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THE NUCLEAR LENGTH OF A CHARACTER

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Abstract

If G is a π-separable group and χ is an irreducible character of G, then Isaacs has defined an associated pair (W, γ), called a *nucleus* of χ. The nucleus is the last term in a certain chain of pairs (I, ψ), where I is a subgroup of G and ψ is an irreducible character of I. The length of this chain is an invariant of χ that we call the *nuclear length*. In this paper we study bounds on the nuclear length of χ as a function of the π-length of G and as a function of the character degree $\chi(1)$.

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1. Introduction

If π is a set of primes, then the character theory of π-separable groups demonstrates a remarkable richness that is not shared by finite groups in general. For example, in [1] Gajendragadkar defined the set $\mathfrak{X}_\pi(G)$ of π-special characters of G and showed that if α is π-special and β is π'-special, then $\alpha\beta$ is irreducible. Because a product of characters is so rarely irreducible, this result seems quite surprising. The idea of factoring characters into a product of π-special and π'-special parts was exploited by Isaacs, who defined a set of irreducible characters $B_\pi(G)$ containing the π-special characters [2]. When $\pi = p'$, the complement of a single prime p, the characters in $B_{p'}(G)$ provide canonical lifts for the irreducible Brauer characters of G. Thus the set $\mathrm{IBr}(G)$ of irreducible Brauer characters of G is obtained simply by restricting the characters in $B_{p'}(G)$ to the p-regular elements of G. More generally, the characters in $B_{\pi'}(G)$ act as a π-analogue of Brauer characters, and Slattery has used them to develop a theory of π-blocks in π-separable groups (see [4, 5, 6]). More recently, Isaacs and Navarro have defined weights and vertices for characters and have proven a π-analogue of Alperin's Weight Conjecture for π-separable groups [3].

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The point of this paper is to return to Isaacs' definition of $B_\pi(G)$ in [2] and study it more closely. If $\chi \in \mathrm{Irr}(G)$, then Isaacs constructs a subgroup I and a character $\psi \in \mathrm{Irr}(I)$ with $\psi^G = \chi$. The pair (I, ψ) is called a *standard inducing pair,* and $(I, \psi) = (G, \chi)$ if and only if χ factors as the product of a π-special and a π'-special character. One can also construct a standard inducing pair for ψ, and this process leads inductively to a pair (W, γ) such that $\gamma^G = \chi$ and γ is the product of a π-special and a π'-special character. The pair (W, γ) is called a *nucleus* for χ and is unique up to G-conjugacy. Isaacs defines the set $B_\pi(G)$ to consist of those irreducible characters χ for which there is a nucleus (W, γ) such that γ is itself π-special.

But it is easy to see that this construction produces more than just the conjugacy class of pairs (W, γ). In particular, associated to the character $\chi \in \mathrm{Irr}(G)$ is a nonnegative integer given by the number of times one must apply the inductive procedure before obtaining a nucleus for χ. We call this number the *nuclear length* of χ, and we write it as $\mathrm{nl}(\chi)$. The purpose of this paper is to study the invariant given by the nuclear length. We will see that it is related to both the character degree $\chi(1)$ and the π-length $\ell_\pi(G)$ of G. Specifically, we will prove the following result.

THEOREM A. *Let G be a π-separable group, and let $\chi \in \mathrm{Irr}(G)$. Then $\mathrm{nl}(\chi)$ is bounded by both of the following numbers:*

(1) $2\ell_\pi(G)$.

(2) *The number of prime divisors (counting multiplicities) of $\chi(1)$.*

In fact, the bound given in (2) is a very easy consequence of the definition of the nucleus. When $\chi \in B_{\pi'}(G)$, however, we can get somewhat better bounds, and in this case neither bound seems to follow directly from the definitions.

THEOREM B. *Let G be a π-separable group, and let $\chi \in B_{\pi'}(G)$. Then $\mathrm{nl}(\chi)$ is bounded by both of the following numbers:*

(1) $\ell_\pi(G)$.

(2) *The number of prime divisors (counting multiplicities) of $\chi(1)_\pi$.*

Here $\chi(1)_\pi$ denotes the π-part of $\chi(1)$.

Section 2 is devoted to a summary of relevant definitions and results from [1, 2], leading to a precise definition of the nuclear length. Theorems A and B are proven in Section 3.

2. The Character Theory of π-Separable Groups

The purpose of this section is to define the nuclear length and to give a brief review of the relevant aspects of character theory for π-separable groups. The results of interest here come from the work of Gajendragadkar [1] and Isaacs [2]. We assume throughout the section that π is a set of primes and G is a π-separable group.

Recall that if χ is a character of G, then the determinantal order $o(\chi)$ is the order of $\det\chi$ in the group of linear characters of G. This concept forms the basis for the following definition from [1].

