\mathrm{d}\tau$. Here it is clear that neither the superlinear growth nor the asymptotically linear growth are required. Throughout the paper the following assumptions are made.

(V_1) $V \in C^1(\mathbb{R}^N, \mathbb{R})$, $V_0 \doteq \inf_{x\in\mathbb{R}^N} V(x) > 0$.

(V_2) There exists $r > 0$ such that for any $M > 0$,

$$\operatorname{meas}(\{x \in B_r(y) \colon V(x) \leqslant M\}) \to 0 \quad \text{as } |y| \to \infty.$$

(V_3) $\|(\nabla V(x), x)^+\|_{N/2s} < 2sS_s$, where S_s is the best Sobolev constant of the embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$, i.e.

$$S_s = \inf_{u\in H^s(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2\,\mathrm{d}x}{(\int_{\mathbb{R}^N} |u(x)|^{2N/(N-2s)}\,\mathrm{d}x)^{(N-2s)/N}}.$$

(g_1) $g \in C^1(\mathbb{R}, \mathbb{R})$ and

$$\lim_{t\to+\infty} \frac{g(t)}{t^{2_s^*-1}} = 0,$$

where $2_s^* = 2N/(N-2s)$.

(g_2) $g(0) = 0$ and there exists $\kappa_0 \in (0, +\infty)$ such that

$$-\kappa_0 \leqslant \liminf_{t\to0^+} \frac{2G(t)}{t^2} \leqslant \limsup_{t\to0^+} \frac{2G(t)}{t^2} < \lambda_1,$$

where

$$\lambda_1 \doteq \inf \sigma((-\Delta)^s + V(x)) = \inf_{u\in H^s(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N}(|(-\Delta)^{s/2}u(x)|^2 + V(x)u^2)\,\mathrm{d}x}{\int_{\mathbb{R}^N} u^2\,\mathrm{d}x}.$$

Here $\sigma((-\Delta)^s + V(x))$ denotes the spectrum of the operator $(-\Delta)^s + V(x)$.

(g_3)

$$\liminf_{t\to+\infty} \frac{2G(t)}{t^2} > \lambda_1.$$

A solution u of (1.1) is usually called a ground state solution if it minimizes the corresponding action functional among all the non-trivial solutions of (1.1).

Our main result is as follows.

Theorem 1.1. *Assume that* (V_1)*–*(V_3) *and* (g_1)*–*(g_3) *hold. Then problem (1.1) has a positive ground state solution.*

Remark 1.2. Note that the assumption $\lim_{|x|\to+\infty} V(x) = +\infty$ (used in [**34**]) implies that, for any $M > 0$, the set $\{x \in \mathbb{R}^N : V(x) \leqslant M\}$ is bounded. There then exists a positive constant R_0 such that $\operatorname{meas}(\{x \in B_r(y) : V(x) \leqslant M\}) = 0$ for any $|y| \geqslant R_0$. In this sense, we can see that (V_2) is a weaker condition than that in [**34**].

Remark 1.3. When $s = 1$, (1.1) becomes the classical Schrödinger equation

$$-\Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

In the past 20 years, the existence and multiplicity of positive solutions of (1.3), when the nonlinearity g is subcritical and satisfies the superlinear or asymptotically linear growth, have been widely investigated (see, for example, [**2**, **3**, **5**, **6**, **30**, **33**, **36**] and references therein). One can see that our result is new, even for the case $s = 1$.

The paper is organized as follows. In § 2, we introduce a variational setting of the problem and present some preliminary results. In § 3, we apply variational methods to give the proof of Theorem 1.1.

2. Preliminaries and functional setting

Consider the Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)\,\mathrm{d}\xi < \infty \right\}.$$

The norm is defined by

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\xi|^{2s}\hat{u}^2 + \hat{u}^2)\,\mathrm{d}\xi \right)^{1/2},$$

and in this paper we consider its subspace

$$E = \left\{ u \in H^s(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x < \infty \right\}.$$

We define the norm in E by

$$\|u\|_E = \left(\int_{\mathbb{R}^N} |\xi|^{2s}\hat{u}^2\,\mathrm{d}\xi + \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x \right)^{1/2}.$$

Clearly,

$$\|u\| = \left(\int_{\mathbb{R}^N} (|(-\Delta)^{s/2}u|^2 + V(x)u^2)\,\mathrm{d}x \right)^{1/2} \quad \forall u \in E.$$

Throughout this paper, we use the norm $\|\cdot\|$ in E. The space $\dot{H}^s(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norms

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s}\hat{u}^2\,\mathrm{d}\xi = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2\,\mathrm{d}x.$$

Turning to problem (1.1), we define the energy functional $I \colon H^s(\mathbb{R}^N) \to \mathbb{R}$ as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{s/2}u(x)|^2 + V(x)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\,\mathrm{d}x \quad \forall u \in E.$$

Definition 2.1. We say that $u \in E$ is a weak solution of (1.1) if

$$\int_{\mathbb{R}^N} ((-\Delta)^{s/2}u(-\Delta)^{s/2}\phi + V(x)u\phi)\,\mathrm{d}x = \int_{\mathbb{R}^N} g(u)\phi\,\mathrm{d}x \quad \forall \phi \in E.$$

