Semi-analytical solution of a McKean–Vlasov equation with feedback through hitting a boundary[†]

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In this paper, we study the nonlinear diffusion equation associated with a particle system where the common drift depends on the rate of absorption of particles at a boundary. We provide an interpretation of this equation, which is also related to the supercooled Stefan problem, as a structural credit risk model with default contagion in a large interconnected banking system. Using the method of heat potentials, we derive a coupled system of Volterra integral equations for the transition density and for the loss through absorption. An approximation by expansion is given for a small interaction parameter. We also present a numerical solution algorithm and conduct computational tests.

Key words: McKean–Vlasov equations, supercooled Stefan problem, Volterra equations, structural credit contagion model, semi-analytic solution

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1 Introduction

In this paper, we derive semi-analytical solutions for the density of interacting particles where the interaction results from shocks to the system when particles hit a boundary. Models of this type have arisen recently as models for the 'integrate and fire' behaviour of neuronal networks and for systemic default risk in networks of interconnected banks. The equation is also closely related to the (supercooled) Stefan problem for heat transfer in the presence of a phase transition.

Structural default models, where a bank's default is triggered by its assets falling below its liabilities, have been studied for decades since the seminal work of Merton [50]. There are several limitations to the basic version of these models: most do not take into account that banks are interconnected, as a result, ignoring the possibility of contagious defaults (but see, e.g., [25, 26]).

To address this, Lipton [45] combined the structural and Eisenberg and Noe [15] framework to consider not only external liabilities but also mutual liabilities.

A further problem is the curse of dimensionality. Numerical and analytical partial differential equation (PDE) techniques are typically applied up to dimension three [32, 33, 35, 36]; for any larger dimension, only Monte Carlo methods are usually considered viable, which are slow to converge and noisy by nature.

When the banking system is large and homogenous, and only macroscopic quantities are of interest, one can consider a large pool approximation of the banks' asset value processes (technically, by taking the limit of their empirical measure for an infinite number of banks). This approach was first studied by Bush *et al.* [7]. Following on, Nadtochiy and Shkolnikov [51] and Hambly *et al.* [23] took into account interaction effects by considering a particle system with positive feedback from the firms' defaults. This leads to McKean–Vlasov type equations, which model a typical representative of the banking system whose dynamics depends on the losses in the wider system. An alternative viewpoint is provided by the Lipton [45] model when taking the number of banks there to infinity. This route leads to the same equation, as shown by our derivation.

Hambly *et al.* [23] assumed zero correlation between banks with linear dependence of the interaction on the loss function, while Hambly and Sojmark [24] and Ledger and Sojmark [42] introduced positive correlation between banks, and Nadtochiy and Shkolnikov [51, 52] considered a nonlinear dependence through the loss function. Very recently, Ichiba *et al.* [31] derived a McKean–Vlasov stochastic differential equation (SDE) in the nonlinear jump-diffusion form for the average bank reserves in an interacting banking system with local and mean-field default intensities.

Earlier, a model similar to the one studied here was found in neuroscience, where a large network of electrically coupled neurons can be described by McKean–Vlasov type equations [8, 9, 11, 12]. If a neuron's potential reaches some fixed threshold, it jumps to a higher potential level and sends a signal to other neurons.

Moreover, a change of coordinates in the PDE for the density leads to the classical supercooled Stefan problem, which describes the freezing of a supercooled (below freezing point) liquid, where our loss function is related to the free boundary separating the phases. The local properties of the PDE solution to the Stefan problem, including finite time blow-up, are analysed in [17, 18, 28, 29]. Dewynne *et al.* [14] study the large time asymptotics of the free boundary. For a survey of early results see also [54].

The detailed behaviour of the solution before and after the blow-up, and the extension past this point, is studied in [20, 27, 38], respectively. A probabilistic representation of a global solution and systematic regularity treatment was only recently established in [13].

McKean–Vlasov type equations were originally suggested by Kac [34] as a stochastic toy model for the Vlasov kinetic equation of plasma, with a detailed study by McKean [49]. In recent years, mean-field problems, and McKean–Vlasov type equations in particular, have become a very popular topic in applied mathematics from both theoretical and practical perspective. Different versions of such problems, apart from the specific form in the papers above, have been applied to mathematical finance, e.g., in portfolio optimisation (Borkar and Suresh Kumar [3] consider optimal allocation into sectors for a large number of stocks) and in game theory (e.g., Huang *et al.* [30] discuss an agent's optimal behaviour with respect to a mass effect).

There are established simulation methods for typical McKean–Vlasov equations (see [2, 4]), and more recently, several authors have analysed multilevel and multi-index schemes (see [22, 57, 59]) and importance sampling [56].

However, none of these methods cover the models described above due to the singular, path-dependent nature of the feedback. Here, we consider the Hambly *et al*. [23] version for simplicity. For this model, Kaushansky and Reisinger [37] proposed an Euler-type particle method and proved convergence with order $1/2$ in the timestep for a sufficiently small time interval in the regular regime, which can be improved to 1 using Brownian bridges. In this paper, we show how to solve these equations by reformulation first as a nonlinear free boundary problem similar to the classical Stefan problem and then as a system of two coupled Volterra equations.

First, for *given* drift term from the mean-field interaction, the problem is formulated as diffusion problem on a semi-infinite domain with curvilinear boundary and its solution represented semi-analytically by the method of heat potentials. A detailed introduction to heat potentials can be found in Refs. [61] and [60] (pp. 530–535). The first use of the method of heat potentials in mathematical finance is found in Ref. [44] for pricing path-dependent options with curvilinear barrier (Section 12.2.3, pp. 462–467).

Second, expressing the interaction term by the solution from the first step results in two coupled Volterra equations. For early applications of heat potentials to versions of the Stefan problem see already, e.g., [58, Part II, Chapter 1]. These singular Volterra equations are then solved by either an expansion method for small interaction parameter or numerically by discretisation and Newton–Raphson iteration.

A direct PDE expansion to the Stefan problem for both small and large nonlinearity parameter was derived in [1]. More recently, an expansion for a certain McKean–Vlasov equation with mean-field interaction through the drift was studied in [19], who perform an iterative two-step procedure which decouples the nonlinearity arising from the dependence of the drift on the law of the process from the standard dependence on the state variables. The present paper differs not only in the solution approach taken but also fundamentally in the considered mean-field interaction (through hitting times rather than the expectation of the drift) and the parameter of the expansion (for small drift interaction rather than small volatility).

To assess the accuracy of the (first-order) perturbation solution and to illustrate the behaviour for strong interactions, where the expansion breaks down, we describe a simple numerical algorithm, but refer the reader to the large and well-established body of literature on more advanced numerical methods for Volterra equations (see Section 5.2).

The rest of the paper is organised as follows: in Section 2, we provide an alternative derivation of the model described in [23] by taking the limiting case for infinitely many firms in the model of Lipton [45], precisely, for the approximation that a small fraction of banks has defaulted, we show that the mean-field limit of the system gives the same McKean–Vlasov equation; in Section 4, we derive a solution for the first passage density of Brownian motion over a curve in terms of a Volterra equation, using the method of heat potentials, and thus obtain the interaction term in the original McKean–Vlasov equation; in Section 5, we consider a perturbation method and a numerical method for the corresponding system of Volterra equations; in Section 6, we show numerical illustrations and compare the methods; in Section 7, we conclude.

2 Mean-field limit for large banking system

Following Lipton [45], we consider a system of N banks with external as well as mutual assets and liabilities. We denote it by L_i the external liabilities for bank i and by L_{ij} the liability from bank i to bank j.

Assume that the dynamics of bank i's total external assets is governed by

$$\frac{dA_t^i}{A_t^i} = \mu\, dt + \sigma_i\, dW_t^i,$$

where μ is the growth rate, W^i is independent standard Brownian motions for $1 \leq i \leq n$ and the liabilities, both external L_i and mutual L_{ij}, are constant.

Bank i is assumed to default when its assets fall below a certain threshold determined by its liabilities, namely at time $\tau_i = \inf\{t : A_t^i \leq \Lambda_t^i\}$, where Λ^i is a default boundary which we now work out. At time $t = 0$, following Section 6 of [45],

$$\Lambda_0^i = R_i \left(L_i + \sum_{j \neq i} L_{ij} \right) - \sum_{j \neq i} L_{ji},$$

where $R_i \in (0, 1)$ is the recovery rate of bank i, i.e., bank i defaults if the recovery value of its liabilities, external and to other banks, exceeds the sum of its assets, external and from other banks.

Since liabilities and recovery rates are assumed constant in time, the default boundary remains constant until some bank defaults. If bank k defaults at time t, bank i has to pay all its mutual liabilities L_{ik}, while k returns only a portion of liabilities $R_k L_{ki}$. This can be viewed as a jump in external liabilities from $L_i \to L_i + L_{ik} - R_k L_{ki}$. Hence, the default boundary of bank i becomes

$$\Lambda_t^i = R_i \left([L_i + L_{ik} - R_k L_{ki}] + \sum_{j \neq i,k} L_{ij} \right) - \sum_{j \neq i,k} L_{ji}.$$

As a result,

$$\Delta \Lambda_t^i = (1 - R_i R_k) L_{ki}.$$

In the following, we assume that the banks have the same parameters, i.e., $\sigma_i = \sigma$ and $R_i = R$, for some μ, σ, R, and $L_i = L$ and $L_{ij} = \frac{\gamma}{N}$, both constant, respectively, for some $L, \gamma > 0$. In particular, this implies that the asset value processes are exchangeable, and we have $\Lambda_0^i = \Lambda_0$ for some Λ_0, which will allow us to take a large pool limit.

These assumptions might seem very strong, since the banking system is highly heterogenous. However, the model is a meaningful approximation to large pools of small banks, and the aim is to demonstrate how even the default of small banks can lead to systemic events. A possible extension in the presence of larger, systemically important banks might be to consider several clusters, where banks are similar within a cluster.

Under the assumptions above, we can write Λ_t^i as

$$\Lambda_t^i = \Lambda_0^i + \frac{\gamma}{N} \sum_{k \neq i} (1 - R^2) \mathbb{1}_{\{\tau_k \leq t\}}.$$

It is more convenient to introduce the distance to default $Y_t^i = \log(A_t^i / \Lambda_t^i)/\sigma$, then

$$Y_t^i = Y_0^i + \frac{1}{\sigma}(\mu - \sigma^2/2)t + W_t^i - \frac{1}{\sigma} \log\left(1 + \frac{\gamma}{N} \sum_{k \neq i}(1 - R^2)\frac{1}{\Lambda_0} \mathbb{1}_{\{\tau_k \leq t\}}\right).$$
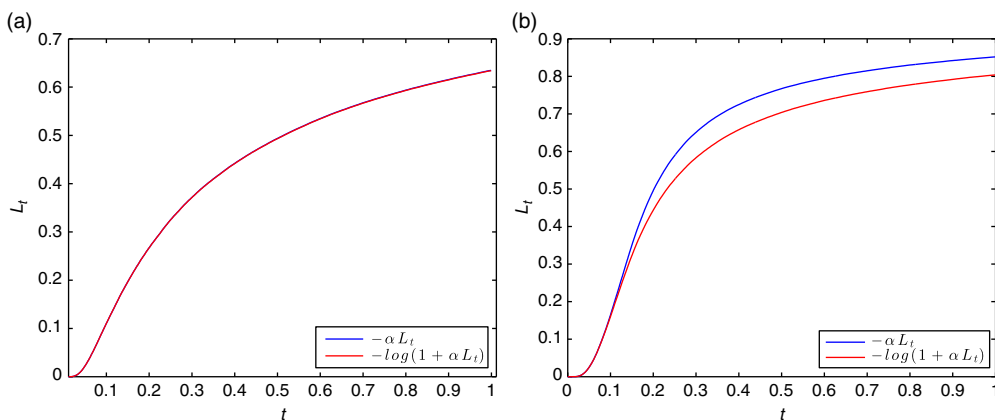
FIGURE 1. Comparison of L_t computed for $Y_t = Y_0 + W_t - \alpha L_t$ and $Y_0 + W_t - \log(1 + \alpha L_t)$, with $z = 0.5$: (a) $\alpha = 0.1$, (b) $\alpha = 0.7$.

Using the approximation $\log(1 + x) \approx x$ for small x (i.e., assuming only a small proportion of the banks have yet defaulted), we get for $t < \tau_i$

$$Y_t^i = Y_t^0 + \frac{1}{\sigma}(\mu - \sigma^2/2)t + W_t^i - \frac{\gamma(1 - R^2)}{\sigma \Lambda_0}L_t^N,$$

$$L_t^N = \frac{1}{N}\sum_k \mathbb{1}_{\{\tau_k \leq t\}}.$$

For simplicity, we take the special case $\mu - \sigma^2/2 = 0$ (in any case, this term will be small for realistic parameter values and not have any qualitative impact on the results).