DEFINITION 2.1. Let G be a π-separable group. A character $\chi \in \operatorname{Irr}(G)$ is said to be *π-special* if the following conditions are satisfied:

(1) $\chi(1)$ is a π-number.
(2) If S is any subnormal subgroup of G and σ is any irreducible constituent of χ_S, then the determinantal order $o(\sigma)$ is a π-number.

The set of π-special characters of G is denoted by $\mathfrak{X}_\pi(G)$.

The following result is almost an immediate consequence of the above definition. It appears as [1, Proposition 4.1].

PROPOSITION 2.2. *Let G be a π-separable group, and let $\chi \in \operatorname{Irr}(G)$ be π-special. If $S \lhd\lhd G$ and $\sigma \in \operatorname{Irr}(S)$ is a constituent of χ_S, then σ is π-special.*

PROPOSITION 2.3. *Let G be a π-separable group, and let N be a normal subgroup of G such that G/N is a π-group. If $\eta \in \operatorname{Irr}(N)$ is π-special, then every irreducible constituent of η^G is π-special.*

PROOF. See [1, Proposition 4.5].

PROPOSITION 2.4. *Let G be a π-separable group, and let N be a normal subgroup of G such that G/N is a π'-group.*

(1) *If $\chi \in \operatorname{Irr}(G)$ is π-special, then χ_N is irreducible and π-special.*
(2) *If $\eta \in \operatorname{Irr}(N)$ is π-special and G-invariant, then η has a π-special extension $\hat{\eta} \in \operatorname{Irr}(G)$. Moreover, $\hat{\eta}$ is the only irreducible constituent of η^G that is π-special.*

PROOF. See [1, Proposition 4.3].

The next result, which appears as Proposition 7.1 of [1], shows that π-special characters have a remarkable multiplicative property.

PROPOSITION 2.5. *Let G be a π-separable group. If $\alpha \in \mathfrak{X}_\pi(G)$ and $\beta \in \mathfrak{X}_{\pi'}(G)$, then $\alpha\beta$ is irreducible. Moreover, if $\alpha' \in \mathfrak{X}_\pi(G)$ and $\beta' \in \mathfrak{X}_{\pi'}(G)$ satisfy $\alpha\beta = \alpha'\beta'$, then $\alpha = \alpha'$ and $\beta = \beta'$.*

As in [2], we will say that a character $\chi \in \operatorname{Irr}(G)$ is *π-factorable* if it can be written as a product $\alpha\beta$ with $\alpha \in \mathfrak{X}_\pi(G)$ and $\beta \in \mathfrak{X}_{\pi'}(G)$. The following result, which is an easy consequence of Propositions 2.2 and 2.5, is given as Corollary 2.6 of [2].

COROLLARY 2.6. *Let G be a π-separable group, and let $\chi \in \mathrm{Irr}(G)$ be π-factorable. If $S \lhd\lhd G$, then every irreducible constituent of χ_S is π-factorable.*

The main point of [2] is to use the concept of π-factorability as the basis for defining a certain subset $B_\pi(G)$ of $\mathrm{Irr}(G)$ containing the π-special characters. Because the definitions involved in the construction of $B_\pi(G)$ are very important to the work presented in the next section, we recall the construction in detail.

Consider the set $\mathscr{P}$ of all pairs (H, θ), where H is a subgroup of G and $\theta \in \mathrm{Irr}(H)$. If $(K, \varphi) \in \mathscr{P}$, then we write $(H, \theta) \leq (K, \varphi)$ if $H \subseteq K$ and θ is a constituent of φ_H. If $g \in G$, let $(H, \theta)^g = (H^g, \theta^g)$. Then $\leq$ is a partial order on $\mathscr{P}$, and the conjugation action of G respects the partial order. We will write $I_G(H, \theta)$ for the stabilizer of (H, θ) in G.

A pair (S, σ) is called a *π-factorable subnormal pair* if $S \lhd\lhd G$ and σ is π-factorable. The set of π-factorable subnormal pairs in G is denoted by $\mathscr{F}(G)$, and we write $\mathscr{F}^*(G)$ for the set of maximal elements of $\mathscr{F}(G)$. The following proposition describes a fundamental property of the partial order on $\mathscr{F}(G)$.

PROPOSITION 2.7. *Let G be a π-separable group, and let $\chi \in \mathrm{Irr}(G)$. Then there is a pair $(S, \sigma) \in \mathscr{F}^*(G)$ with $(S, \sigma) \leq (G, \chi)$. Moreover, if $(U, \varphi) \in \mathscr{F}(G)$ with $(U, \varphi) \leq (G, \chi)$, then $(U, \varphi)^g \leq (S, \sigma)$ for some $g \in G$.*

PROOF. See [2, Theorem 3.2].