Lemma 2.2 (Lions [31]). *$H^s(\mathbb{R}^N)$ is continuously embedded into $L^r(\mathbb{R}^N)$ for $r \in [2, 2N/(N-2s)]$ and compactly embedded into $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for $r \in [2, 2N/(N-2s))$.*

By (g_1)–(g_3) and Lemma 2.2 it follows that $g(u) \in L^{2N/(N+2s)}(\mathbb{R}^N)$ and $\phi \in L^{2N/(N-2s)}(\mathbb{R}^N)$. Then $g(u)\phi \in L^1(\mathbb{R}^N)$. On the other hand, for any $u, \phi \in E$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}\phi\,\mathrm{d}x &\leqslant \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\phi|^2\,\mathrm{d}x \right)^{1/2} \\ &\leqslant \|u\|\,\|\phi\| \\ &< \infty,\end{aligned}$$

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hence the identity in Definition 2.1 is well defined. Furthermore, by arguments similar to those in [**32**], it is easily seen that $I \in C^1(E, \mathbb{R})$ and that the critical points of I correspond to weak solutions of (1.1).

Note that the operator $(-\Delta)^s$ is non-local, but we can apply the s-harmonic extension technique as in [**11**] to transform (1.1) into a local problem. In fact, for a given function $u \in \dot{H}^s(\mathbb{R}^N)$, by a minimization procedure to the problem

$$\min\left\{ \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}y\colon w(x,0) = u \text{ on } \mathbb{R}^N \right\},$$

we can obtain a unique function $w \in X^s(\mathbb{R}^{N+1}_+)$ satisfying

$$\left.\begin{aligned} -\operatorname{div}(y^{1-2s}\nabla w) &= 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\ w(x,0) &= u \quad \text{on } \mathbb{R}^N. \end{aligned}\right\} \tag{2.1}$$

Here, w is usually called the s-harmonic extension of u to the upper half-space $\mathbb{R}^{N+1}_+$ (see [**11**]), and $X^s(\mathbb{R}^{N+1}_+)$ is the completion of $C_0^\infty(\mathbb{R}^{N+1}_+)$ under the norms

$$\|w\|_{X^s(\mathbb{R}^{N+1}_+)} = \left(\kappa_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{1/2},$$

where $\kappa_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$. By Lemma A.2 in [**7**] it follows that

$$\|w\|_{X^s(\mathbb{R}^{N+1}_+)} = \|u\|_{\dot{H}^s(\mathbb{R}^N)}.$$

Conversely, if we let $w \in X^s(\mathbb{R}^{N+1}_+)$ and denote by $\mathrm{Tr}(w)$ its trace on $\mathbb{R}^N \times \{y = 0\}$, then the following trace inequality holds:

$$\|\mathrm{Tr}(w)\|_{\dot{H}^s(\mathbb{R}^N)} \leqslant \|v\|_{X^s(\mathbb{R}^{N+1}_+)}$$

(see [**7**]). Thus, as shown in [**11**], the fractional Laplacian operator $(-\Delta)^s$ can be defined by the following Dirichlet-to-Neumann map:

$$(-\Delta)^s u(x) = -\frac{1}{\kappa_s}\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y) \quad \forall u \in \dot{H}^s(\mathbb{R}^N), \tag{2.2}$$

where $w = E_s(u)$. In view of the arguments put forward in [**11**] (see also [**9**]) we can see that the operator $(-\Delta)^s$, when defined by the Dirichlet-to-Neumann map (2.2), is equivalent to that obtained from the Fourier transform. Moreover, the map $E_s(\cdot)$ is a one-to-one map from $\dot{H}^s(\mathbb{R}^N)$ into $X^s(\mathbb{R}^{N+1}_+)$.

We can now see that (1.1) can be transformed into the following local problem:

$$\left.\begin{aligned} -\operatorname{div}(y^{1-2s}\nabla w) &= 0 && \text{in } \mathbb{R}^{N+1}_+, \\ \partial^s_\nu w + V(x)w &= g(w) && \text{on } \mathbb{R}^N, \end{aligned}\right\} \tag{2.3}$$

where

$$\partial^s_\nu w(x,0) \doteq -\frac{1}{\kappa_s}\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y) \quad \forall x \in \mathbb{R}^N.$$

Clearly, if w is a weak solution of (2.3), then $u = w(\cdot, 0) = \mathrm{Tr}(w)$ is a weak solution of (1.1). Thus, by similar arguments as those given in [**15**], we can obtain the Pohozaev identity for (1.1) as follows.

Theorem 2.3. *Assume that $V \in C^1(\mathbb{R}^N, \mathbb{R})$, $g \in C^1(\mathbb{R}^N, \mathbb{R})$ and $g(0) = 0$. If $u \in H^s(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of (1.1), then*

$$\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x = N\int_{\mathbb{R}^N} G(u)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x), x)u^2 \,\mathrm{d}x.$$

Remark 2.4. Frank and Lenzmann [**24**] established the Pohozaev identity of (1.1) for $g(t) = -t + |t|^{p-2}t$ with $p \in (2, (2N+s)/(2N-s))$ and $N = 1$. As pointed out in [**20**], the arguments in [**24**] can be modified to the high-dimensional case where $N \geqslant 2$. In [**15**], the Pohozaev identity for (1.1) with $V(x) \equiv 0$ is obtained by different arguments from [**24**]. For the Pohozaev identity of fractional Laplacian equations on bounded domains, see [**35**].