Then, using propagation of chaos as in [12], one can obtain that in the limit for $N \to \infty$, all Y_t^i have the same distribution as Y given by[1]

$$Y_t = Y_0 + W_t - \alpha L_t,$$
$$L_t = \mathbb{P}(\tau \leq t), \tag{2.1}$$
$$\tau = \inf\{t \in [0, T] : Y_t \leq 0\},$$

where $\alpha = \frac{\gamma(1-R^2)}{\sigma \Lambda_0}$.

To analyse the effect of the approximation $\log(1 + x) \approx x$ above, we compute L_t using particle system simulations (as in [37]) for both αL_t and $-\log(1 + \alpha L_t)$. We deduce from Figure 1 that for small α, the results are almost indistinguishable, while for larger α, the losses accumulate slightly faster for the linearised version, as is to be expected from the expressions. Note also that the effect would be largely compensated for if we used this model to calibrate α to credit market data. As the linear loss term coincides with the model in [23], we will use that model subsequently.

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[1]Note the slight ambiguity between the liabilities above and the loss function below, which are both denoted by L. It should be clear from the context and indices applied which one is being referred to, hence for consistency with the literature we keep the notation.

The derivation above from first principles allows us to estimate economically meaningful values of α; see also [6] for the estimation of the initial values Y_0^i from credit spreads. According to David and Lehar [10], on average, the fraction of interbank liabilities in comparison to total liabilities is 12% for the EU, 8% for Canada and, as per Economic Research website of the Federal Reserve Bank of St. Louis, 4.5% for the USA. Consider, for example, the EU area. In our notation, $\sum_{j \neq i} L_{ij} \approx \gamma \approx \frac{0.12}{1-0.12}L \approx 0.14L$, where $L_i = L$ is the external liabilities for each bank, assumed identical. We can write α as

$$\alpha \approx \frac{1}{\sigma} \frac{(1-R^2)\gamma}{RL-(1-R)\gamma} \approx \frac{1}{\sigma} \frac{(1-R^2)0.14}{R-(1-R)0.14}.$$

The typical volatility of assets varies from 1% to 8%, which can be confirmed, for example, by calibration of the one-dimensional Lipton and Sepp [46] model. Even for a conservative case, when the recovery rate is close to 1, we get a significant value of α. To be precise, for $R = 0.9$ and $\sigma = 0.08$, we get $\alpha \approx 0.3$. On the other hand, for typical recovery rates of $R \approx 0.4$ and volatility at the lower end, one can easily get $\alpha > 5$.

We will illustrate the impact of α on the loss process in the next section.

3 Transition density and known regularity results

Hambly *et al*. [23] and Kaushansky and Reisinger [37] considered Y_0 as a random variable from a given distribution with known density. For simplicity, we assume in some of our derivations $Y_0 = z$ a.s. for some $z > 0$, by taking A_0^i and Λ_0^i the same for all i, but the results can be extended by making A_0^i random and drawn from the same distribution, as outlined in Section 4.4.

Writing the increasing process L as $\alpha L_t = -\int_0^t \mu(t')\,dt'$ for some negative μ, $p(t,x)$, the Radon–Nikodym density of the distribution with respect to Lebesgue measure of the stopped process $Y_{t \wedge \tau}$ satisfies the Kolmogorov forward equation

$$
\begin{aligned}
&p_t(t,x;z) = -\mu(t)p_x(t,x;z) + \frac{1}{2}p_{xx}(t,x;z),\\
&p(0,x;z) = \delta(x-z),\\
&p(t,0;z) = 0,
\end{aligned}
\tag{3.1}
$$

where the subscripts t and x denote partial derivatives. Given the continuous differentiability assumption of L, we can consider classical solutions of (3.1).

Using the relation $L_t = 1 - \int_0^\infty p(t,x;z)\,dx$, we can express μ in terms of p by

$$g(t;z) := \frac{dL_t}{dt} = -\int_0^\infty p_t\,dx = \mu(t)\int_0^\infty p_x\,dx - \frac{1}{2}\int_0^\infty p_{xx}\,dx = \frac{1}{2}p_x(t,0;z), \tag{3.2}$$

where we have used the PDE (3.1) as well as $p(t,0;z) = 0$ and $\lim_{x \to \infty} p(t,x;z) = \lim_{x \to \infty} p_x(t,x;z) = 0$. Hence, (3.1) can be written in the self-consistent form

$$
\begin{aligned}
&p_t(t,x;z) = \frac{\alpha}{2}p_x(t,0;z)p_x(t,x;z) + \frac{1}{2}p_{xx}(t,x;z),\\
&p(0,x;z) = \delta(x-z),\\
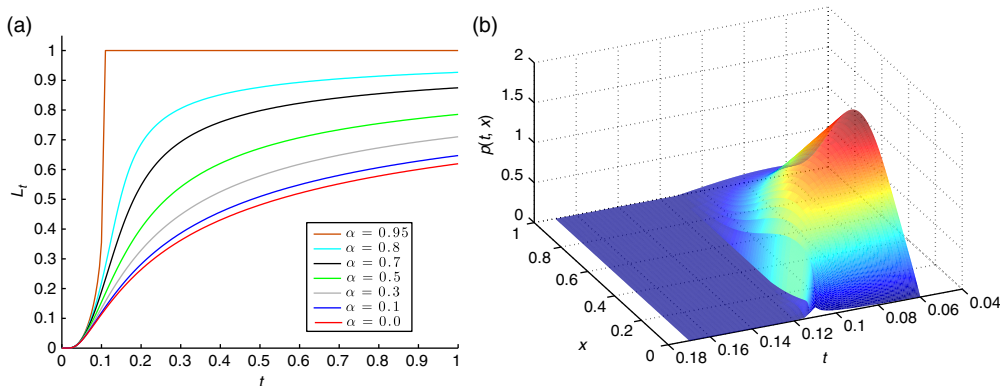&p(t,0;z) = 0.
\end{aligned}
\tag{3.3}
$$

FIGURE 2. (a) L_t for different values of α and $z = 0.5$. (b) The density $p(t, x)$ for $\alpha = 0.95$.

From the second equation in (2.1), g is also the density of the first passage time of Y. Noting a result from [55] for the first-passage problem of Brownian motion, applied to $Y_t - L_t$, Hambly *et al.* [23] give the following Volterra equation for L:

$$\Phi\left(-\frac{z - \alpha L_t}{\sqrt{t}}\right) = \int_0^t \Phi\left(\alpha\frac{L_t - L_s}{\sqrt{t - s}}\right) \, dL_s,$$

where $\Phi(x)$ is the cumulative density function (CDF) of the standard normal distribution.

In contrast to this equation, we will derive a coupled system of Volterra equations which give both p and g (i.e., not the cumulative distribution). These are in a more standard form without the nonlinearity Φ on the left-hand side and integration over L on the right-hand side, and hence numerically more amenable.

In Figure 2(a), we plot the loss function $t \to L_t$, computed by the methods described in the remainder of this paper, for different values of α, where α measures the interconnectivity of the banking system. The losses increase dramatically because of interbank liabilities, which may even lead to systemic events, here for α larger than around 0.9. Hereby, the rate of losses increases to infinity, as seen in Figure 2(b) from the large gradient $p_x(t, 0)$ for t immediately before the blow-up, and then triggers a jump in L_t. [51] first gives a rigorous mathematical characterisation of this type of behaviour in their model, and Theorem 1.1 in [23] shows the necessity of such "blow-ups" for large enough α, depending on the initial distribution (see also [18] for this property phrased in terms of the Stefan problem), while Theorem 1.1 in [13] gives a detailed characterisation of the regularity of the solution and conditions for jumps. We note from there that the derivative of L_t' may explode without necessarily triggering a jump, and instead the PDE solution immediately returns to the smooth case.

In this paper, we consider the solution in an initial interval where the loss function is differentiable. It is well known that if $\inf\{x \geq 0 : m(0, \alpha x) < x\} = 0$, m the initial measure, a unique classical solution (with continuously differentiable L_t) exists for small times under mild regularity assumptions on the initial density (see [16]).

In the rest of the paper, we will often take the initial distribution as a delta function centred at some point z and assume that continuous differentiability of L still holds (possibly without a control on the derivative) in this case. Then, μ is continuous and the solutions to (3.1) are classical, so that the derivations of the integral equations will be justified.

Lastly, a comparison of the data for the loss function with $\alpha = 0.1$ and $\alpha = 0.7$ in Figure 2(a), computed by numerical solution (see Section 5.2) of the integral equations derived by the techniques in Section 5, with the data in Figure 1 (blue curves) for the same values of α and computed by the particle method in [37] applied directly to the McKean–Vlasov SDE, shows that they differ only by numerical errors. This provides an empirical verification of the correctness of the derivations.

4 The method of heat potentials

In this section, we compute the transition density and the first passage density of Brownian motion with a known time-dependent drift $\mu(t)$ on the positive semi-axis. The transition probability density $p(t, x, z)$ satisfies the Kolmogorov forward equation (3.1).

We first derive an expression for $p(t, x; z)$ using the method of heat potentials with an unknown weight function which can be found as a solution of a Volterra equation of the second kind. Next, we differentiate the expression for $p(t, x; z)$ with respect to x and take its limit to 0 in order to find the first passage density of Y in (2.1), or, equivalently, g in (3.2). This limit is less well known, so we calculate it for completeness.

Below we omit z when possible.

4.1 Semi-closed formula for the transition density

Consider the case of a continuous and bounded drift $\mu(t)$ on $[0, T^*]$.

Let $M(t) = \int_0^t \mu(t')\, dt'$, which is a differentiable function on $[0, T^*]$. The change of variables $\tilde{p}(t, y) = p(t, y - M(t))$ yields the following Cauchy–Dirichlet problem:

$$
\begin{aligned}
&\tilde{p}_t(t, y) = \frac{1}{2}\tilde{p}_{yy}(t, y), \qquad y > -M(t),\ t > 0, \\
&\tilde{p}(0, y) = \delta(y - z), \\
&\tilde{p}(t, -M(t)) = 0.
\end{aligned}
\tag{4.1}
$$

With the boundary condition $\tilde{p}_y(t, -M(t)) = -2/\alpha\,(\mathrm{d}M/\mathrm{d}t)(t)$, this is the supercooled Stefan problem (see, e.g., [13, 18]).

In the following, we omit tilde for brevity. We split p from the last equation into

$$
p(t, y) = q(t, y) + u(t, y),
$$

where u is a solution of

$$
\begin{aligned}
&u_t = \frac{1}{2}u_{yy}, \qquad y \in \mathbb{R},\ t > 0, \\
&u(0, y) = \delta(x - z).
\end{aligned}
\tag{4.2}
$$

The solution of this equation is clearly the heat kernel

$$
u(t, y) = \frac{\exp\left(-\frac{(y-z)^2}{2t}\right)}{\sqrt{2\pi t}}.
$$

The corresponding problem for q has the form

$$q_t(t, y) = \frac{1}{2} q_{yy}(t, y),$$

$$q(0, y) = 0,$$

$$q(t, -M(t)) = -u(t, M(t)).$$

We use the method of heat potentials (see [44, pp. 462–468]). Thereby, we represent q in the form

$$q(t, y) = \int_0^t \frac{(y + M(t')) \exp\left(-\frac{(y+M(t'))^2}{2(t-t')}\right)}{\sqrt{2\pi (t-t')^3}} v(t') \, dt',$$

where v is a suitable weight which will be determined to match the boundary condition. Assuming that $v(t)$ is known, we can revert to the original variables and get

$$p(t, x) = \int_0^t \frac{(x - \Psi(t, t')) \exp\left(-\frac{(x-\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi (t-t')^3}} v(t') \, dt' + \frac{\exp\left(-\frac{(x-M(t)-z)^2}{2t}\right)}{\sqrt{2\pi t}}, \tag{4.3}$$

where $\Psi(t, t') = M(t) - M(t')$.