PROPOSITION 2.8. *Let G be a π-separable group, let $(S, \sigma) \in \mathscr{F}^*(G)$, and set $I = I_G(S, \sigma)$. Then induction defines a bijection $\mathrm{Irr}(I|\sigma) \to \mathrm{Irr}(G|\sigma)$. Moreover, if $S < G$, then $I < G$.*

PROOF. See [2, Theorem 4.4 and Lemma 4.5].

We are now ready to present Isaacs' definition of the nucleus of a character. Suppose that G is π-separable, and let χ be any irreducible character of G. By Proposition 2.7 there is a pair $(S, \sigma) \in \mathscr{F}^*(G)$ with $(S, \sigma) \leq (G, \chi)$. Let $I = I_G(S, \sigma)$. Then Proposition 2.8 implies that there is a unique character $\psi \in \mathrm{Irr}(I|\sigma)$ such that $\psi^G = \chi$. In this way we associate a pair (I, ψ) with the original pair (G, χ), and (I, ψ) is unique up to G-conjugacy. The pair (I, ψ) is called a *standard inducing pair* for χ. If χ is π-factorable, then $S = G$ so that $(I, \psi) = (G, \chi)$. But if χ is not π-factorable, then $S < G$ and hence $I < G$ by Proposition 2.8. We can then repeat the process by finding a standard inducing pair for ψ. Proceeding by induction, we see that there is a chain

$$(G, \chi) = (I_0, \psi_0) > (I_1, \psi_1) > \cdots > (I_n, \psi_n) = (W, \gamma)$$

such that (I_j, ψ_j) is a standard inducing pair for ψ_{j-1} and γ is π-factorable. Any such pair (W, γ) is called a *nucleus* for χ, and the set of all nuclei for χ is denoted $\mathrm{nuc}(\chi)$.

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Thus $\operatorname{nuc}(\chi)$ consists of a single G-conjugacy class of pairs, and if (W, γ) is a nucleus for χ, then $\gamma^G = \chi$. Finally, $B_\pi(G)$ denotes the set of all $\chi \in \operatorname{Irr}(G)$ such that there is a pair $(W, \gamma) \in \operatorname{nuc}(\chi)$ with $\gamma \in \mathfrak{X}_\pi(G)$. Note that if (I, ψ) is a standard inducing pair for χ, then $\operatorname{nuc}(\psi) \subseteq \operatorname{nuc}(\chi)$. Thus $\chi \in B_\pi(G)$ if and only if $\psi \in B_\pi(I)$.

To reach the main subject of this paper, we now need only one simple observation about Isaacs' definition of the nucleus. Let $\chi \in \operatorname{Irr}(G)$. Because the standard inducing pair (I, ψ) is unique up to G-conjugacy, so is the pair (W, γ). But we can say more: in fact, the entire chain of pairs from (G, χ) to (W, γ) is unique up to G-conjugacy. In particular, the length of the chain is uniquely determined by χ. This observation suggests the following definition.

DEFINITION 2.9. Let G be a π-separable group, and let $\chi \in \operatorname{Irr}(G)$. Let $(W, \gamma) \in \operatorname{nuc}(\chi)$, and let

$$(G, \chi) = (I_0, \psi_0) > (I_1, \psi_1) > \cdots > (I_n, \psi_n) = (W, \gamma)$$

be a chain such that (I_j, ψ_j) is a standard inducing pair for ψ_{j-1}. Then we say that χ has *nuclear length n*. We write $\operatorname{nl}(\chi)$ for the nuclear length of χ.

The nuclear length can be regarded as a number that measures how close χ is to being π-factorable. Indeed, it is immediate from the definition that χ is π-factorable if and only if $\operatorname{nl}(\chi) = 0$.

We end this section with a few more results that will be needed to study nuclear lengths in the next section. Because it seems to be convenient for our purposes to focus on $B_{\pi'}(G)$ instead of $B_\pi(G)$, we have stated the results below for this set of characters.

PROPOSITION 2.10. *Let G be a π-separable group, and let N be a normal subgroup of G. Suppose that $\eta \in \operatorname{Irr}(N)$.*

(1) *Assume that G/N is a π'-group and $\chi \in \operatorname{Irr}(G|\eta)$. Then $\eta \in B_{\pi'}(N)$ if and only if $\chi \in B_{\pi'}(G)$.*
(2) *Assume that G/N is a π-group. Then $\eta \in B_{\pi'}(N)$ if and only if there exists a character $\chi \in B_{\pi'}(G)$ lying over η. Moreover, in this case χ is the unique irreducible constituent of η^G in $B_{\pi'}(G)$.*

PROOF. See [2, Theorems 6.2 and 7.1].

The following proposition is essentially proven in [2, Theorem 6.2 and Proposition 7.3]. Because not all parts are explicitly stated there, we have sketched a proof for the convenience of the reader.