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. To begin with, we have the following result.

Lemma 3.1. *Under the assumptions (V_1)–(V_3), the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for any $p \in [2, 2_s^*)$.*

Proof. Assume that $\{u_n\} \subset E$ is bounded, i.e. $\|u_n\| \leqslant C_1$ for all $n \in \mathbb{N}$, for some $C_1 > 0$. By Lemma 2.2, there exists $u_0 \in E$ such that $u_n \to u_0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $p \in [2, 2N/(N-2s))$. Without loss of generality, we assume that $u_0 \equiv 0$. We prove that

$$u_n \to 0 \quad \text{in } L^2(\mathbb{R}^N). \tag{3.1}$$

For any $y \in \mathbb{R}^N$, set

$$A_M(y) \doteq \{x \in B_r(y) \colon V(x) \leqslant M\} \quad \text{and} \quad D_M(y) \doteq \{x \in B_r(y) \colon V(x) > M\} \quad \forall M > 0.$$

Choose $\{y_i\} \subset \mathbb{R}^N$ such that $\mathbb{R}^N \subset \bigcup_{i=1}^\infty B_r(y_i)$ and each $x \in \mathbb{R}^N$ is covered by at most 2^N such balls. Then, for any $M > 0$ and $R > 2r$, we have

$$\begin{aligned}\int_{B_R^c} u_n^2 \,\mathrm{d}x &\leqslant \sum_{|y_i| \geqslant R-r}^{\infty} \int_{B_r(y_i)} u_n^2 \,\mathrm{d}x \\ &= \sum_{|y_i| \geqslant R-r}^{\infty} \left[\int_{B_r(y_i)\cap A_M(y_i)} u_n^2 \,\mathrm{d}x + \int_{B_r(y_i)\cap D_M(y_i)} u_n^2 \,\mathrm{d}x\right] \\ &\leqslant \sum_{|y_i| \geqslant R-r}^{\infty} \left[\int_{A_M(y_i)} u_n^2 \,\mathrm{d}x + \frac{1}{M}\int_{B_r(y_i)} V(x)u_n^2 \,\mathrm{d}x\right]\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{|y_i|\geqslant R-r}^{\infty}\biggl[(\operatorname{meas}(A_M(y_i)))^{2s/N}\|u_n\|^2_{L^{2N/(N-2s)}(B_r(y_i))}+\frac{1}{M}\int_{B_r(y_i)}V(x)u_n^2\,\mathrm{d}x\biggr]\\
&\leqslant \sum_{|y_i|\geqslant R-r}^{\infty}\biggl[C_2(\operatorname{meas}(A_M(y_i)))^{2s/N}\|u_n\|^2_{H^s(B_r(y_i))}+\frac{1}{M}\int_{B_r(y_i)}V(x)u_n^2\,\mathrm{d}x\biggr]\\
&\leqslant 2^N\biggl[C_2\sup_{|y|\geqslant R-r}(\operatorname{meas}(A_M(y)))^{2s/N}\|u_n\|^2_{H^s(B^c_{R-2r})}+\frac{1}{M}\int_{B^c_{R-2r}}V(x)u_n^2\,\mathrm{d}x\biggr]\\
&\leqslant 2^N\|u_n\|^2\biggl[C_2\sup_{|y|\geqslant R-r}(\operatorname{meas}(A_M(y)))^{2s/N}+\frac{1}{M}\biggr].
\end{aligned}
$$

Together with (V_1)–(V_3) and Lemma 2.2, we obtain that (3.1) holds. By the interpolation inequality we find that, for $p\in(2,2N/(N-2s))$,

$$
\begin{aligned}
\|u_n\|_p &= \|u_n\|_2^t\|u_n\|^{1-t}_{2N/(N-2s)}\\
&= C_2^{1-t}\|u_n\|_2^t\|u_n\|^{1-t}\\
&\leqslant (C_1C_2)^{1-t}\|u_n\|_2^t\\
&\to 0.
\end{aligned}
$$

We then get the compact embedding $E\hookrightarrow L^p(\mathbb{R}^N)$ for $p\in[2,2_s^*)$, which completes the proof. □

Lemma 3.2. *Under the assumptions* (V_1)–(V_3), λ_1 *is an eigenvalue of the operator* $(-\Delta)^s+V(x)$ *and there exists a corresponding eigenfunction* φ_1 *with* $\varphi_1(x)>0$ *for all* $x\in\mathbb{R}^N$.