By construction, p in (4.3) satisfies the first two equations in (4.1). We also need to satisfy the boundary condition in (4.1). It is easy to show (see Appendix A) that

$$\mathbb{L}_1 := \lim_{x \to 0} \int_0^t \frac{(x - \Psi(t, t')) \exp\left(-\frac{(x-\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi (t-t')^3}} v(t') \, dt'$$

$$= v(t) - \int_0^t \frac{\Psi(t, t') \, \Xi(t, t')}{\sqrt{2\pi (t-t')^3}} v(t') \, dt'.$$

The requirement $\lim_{x \to 0} p(t, x) = 0$ thus leads to the following Volterra integral equation of the second kind for v,

$$v(t) - \int_0^t \frac{\Psi(t, t') \, \Xi(t, t')}{\sqrt{2\pi (t-t')^3}} v(t') \, dt' + \frac{\exp\left(-\frac{(M(t)+z)^2}{2t}\right)}{\sqrt{2\pi t}} = 0, \tag{4.4}$$

where

$$\Xi(t, t') = \exp\left(-\frac{\Psi(t, t')^2}{2(t-t')}\right), \quad t \neq t',$$

$$\Xi(t, t) = 1.$$

4.2 Computation of loss rate over boundary

In this section, we compute $g(t)$ from (3.2), suppressing z, by first differentiating (4.3),

$$p_x(t, x) = \int_0^t \left(1 - \frac{(x - \Psi(t, t'))^2}{(t - t')}\right) \frac{\exp\left(-\frac{(x - \Psi(t, t'))^2}{2(t - t')}\right)}{\sqrt{2\pi (t - t')^3}} v(t') \, dt'$$

$$- (x - M(t) - z) \frac{\exp\left(-\frac{(x - M(t) - z)^2}{2t}\right)}{\sqrt{2\pi t^3}}.$$

In Appendix A, we show that

$$\mathbb{L}_2 := \lim_{x \to 0} \int_0^t \left(1 - \frac{(x - \Psi(t, t'))^2}{(t - t')}\right) \frac{\exp\left(-\frac{(x - \Psi(t, t'))^2}{2(t - t')}\right)}{\sqrt{2\pi (t - t')^3}} v(t') \, dt'$$

$$= 2\left(M'(t) - \frac{1}{\sqrt{2\pi t}}\right) v(t) + \int_0^t \frac{\left(\left(1 - \frac{\Psi(t, t')^2}{(t - t')}\right) \Xi(t, t') v(t') - v(t)\right)}{\sqrt{2\pi (t - t')^3}} \, dt.$$

Accordingly,

$$g(t) = \left(M'(t) - \frac{1}{\sqrt{2\pi t}}\right) v(t)$$

$$+ \frac{1}{2} \int_0^t \frac{\left(\left(1 - \frac{\Psi(t, t')^2}{(t - t')}\right) \Xi(t, t') v(t') - v(t)\right)}{\sqrt{2\pi (t - t')^3}} \, dt' + \frac{(M(t) + z) \exp\left(-\frac{(M(t) + z)^2}{2t}\right)}{2\sqrt{2\pi t^3}}. \tag{4.5}$$

4.3 Direct computation of loss rate

Alternatively, we can represent $g(t)$ using (3.2) by

$$g(t) = -\frac{d}{dt} \int_0^\infty p(t, x) \, dx,$$

so that

$$g(t) = -\frac{d}{dt} \int_0^t \left(\int_0^\infty \frac{(x - \Psi(t, t')) \exp\left(-\frac{(x - \Psi(t, t'))^2}{2(t - t')}\right)}{\sqrt{2\pi (t - t')^3}} \, dx\right) v(t') \, dt'$$

$$- \frac{d}{dt} \int_0^\infty \frac{\exp\left(-\frac{(x - M(t) - z)^2}{2t}\right)}{\sqrt{2\pi t}} \, dx$$

$$= -\frac{d}{dt} \int_0^t \left(\int_{-\Psi(t,t')}^\infty \frac{\xi \exp\left(-\frac{\xi^2}{2(t-t')}\right)}{\sqrt{2\pi (t-t')^3}} \, d\xi \right) v\left(t'\right) dt' - \frac{d}{dt} \int_{-\frac{(M(t)+z)}{\sqrt{t}}}^\infty \frac{\exp\left(-\frac{\xi^2}{2}\right)}{\sqrt{2\pi}} \, d\xi$$

$$= -\frac{d}{dt} \int_0^t \frac{\Xi\left(t,t'\right)}{\sqrt{2\pi (t-t')}} v\left(t'\right) dt' - \left(M'\left(t\right) - \frac{(M(t)+z)}{2t} \right) \frac{\exp\left(-\frac{(M(t)+z)^2}{2t}\right)}{\sqrt{2\pi t}}.$$

As an aside, we can verify easily by direct computation that the two expressions for g agree. We apply the following lemma, which will also be useful in Section 5.1.

Lemma 4.1 *Consider a differentiable function* $\Xi(t,t')$ *such that* $\Xi(t,t) = 1$. *Then,*

$$\frac{d}{dt} \int_0^t \frac{\Xi\left(t,t'\right) v\left(t'\right)}{\sqrt{2\pi (t-t')}} \, dt' = \frac{v\left(t\right)}{\sqrt{2\pi t}} + \frac{1}{2} \int_0^t \frac{v\left(t\right) - \left(\Xi\left(t,t'\right) - 2\left(t-t'\right) \Xi_t\left(t,t'\right)\right) v\left(t'\right)}{\sqrt{2\pi (t-t')^3}} \, dt'$$

$$= \int_0^t \frac{\frac{\partial}{\partial t'}\left(\left(\Xi\left(t,t'\right) - 2\left(t-t'\right) \Xi_t\left(t,t'\right)\right) v\left(t'\right)\right)}{\sqrt{2\pi (t-t')}} \, dt'.$$

Proof See Appendix C. □

We use (4.4) and rewrite the second term in the form

$$M'\left(t\right) \frac{\exp\left(-\frac{(M(t)+z)^2}{2t}\right)}{\sqrt{2\pi t}} = -M'\left(t\right) v\left(t\right) + M'\left(t\right) \int_0^t \frac{\Psi\left(t,t'\right) \Xi\left(t,t'\right)}{\sqrt{2\pi (t-t')^3}} v\left(t'\right) dt'.$$

Using the first equality in the lemma, we arrive at the following expression

$$g\left(t\right) = \left(M'\left(t\right) - \frac{1}{\sqrt{2\pi t}} \right) v\left(t\right)$$

$$- \frac{1}{2} \int_0^t \frac{v\left(t\right) - \left(\Xi\left(t,t'\right) - 2\left(M'\left(t\right) \Psi\left(t,t'\right) \Xi\left(t,t'\right) + \left(t-t'\right) \Xi_t\left(t,t'\right)\right)\right) v\left(t'\right)}{\sqrt{2\pi (t-t')^3}} \, dt'$$

$$+ \frac{(M\left(t\right)+z) \exp\left(-\frac{(M(t)+z)^2}{2t}\right)}{2\sqrt{2\pi t^3}}.$$

We notice that

$$\Xi_t\left(t,t'\right) = \left(-\frac{M'\left(t\right) \Psi\left(t,t'\right)}{(t-t')} + \frac{\Psi\left(t,t'\right)^2}{2\left(t-t'\right)^2} \right) \Xi\left(t,t'\right),$$

so that

$$\Xi\left(t,t'\right) - 2\left(M'\left(t\right) \Psi\left(t,t'\right) \Xi\left(t,t'\right) + \left(t-t'\right) \Xi_t\left(t,t'\right)\right) = \left(1 - \frac{\Psi\left(t,t'\right)^2}{(t-t')} \right) \Xi\left(t,t'\right),$$

from which (4.5) follows.

4.4 General initial data

We now consider the problem with general initial condition $p(0, x) = f(x)$ in (3.1),

$$p_t(t, x) = \frac{\alpha}{2} p_x(t, 0) p_x(t, x) + \frac{1}{2} p_{xx}(t, x),$$

$$p(0, x) = f(x),$$

$$p(t, x) = 0.$$

Recalling from the earlier definitions

$$M(t) = -\alpha \int_0^t g(t')\, dt', \quad \Psi(t, t') = M(t) - M(t'), \quad \Omega(t, t') = \int_{t'}^t g(t'')\, dt'', \tag{4.6}$$

we can represent an expression for $p(t, x)$, and equations for $v(t)$ and $g(t) = \frac{1}{2} p_x(t, 0)$, by solving (4.2) with initial datum f instead of δ by way of convolution. Instead of (4.3)–(4.5), we thus obtain

$$p(t, x) = \int_0^t \frac{(x - \Psi(t, t')) \exp\left(-\frac{(x - \alpha\Omega(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t - t')^3}} v(t')\, dt'$$

$$+ \int_0^\infty \frac{\exp\left(-\frac{(x + \alpha\Omega(t,0) - z')^2}{2t}\right)}{\sqrt{2\pi t}} f(z')\, dz',$$

$$v(t) + \int_0^t \frac{\alpha\Omega(t, t') \exp\left(-\frac{\alpha^2\Omega(t,t')^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t - t')^3}} v(t')\, dt' + \int_0^\infty \frac{\exp\left(-\frac{(\alpha\Omega(t,0) - z')^2}{2t}\right)}{\sqrt{2\pi t}} f(z')\, dz' = 0,$$

$$g(t) + \left(\alpha g(t) + \frac{1}{\sqrt{2\pi t}}\right) v(t) + \int_0^\infty \frac{(\alpha\Omega(t, 0) - z') \exp\left(-\frac{(\alpha\Omega(t,0) - z')^2}{2t}\right)}{2\sqrt{2\pi t^3}} f(z')\, dz$$

$$+ \frac{1}{2} \int_0^t \frac{\left(v(t) - \left(1 - \frac{\alpha^2\Omega(t,t')^2}{(t-t')}\right) \exp\left(-\frac{\alpha^2\Omega(t,t')^2}{2(t-t')}\right) v(t')\right)}{\sqrt{2\pi\,(t - t')^3}}\, dt' = 0.$$

Due to the nonlinearity, the solution for different f cannot simply be found by convolution, but a different system of equations for v and g must be solved for each f.

As the convolution terms involving f are non-singular, they would not add substantial numerical difficulties for this Volterra equation. For simplicity, we focus on $f(x) = \delta(x - z)$ in the sequel.

5 Solution of the McKean–Vlasov equation

Now, for the McKean–Vlasov equation (3.3), we obtain from (4.3) to (4.5) the following.

Proposition 5.1 *If the loss function L is continuously differentiable on* $[0, t]$*, then*

$$p(t, x) = \int_0^t \frac{(x - \Psi(t, t')) \exp\left(-\frac{(x - \Psi(t, t'))^2}{2(t - t')}\right)}{\sqrt{2\pi} \, (t - t')^3} \nu(t') \, dt' + \frac{\exp\left(-\frac{(x - M(t) - z)^2}{2t},\right)}{\sqrt{2\pi t}},$$

where ν and g satisfy the system of integral equations

$$
\begin{cases}
\nu(t) + \displaystyle\int_0^t \frac{\alpha\Omega(t, t') \exp\left(-\frac{\alpha^2 \Omega(t, t')^2}{2(t - t')}\right)}{\sqrt{2\pi} \, (t - t')^3} \nu(t') \, dt' + \frac{\exp\left(-\frac{(\alpha\Omega(t, 0) - z)^2}{2t}\right)}{\sqrt{2\pi t}} = 0, \\[3mm]
g(t) + \left(\alpha g(t) + \dfrac{1}{\sqrt{2\pi t}}\right) \nu(t) + \dfrac{(\alpha\Omega(t, 0) - z) \exp\left(-\frac{(\alpha\Omega(t, 0) - z)^2}{2t}\right)}{2\sqrt{2\pi t^3}} \\[3mm]
\quad + \dfrac{1}{2} \displaystyle\int_0^t \frac{\left(\nu(t) - \left(1 - \frac{\alpha^2 \Omega(t, t')^2}{(t - t')}\right) \exp\left(-\frac{\alpha^2 \Omega(t, t')^2}{2(t - t')}\right) \nu(t')\right)}{\sqrt{2\pi} \, (t - t')^3} \, dt' = 0,
\end{cases}
\tag{5.1}
$$

where M, Ψ and Ω are given in (4.6). *Moreover, $L_t = -M(t)/\alpha$.*

In Appendix B, we give the explicit solution for special cases, in particular when there is no feedback, $\alpha = 0$, $M(t) = 0$. In general, only approximations to the solution can be found. We give an asymptotic and a numerical approach in the remainder of this section.

5.1 Perturbation solution

We expand the solution of (5.1) formally in powers of α:

$$(\nu(t), g(t)) = (\nu_0(t), g_0(t)) + \alpha(\nu_1(t), g_1(t)) + \alpha^2(\nu_2(t), g_2(t)) + \cdots, \tag{5.2}$$

which we will truncate after the first two terms. This will give us an analytical expression which can be expected to be a good approximation for small values of α.