PROPOSITION 2.11. *Let G be a π-separable group and $N \lhd G$ such that G/N is a π-group. Let $\eta \in \mathrm{Irr}(N)$, and let $\chi \in \mathrm{Irr}(G|\eta)$. Suppose that $(U, \varphi) \in \mathscr{F}^*(N)$ with $(U, \varphi) \leq (N, \eta)$ and $(S, \sigma) \in \mathscr{F}^*(G)$ with $(U, \varphi) \leq (S, \sigma) \leq (G, \chi)$. Let (J, ξ) and (I, ψ) be the corresponding standard inducing pairs for η and for χ respectively. Then*

(1) $U = S \cap N$.
(2) *I normalizes J.*
(3) $I \cap J \lhd IJ$.
(4) *$I/(I \cap J)$ and $J/(I \cap J)$ are π-groups.*

Moreover, if $\chi \in B_{\pi'}(G)$, then

(5) *$I = I_G(U, \varphi)$ and $J = I \cap N$.*
(6) *ψ lies over ξ.*

PROOF. The fact that $U = S \cap N$ is [2, Lemma 3.3]. Moreover, [2, Lemma 6.1 and Proposition 7.3] imply that I normalizes J, $I \cap J \lhd IJ$, $|I : I \cap J|$ divides $|G : N|^2$, and $|J : I \cap J|$ divides $|G : N|$. Hence $I/(I \cap J)$ and $J/(I \cap J)$ are π-groups.

Now assume that $\chi \in B_{\pi'}(G)$. Then σ is π'-special by [2, Lemma 5.2], and S/U is a π-group because it is isomorphic to a subgroup of G/N. Thus Proposition 2.4 implies that σ_U is irreducible and π'-special, so it follows that $\sigma_U = \varphi$. Moreover, $U = S \cap N \lhd I$ and I stabilizes $\sigma_U = \varphi$, so $I \subseteq I_G(U, \varphi)$. By [2, Lemma 6.1] we know that $I_G(U, \varphi)$ normalizes S. But $I_G(U, \varphi)$ must also stabilize σ because σ is the unique π'-special extension of φ by Proposition 2.4. Hence $I = I_G(U, \varphi)$, and $J = I \cap N$.

Now let ξ_0 be an irreducible constituent of ψ_J lying over φ, and set $\eta_0 = \xi_0^N$. Then Proposition 2.8 implies that η_0 is an irreducible constituent of χ_N, so $\eta_0^x = \eta$ for some $x \in G$. Then $(U, \varphi)^x \leq (N, \eta)$ and $(U, \varphi)^x \in \mathscr{F}^*(N)$, so Proposition 2.7 shows that there is an $n \in N$ with $(U, \varphi)^{xn} = (U, \varphi)$. It follows that $xn \in I$, so we may assume without loss of generality that $x \in I$. Then $\xi_0^x \in \mathrm{Irr}(J|\varphi)$ and $(\xi_0^x)^N = \eta = \xi^N$, so Proposition 2.8 implies that $\xi_0^x = \xi$. Hence ξ is a constituent of ψ_J, and (6) follows. This completes the proof.

3. Bounding the Nuclear Length

In this section we study the nuclear length of a character $\chi \in \mathrm{Irr}(G)$ and obtain the bounds given in Theorems A and B. Because the strategy is to proceed by induction on the order of the group G, we begin by considering a normal subgroup N of G, and we try to relate the nuclear length of χ to that of an irreducible constituent of χ_N. The first step in this direction is a technical lemma.

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LEMMA 3.1. *Let G be a π-separable group, and let N be a normal subgroup such that G/N is either a π-group or a π'-group. Let $\eta \in \mathrm{Irr}(N)$, and let $\chi \in \mathrm{Irr}(G|\eta)$. Let $(U, \varphi) \in \mathscr{F}^*(N)$ with $(U, \varphi) \leq (N, \eta)$, and let (J, ξ) be the corresponding standard inducing pair for η. Then there is a pair $(S, \sigma) \in \mathscr{F}^*(G)$ with $(U, \varphi) \leq (S, \sigma) \leq (G, \chi)$ such that if (I, ψ) is the corresponding standard inducing pair for χ, then $(\psi_{I \cap J}, \xi_{I \cap J}) \neq 0$.*

PROOF. It suffices to consider the case in which G/N is a π-group. We prove the result by induction on $\mathrm{nl}(\chi)$. If $\mathrm{nl}(\chi) = 0$, then χ and η are both π-factorable by Corollary 2.6. Hence $(J, \xi) = (N, \eta)$ and $(I, \psi) = (G, \chi)$, so the conclusion of the lemma is satisfied.