Proof. By Ekeland's variational principle, there exist sequences $\{u_n\}\subset E$ with $\|u_n\|_2=1$, $\|u_n\|^2\to\lambda_1$, and $\{\mu_n\}\subset\mathbb{R}^+$ such that

$$
\int_{\mathbb{R}^N}((-\Delta)^{s/2}u_n(-\Delta)^{s/2}\phi+V(x)u_n\phi)\,\mathrm{d}x=\mu_n\int_{\mathbb{R}^N}u_n\phi\,\mathrm{d}x+o(1)\|\phi\|\quad\forall\phi\in E. \tag{3.2}
$$

Taking $\phi=u_n$ in (3.2), we get

$$
\|u_n\|^2=\int_{\mathbb{R}^N}(|(-\Delta)^{s/2}u_n|^2+V(x)u_n^2)\,\mathrm{d}x=\mu_n+o(1)\|u_n\|. \tag{3.3}
$$

Clearly, $\mu_n\to\lambda_1$. By Lemma 3.1, passing to a subsequence if necessary, there exists $\varphi_1\in E$ with $\|\varphi_1\|_2=1$ such that $u_n\to\varphi_1$ in E. We then obtain that

$$
\int_{\mathbb{R}^N}((-\Delta)^{s/2}\varphi_1(-\Delta)^{s/2}\phi+V(x)\varphi_1\phi)\,\mathrm{d}x=\lambda_1\int_{\mathbb{R}^N}\varphi_1\phi\,\mathrm{d}x\quad\forall\phi\in E,
$$

i.e. φ_1 is a non-trivial weak solution of the equation

$$
(-\Delta)^s u+V(x)u=\lambda_1 u\quad\text{in }\mathbb{R}^N.
$$

Note that $|\varphi_1|$ is also a solution of

$$\lambda_1 = \inf_{u\in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|(-\Delta)^{s/2}u(x)|^2 + V(x)u^2)\,\mathrm{d}x}{\int_{\mathbb{R}^N} u^2\,\mathrm{d}x}.$$

Hence, $\varphi_1 \geqslant 0$ and $\varphi_1 \not\equiv 0$. Following the arguments of § 2, it follows that $w_1 \doteq E_s(\varphi_1)$ is a non-trivial non-negative weak solution of

$$\begin{aligned} -\operatorname{div}(y^{1-2s}\nabla w) = 0 &\quad \text{in } \mathbb{R}^{N+1}_+, \\ \partial_\nu^s w + V(x)w = \lambda_1 w &\quad \text{on } \mathbb{R}^N. \end{aligned}$$

By the strong maximum principle for A_2 weight (see [**21**] or [**9**]) it follows that $w_1(x,y) > 0$ for all $(x,y) \in \mathbb{R}^{N+1}_+$, which implies that $\varphi_1(x) > 0$ for all $x \in \mathbb{R}^N$. This completes the proof. □

Set $g_1(s) = (g(s) + \kappa_0 s)^+$ and define $G_1(s) = \int_0^s g_1(t)\,\mathrm{d}t$. By (g_1)–(g_3) it then follows that

$$\lim_{s\to+\infty} \frac{g_1(s)}{s^{2_s^*-1}} = 0, \tag{3.4}$$

$$0 \leqslant \liminf_{s\to 0^+} \frac{g_1(s)}{s} \leqslant \limsup_{s\to 0^+} \frac{g_1(s)}{s} < \lambda_1 + \kappa_0, \tag{3.5}$$

$$\liminf_{s\to+\infty} \frac{2G_1(s)}{s^2} > \lambda_1 + \kappa_0. \tag{3.6}$$

Define the perturbed functional $I_\mu\colon E \to \mathbb{R}$ with $\mu \in [\frac{1}{2}, 1]$ by

$$I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} [|(-\Delta)^{s/2}u|^2 + (V(x) + \kappa_0)u^2]\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} G_1(u)\,\mathrm{d}x.$$

Clearly, a critical point of I_1 is a weak solution of (1.1).

Under the assumptions of Theorem 1.1, we see that there exists $\eta \in (0,1)$ such that I_μ with $\mu \in [\eta, 1]$ has the mountain pass geometry.

Lemma 3.3.

(i) *There exists* $v_0 \in E$ *and* $\eta \in (0,1)$ *such that*

$$I_\mu(v_0) < 0 \quad \forall \mu \in [\eta, 1].$$

(ii) $c_\mu(u) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)) > 0$ *for all* $\mu \in [\eta, 1]$, *where*

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0,\ \gamma(0) = v_0\}.$$

Proof. (i) By (3.6), there exist $\delta_1, M_1 > 0$ and $\eta \in (0,1)$ such that, for any $\mu \in [\eta, 1]$, we have

$$2\mu G_1(s) \geqslant (\lambda_1 + \kappa_0 + \delta_1)s^2 \quad \forall s \geqslant M_1. \tag{3.7}$$

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By $\varphi_1 \in E$ and Lemma 3.1 it follows that $\varphi_1 \in L^2(\mathbb{R}^N)$, taking $R_1 > 0$ large enough such that

$$\|\varphi_1\|^2_{L^2(B_{R_1})} \geqslant \frac{\lambda_1 + \kappa_0 + \delta_1/2}{\lambda_1 + \kappa_0 + \delta_1}\|\varphi_1\|_2^2.$$