By plugging the formal expansion for (ν, g) into (5.1) and collecting the terms of the same order in α, we get the following systems for $(\nu_0(t), g_0(t))$ and $(\nu_1(t), g_1(t))$:

$$
\begin{cases}
\nu_0(t) + \dfrac{\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t}} = 0, \\[4mm]
g_0(t) + \dfrac{1}{\sqrt{2\pi t}} \nu_0(t) + \dfrac{1}{2} \displaystyle\int_0^t \frac{(\nu_0(t) - \nu_0(t'))}{\sqrt{2\pi} \, (t - t')^3} \, dt' - \dfrac{z \exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}} = 0.
\end{cases}
$$

$$\begin{cases} v_1\left(t\right) + \int_0^t \dfrac{\Omega_0\left(t, t'\right)}{\sqrt{2\pi\left(t - t'\right)^3}} v_0\left(t'\right) dt' + \dfrac{z\Omega_0\left(t, 0\right)\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}} = 0, \\[4mm] g_1\left(t\right) + g_0\left(t\right)v_0\left(t\right) + \dfrac{1}{\sqrt{2\pi t}}v_1\left(t\right) + \dfrac{1}{2}\int_0^t \dfrac{\left(v_1\left(t\right) - v_1\left(t'\right)\right)}{\sqrt{2\pi\left(t - t'\right)^3}}\,dt' \\[4mm] \qquad + \left(1 - \dfrac{z^2}{t}\right)\dfrac{\Omega_0\left(t, 0\right)\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}} = 0, \end{cases}$$

where $\Omega_0\left(t, t'\right) = \int_{t'}^t g_0\left(t''\right) dt''$. The equations for g_0 and g_1 can be written as

$$g_0\left(t\right) + \int_0^t \dfrac{\dot{v}_0\left(t'\right)}{\sqrt{2\pi\left(t - t'\right)^3}}\,dt' - \dfrac{z\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}} = 0,$$

$$g_1\left(t\right) + g_0\left(t\right)v_0\left(t\right) + \int_0^t \dfrac{\dot{v}_1\left(t'\right)}{\sqrt{2\pi\left(t - t'\right)}}\,dt' + \left(1 - \dfrac{z^2}{t}\right)\dfrac{\Omega_0\left(t, 0\right)\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}} = 0.$$

Thus, using the results in Appendix B for $\alpha = 0$,

$$v_0\left(t\right) = -\dfrac{\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t}}, \qquad g_0\left(t\right) = \dfrac{z\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}},$$

$$\Omega_0\left(t, t'\right) = 2\left(\Phi\left(\dfrac{z}{\sqrt{t'}}\right) - \Phi\left(\dfrac{z}{\sqrt{t}}\right)\right), \tag{5.3}$$

$$\dfrac{\partial}{\partial t}\Omega_0\left(t, t'\right) = \dfrac{z\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}} = g_0(t), \qquad \dfrac{\partial}{\partial t'}\Omega_0\left(t, t'\right) = -g_0(t').$$

Next,

$$v_1\left(t\right) = -\int_0^t \dfrac{\Omega_0\left(t, t'\right)}{\sqrt{2\pi\left(t - t'\right)^3}} v_0\left(t'\right) dt' - \dfrac{z\Omega_0\left(t, 0\right)\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}},$$

$$= 2\int_0^t \dfrac{\left(\Phi\left(\frac{z}{\sqrt{t}}\right) - \Phi\left(\frac{z}{\sqrt{t'}}\right)\right)}{\sqrt{2\pi\left(t - t'\right)^3}}\dfrac{\exp\left(-\frac{z^2}{2t'}\right)}{\sqrt{2\pi t'}}\,dt' + \dfrac{2z\left(\Phi\left(\frac{z}{\sqrt{t}}\right) - 1\right)\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}}, \tag{5.4}$$

$$g_1\left(t\right) = -g_0\left(t\right)v_0\left(t\right) - \int_0^t \dfrac{\dot{v}_1\left(t'\right)}{\sqrt{2\pi\left(t - t'\right)}}\,dt' - \left(1 - \dfrac{z^2}{t}\right)\dfrac{\Omega_0\left(t, 0\right)\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}},$$

where

$$\Omega_0\left(t, 0\right) = 2\left(1 - \Phi\left(\dfrac{z}{\sqrt{t}}\right)\right).$$

We can write $v_1\left(t\right)$ in the form

$$v_1\left(t\right) = -\int_0^t \dfrac{\omega_0\left(t, t'\right)}{\sqrt{2\pi\left(t - t'\right)}} g_0\left(t'\right)v_0\left(t'\right) dt' - \dfrac{z\Omega_0\left(t, 0\right)\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}},$$

where

$$\omega_0\left(t,t'\right) = \frac{\Omega_0\left(t,t'\right)}{g_0\left(t'\right)\left(t-t'\right)} = -\frac{2\left(\Phi\left(\frac{z}{\sqrt{t}}\right)-\Phi\left(\frac{z}{\sqrt{t'}}\right)\right)\sqrt{2\pi t'^3}\exp\left(\frac{z^2}{2t'}\right)}{z\left(t-t'\right)}, \quad t\neq t',$$

$$\omega_0\left(t,t\right) = 1,$$

and obtain an expression for $\dot{v}_1$:

$$\dot{v}_1\left(t\right) = -\frac{d}{dt}\int_0^t \frac{\omega_0\left(t,t'\right)}{\sqrt{2\pi\left(t-t'\right)}}g_0(t')v_0\left(t'\right)dt' - \frac{d}{dt}\left(\frac{z\Omega_0\left(t,0\right)\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}}\right). \tag{5.5}$$

To compute the first term of (5.5), we use the second equality in Lemma 4.1 with $v(t)=v_0(t)g_0(t)$ and $\Xi(t,t')=\omega_0(t,t')$, to get

$$\dot{v}_1\left(t\right) = -\int_0^t \frac{\left(\left(\omega_0\left(t,t'\right)-2\left(t-t'\right)\frac{\partial}{\partial t}\omega_0\left(t,t'\right)\right)g_0\left(t'\right)v_0\left(t'\right)\right)_{t'}}{\sqrt{2\pi\left(t-t'\right)}}dt'$$

$$-\frac{d}{dt}\left(\frac{z\Omega_0\left(t,0\right)\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}}\right),$$

$$= -\int_0^t \frac{\left(\left(\frac{3\Omega_0(t,t')}{(t-t')}-2\frac{\partial}{\partial t}\Omega_0\left(t,t'\right)\right)v_0\left(t'\right)\right)_{t'}}{\sqrt{2\pi\left(t-t'\right)}}dt' - \frac{d}{dt}\left(\frac{z\Omega_0\left(t,0\right)\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}}\right), \tag{5.6}$$

$$= \int_0^t \frac{\left(\frac{3g_0(t')}{(t-t')}-3\frac{\Omega_0(t,t')}{(t-t')^2}\right)v_0(t')-\left(3\frac{\Omega_0(t,t')}{(t-t')}-2g_0\left(t\right)\right)\dot{v}_0\left(t'\right)}{\sqrt{2\pi\left(t-t'\right)}}dt'$$

$$-\frac{z^2\exp\left(-\frac{z^2}{t}\right)}{2\pi t^3}+2z\left(1-\Phi\left(\frac{z}{\sqrt{t}}\right)\right)\left(3-\frac{z^2}{t}\right)\frac{\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^5}}.$$

Substituting this in the second equation (5.4) yields an expression for $g_1\left(t\right)$, which can be evaluated by numerical integration.

In summary, in the formal expansions (5.2), v_0 and g_0 are given by (5.3) and v_1 and g_1 are given by (5.4), where $\dot{v}_1$ can be evaluated (without differentiation) by (5.6).

Finally, we evaluate the complexity of the computation of $g_1(t)$, the most expensive expression to compute. Consider numerical quadrature with N points. First, we precompute $\dot{v}_1(t)$ using (5.6); it can be done in $O(N^2)$ operations. Then, we can compute $g_1(t)$ using the second equation in (5.4) with precomputed $\dot{v}_1(t)$ again in $O(N^2)$. Thus, the total complexity of the perturbation method is $O(N^2)$.

5.2 Numerical solution

In this section, we present (without convergence analysis) a simple method for the numerical approximation of the solution to the coupled Volterra equations (5.1). We note that Volterra equations and their numerical solution are a well-established research field. For a relevant discussion of the stability and convergence of some methods for equations with a weak singularity see [43]. Noble [53] discusses possible instabilities of multi-step methods in the presence of weak singularities.

A number of papers propose higher-order methods and collocation techniques to improve the convergence and treat instabilities. For example, Brunner [5] proved the convergence for a polynomial spline collocation method with quadratures; Kolk *et al.* [41], Kolk and Pedas [39] and Kolk and Pedas [40] used a piecewise polynomial collocation method to solve a Volterra equation with weak singularity and derived optimal global convergence estimates and a local superconvergence result. An alternative is to consider a special functional basis, such as Chebyshev polynomials and Bernstein polynomials (see [47, 48], respectively). In both cases, the approximation leads to a system of linear or nonlinear algebraic equations. Hairer *et al.* [21] developed a method based on fast Fourier transform to reduce the number of kernel evaluations on an N-point grid from $O(N^2)$ to $O(N(\log N)^2)$.

In this paper, for simplicity, we consider trapezoidal quadrature, with a special treatment of the interval containing the singularity, to obtain the numerical solution recursively. We divide the interval $[0, T]$ into equally spaced subintervals of length Δ and discretize (5.1) appropriately. To this end, we assume that v and g are piecewise linear with $v(l\Delta) = v_l$ and $g(l\Delta) = g_l$, so that on the interval $[(l-1)\,\Delta, l\Delta]$ we have

$$v\,(t) = \frac{v_{l-1}\,(l\Delta - t) + v_l\,(t - (l-1)\,\Delta)}{\Delta} = v_l - \frac{(v_l - v_{l-1})}{\Delta}\,(l\Delta - t),$$

$$g\,(t) = \frac{g_{l-1}\,(l\Delta - t) + g_l\,(t - (l-1)\,\Delta)}{\Delta} = g_l - \frac{(g_l - g_{l-1})}{\Delta}\,(l\Delta - t).$$

Accordingly,

$$\Omega_{nl} = \int_{l\Delta}^{n\Delta} g\,(t'')\,dt' = \frac{\Delta}{2}\,(g_l + 2g_{l+1} + \cdots + 2g_{n-1} + g_n),$$

$$\Omega_{nl} = \Omega_{(n-1)l} + \frac{\Delta}{2}\,(g_n + g_{n-1})\,.$$

Inserting in (5.1), the discretised system of equations has the form

$$\begin{cases} v_n + \alpha \sum_{l=1}^{n} \mathcal{I}_l^n + \dfrac{\exp\left(-\frac{(\alpha\Omega_{n0}-z)^2}{2n\Delta}\right)}{\sqrt{2\pi n\Delta}} = 0, \\[4mm] g_n + \left(\alpha g_n + \dfrac{1}{\sqrt{2\pi n\Delta}}\right) v_n + \dfrac{1}{2} \sum_{l=1}^{n} \mathcal{J}_l^n + \dfrac{(\alpha\Omega_{n0}-z)\exp\left(-\frac{(\alpha\Omega_{n0}-z)^2}{2n\Delta}\right)}{2\sqrt{2\pi n^3\Delta^3}} = 0. \end{cases} \tag{5.7}$$

For a given n, all the relevant integrals $\mathcal{I}_l$, $\mathcal{J}_l$, $1 \leq l < n$ can be approximated by the trapezoidal rule (or via more accurate composite formulas, if necessary). Accordingly,

$$\mathcal{I}_l^n = \int_{(l-1)\Delta}^{l\Delta} \frac{\Omega\,(n\Delta, t')\exp\left(-\frac{\alpha^2\Omega(n\Delta,t')^2}{2(n\Delta-t')}\right) v\,(t')}{\sqrt{2\pi\,(n\Delta - t')^3}}\,dt'$$

$$\approx \frac{1}{\sqrt{8\pi\Delta}}\left(\frac{\Omega_{nl}\exp\left(-\frac{\alpha^2\Omega_{nl}^2}{2(n-l)\Delta}\right) v_l}{(n-l)^{3/2}} + \frac{\Omega_{n(l-1)}\exp\left(-\frac{\alpha^2\Omega_{n(l-1)}^2}{2(n-l+1)\Delta}\right) v_{l-1}}{(n-l+1)^{3/2}}\right), \tag{5.8}$$

and

$$\mathcal{J}_l^n = \int_{(l-1)\Delta}^{l\Delta} \frac{\left(v_n - \left(1 - \frac{\alpha^2 \Omega(n\Delta,t')^2}{(n\Delta-t')}\right) \exp\left(-\frac{\alpha^2 \Omega(n\Delta,t')^2}{2(n\Delta-t')}\right) v\left(t'\right)\right)}{\sqrt{2\pi} \ (n\Delta - t')^3} \, dt'$$

$$\approx \frac{1}{\sqrt{8\pi\,\Delta}} \left(\frac{\left(v_n - \left(1 - \frac{\alpha^2 \Omega_{nl}^2}{(n-l)\Delta}\right) \exp\left(-\frac{\alpha^2 \Omega_{nl}^2}{2(n-l)\Delta}\right) v_l\right)}{(n-l)^{3/2}} \right. \tag{5.9}$$

$$\left. + \frac{\left(v_n - \left(1 - \frac{\alpha^2 \Omega_{n(l-1)}^2}{(n-l+1)\Delta}\right) \exp\left(-\frac{\alpha^2 \Omega_{n(l-1)}^2}{2(n-l+1)\Delta}\right) v_{l-1}\right)}{(n-l+1)^{3/2}} \right).$$