Now assume that $\mathrm{nl}(\chi) > 0$ so that χ is not π-factorable. Choose $(S, \sigma) \in \mathscr{F}^*(G)$ with $(U, \varphi) \leq (S, \sigma) \leq (G, \chi)$, and let (I, ψ) be the standard inducing pair for χ determined by (S, σ). Then Proposition 2.11 implies that I normalizes J, $I \cap J \lhd IJ$, and $I/(I \cap J)$ and $J/(I \cap J)$ are π-groups.

Let $\theta = \psi^{IJ}$, and let $\xi_0 \in \mathrm{Irr}(J|\varphi)$ be a constituent of θ_J. Then $\xi_0^N \in \mathrm{Irr}(N)$ by Proposition 2.8, and χ lies over ξ_0^N because it lies over ξ_0. Hence $\xi_0^N = \eta^x$ for some $x \in G$. It follows that $(U, \varphi) \leq (N, \eta^x)$, so $(U, \varphi)^{x^{-1}} \leq (N, \eta)$ and $(U, \varphi)^{x^{-1}} \in \mathscr{F}^*(N)$. Thus there is an $n \in N$ such that $(U, \varphi)^{x^{-1}} = (U, \varphi)^n$, and $nx \in I_G(U, \varphi)$. Hence we may assume that $x \in I_G(U, \varphi)$ so that x normalizes J. Moreover, ξ^x lies over $\varphi^x = \varphi$ and $(\xi^x)^N = (\xi^N)^x = \eta^x = \xi_0^N$, so $\xi^x = \xi_0$ by Proposition 2.8. Thus $(J, \xi^x) \leq (IJ, \theta)$, and $(J, \xi) \leq (IJ, \theta)^{x^{-1}}$.

If we now replace (S, σ) by $(S, \sigma)^{x^{-1}}$, then (I, ψ) is replaced by $(I, \psi)^{x^{-1}}$ and (IJ, θ) by $(IJ, \theta)^{x^{-1}}$. Hence we may assume that $(J, \xi) \leq (IJ, \theta)$. Then ξ is a constituent of $\theta_J = (\psi^{IJ})_J = (\psi_{I \cap J})^J$, so there is a character $\zeta \in \mathrm{Irr}(I \cap J)$ that is a constituent of both $\psi_{I \cap J}$ and $\xi_{I \cap J}$. This completes the proof.

LEMMA 3.2. *Let G be a π-separable group, and let N be a normal subgroup of G such that G/N is either a π-group or a π'-group. Let $\eta \in \mathrm{Irr}(N)$, and suppose that there is a subgroup $U \lhd\lhd N$ such that $(U, \varphi) \in \mathrm{nuc}(\eta)$. If $\chi \in \mathrm{Irr}(G|\eta)$, then* $\mathrm{nl}(\chi) \leq 1$.

PROOF. It suffices to consider the case in which G/N is a π-group. Let $(S, \sigma) \in \mathscr{F}^*(G)$ with $(U, \varphi) \leq (S, \sigma) \leq (G, \chi)$, and let (I, ψ) be the standard inducing pair for χ determined by (S, σ). Since $U \lhd\lhd N$ and $(U, \varphi) \in \mathrm{nuc}(\eta)$, it follows that $(U, \varphi) \in \mathscr{F}^*(N)$ and $I_N(U, \varphi) = U$. Then Proposition 2.11 shows that $U \lhd I$ and I/U is a π-group, so I/S is a π-group.

Write $\sigma = \alpha\beta$, where $\alpha \in \mathfrak{X}_\pi(S)$ and $\beta \in \mathfrak{X}_{\pi'}(S)$. Then β is invariant in I and $|I : S|$ is a π-number, so β extends to $\hat{\beta} \in \mathfrak{X}_{\pi'}(I)$ by Proposition 2.4. Hence $\sigma^I = (\alpha\beta)^I = \alpha^I \hat{\beta}$. All irreducible constituents of α^I are π-special by Proposition 2.3,

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so all irreducible constituents of σ^I are π-factorable. In particular, ψ is π-factorable, and $\mathrm{nl}(\chi) \leq 1$.

PROPOSITION 3.3. *Let G be a π-separable group, and let N, H_1, and H_2 be subgroups of G such that $N \lhd H_1$ and $N \lhd H_2$. Suppose that H_1/N and H_2/N are π-groups. Let $\eta \in \mathrm{Irr}(N)$, and let $\chi_1 \in \mathrm{Irr}(H_1|\eta)$ and $\chi_2 \in \mathrm{Irr}(H_2|\eta)$. Then* $|\mathrm{nl}(\chi_1) - \mathrm{nl}(\chi_2)| \leq 1$.