Since $\varphi_1(x) > 0$ in $\mathbb{R}^N$, there exists $t_1 > 0$ such that

$$t_1\varphi_1(x) > M_1 \quad \forall |x| \leqslant R_1.$$

Then, together with (3.7), we can see that, for $t \geqslant t_1$,

$$\begin{aligned} I_\mu(t\varphi_1) &= \frac{t^2}{2}\int_{\mathbb{R}^N}[|(-\Delta)^{s/2}\varphi_1|^2 + (V(x)+\kappa_0)\varphi_1^2]\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} G_1(t\varphi_1)\,\mathrm{d}x \\ &= \frac{t^2}{2}(\lambda_1+\kappa_0)\|\varphi_1\|_2^2 - \int_{B_{R_1}}\mu G_1(t\varphi_1)\,\mathrm{d}x - \int_{B^c_{R_1}}\mu G_1(t\varphi_1)\,\mathrm{d}x \\ &\leqslant \frac{t^2}{2}(\lambda_1+\kappa_0)\|\varphi_1\|_2^2 - \frac{t^2}{2}(\lambda_1+\kappa_0+\delta_1)\int_{B_{R_1}}\varphi_1^2\,\mathrm{d}x \\ &\leqslant -\frac{t^2}{4}\delta_1\|\varphi_1\|_2^2. \end{aligned}$$

Take $v_0 = \bar{t}\varphi_1$ with $\bar{t} > t_1$. It is easily seen that $I_\mu(v_0) < 0$ for all $\mu \in [\eta, 1]$.

(ii) By (3.4)–(3.6), there exist $\delta_0, C_0 > 0$ such that

$$g_1(s) \leqslant (\lambda_1+\kappa_0-\delta_0)s + C_0 s^{2^*_s-1} \quad \forall s \geqslant 0, \tag{3.8}$$

which implies that

$$2\mu G_1(s) \leqslant (\lambda_1+\kappa_0-\delta_0)s^2 + C_0 s^{2^*_s} \quad \forall s \geqslant 0,\ \forall \mu \in [\eta, 1].$$

Then, for any $\mu \in [\eta, 1]$,

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int_{\mathbb{R}^N}[|(-\Delta)^{s/2}u|^2 + (V(x)+\kappa_0)u^2]\,\mathrm{d}x - \mu\int_{\mathbb{R}^N} G_1(u)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N}[|(-\Delta)^{s/2}u|^2 + (V(x)+\kappa_0)u^2]\,\mathrm{d}x \\ &\qquad\qquad - \frac{\lambda_1+\kappa_0-\delta_0}{2}\int_{\mathbb{R}^N}u^2\,\mathrm{d}x - C_0\int_{\mathbb{R}^N}u^{2^*_s}\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\left(1 - \frac{\lambda_1+\kappa_0-\delta_0}{\lambda_1+\kappa_0}\right)\int_{\mathbb{R}^N}[|(-\Delta)^{s/2}u|^2 + (V(x)+\kappa_0)u^2]\,\mathrm{d}x - C_0\int_{\mathbb{R}^N}u^{2^*_s}\,\mathrm{d}x \\ &\geqslant \frac{\delta_0}{2(\lambda_1+\kappa_0)}\|u\|^2 - \frac{C_0}{S_s^{N/(N-2s)}}\left(\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2\,\mathrm{d}x\right)^{N/(N-2s)} \\ &\geqslant \frac{\delta_0}{2(\lambda_1+\kappa_0)}\|u\|^2 - \frac{C_0}{S_s^{N/(N-2s)}}\|u\|^{2N/(N-2s)} \\ &= \frac{\delta_0}{2(\lambda_1+\kappa_0)}\|u\|^2\left[1 - \frac{2(\lambda_1+\kappa_0)C_0}{\delta_0 S_s^{N/(N-2s)}}\|u\|^{4s/(N-2s)}\right]. \end{aligned}$$

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It is not difficult to see that there exists a sufficiently small $\rho > 0$ such that

$$I_\mu(u) > 0 \quad \forall u \in E$$

with

$$\|u\| = \rho \quad \forall \mu \in [\eta, 1].$$

Hence,

$$c_\mu(u) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\mu(\gamma(t)) \geqslant \inf_{u \in E, \|u\| = \rho} I_\mu(u) > 0 \quad \forall \mu \in [\eta, 1].$$

□

To prove Theorem 1.1, we need the following abstract result.

Theorem 3.4 (Jeanjean [27, Theorem 1.1]). *Let $(X, \|\cdot\|_X)$ be a Banach space and let $J \subset \mathbb{R}^+$ be an interval. Consider a family $\{I_\mu\}_{\mu \in J}$ of C^1-functionals on X with the form*

$$I_\mu(u) = A(u) - \mu B(u) \quad \forall \mu \in J,$$

where $B(u) \geqslant 0$ for all $u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. If there are two points $v_1, v_2 \in X$ such that

$$c_\mu(u) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(v_1), I_\mu(v_2)\} \quad \forall \mu \in J,$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = v_1,\ \gamma(0) = v_2\},$$

then, for almost every $\mu \in J$, there exists a sequence $\{v_n\} \subset X$ such that

(i) *$\{v_n\}$ is bounded,*

(ii) *$I_\mu(v_n) \to c_\mu$,*

(iii) *$I'_\mu(v_n) \to 0$ in the dual X^{-1} of X.*

By Lemma 3.3 and Theorem 3.4 it follows that I_μ has a bounded Palais–Smale (PS) sequence at the level of c_μ for almost every (a.e.) $\mu \in [\eta, 1]$. Furthermore, we have the following result.