However, the last two integrals, $\mathcal{I}_n^n, \mathcal{J}_n^n$, require special care because they have weak singularities. Consider the integral $\mathcal{I}_n$, which has the form

$$\mathcal{I}_n^n = \int_{(n-1)\Delta}^{n\Delta} \frac{\Omega\left(n\Delta, t'\right) \exp\left(-\frac{\alpha^2 \Omega(n\Delta,t')^2}{2(n\Delta-t')}\right)}{\sqrt{2\pi} \ (n\Delta - t')^3} v\left(t'\right) dt'.$$

In view of our piecewise linearity assumption, we have

$$\Omega\left(n\Delta, t'\right) = g_n \tau - \frac{g_n - g_{n-1}}{2\Delta} \tau^2, \tag{5.10}$$

where $\tau = n\Delta - t'$. Accordingly,

$$\mathcal{I}_n^n = \int_0^\Delta \frac{\left(g_n - \frac{g_n-g_{n-1}}{2\Delta}\tau\right) \exp\left(-\frac{\alpha^2 \left(g_n - \frac{(g_n-g_{n-1})}{2\Delta}\tau\right)^2 \tau}{2}\right) \left(v_n - \frac{(v_n-v_{n-1})}{\Delta}\tau\right)}{\sqrt{2\pi\tau}} \, d\tau.$$

A standard change of variables $\tau = u^2$ yields

$$\mathcal{I}_n^n = \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{\Delta}} \left(g_n - \frac{(g_n-g_{n-1})}{2\Delta}u^2\right) \exp\left(-\frac{\alpha^2 \left(g_n - \frac{(g_n-g_{n-1})}{2\Delta}u^2\right)^2 u^2}{2}\right)$$

$$\times \left(v_n - \frac{(v_n-v_{n-1})}{\Delta}u^2\right) du. \tag{5.11}$$

The latter integral is now non-singular and can be approximated by the trapezoidal rule:

$$\mathcal{I}_n^n \approx \frac{1}{\sqrt{2\pi\,\Delta}} \left(\Delta g_n v_n + \Gamma_n \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta}\right) v_{n-1}\right) = \sqrt{\frac{\Delta}{2\pi}} g_n v_n + \frac{\Gamma_n \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta}\right)}{\sqrt{2\pi\,\Delta}} v_{n-1}, \tag{5.12}$$

where

$$\Gamma_n = \frac{\Delta}{2} \left(g_n + g_{n-1} \right).$$

Similarly,

$$\mathcal{J}_n^n = \int_{(n-1)\Delta}^{n\Delta} \frac{\left(v_n - \exp\left(-\frac{\alpha^2 \Omega(n\Delta, t')^2}{2(n\Delta - t')} \right) v\left(t'\right) \right)}{\sqrt{2\pi \, (n\Delta - t')^3}} \, dt'$$

$$+ \alpha^2 \int_{(n-1)\Delta}^{n\Delta} \frac{\Omega\left(n\Delta, t'\right)^2 \exp\left(-\frac{\alpha^2 \Omega(n\Delta, t')^2}{2(n\Delta - t')} \right) v\left(t'\right)}{(n\Delta - t') \sqrt{2\pi \, (n\Delta - t')^3}} \, dt'$$

$$= \mathcal{J}_n^{n,1} + \alpha^2 \mathcal{J}_n^{n,2},$$

for $l = 1, \ldots, n - 1$.

Computing the integrals in $\mathcal{J}_n^{n,1}$ and $\mathcal{J}_n^{n,2}$, we have (detailed derivations are in Appendix D)

$$\mathcal{J}_n^{n,1} \approx \frac{\alpha^2}{2} \frac{1}{\sqrt{2\pi \Delta^3}} \left(\Delta^2 g_n^2 + \Gamma_n^2 \right) v_n + \frac{1}{\sqrt{2\pi \Delta}} \left(1 + \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta} \right) \right) (v_n - v_{n-1}), \quad (5.13)$$

and

$$\mathcal{J}_n^{n,2} \approx \frac{1}{\sqrt{2\pi \Delta^3}} \left(\Delta^2 g_n^2 v_n + \Gamma_n^2 \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta} \right) v_{n-1} \right). \quad (5.14)$$

Thus,

$$\mathcal{J}_n^n \approx \left(\frac{\left(1 + \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta} \right) \right)}{\sqrt{2\pi \Delta}} + \frac{\alpha^2 \left(\frac{3}{2} \Delta^2 g_n^2 + \Gamma_n^2 \right)}{\sqrt{2\pi \Delta^3}} \right) v_n$$

$$- \left(\frac{\left(1 + \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta} \right) \right)}{\sqrt{2\pi \Delta}} - \frac{\alpha^2 \Gamma_n^2 \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta} \right)}{\sqrt{2\pi \Delta^3}} \right) v_{n-1}. \quad (5.15)$$

By using (5.10), we can represent expressions (5.8), (5.9) in a recurrent form, neglecting now quadrature errors:

$$\mathcal{I}_l^n = \frac{1}{\sqrt{8\pi \Delta}} \left(\frac{\Omega_{nl} \exp\left(-\frac{\alpha^2 \Omega_{nl}^2}{2(n-l)\Delta} \right) v_l}{(n-l)^{3/2}} + \frac{\Omega_{n(l-1)} \exp\left(-\frac{\alpha^2 \Omega_{n(l-1)}^2}{2(n-l+1)\Delta} \right) v_{l-1}}{(n-l+1)^{3/2}} \right)$$

$$= \mathbb{I}_l^n \left(g_n | \, v_{l-1}, v_l, g_{l-1}, \ldots, g_{n-1} \right),$$

$$\mathcal{J}_l^n = \left(\frac{1}{\sqrt{8\pi\Delta}\,(n-l)^{3/2}} + \frac{1}{\sqrt{8\pi\Delta}\,(n-l+1)^{3/2}} \right) v_n$$

$$- \frac{\left(1 - \frac{\alpha^2\Omega_{nl}^2}{(n-l)\Delta}\right)\exp\left(-\frac{\alpha^2\Omega_{nl}^2}{2(n-l)\Delta}\right)v_l}{\sqrt{8\pi\Delta}\,(n-l)^{3/2}} - \frac{\left(1 - \frac{\alpha^2\Omega_{n(l-1)}^2}{(n-l+1)\Delta}\right)\exp\left(-\frac{\alpha^2\Omega_{n(l-1)}^2}{2(n-l+1)\Delta}\right)v_{l-1}}{\sqrt{8\pi\Delta}\,(n-l+1)^{3/2}}$$

$$= \mathbb{J}_l^n\left(v_n, g_n|\, v_{l-1}, v_l, g_{l-1}, \dots, g_{n-1}\right) = \mathbb{U}_l^n v_n + \mathbb{V}_l^n\left(g_n|\, v_{l-1}, v_l, g_{l-1}, \dots, g_{n-1}\right).$$

By the same token, $\mathcal{I}_n^n$, $\mathcal{J}_n^n$ given by (5.12), (5.15) can be written in the form

$$\mathcal{I}_n^n = \mathbb{I}_n^n\left(v_n, g_n|\, v_{n-1}, g_{n-1}\right)$$

$$= \mathbb{A}_n^n\left(g_n|\, g_{n-1}\right)v_n + \mathbb{B}_n^n\left(g_n|\, v_{n-1}, g_{n-1}\right),$$

$$\mathcal{J}_n^n = \mathbb{J}_n^n\left(v_n, g_n|\, v_{n-1}, g_{n-1}\right)$$

$$= \mathbb{U}_n^n\left(g_n|\, g_{n-1}\right)v_n + \mathbb{V}_n^n\left(g_n|\, v_{n-1}, g_{n-1}\right).$$

In view of the above, system (5.7) can be written as follows:

$$\begin{cases} v_n + \alpha\sum_{l=1}^{n-1}\mathbb{I}_l^n\left(g_n\right) + \alpha\mathbb{I}_n^n\left(v_n, g_n\right) + \dfrac{\exp\left(-\frac{(\alpha\Omega_{n0}-z)^2}{2n\Delta}\right)}{\sqrt{2\pi n\Delta}} = 0, \\[3mm] g_n + \left(\alpha g_n + \dfrac{1}{\sqrt{2\pi n\Delta}}\right)v_n + \dfrac{1}{2}\sum_{l=1}^{n-1}\mathbb{J}_l^n\left(v_n, g_n\right) + \dfrac{1}{2}\mathbb{J}_n^n\left(v_n, g_n\right) \\[3mm] + \dfrac{(\alpha\Omega_{n0}-z)\exp\left(-\frac{(\alpha\Omega_{n0}-z)^2}{2n\Delta}\right)}{2\sqrt{2\pi n^3\Delta^3}} = 0, \end{cases} \tag{5.16}$$

where we suppress explicit dependencies on $g_1, \dots, g_{n-1}, v_1, \dots, v_{n-1}$ for brevity provided that $g_1, \dots, g_{n-1}, v_1, \dots, v_{n-1}$ are given. This system can be solved by using the Newton–Raphson method, say. As a result, the new pair (v_n, g_n) can be found and the recurrence advanced by one more step as required.

If so desired, system (5.16) can be simplified further. We notice that the dependence on v_n is linear, eliminate v_n in favour of g_n from the first equation,

$$v_n = -\frac{\alpha\left(\sum_{l=1}^{n-1}\mathbb{I}_l^n\left(g_n\right) + \mathbb{B}_n^n\left(g_n\right)\right) + \dfrac{\exp\left(-\frac{(\alpha\Omega_{n0}-z)^2}{2n\Delta}\right)}{\sqrt{2\pi n\Delta}}}{\left(1 + \alpha\mathbb{A}_n^n\left(g_n\right)\right)},$$
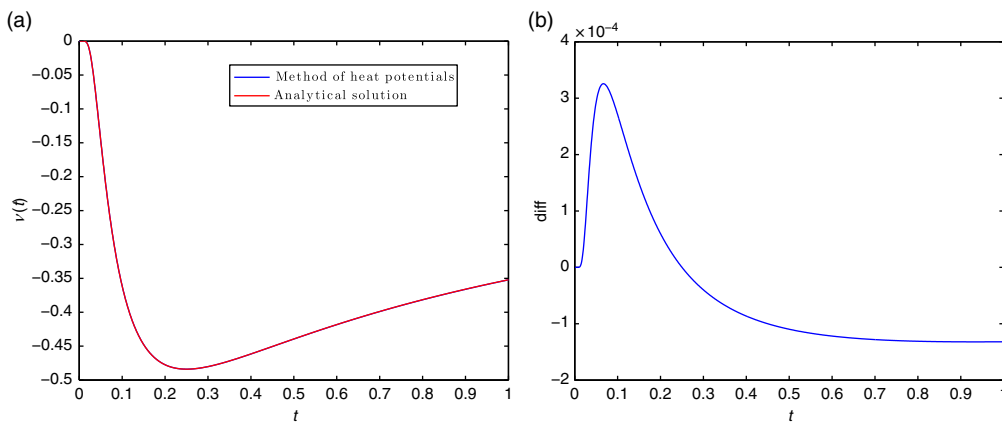
(a)

(b)

FIGURE 3. $v(t)$: (a) Numerical and analytical solution for $\alpha = 0$, $N = 1000$ (visually indistinguishable). (b) The difference with the exact solution for $\alpha = 0$.

and substitute this expression in the second equation, obtaining a scalar recursive nonlinear equation of the form

$$
\begin{aligned}
g_n + \frac{1}{2} \left(\sum_{l=1}^{n-1} \mathbb{V}_l^n (g_n) + \mathbb{V}_n^n (g_n) \right) &- \frac{\left(\alpha g_n + \frac{1}{\sqrt{2\pi n \Delta}} + \frac{1}{2} \left(\sum_{l=1}^{n-1} \mathbb{U}_l^n + \mathbb{U}_n^n (g_n) \right) \right)}{\left(1 + \alpha \mathbb{A}_n^n (g_n) \right)} \\
&\times \left(\alpha \left(\sum_{l=1}^{n-1} \mathbb{I}_l^n (g_n) + \mathbb{B}_n^n (g_n) \right) + \frac{\exp \left(-\frac{(\alpha \Omega_{n0} - z)^2}{2n\Delta} \right)}{\sqrt{2\pi n \Delta}} \right) \\
&+ \frac{(\alpha \Omega_{n0} - z) \exp \left(-\frac{(\alpha \Omega_{n0} - z)^2}{2n\Delta} \right)}{2\sqrt{2\pi n^3 \Delta^3}} = 0.
\end{aligned}
\tag{5.17}
$$

6 Numerical tests and results

In this section, we first analyse the convergence (order) of the numerical method, then test the accuracy of the first-order expansion against the numerical solution, and finally perform parameter studies (in α) to investigate the influence of the mean-field interaction on the behaviour of the solution.

6.1 Numerical method

To demonstrate the performance of the discretisation scheme, we compare in Figures 3 to 5 the solution with (B.1), the analytical solution, in the case $\alpha = 0$.