PROOF. We prove the result by induction on $\mathrm{nl}(\chi_1)$. If $\mathrm{nl}(\chi_1) = 0$, then χ_1 and η are π-factorable by Corollary 2.6. Hence $(N, \eta) \in \mathrm{nuc}(\eta)$, and Lemma 3.2 shows that $\mathrm{nl}(\chi_2) \leq 1$. Thus $|\mathrm{nl}(\chi_1) - \mathrm{nl}(\chi_2)| \leq 1$ in this case. Similarly, we get the desired result if $\mathrm{nl}(\chi_2) = 0$.

Thus we may assume that $\mathrm{nl}(\chi_1) > 0$ and $\mathrm{nl}(\chi_2) > 0$. Let $(U, \varphi) \in \mathcal{F}^*(N)$ with $(U, \varphi) \leq (N, \eta)$, and let (J, ξ) be the corresponding standard inducing pair for η. For $i = 1, 2$ Lemma 3.1 implies that there is a pair $(S_i, \sigma_i) \in \mathcal{F}^*(H_i)$ with $(U, \varphi) \leq (S_i, \sigma_i) \leq (H_i, \chi_i)$ such that if (I_i, ψ_i) is the corresponding standard inducing pair for χ_i, then $((\psi_i)_{I_i \cap J}, \xi_{I_i \cap J}) \neq 0$. Moreover, Proposition 2.11 shows that $I_i \cap J \lhd I_i J$, and $I_i/(I_i \cap J)$ and $J/(I_i \cap J)$ are π-groups. In particular, it follows that $O^\pi(I_i) = O^\pi(I_i \cap J) = O^\pi(J)$ for $i = 1, 2$. Set $M = O^\pi(J)$. Let $\theta_1 \in \mathrm{Irr}(M)$ be a common constituent of $(\psi_1)_M$ and ξ_M, and let $\theta_2 \in \mathrm{Irr}(M)$ be a common constituent of $(\psi_2)_M$ and ξ_M. Then there is an $x \in J$ such that $\theta_1 = \theta_2^x$. If we replace (H_2, χ_2) by $(H_2, \chi_2)^x$ and (S_2, σ_2) by $(S_2, \sigma_2)^x$, then we may assume that $\theta_1 = \theta_2$.

Since $M = O^\pi(I_1) = O^\pi(I_2)$, we know that $M \lhd I_1$ and $M \lhd I_2$, and I_1/M and I_2/M are π-groups. Moreover, $\psi_1 \in \mathrm{Irr}(I_1|\theta_1)$ and $\psi_2 \in \mathrm{Irr}(I_2|\theta_1)$. Thus it follows by induction that

$$|\mathrm{nl}(\chi_1) - \mathrm{nl}(\chi_2)| = |\mathrm{nl}(\psi_1) - \mathrm{nl}(\psi_2)| \leq 1,$$

and this completes the proof.

We will use Proposition 3.3 in the following form.

COROLLARY 3.4. *Let G be a π-separable group, and let N be a normal subgroup of G such that G/N is either a π-group or a π'-group. If $\eta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G|\eta)$, then* $\mathrm{nl}(\chi) \leq 1 + \mathrm{nl}(\eta)$.

PROOF. It suffices to consider the case in which G/N is a π-group. The result then follows by applying Proposition 3.3 with $H_1 = G$, $H_2 = N$, $\chi_1 = \chi$, and $\chi_2 = \eta$.

THEOREM 3.5. *Let $G \neq 1$ be a π-separable group, and let $\chi \in \mathrm{Irr}(G)$. Let $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ be any chain of subgroups such that G_i/G_{i-1} is either a π-group or a π'-group for $1 \leq i \leq n$. Then* $\mathrm{nl}(\chi) \leq n - 1$.

PROOF. We prove the result by induction on n. If $n = 1$, then G is either a π-group or a π'-group, so $\mathrm{nl}(\chi) = 0 = n - 1$.

Now assume that $n > 1$, and let $\eta \in \mathrm{Irr}(G_{n-1})$ be a constituent of $\chi_{G_{n-1}}$. Then Corollary 3.4 shows that $\mathrm{nl}(\chi) \leq 1 + \mathrm{nl}(\eta)$. Moreover, we know by induction that $\mathrm{nl}(\eta) \leq n - 2$, so it follows that $\mathrm{nl}(\chi) \leq n - 1$.

We now obtain Theorem A as an easy consequence of this result.

PROOF OF THEOREM A. To prove (1), consider the chain of subgroups

$$1 \lhd O_{\pi'}(G) \lhd O_{\pi'\pi}(G) \lhd O_{\pi'\pi\pi'}(G) \lhd \cdots \lhd G.$$

The number of successive quotients in this series that are π-groups is $\ell_\pi(G)$, and the number that are π'-groups is at most $\ell_\pi(G) + 1$. Thus the total number of quotients in the series is at most $2\ell_\pi(G) + 1$, and Theorem 3.5 implies that $\mathrm{nl}(\chi) \leq 2\ell_\pi(G)$.