Lemma 3.5. *For any $\mu \in [\eta, 1]$, each bounded (PS) sequence for the functional I_μ admits a convergent subsequence.*

Proof. Assume that $\{u_n\} \subset E$ is a bounded (PS) sequence (i.e. $\|u_n\| \leqslant C_1$ for all $n \in \mathbb{N}$ for some C_1 greater than 0) and that

$$\{I_\mu(u_n)\} \quad \text{is bounded,} \tag{3.9}$$

$$\lim_{n \to \infty} I'_\mu(u_n) = 0 \quad \text{in } E'. \tag{3.10}$$

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By Lemma 2.2, there exists $u_0 \in E$ such that $u_n \to u_0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for all $p \in [2, 2N/(N-2s))$. Without loss of generality, we assume that $u_0 \equiv 0$. By Lemma 3.1, passing to a subsequence if necessary, we get some $u_0 \in E$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0 && \text{in } E, \\ u_n &\to u_0 && \text{in } L^p(\mathbb{R}^N) \text{ for } p \in [2, 2_s^*), \\ u_n(x) &\to u_0(x) && \text{a.e. } x \in \mathbb{R}^N. \end{aligned}$$

Together with (3.10) it follows that, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, $I'_\mu(u_0)\varphi = 0$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in E, it follows that $I'_\mu(u_0) = 0$ in E, which implies that

$$\int_{\mathbb{R}^N} [|(-\Delta)^{s/2} u_0|^2 + (V(x) + \kappa_0) u_0^2] \, dx = \mu \int_{\mathbb{R}^N} g_1(u_0) u_0 \, dx. \tag{3.11}$$

We now claim that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g_1(u_n) u_n \, dx = \int_{\mathbb{R}^N} g_1(u_0) u_0 \, dx. \tag{3.12}$$

If this is correct, using (3.10) and (3.11) we get

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^N} [|(-\Delta)^{s/2} u_n|^2 + V(x) u_n^2] \, dx &= \lim_{n\to\infty} \int_{\mathbb{R}^N} [\mu g_1(u_n) u_n - \kappa_0 u_n^2] \, dx \\ &= \int_{\mathbb{R}^N} [\mu g_1(u_0) u_0 - \kappa_0 u_0^2] \, dx \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} [|(-\Delta)^{s/2} u_0|^2 + V(x) u_0^2] \, dx, \end{aligned}$$

i.e. $u_n \to u_0$ in E.

In what follows we prove (3.12). In fact, we define, for $0 \leqslant a < b \leqslant +\infty$, that

$$\Omega_n(a, b) = \{x \in \mathbb{R}^N : a \leqslant |u_n(x)| < b\}.$$

By (3.4) and (3.5), for any $\epsilon > 0$, there exist $b_\epsilon > 0$ sufficiently large and $C_\epsilon > 0$ such that

$$|g_1(u_n) u_n| \leqslant \begin{cases} \epsilon |u_n(x)|^{2_s^*} & \forall x \in \Omega_n(b_\epsilon, +\infty), \\ C_\epsilon |u_n(x)|^p & \forall x \in \Omega_n(0, b_\epsilon) \end{cases}$$

for some $p \in (2, 2_s^*)$. Then,

$$\begin{aligned} \int_{\mathbb{R}^N} |g_1(u_n) u_n| \, dx &= \int_{\Omega_n(b_\epsilon, +\infty)} |g_1(u_n) u_n| \, dx + \int_{\Omega_n(0, b_\epsilon)} |g_1(u_n) u_n| \, dx \\ &= \epsilon \int_{\Omega_n(b_\epsilon, +\infty)} |u_n(x)|^{2_s^*} \, dx + C_\epsilon \int_{\Omega_n(0, b_\epsilon)} |u_n(x)|^p \, dx \\ &\leqslant \epsilon \int_{\mathbb{R}^N} |u_n(x)|^{2_s^*} \, dx + C_\epsilon \int_{\mathbb{R}^N} |u_n(x)|^p \, dx. \end{aligned}$$

Using (3.4) and (3.5) again, we can see that

$$(|g_1(u_n)u_n - g_1(u_0)u_0| - \epsilon|u_n(x)|^{2_s^*})^+ \leqslant C_\epsilon|u_n(x)|^p + |u(x)|.$$

By the compact embedding $E \hookrightarrow L^p(\mathbb{R}^m)$ and Theorem 4.9 in [**8**], there exists a subsequence $\{u_{n_k}\}$ and functions $\bar{u} \in E$, $h \in L^p(\mathbb{R}^N)$ such that

(i) $u_{n_k}(x) \to \bar{u}(x)$ a.e. $x \in \mathbb{R}^m$,

(ii) $|u_{n_k}(x)| \leqslant h(x)$ for all k a.e. $x \in \mathbb{R}^m$.

