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ON THE RUIN PROBABILITY OF A GENERALIZED CRAMÉR–LUNDBERG MODEL DRIVEN BY MIXED POISSON PROCESSES

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Abstract

We propose a generalized Cramér–Lundberg model of the risk theory of non-life insurance and study its ruin probability. Our model is an extension of that of Dubey (1977) to the case of multiple insureds, where the counting process is a mixed Poisson process and the continuously varying premium rate is determined by a Bayesian rule on the number of claims. We use two proofs to show that, for each fixed value of the safety loading, the ruin probability is the same as that of the classical Cramér–Lundberg model and does not depend on either the distribution of the mixing variable of the driving mixed Poisson process or the number of claim contracts.

Keywords: Risk theory; varying insurance premium; conditional distribution; Bayesian estimator; adjustment coefficient

2020 Mathematics Subject Classification: Primary 91G05
Secondary 60G44; 62P05

1. Introduction

Risk theory plays an essential role in non-life insurance mathematics and dates back to the work of Lundberg (1903). Some of the main purposes of the theory are to model the surplus of an insurance company, derive the ruin probability, and measure various risks such as Gerber–Shiu functions (Gerber and Shiu, 1998). In the classical Cramér–Lundberg model, the loss process is a compound Poisson process with constant intensity, and the premium parameter is also a constant. There is a vast literature on the classical Cramér–Lundberg model, its generalizations, and risk measures; see, e.g., Rolski et al. (1998), Asmussen and Albrecher (2010), and Klugman et al. (2012).

The study by Dubey (1977) is an important work on this topic because he proposed several ruin models driven by a mixed Poisson process instead of a Poisson process. Dubey's models describe the scenario in which the true intensity of the insured is unknown at the beginning of the insurance contract. He studied the ruin probability of some models with a mixed Poisson

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process and adaptive premium. For a single insured and continuously varying premium, Dubey studied the ruin probability for each of the following methods of changing insurance premiums: the Bayesian estimator, historical frequency, and Bühlmann estimator. Dubey derived the exact ruin probability theoretically in the case of the Bayesian estimator and historical frequency, and derived the asymptotic result in the case of the Bühlmann estimator. He also proposed discrete-time models of a single insured and a continuous model of multiple insureds with the premium determined by the Bühlmann estimator. Other scholars that have studied ruin models using a mixed Poisson process include Grandell (1997) and Minkova (2004). The latter proposed a ruin model with a gamma-distributed intensity.

There are both discrete- and continuous-time models in the literature for which the premium is adapted to the claims. Tsai and Parker (2004) used a discrete-time model with adapted premiums determined by Bühlmann's method and compared, through Monte Carlo simulations, the ruin probability with that of the classical ruin model. Trufin and Loisel (2013) used a similar model, and proved that the adaptive premium policy decreases the ruin probability under some assumptions. Afonso (2008) and Afonso et al. (2010) studied a continuous-time model with discretely varying premiums, and presented numerical evaluations of the ruin probability. Asmussen (1999) assumed that the company continuously modifies the premium using the sample mean of the number of observed claims, even though credibility theory or the Bayes method is not incorporated. Landriault et al. (2012) and Li et al. (2015) used models in which the premium is determined not only by the number of claims but also by the surplus. Igarashi (2015) considered a model in which both the intensity of the driving mixed Poisson process and the premium rate have two values, and derived the optimal timing for switching the premium rate from the higher value to the lower value.

In previous studies, the surplus processes of an insurance portfolio with dependent business classes have also been investigated. Müller and Pflug (2001) derived a Lundberg-type result for the asymptotic ruin probability of risk models with dependent increments under the assumption of the existence of the probability-generating function. Cossette and Marceau (2000) modeled the dependence using a Poisson shock process, and Boudreault et al. (2006) proposed a model that uses the dependence structure between interclaim times and claim sizes.