For $\alpha > 0$, no closed-form solution is available and we therefore use the Euler timestepping particle method from [37] with sufficiently many particles and timesteps as benchmark.

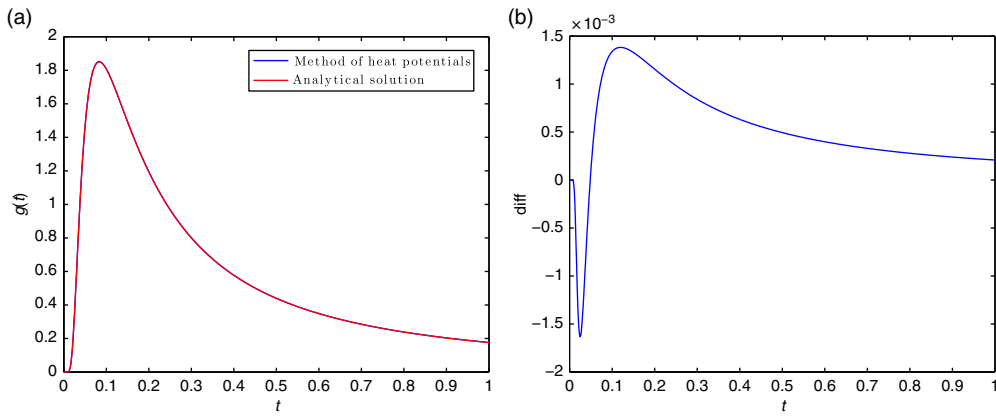We illustrate the difference between our method and [37] in Figure 6.

FIGURE 4. $g(t)$: (a) Numerical and analytical solution for $\alpha = 0$, $N = 1000$ (visually indistinguishable). (b) The difference with the exact solution for $\alpha = 0$.
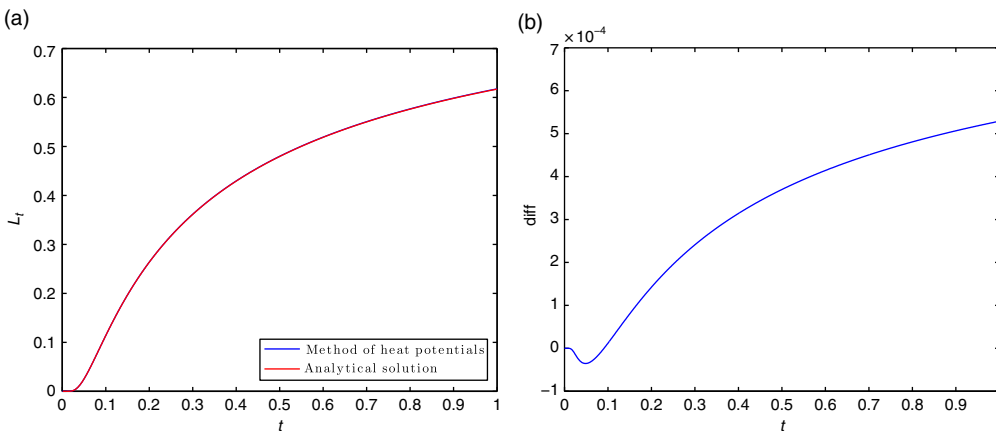

FIGURE 5. L_t: (a) Numerical and analytical solution for $\alpha = 0$, $N = 1000$ (visually indistinguishable). (b) The difference with the exact solution for $\alpha = 0$.

We now analyse the convergence order of the discretisation scheme for the Volterra equation empirically. With N timesteps, the error of the approximation (5.17) is expected to be $O(N^{-1})$ because the trapezoidal integration in (5.11), (5.13) and (5.14) is on intervals $(0, \sqrt{\Delta})$, and the result is divided by $\sqrt{\Delta}$ after that. We empirically confirm this in Figure 7.

The complexity of our method is $O(N^2)$. Hence, in order to achieve precision ε, we need $O(\varepsilon^{-2})$ operations. In comparison, the particle method with Brownian bridge in [37] requires $O(\varepsilon^{-3})$ operations. The latter could be improved to $O(\varepsilon^{-2})$ by multilevel simulation, but equally a higher-order method for the Volterra equation would bring the complexity down. Another advantage of the method above is that we automatically get directly the derivative g of the loss function, which is harder to obtain in [37] because of Monte Carlo noise.
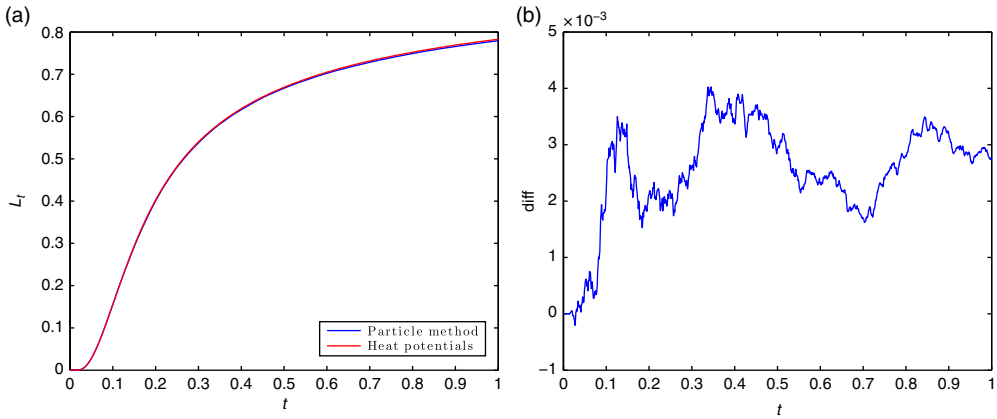
FIGURE 6. Comparison of the numerical solution of L_t by Volterra equations ($N = 1000$) with that of the particle method in [37] for $\alpha = 0.5$.
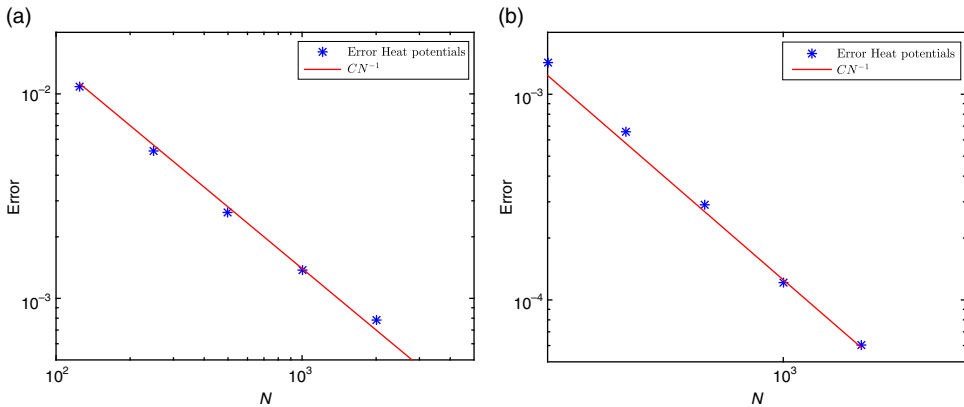

FIGURE 7. Error in the maximum norm for numerical method in Section 5.2: (a) for g compared to the exact solution for $\alpha = 0$; (b) for L compared to the simulation solution for $\alpha = 0.5$.

6.2 Comparison of perturbation and numerical methods

Here, we compare the numerical and perturbation solutions described above. We fix $T = 1$, $z = 0.5$ and choose $N = 1000$, the number of grid points, sufficiently large so that the numerical error is negligible. In Figure 8, we plot g, the hitting time density, computed with numerical and perturbation methods as well as $g_0(t)$, the solution for $\alpha = 0$, to measure the impact of the nonlinear term, for different values of α. For $\alpha = 0.1$, the two solutions are visually indistinguishable; for $\alpha = 0.3$, there is small but visible difference between the solutions, which increases further for $\alpha = 0.5$ and arises from the higher-order terms.[2] For $\alpha = 1$, where the numerical solution

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[2]In our implementation of the perturbation solution, we also perform a scaling to ensure the correct cumulative density at T (which in practice is unknown) to improve the results slightly.
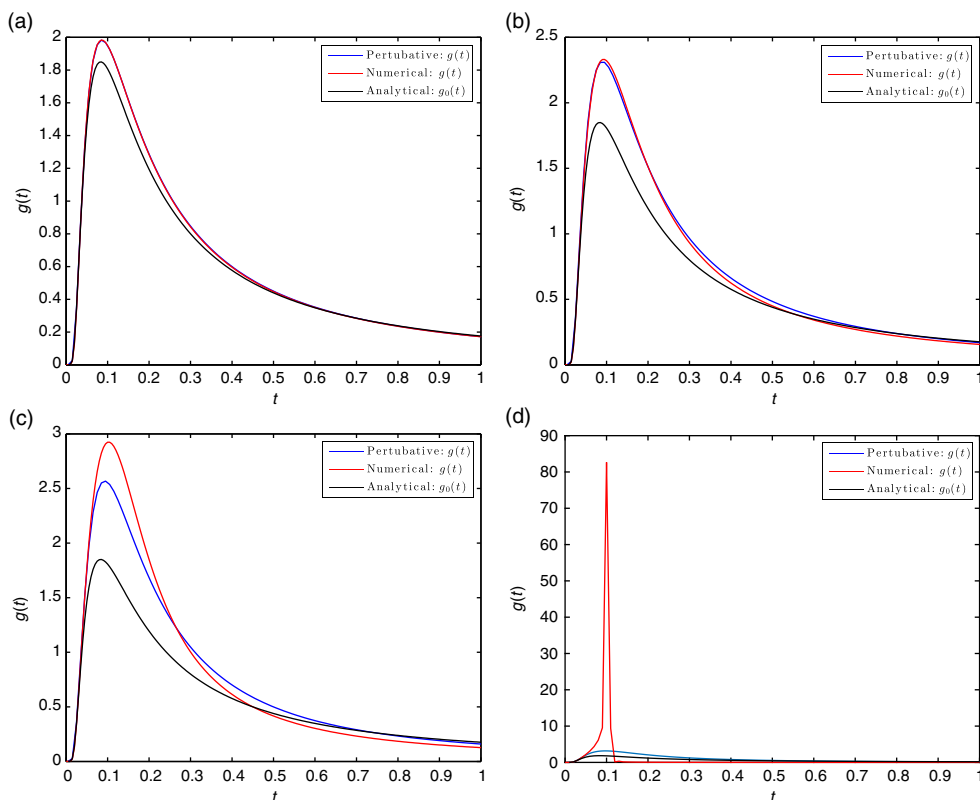
FIGURE 8. Comparison of the numerical and perturbative methods for different values of α.: (a) $\alpha = 0.1$; (b) $\alpha = 0.3$; (c) $\alpha = 0.5$; (d) $\alpha = 1$. For visibility, we take $N = 100$ in the numerical method in the last plot.

shows a jump in the loss function at around $t = 0.1$, the first-order expansion approximation breaks down.

6.3 Parameter studies

We now assess the impact of the mean-field interaction by varying the parameter α.

We fix $T = 1, z = 0.5$ and choose $N = 1000$, the number of the grid points. Figure 9 demonstrates the behaviour of $\nu(t)$ and $g(t)$ for different values of α, starting with $\alpha = 0$; for L_t, including a case with discontinuity, see already Figure 2.