Then, up to a subsequence, we have that

$$(|g_1(u_n)u_n - g_1(u_0)u_0| - \epsilon|u_n(x)|^{2_s^*})^+ \leqslant C_\epsilon|h(x)|^p + |g_1(u_0)u_0|,$$

which together with Lebesgue's dominated convergence theorem implies that

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N}&(|g_1(u_n)u_n - g_1(u_0)u_0| - \epsilon|u_n(x)|^{2_s^*})^+\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\lim_{n\to\infty}(|g_1(u_n)u_n - g_1(u_0)u_0| - \epsilon|u_n(x)|^{2_s^*})^+\,\mathrm{d}x\\ &= 0.\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|g_1(u_n)u_n - g_1(u_0)u_0|\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^m}\limsup_{n\to\infty}|g_1(u_n)u_n - g_1(u_0)u_0|\,\mathrm{d}x\\ &\leqslant \epsilon\int_{\mathbb{R}^m}|u_n(x)|^{2_s^*}\,\mathrm{d}x\\ &\leqslant \epsilon C\end{aligned}$$

for some $C > 0$. Since ϵ is arbitrary, we can see that (3.12) holds. This completes the proof. □

Lemma 3.6. *The set* $K \doteq \{u \in E \setminus \{0\}\colon I'(u) = 0\}$ *is not empty.*

Proof. By Theorem 3.4, for almost every $\mu \in [\eta, 1]$, there exists a bounded sequence $\{u_{\mu,n}\} \subset E$ such that $I_\mu(u_{\mu,n}) = c_\mu$ and $I'_\mu(u_{\mu,n}) \to 0$ in E'. Applying Lemma 3.5, passing to a subsequence if possible, there exists $u_\mu \in E \setminus \{0\}$ such that $u_{\mu,n} \to u_\mu$ in E, which implies that $I_\mu(u_\mu) = c_\mu$ and $I'_\mu(u_\mu) = 0$. Taking $\{\mu_n\} \subset [\eta, 1)$ with $\mu_n \nearrow 1$ such that, for any $n \geqslant 1$, there exists $u_n \in E \setminus \{0\}$ satisfying

$$I_{\mu_n}(u_n) = c_{\mu_n}, \tag{3.13}$$

$$I'_{\mu_n}(u_n) = 0, \tag{3.14}$$

it follows that each u_n is a weak solution of the equation

$$(-\Delta)^s u_n + (V(x) + \kappa_0)u_n = \mu_n g_1(u_n) \quad \text{in } \mathbb{R}^N. \tag{3.15}$$

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Using Theorem 2.3, we can see that u_n satisfies the Pohozaev identity

$$\frac{N-2s}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x+\frac{N}{2}\int_{\mathbb{R}^N}(V(x)+\kappa_0)u_n^2\,\mathrm{d}x \\ = N\int_{\mathbb{R}^N}\mu_n G_1(u_n)\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)u_n^2\,\mathrm{d}x.$$

Then, denoting $\|(\nabla V(x),x)^+\|_{N/2s}=2s(S_s-\sigma)$ for some $\sigma\in(0,S_s)$, by (V_3), Lemma 2.2 and the fact that c_μ is decreasing with respect to μ (see [**27**]), we get

$$\begin{aligned} s\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x &= \frac{N}{2}\int_{\mathbb{R}^N}[|(-\Delta)^{s/2}u_n|^2+(V(x)+\kappa_0)u_n^2]\,\mathrm{d}x \\ &\quad - N\int_{\mathbb{R}^N}\mu_n G_1(u_n)\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)u_n^2\,\mathrm{d}x \\ &= NI_{\mu_n}(u_n)+\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)u_n^2\,\mathrm{d}x \\ &\leqslant Nc_{\mu_n}+\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)^+u_n^2\,\mathrm{d}x \\ &\leqslant Nc_\eta+\tfrac{1}{2}\|(\nabla V(x),x)^+\|_{N/2s}\|u_n\|_{2_s^*}^2 \\ &\leqslant Nc_\eta+s\biggl(1-\frac{\sigma}{S_s}\biggr)\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x, \end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x\leqslant\frac{Nc_\eta S_s}{s\sigma}. \tag{3.16}$$

Hence, by (3.4), (3.5), (3.8) and (3.15), we have

$$\begin{aligned} (\lambda_1+\kappa_0)\int_{\mathbb{R}^N}u_n^2\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^N}[|(-\Delta)^{s/2}u_n|^2+(V(x)+\kappa_0)u_n^2]\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\mu_n g_1(u_n)u_n\,\mathrm{d}x \\ &\leqslant (\lambda_1+\kappa_0-\delta_0)\int_{\mathbb{R}^N}u_n^2\,\mathrm{d}x+C_0\int_{\mathbb{R}^N}u_n^{2_s^*}\,\mathrm{d}x \\ &\leqslant (\lambda_1+\kappa_0-\delta_0)\int_{\mathbb{R}^N}u_n^2\,\mathrm{d}x \\ &\qquad +\frac{C_0}{S_s^{N/(N-2s)}}\biggl(\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x\biggr)^{N/(N-2s)}. \end{aligned}$$