To prove (2), we merely use the definition of the nucleus. If $(W, \gamma) \in \mathrm{nuc}(\chi)$, then there is a chain

$$(W, \gamma) = (I_n, \psi_n) < (I_{n-1}, \psi_{n-1}) < \cdots < (I_0, \psi_0) = (G, \chi)$$

in which (I_j, ψ_j) is a standard inducing pair for ψ_{j-1}. Since $W = I_n < I_{n-1} < \cdots < I_0 = G$, we see that $n = \mathrm{nl}(\chi)$ is at most the number of prime divisors (counting multiplicities) of $|G : W|$. Because $|G : W|$ divides $\gamma^G(1) = \chi(1)$, the result follows.

We now turn our attention to characters in $B_{\pi'}(G)$. The assumption that $\chi \in B_{\pi'}(G)$ allows us to obtain better bounds on $\mathrm{nl}(\chi)$ with rather little additional effort. Indeed, the following proposition already improves upon the conclusion of Corollary 3.4.

PROPOSITION 3.6. *Let G be a π-separable group, and let N be a normal subgroup such that G/N is a π'-group. Let $\chi \in B_{\pi'}(G)$, and let η be an irreducible constituent of χ_N. Then* $\mathrm{nl}(\chi) = \mathrm{nl}(\eta)$.

PROOF. We prove the result by induction on $|G|$. If $\mathrm{nl}(\chi) = 0$, then $\chi \in B_{\pi'}(G)$ is π-factorable and hence χ is π'-special. Then η is also π'-special by Proposition 2.2, so $\mathrm{nl}(\eta) = 0$. Conversely, suppose that $\mathrm{nl}(\eta) = 0$. Because $\eta \in B_{\pi'}(N)$ by Proposition 2.10, it follows that η is π'-special. Then Proposition 2.3 implies that χ is π'-special, so $\mathrm{nl}(\chi) = 0$.

Now suppose that $\mathrm{nl}(\chi) > 0$ so that $\mathrm{nl}(\eta) > 0$ as well. Let $(U, \varphi) \in \mathscr{F}^*(N)$ with $(U, \varphi) \leq (N, \eta)$, and let (J, ξ) be the corresponding standard inducing pair for η. Then by Lemma 3.1 there is a pair $(S, \sigma) \in \mathscr{F}^*(G)$ with $(U, \varphi) \leq (S, \sigma) \leq (G, \chi)$ such that if (I, ψ) is the corresponding standard inducing pair for χ, then $(\psi_{I \cap J}, \xi_{I \cap J}) \neq 0$. Let $\zeta \in \mathrm{Irr}(I \cap J)$ be a constituent of both $\psi_{I \cap J}$ and $\xi_{I \cap J}$. Because $\mathrm{nl}(\chi) > 0$

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and $\mathrm{nl}(\eta) > 0$, we have $I < G$ and $J < N \subseteq G$. Moreover, Proposition 2.11 shows that I normalizes J, $I \cap J \lhd IJ$, and $I/(I \cap J)$ and $J/(I \cap J)$ are both π'-groups. Thus by induction it follows that $\mathrm{nl}(\psi) = \mathrm{nl}(\zeta)$ and $\mathrm{nl}(\xi) = \mathrm{nl}(\zeta)$. Hence $\mathrm{nl}(\chi) = 1 + \mathrm{nl}(\psi) = 1 + \mathrm{nl}(\xi) = \mathrm{nl}(\eta)$, and this completes the proof.

PROPOSITION 3.7. *Let G be a π-separable group and N a normal subgroup of G such that G/N is a π-group. Let $\chi \in B_{\pi'}(G)$, and let η be an irreducible constituent of χ_N. If η is G-invariant, then* $\mathrm{nl}(\chi) = \mathrm{nl}(\eta)$.

PROOF. We prove the result by induction on $\mathrm{nl}(\eta)$. If $\mathrm{nl}(\eta) = 0$, then η is π'-special because $\eta \in B_{\pi'}(G)$ by Proposition 2.10. Since η is G-invariant by assumption, Proposition 2.4 shows that η has a unique π'-special extension $\hat{\eta} \in \mathrm{Irr}(G)$. In particular, $\hat{\eta} \in B_{\pi'}(G)$. Thus $\chi = \hat{\eta}$ by Proposition 2.10, and $\mathrm{nl}(\chi) = 0$.