To illustrate the impact of the interaction further, we consider the expectation and variance of the default time depending on α. Since the expectation is infinite, we restrict it to the interval $(0, T)$, i.e., consider $\mathbb{E}[\tau | \tau < T]$ and $\mathbb{V}[\tau | \tau < T]$. These expectations must be finite for any fixed T and go to infinity when $T \to \infty$. The conditional density is then $p_{\tau | \tau < T}(t) = \frac{p_\tau(t)}{\int_0^T p_\tau(s) \, ds}$ for $t \in [0, T]$. Using this, one can easily evaluate the moments numerically. We present the results in Figure 10. As expected, we observe that the expected default time and its variance become smaller with increasing of α, and grow with increasing T.
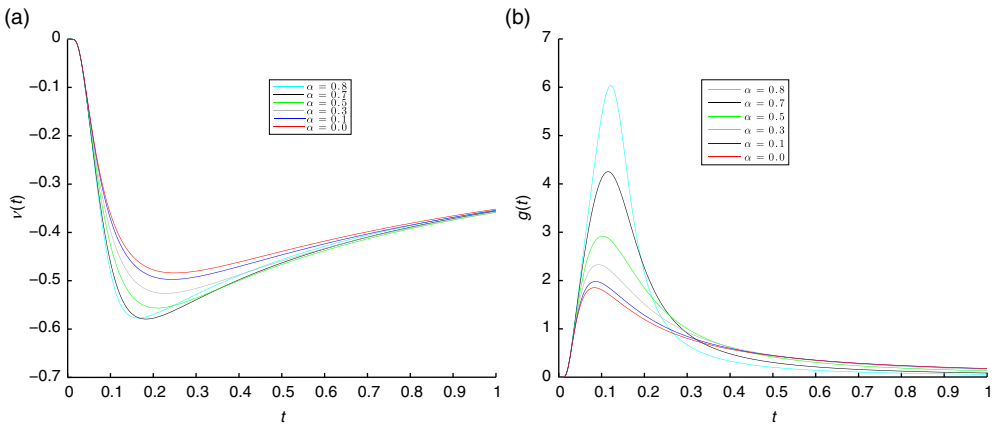
(a)

(b)

FIGURE 9. (a) $\nu(t)$ for different values of α. (b) $g(t)$ for different values of α. Both computed by solving (5.1) via numerical method.
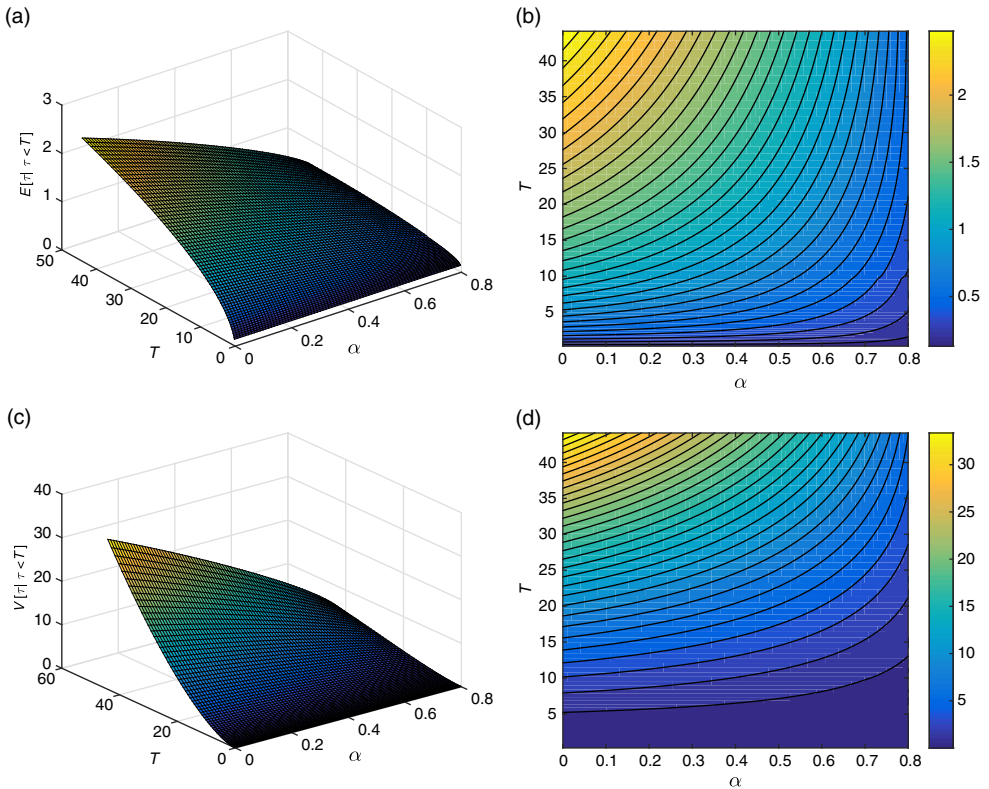
(a)

(b)

(c)

(d)

FIGURE 10. (a) and (b): $\mathbb{E}[\tau | \tau < T]$; (c) and (d) $\mathbb{V}[\tau | \tau < T]$ for different values of α and T.

7 Conclusion

We have developed a semi-analytical approach to finding the density of interacting particles where their common downwards drift increases in magnitude when particles hit a lower boundary, thus creating a positive feedback effect. This leads to a nonlinear and nonlocal parabolic equation. Using the method of heat potentials, we derived an equivalent coupled system of Volterra integral equations and solved it numerically, or by expansion for a small interaction parameter α. We confirmed empirically the convergence of order 1 of the numerical method and demonstrated its better complexity in comparison to the particle method in [37]. There are striking financial implications as the computations uncover, in a very simplified setting, how mutual liabilities accelerate defaults of individual banks.

This paper raises several open questions. The numerical method for the system of Volterra equations can be improved using the methods we described at the beginning of Section 5.2; one can potentially analyse the convergence of the method for the blow-up case. Another interesting direction is to study a model with common noise as in [24, 42] using the method of heat potentials. Lastly, it would be interesting to investigate an extension of the current paper for more complicated diffusion equations such as those in [9, 51].

Acknowledgements

Conflicts of interest

None.

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Appendix A. Derivation of limits in Section 4

We start with (4.1). We split $\mathbb{L}_1$ into two parts,

$$\mathbb{L}_1 = \mathbb{L}_1^{(1)} - \mathbb{L}_1^{(2)},$$

where

$$\mathbb{L}_1^{(1)} = \lim_{x\to 0} \left(x \int_0^t \frac{\exp\left(-\frac{(x-\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t-t')^3}} \nu\left(t'\right) dt' \right),$$

$$\mathbb{L}_1^{(2)} = \lim_{x\to 0} \int_0^t \frac{\Psi\left(t,t'\right)\exp\left(-\frac{(x-\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t-t')^3}} \nu\left(t'\right) dt'.$$

We represent $\nu\left(t'\right) = \nu\left(t\right) + \left(\nu\left(t'\right) - \nu\left(t\right)\right)$ and write

$$\mathbb{L}_1^{(1)} = \lim_{x\to 0} \left(x\nu\left(t\right) \int_0^t \frac{\exp\left(-\frac{(x-\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t-t')^3}} dt' \right)$$

$$+ \lim_{x\to 0} \left(x \int_0^t \frac{\exp\left(-\frac{(x-\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t-t')^3}} \left(\nu\left(t'\right) - \nu\left(t\right)\right) dt' \right)$$

$$= \nu\left(t\right) \lim_{x\to 0} \left(x \int_0^t \frac{\exp\left(-\frac{(x-\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t-t')^3}} dt' \right),$$

since the second integral converges. We use the change of variables $t - t' = x^2 u$, so that

$$\lim_{x\to 0} \left(x \int_0^t \frac{\exp\left(-\frac{(x-\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t-t')^3}} dt' \right) = \lim_{x\to 0} \int_0^{t/x^2} \frac{\exp\left(-\frac{1}{2u}\right)}{\sqrt{2\pi u^3}} du = \int_0^\infty \frac{\exp\left(-\frac{1}{2u}\right)}{\sqrt{2\pi u^3}} du = 1.$$

The first equality is valid since $\Psi(t, t')$ is differentiable on $[0, T^*]$.

Since the second integral converges, it is clear that

$$\mathbb{L}_1^{(2)} = \int_0^t \frac{\Psi\left(t, t'\right) \Xi\left(t, t'\right)}{\sqrt{2\pi\left(t - t'\right)^3}} v\left(t'\right) dt'.$$

Thus, (4.1) is valid.

The second limit $\mathbb{L}_2$ in (4.2) is less standard and more difficult to evaluate. We introduce a non-singular function ϕ,

$$\phi\left(t, t'\right) = \frac{\Psi\left(t, t'\right)}{t - t'}, \quad t \neq t', \qquad \phi\left(t, t\right) = M'\left(t\right),$$

and write

$$\mathbb{L}_2 = \lim_{x \to 0} \int_0^t \left(1 - \frac{x^2}{(t - t')} + 2x\phi\left(t, t'\right) - \phi\left(t, t'\right)^2 \left(t - t'\right)\right)$$

$$\times \frac{\exp\left(-\frac{x^2}{2(t - t')} + x\phi\left(t, t'\right)\right) \Xi\left(t, t'\right)}{\sqrt{2\pi\left(t - t'\right)^3}} v\left(t'\right) dt',$$

$$= \mathbb{L}_2^{(1)} + \mathbb{L}_2^{(2)} + 2\mathbb{L}_2^{(3)} - \mathbb{L}_2^{(4)},$$

where

$$\mathbb{L}_2^{(1)} = \lim_{x \to 0} v\left(t\right) \int_0^t \left(1 - \frac{x^2}{(t - t')}\right) \frac{\exp\left(-\frac{x^2}{2(t - t')} + 2x\phi\left(t, t'\right)\right)}{\sqrt{2\pi\left(t - t'\right)^3}} dt',$$

$$\mathbb{L}_2^{(2)} = \lim_{x \to 0} \int_0^t \left(1 - \frac{x^2}{(t - t')}\right) \frac{\exp\left(-\frac{x^2}{2(t - t')} + 2x\phi\left(t, t'\right)\right)}{\sqrt{2\pi\left(t - t'\right)^3}} \left(\Xi\left(t, t'\right) v\left(t'\right) - v\left(t\right)\right) dt',$$

$$\mathbb{L}_2^{(3)} = \lim_{x \to 0} x \int_0^t \phi\left(t, t'\right) \frac{\exp\left(-\frac{x^2}{2(t - t')} + 2x\phi\left(t, t'\right)\right) \Xi\left(t, t'\right)}{\sqrt{2\pi\left(t - t'\right)^3}} v\left(t'\right) dt',$$

$$\mathbb{L}_2^{(4)} = \lim_{x \to 0} \int_0^t \phi\left(t, t'\right)^2 \frac{\exp\left(-\frac{x^2}{2(t - t')} + x\phi\left(t, t'\right)\right) \Xi\left(t, t'\right)}{\sqrt{2\pi\left(t - t'\right)}} v\left(t'\right) dt'.$$

We have

$$\mathbb{L}_2^{(1)} = \lim_{x \to 0} v\left(t\right) \int_0^t \left(1 - \frac{x^2}{(t - t')}\right) \frac{\exp\left(-\frac{x^2}{2(t - t')} + 2x\phi\left(t, t'\right)\right)}{\sqrt{2\pi\left(t - t'\right)^3}} dt'$$

$$= -2v\left(t\right) \lim_{x \to 0} \frac{1}{x} \int_0^{x/\sqrt{t}} \left(1 - v^2\right) \frac{\exp\left(-\frac{v^2}{2}\right)}{\sqrt{2\pi}} dv$$

$$= -\frac{2}{\sqrt{2\pi t}} v\left(t\right),$$

where $\left(t - t'\right) = u$, $u = 1/v^2$ and we have used the fact that

$$\int_0^\infty \left(1 - v^2\right) \frac{\exp\left(-\frac{v^2}{2}\right)}{\sqrt{2\pi}} dv = 0.$$

Further,

$$\mathbb{L}_2^{(2)} = \lim_{x \to 0} \int_0^t \left(1 - \frac{x^2}{(t - t')}\right) \frac{\exp\left(-\frac{x^2}{2(t-t')} + 2x\phi(t,t')\right)}{\sqrt{2\pi(t-t')^3}} \left(\Xi(t,t')\, v(t') - v(t)\right)\, dt'$$

$$= \lim_{x \to 0} \int_0^t \frac{\exp\left(-\frac{x^2}{2(t-t')} + 2x\phi(t,t')\right)}{\sqrt{2\pi(t-t')^3}} \left(\Xi(t,t')\, v(t') - v(t)\right)\, dt'$$

$$= \int_0^t \frac{\left(\Xi(t,t')\, v(t') - v(t)\right)}{\sqrt{2\pi(t-t')^3}}\, dt',$$

where we have dropped the higher-order x^2 term in the integral in the second line,

$$\mathbb{L}_2^{(3)} = \lim_{x \to 0} x \int_0^t \phi(t,t') \frac{\exp\left(-\frac{x^2}{2(t-t')} + 2x\phi(t,t')\right)}{\sqrt{2\pi(t-t')^3}} \Xi(t,t')\, v(t')\, dt'$$

$$= \phi(t,t)\, \Xi(t,t)\, v(t) = M'(t)\, v(t),$$

and

$$\mathbb{L}_2^{(4)} = \lim_{x \to 0} \int_0^t \phi(t,t')^2 \frac{\exp\left(-\frac{x^2}{2(t-t')} + x\phi(t,t')\right)}{\sqrt{2\pi(t-t')}} \Xi(t,t')\, v(t')\, dt'$$

$$= \int_0^t \phi(t,t')^2 \frac{\Xi(t,t')}{\sqrt{2\pi(t-t')}} v(t')\, dt' = \int_0^t \frac{\Psi(t,t')^2}{(t-t')} \frac{\Xi(t,t')}{\sqrt{2\pi(t-t')^3}} v(t')\, dt'.$$

Finally,

$$\mathbb{L}_2 = \mathbb{L}_2^{(1)} + \mathbb{L}_2^{(2)} + 2\mathbb{L}_2^{(3)} - \mathbb{L}_2^{(4)}$$

$$= -2\frac{1}{\sqrt{2\pi t}} v(t) + \int_0^t \frac{\left(\Xi(t,t')\, v(t') - v(t)\right)}{\sqrt{2\pi(t-t')^3}}\, dt'$$

$$+ 2M'(t)\, v(t) - \int_0^t \frac{\Psi(t,t')^2}{(t-t')} \frac{\Xi(t,t')}{\sqrt{2\pi(t-t')^3}} v(t')\, dt'$$

$$= 2\left(M'(t) - \frac{1}{\sqrt{2\pi t}}\right) v(t) + \int_0^t \frac{\left(\left(1 - \frac{\Psi(t,t')^2}{(t-t')}\right) \Xi(t,t')\, v(t') - v(t)\right)}{\sqrt{2\pi(t-t')^3}}\, dt',$$

as stated.