From the above and (3.16), it follows that there exists $M_1>0$ such that

$$\int_{\mathbb{R}^N}u_n^2\,\mathrm{d}x\leqslant M_1.$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^N}[|(-\Delta)^{s/2}u_n|^2+V(x)u_n^2]\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^N}\mu_n g_1(u_n)u_n\,\mathrm{d}x-\int_{\mathbb{R}^N}\kappa_0 u_n^2\,\mathrm{d}x\\ &\leqslant(\lambda_1-\delta_0)\int_{\mathbb{R}^N}u_n^2\,\mathrm{d}x+C_0\int_{\mathbb{R}^N}u_n^{2_s^*}\,\mathrm{d}x\\ &\leqslant(\lambda_1-\delta_0)\int_{\mathbb{R}^N}u_n^2\,\mathrm{d}x\\ &\quad+\frac{C_0}{S_s^{N/(N-2s)}}\biggl(\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x\biggr)^{N/(N-2s)}\\ &\leqslant M_2\end{aligned}$$

for some $M_2>0$. Thus, $\{u_n\}$ is bounded in E. By (3.4), (3.5) and the Sobolev embedding theorem we can see that $\{\int_{\mathbb{R}^N}g_1(u_n)u_n\,\mathrm{d}x\}$ and $\{\int_{\mathbb{R}^N}G_1(u_n)\,\mathrm{d}x\}$ are bounded uniformly. Then, by (3.13), (3.14) and $\mu_n\to 1$ it follows that

$$\begin{aligned}I(u_n)&=I_{\mu_n}(u_n)+(\mu_n-1)\int_{\mathbb{R}^N}G_1(u_n)\,\mathrm{d}x\to c_1,\\ I'(u_n)&=I'_{\mu_n}(u_n)+(\mu_n-1)\int_{\mathbb{R}^N}g_1(u_n)u_n\,\mathrm{d}x\to 0.\end{aligned}$$

That is, $\{u_n\}$ is a bounded (PS) sequence for the functional I, and thus, by similar arguments to those in Lemma 3.5, we obtain that there exists $u_0\in E\setminus\{0\}$ such that $u_n\to u_0$ in E. Hence, $I'(u_0)=0$ and u_0 is a non-trivial solution of (1.1) of mountain pass type. Using the arguments of § 2 it follows that $w_0\doteq E_s(u_0)$ is a non-trivial non-negative weak solution of

$$\begin{aligned}-\operatorname{div}(y^{1-2s}\nabla w)&=0 &&\text{in } \mathbb{R}^{N+1}_+,\\ \partial^s_\nu w+V(x)w&=g(w) &&\text{on } \mathbb{R}^N.\end{aligned}$$

By the strong maximum principle for A_2 weight (see [**21**] or [**9**]) we can see that $w_0(x,y)>0$ for all $(x,y)\in\mathbb{R}^{N+1}_+$, which implies that $u_0>0$ is a positive solution of (1.1). □

Proof of Theorem 1.1. Let $\{u_n\}\subset K$ be a minimizing sequence for $c_{\min}\doteq\inf_{u\in K}I(u)$, i.e. $I(u_n)\to c_{\min}$ and $I'(u_n)=0$. Similar arguments to those in Lemma 3.5 and Lemma 3.6 imply that $\{u_n\}$ is bounded in E and there exists $\bar{u}\in E\setminus\{0\}$ such that $u_n\to\bar{u}$. Hence, $\bar{u}$ is a non-trivial critical point of I with $I(\bar{u})=c_{\min}$.

In what follows, we show that $c_{\min}>0$. In fact, if $u\in K$, then

$$\|u\|^2=\int_{\mathbb{R}^N}g(u)u\,\mathrm{d}x. \tag{3.17}$$

By (g_1)–(g_3), there exist $\delta>0$ and $C_\delta>0$ such that

$$|g(t)|\leqslant(\lambda_1-\delta)|t|+C_\delta|t|^{2^*-1}\quad\forall t\in\mathbb{R}. \tag{3.18}$$

Then, by (3.17), (3.18), Lemmas 2.2 and 3.2, there exist $C_1, C_2 > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x &\leqslant \|u\|^2 \\ &\leqslant (\lambda_1 - \delta)\|u\|_2^2 + C_\delta \|u\|_{2^*}^{2^*} \\ &\leqslant C_1 \int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x \\ &\leqslant C_2 \biggl(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x \biggr)^{N/(N-2s)},\end{aligned}$$

which implies that

$$\inf_{u \in K} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x > 0.$$

By Theorem 2.3 and (V_3) it follows that u satisfies the Pohozaev identity, and hence

$$\begin{aligned}I(u) &= \frac{s}{N} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x - \frac{1}{2N} \int_{\mathbb{R}^N} (\nabla V(x), x) u^2 \,\mathrm{d}x \\ &\geqslant \frac{s}{N} \biggl[\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x - \frac{1}{2s} \|(\nabla V(x), x)^+\|_{N/2s} \|u\|_{2_s^*}^2 \biggr] \\ &\geqslant \frac{s}{N} \biggl[1 - \biggl(1 - \frac{\sigma}{S_s} \biggr) \biggr] \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x \\ &= \frac{s\sigma}{N S_s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x \\ &> 0,\end{aligned}$$

where $\sigma \in (0, S_s)$ is defined as in Lemma 3.6. The proof is complete. □

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