Now assume that $\mathrm{nl}(\eta) \geq 1$. Choose $(U, \varphi) \in \mathscr{F}^*(N)$ with $(U, \varphi) \leq (N, \eta)$ and $(S, \sigma) \in \mathscr{F}^*(G)$ with $(U, \varphi) \leq (S, \sigma) \leq (G, \chi)$. Let (J, ξ) and (I, ψ) be the corresponding standard inducing pairs for η and for χ respectively. Then Proposition 2.11 shows that $U = S \cap N$, $J = I \cap N$, $I = I_G(U, \varphi)$, and ψ lies over ξ. Hence I/J is a π-group. Moreover, if $x \in I$, then x stabilizes (U, φ) and (N, η), and x normalizes J. Hence x must stabilize ξ, and we see that ξ is I-invariant. Thus it follows by induction that $\mathrm{nl}(\chi) = \mathrm{nl}(\psi) + 1 = \mathrm{nl}(\xi) + 1 = \mathrm{nl}(\eta)$, as desired.

THEOREM 3.8. *Let G be a π-separable group and $\chi \in B_{\pi'}(G)$. Let $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ be a chain of subgroups such that G_i/G_{i-1} is either a π-group or a π'-group for $1 \leq i \leq n$. Choose a corresponding chain $(G_0, \chi_0) \leq (G_1, \chi_1) \leq \cdots \leq (G_n, \chi_n) = (G, \chi)$. Then $\mathrm{nl}(\chi)$ is at most the number of indices i with $1 \leq i \leq n$ for which the following two conditions are satisfied:*

(1) *G_i/G_{i-1} is a π-group.*
(2) *χ_{i-1} is not G_i-invariant.*

PROOF. We prove the result by induction on n. If $n \leq 1$, then G is either a π-group or a π'-group, so $\mathrm{nl}(\chi) = 0$.

Now suppose that $n > 1$. For $1 \leq m \leq n$ let $f(m)$ be the number of indices i with $1 \leq i \leq m$ such that (1) and (2) hold. Then we know by induction that $\mathrm{nl}(\chi_{n-1}) \leq f(n-1)$. If G_n/G_{n-1} is a π'-group, then $f(n-1) = f(n)$ and $\mathrm{nl}(\chi) = \mathrm{nl}(\chi_{n-1})$ by Proposition 3.6. Similarly, when G_n/G_{n-1} is a π-group and χ_{n-1} is G-invariant, it follows that $f(n-1) = f(n)$ and $\mathrm{nl}(\chi) = \mathrm{nl}(\chi_{n-1})$ by Proposition 3.7. Thus we may assume that G_n/G_{n-1} is a π-group and χ_{n-1} is not G-invariant. In this case $f(n) = f(n-1) + 1$ and $\mathrm{nl}(\chi) \leq \mathrm{nl}(\chi_{n-1}) + 1$ by Corollary 3.4, so we again conclude that $\mathrm{nl}(\chi) \leq f(n)$.

Finally, we can now prove Theorem B.

PROOF OF THEOREM B. The bound given in (1) follows by applying Theorem 3.8 to the chain of subgroups

$$1 \lhd O_{\pi'}(G) \lhd O_{\pi'\pi}(G) \lhd O_{\pi'\pi\pi'}(G) \lhd \cdots \lhd G.$$

We prove by induction on $|G|$ that the number described in (2) also bounds $\mathrm{nl}(\chi)$. Let N be a maximal normal subgroup of G, and let $\eta \in \mathrm{Irr}(N)$ be a constituent of χ_N. Then there is an integer $e \geq 1$ such that

$$\chi(1) = e|G : I_G(\eta)|\eta(1),$$

and e divides $|G : N|$. If G/N is a π'-group, then $\mathrm{nl}(\chi) = \mathrm{nl}(\eta)$ by Proposition 3.6. But $\chi(1)_\pi = \eta(1)_\pi$, so the result follows by induction.

Now suppose that G/N is a π-group. Then $\mathrm{nl}(\chi) \leq \mathrm{nl}(\eta) + 1$ by Corollary 3.4, and $\eta(1)_\pi$ divides $\chi(1)_\pi$, so it suffices by induction to show that if $\mathrm{nl}(\chi) = \mathrm{nl}(\eta) + 1$, then $\chi(1)_\pi \neq \eta(1)_\pi$. But we know by Proposition 3.7 that if $\mathrm{nl}(\chi) = \mathrm{nl}(\eta) + 1$, then η is not G-invariant. Hence $\chi(1)_\pi = e|G : I_G(\eta)|\eta(1)_\pi \neq \eta(1)_\pi$, and this completes the proof.

It would be interesting to know whether the bounds on $\mathrm{nl}(\chi)$ given in Theorems A and B are the best possible in any reasonable sense. In the case of Theorem B, for instance, it is easy to find individual examples of a π-separable group G with a character $\chi \in B_{\pi'}(G)$ such that $\mathrm{nl}(\chi) = \ell_\pi(G)$. But for every nonnegative integer n we can ask whether there is a π-separable group G of π-length n and a character $\chi \in B_{\pi'}(G)$ of nuclear length n. This question, along with the analogous questions for the other bounds on $\mathrm{nl}(\chi)$, seems to be difficult to answer.

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