In this study, we extend the work of Dubey (1977) to study the ruin probability. In one of Dubey's models it is assumed that there is only one insured, the counting process is a mixed Poisson process, and the continuously varying premium rate is determined by a Bayesian rule based on the number of claims. He showed that, for each fixed value of the safety loading, the ruin probability of the model is the same as that of the classical Cramér–Lundberg model, and does not depend on the distribution of the mixing variable of the driving mixed Poisson process. We propose an extended model in which the insurance portfolio has one or more insureds. The counting process for each insured is a mixed Poisson process, and the premium rate of each insured is determined by a Bayesian rule on the number of claims. This describes the scenario in which the true intensity of each insured is unknown at the start of the insurance contract and is estimated by the observed number of claims. In this scenario, the interclaim times of each insured are autocorrelated because the intensity is a random variable, but the interclaim times of different insureds are independent.

In the proposed model, we show that, for each fixed value of the safety loading, the ruin probability of the model is the same as that of the classical Cramér–Lundberg model, and does not depend on either the distribution of the mixing variable of the driving mixed Poisson process or the number of insurance contracts; that is, we show that the result of Dubey's model with the premium based on the Bayesian estimator can be extended to multiple insureds. This result is also consistent with the result of Dubey's model that the ruin probability of multiple insureds with the premium based on the Bühlmann estimator converges to the ruin probability of the classical model when the number of insureds approaches infinity. The key lemma for the proof of our theorem is that the conditional distribution of a random variable associated with the mixed Poisson process, given the history of the process up to an arrival time, is an exponential distribution with parameter 1, regardless of the distribution of the mixing variable of the process. We provide two proofs of the result: the first uses Bayes' theorem in the same manner as Dubey (1977), and the second uses stochastic calculus.

This paper is organized as follows. In the next section we define our generalized Cramér–Lundberg model, state our results, and present a numerical example. We provide the first and second proofs in Sections 3 and 4, respectively.

2. Our setting and results

We consider the following generalization of the classical Cramér–Lundberg model of the risk theory of non-life insurance.

Definition 1. We consider an insurance portfolio of n insurance policies. We consider a surplus process in the form $U(t) := u_0 + \int_0^t c(s)\,\mathrm{d}s - \sum_{k=1}^{N(t)} X_k$, $t \geq 0$, where the variables are defined as follows:

- The claim number processes of the n insured people, $N_i = \{N_i(t)\}_{t\in[0,\infty)}$, $i = 1, \ldots, n$, are independent mixed Poisson processes, and their aggregation is denoted by $N(t) := \sum_{i=1}^{n} N_i(t)$, $t \geq 0$. For each $i \in \{1, \ldots, n\}$, the mixing random variable Λ_i of N_i, obtained by $\Lambda_i = \lim_{t\to\infty} N_i(t)/t$ almost surely (a.s.), is assumed to satisfy $\mathbb{E}[\Lambda_i] < \infty$. We do not assume that the Λ_is are identically distributed. The distribution function of Λ_i is denoted by $F_i(\,\cdot\,)$. Moreover, the filtration generated by process N_i is denoted by $\mathcal{F}_i = \{\mathcal{F}_i(t)\}_{t\in[0,\infty)}$.
- Let the claim sizes $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) positive random variables that are independent of the processes N_i. We assume that $\mu := \mathbb{E}[X_k] < \infty$.
- The instantaneous premium rate $c = \{c(t)\}_{t\in[0,\infty)}$ is defined as

$$c(t) := (1+\theta)\mu \sum_{i=1}^{n} \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(t)], \qquad t \geq 0,$$

  where the safety loading θ is a positive constant.
- The initial surplus u_0 is a positive constant.

Furthermore, we define the following four concepts:

- The filtration $\mathcal{F} = \{\mathcal{F}(t)\}_{t\in[0,\infty)}$ is defined as $\mathcal{F}(t) := \mathcal{F}_1(t) \vee \cdots \vee \mathcal{F}_n(t)$.
- Let τ_k be the kth arrival time of N for each positive integer k. We set $\tau_0 := 0$.
- Let τ be the ruin time defined as $\tau := \inf\{t \mid U(t) < 0\}$, where $\inf \emptyset := \infty$ by convention.
- Finally, for each $k \in \{1, 2, \ldots\}$, the random variable I_k is defined as

$$I_k := \int_{\tau_{k-1}}^{\tau_k} \sum_{i=1}^{n} \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(t)]\,\mathrm{d}t.$$