Appendix B. Special cases

For illustration, we work out the solution from the formula obtained in Section 4 for two special cases which are also accessible by the standard reflection principle for Brownian motion or method of images for parabolic equations.

B.1 $M(t) = 0$

When $M(t) = 0$, we get

$$
\begin{cases}
v(t) + \dfrac{\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t}} = 0, \\[4mm]
g(t) + \dfrac{1}{\sqrt{2\pi t}}v(t) + \dfrac{1}{2}\displaystyle\int_0^t \dfrac{\left(v(t) - v(t')\right)}{\sqrt{2\pi}\,(t-t')^3}\,dt' - \dfrac{z\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}} = 0.
\end{cases}
$$

Integration by parts of the second equation and use of the first yields

$$
\begin{aligned}
g(t) &= -\int_0^t \frac{\dot{v}(t')}{\sqrt{2\pi}\,(t-t')}\,dt' + \frac{z\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}} \\[3mm]
&= \frac{z\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}} + \frac{z\exp\left(-\frac{z^2}{2t}\right)}{2\sqrt{2\pi t^3}} \\[3mm]
&= \frac{z\exp\left(-\frac{z^2}{2t}\right)}{\sqrt{2\pi t^3}},
\end{aligned}
$$

as expected.

B.2 $M(t) = \mu t$

When $M(t) = \mu t$, we get

$$
\begin{cases}
v(t) - \mu\displaystyle\int_0^t \dfrac{\exp\left(-\frac{\mu^2(t-t')}{2}\right)}{\sqrt{2\pi(t-t')}}v(t')dt' + \dfrac{\exp\left(-\frac{(\mu t+z)^2}{2t}\right)}{\sqrt{2\pi t}} = 0, \\[5mm]
g(t) = \dfrac{1}{2\sqrt{2\pi}}\displaystyle\int_0^t \dfrac{1}{\sqrt{(t-t')^3}}\left(\exp\left(-\frac{\mu^2(t-t')}{2}\right)v(t') - v(t)\right)dt' \\[5mm]
\qquad - \dfrac{\mu^2}{2\sqrt{2\pi}}\displaystyle\int_0^t \dfrac{1}{\sqrt{t-t'}}\exp\left(-\frac{\mu^2(t-t')}{2}\right)\dot{v}(t')\,dt' + \left(\mu - \dfrac{1}{\sqrt{2\pi t}}\right)v(t) \\[5mm]
\qquad + \dfrac{(\mu t + z)\exp\left(-\frac{(\mu t+z)^2}{2t}\right)}{\sqrt{2\pi t^3}}.
\end{cases}
\tag{B1}
$$

Taking the Laplace transform of the first equation in (B1), we have

$$
\hat{v}(s) - \mu\hat{v}(s)\frac{1}{\sqrt{2s+\mu^2}} + \frac{\exp(-z\sqrt{2s+\mu^2} - \mu z)}{\sqrt{2s+\mu^2}} = 0.
$$

Hence,

$$
\hat{v}(s) = -\frac{\exp(-z\sqrt{2s+\mu^2} - \mu z)}{\sqrt{2s+\mu^2} - \mu}.
$$

The inverse Laplace transform of $\hat{v}(s)$ can be found analytically, but we do not need it to compute $g(t)$. Consider the second equation of (B1). The first integral can be rewritten as

$$
\frac{1}{2\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{(t-t')^3}} \left(\exp\left(-\frac{\mu^2(t-t')}{2}\right) v\left(t'\right) - v(t) \right) dt'
$$

$$
= \frac{1}{\sqrt{2\pi}} \int_0^t \left(\exp\left(-\frac{\mu^2(t-t')}{2}\right) v\left(t'\right) - v(t) \right) d\left(\frac{1}{\sqrt{(t-t')}}\right)
$$

$$
= -\frac{1}{\sqrt{2\pi t}} \left(\exp\left(-\frac{\mu^2 t}{2}\right) v\left(0\right) - v(t) \right) - \frac{1}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{t-t'}} \exp\left(-\frac{\mu^2(t-t')}{2}\right) \dot{v}\left(t'\right) dt'
$$

$$
- \frac{\mu^2}{2\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{(t-t')}} \exp\left(-\frac{\mu^2(t-t')}{2}\right) v\left(t'\right) dt'.
$$

Thus,

$$
g(t) = \mu v(t) - \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\mu^2 t}{2}\right) v\left(0\right) - \frac{1}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{t-t'}} \exp\left(-\frac{\mu^2(t-t')}{2}\right) \dot{v}\left(t'\right) dt'
$$

$$
- \frac{\mu^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{(t-t')}} \exp\left(-\frac{\mu^2(t-t')}{2}\right) v\left(t'\right) dt' + \frac{(\mu t + z) \exp\left(-\frac{(\mu t + z)^2}{2t}\right)}{2\sqrt{2\pi t^3}}.
$$

Taking Laplace transform of the last equation, we get

$$
\hat{g}(s) = \mu \hat{v}(s) - \frac{1}{\sqrt{2s + \mu^2}} v(0) - \frac{1}{\sqrt{2s + \mu^2}} \left(s\hat{v}(s) - v(0) \right) - \frac{\mu^2}{\sqrt{2s + \mu^2}} \hat{v}(s)
$$

$$
+ \left(\frac{1}{2} + \frac{\mu}{2\sqrt{2s + \mu^2}} \right) \exp(-z\sqrt{2s + \mu^2} + \mu z)
$$

$$
= \left(-\frac{\mu}{\sqrt{2s + \mu^2} - \mu} + \frac{s + \mu^2}{\sqrt{2s + \mu^2}} \frac{1}{\sqrt{2s + \mu^2} - \mu^2} \right.
$$

$$
\left. + \frac{1}{2} + \frac{\mu}{2\sqrt{2s + \mu^2}} \right) \exp(-z\sqrt{2s + \mu^2} + \mu z) = \exp(-z\sqrt{2s + \mu^2} + \mu z).
$$

The inverse Laplace transform yields the final expression for $g(t)$

$$
g(t) = \frac{z \exp\left(-\frac{(z - \mu t)^2}{2t}\right)}{\sqrt{2\pi t^3}},
$$

as expected.

Appendix C. Proof of Lemma 4.1

Proof of Lemma 4.1 We start with the first term and judiciously use integration by parts several times to get

$$\frac{d}{dt} \int_0^t \frac{\Xi\left(t, t'\right) v\left(t'\right)}{\sqrt{2\pi\left(t - t'\right)}} \, dt'$$

$$= 2\frac{d}{dt}\left(\Xi\left(t, 0\right) v\left(0\right) \sqrt{\frac{t}{2\pi}} + \int_0^t \sqrt{\frac{\left(t - t'\right)}{2\pi}} d\left(\Xi\left(t, t'\right) v\left(t'\right)\right)\right)$$

$$= \frac{\left(2t\Xi_t\left(t, 0\right) + \Xi\left(t, 0\right)\right) v\left(0\right)}{\sqrt{2\pi t}} + \int_0^t \frac{1}{\sqrt{2\pi\left(t - t'\right)}} d\left(\Xi\left(t, t'\right) v\left(t'\right) - v\left(t\right)\right)$$

$$\quad + 2\int_0^t \sqrt{\frac{\left(t - t'\right)}{2\pi}} d\left(\Xi_t\left(t, t'\right) v\left(t'\right)\right)$$

$$= \frac{v\left(t\right)}{\sqrt{2\pi t}} + \frac{1}{2}\int_0^t \frac{v\left(t\right) - \left(\Xi\left(t, t'\right) - 2\left(t - t'\right) \Xi_t\left(t, t'\right)\right) v\left(t'\right)}{\sqrt{2\pi\left(t - t'\right)^3}} \, dt'$$

$$= \frac{v\left(t\right)}{\sqrt{2\pi t}} + \int_0^t \left(v\left(t\right) - \left(\Xi\left(t, t'\right) - 2\left(t - t'\right) \Xi_t\left(t, t'\right)\right) v\left(t'\right)\right) d\left(\frac{1}{\sqrt{2\pi\left(t - t'\right)}}\right)$$

$$= \int_0^t \frac{\left(\left(\Xi\left(t, t'\right) - 2\left(t - t'\right) \Xi_t\left(t, t'\right)\right) v\left(t'\right)\right)_{t'}}{\sqrt{2\pi\left(t - t'\right)}} \, dt'.$$

$$\square$$

Appendix D. Derivation of $\mathcal{J}_n^{n,1}$ and $\mathcal{J}_n^{n,2}$

We have

$$\mathcal{J}_n^{n,1} = \int_0^\Delta \frac{\left(v_n - \exp\left(-\frac{\alpha^2\left(g_n - \frac{\left(g_n - g_{n-1}\right)}{2\Delta}\tau\right)^2 \tau}{2}\right)\right)\left(v_n - \frac{\left(v_n - v_{n-1}\right)}{\Delta}\tau\right)}{\sqrt{2\pi\tau^3}} \, d\tau$$

$$= \int_0^\Delta \frac{\left(\left(1 - \exp\left(-\frac{\alpha^2\left(g_n - \frac{\left(g_n - g_{n-1}\right)}{2\Delta}\tau\right)^2 \tau}{2}\right)\right) v_n\right)}{\sqrt{2\pi\tau^3}} \, d\tau$$

$$\quad + \frac{\left(v_n - v_{n-1}\right)}{\Delta}\int_0^\Delta \frac{\exp\left(-\frac{\alpha^2\left(g_n - \frac{\left(g_n - g_{n-1}\right)}{2\Delta}\tau\right)^2 \tau}{2}\right)}{\sqrt{2\pi\tau}} \, d\tau$$

$$\approx \frac{\alpha^2 v_n}{2}\int_0^\Delta \frac{\left(g_n - \frac{\left(g_n - g_{n-1}\right)}{2\Delta}\tau\right)^2}{\sqrt{2\pi\tau}} \, d\tau$$

$$
+ \frac{(\nu_n - \nu_{n-1})}{\Delta} \int_0^{\Delta} \frac{\exp\left(-\frac{\alpha^2\left(g_n - \frac{(g_n - g_{n-1})}{2\Delta}\tau\right)^2 \tau}{2}\right)}{\sqrt{2\pi\tau}} \, d\tau
$$

$$
= \frac{\alpha^2 \nu_n}{\sqrt{2\pi}} \int_0^{\sqrt{\Delta}} \left(g_n - \frac{(g_n - g_{n-1})}{2\Delta}u^2\right)^2 du
$$

$$
+ \frac{2(\nu_n - \nu_{n-1})}{\sqrt{2\pi}\,\Delta} \int_0^{\sqrt{\Delta}} \exp\left(-\frac{\alpha^2\left(g_n - \frac{(g_n - g_{n-1})}{2\Delta}u^2\right)^2 u^2}{2}\right) du
$$

$$
\approx \frac{\alpha^2}{2} \frac{1}{\sqrt{2\pi\,\Delta^3}} \left(\Delta^2 g_n^2 + \Gamma_n^2\right) \nu_n + \frac{1}{\sqrt{2\pi\,\Delta}} \left(1 + \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta}\right)\right) (\nu_n - \nu_{n-1}).
$$

Similarly,

$$
\mathcal{J}_n^{n,2} = \int_0^{\Delta} \frac{\left(g_n - \frac{g_n - g_{n-1}}{2\Delta}\tau\right)^2 \exp\left(-\frac{\alpha^2\left(g_n - \frac{(g_n - g_{n-1})}{2\Delta}\tau\right)^2 \tau}{2}\right)\left(\nu_n - \frac{(\nu_n - \nu_{n-1})}{\Delta}\tau\right)}{\sqrt{2\pi\tau}} \, d\tau
$$

$$
= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{\Delta}} \left(g_n - \frac{(g_n - g_{n-1})}{2\Delta}u^2\right)^2 \exp\left(-\frac{\alpha^2\left(g_n - \frac{(g_n - g_{n-1})}{2\Delta}u^2\right)^2 u^2}{2}\right)
$$

$$
\times \left(\nu_n - \frac{(\nu_n - \nu_{n-1})}{\Delta}u^2\right) du
$$

$$
\approx \frac{1}{\sqrt{2\pi\,\Delta^3}} \left(\Delta^2 g_n^2 \nu_n + \Gamma_n^2 \exp\left(-\frac{\alpha^2 \Gamma_n^2}{2\Delta}\right) \nu_{n-1}\right).
$$