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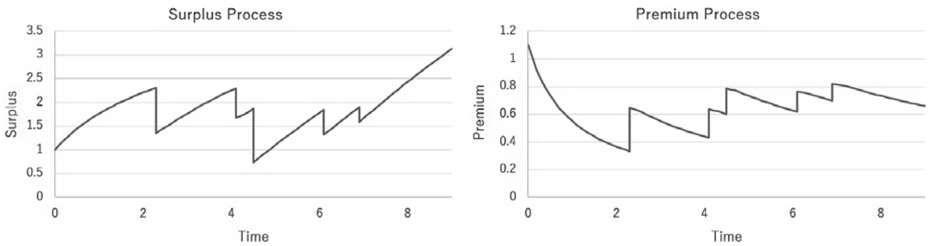

FIGURE 1. Typical sample path of the surplus process and premium rate.

For comparison with our model, we introduce the surplus of the classical Cramér–Lundberg model: $U^0(t) := u_0 + ct - \sum_{k=1}^{N^0(t)} X_k,\ t \geq 0$. The intensity of the Poisson process N^0 is the constant λ, and the premium rate c is defined as $c := (1+\theta)\mu\lambda$. The setting is the same as in Definition 1 except for the claim number process and premium rate. The ruin probability of the classical Cramér–Lundberg model and the ruin time are defined as

$$\psi^0(u) := \mathbb{P}\Big(\inf_{t\in[0,\infty)} U^0(t) < 0\Big),$$

$$\tau^0 := \inf\big\{t \mid U^0(t) < 0\big\},$$

where $\inf \emptyset := \infty$. Additionally, τ_k^0 is defined as the kth arrival time of N^0 for each positive integer k. Note that the ruin probability $\psi^0(u)$ does not depend on the constant intensity λ.

Remark 1. Compared with the classical Cramér–Lundberg model, where intensity is a constant, the intensity Λ_i in our generalized model is a random variable that describes the scenario in which the true intensity of each insured is unknown at the start of the insurance contract. The instantaneous premium rate $c(t)$ at time t is determined as the aggregation of the best estimators for each Λ_i using the history of N_i up to the time (multiplied by the constant $(1+\theta)\mu$).

Remark 2. Using Bayes' theorem, Dubey (1977) showed that the expression $\mathbb{E}[\Lambda_i \mid \mathcal{F}_i(t)]$ equals

$$\frac{\int_0^\infty \lambda^{1+N_i(t)} \mathrm{e}^{-\lambda t}\, \mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(t)} \mathrm{e}^{-\lambda t}\, \mathrm{d}F_i(\lambda)}.$$

See also Proposition 4.1 of Grandell (1997).

Remark 3. Figure 1 shows a typical sample path of our surplus process $U(t)$ and the corresponding path of the premium rate $c(t)$, where $n=1$, $\Lambda_1 \sim \Gamma(1, 1)$, $X_k \sim \Gamma(2, 2)$, $u_0 = 1$, and $\theta = \frac{1}{10}$.

For our generalized model, we have the following theorem; the second assertion addresses the ruin probability.

Theorem 1. *(i)* $\{U(\tau_k)\}_{k\in\{0,1,\dots\}} \overset{\mathrm{d}}{=} \big\{U^0\big(\tau_k^0\big)\big\}_{k\in\{0,1,\dots\}}$*; that is, the distribution of the discrete-time process* $\{U(\tau_k)\}_{k\in\{0,1,\dots\}}$ *depends on neither the number n nor the distributions of mixing variables* Λ_i.

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(ii) Furthermore, the ruin probability of our model satisfies $\mathbb{P}\big(\inf_{t\in[0,\infty)} U_t < 0\big) = \mathbb{P}\big(\inf_{t\in[0,\infty)} U_t^0 < 0\big)$*; that is, the ruin probability depends on neither the number n nor the distributions of mixing variables* Λ_i*.*

Remark 4. The distribution of $\{U(\tau_k)\}_{k\in\{1,2,\dots\}}$ and ruin probability $\mathbb{P}(\inf_t U_t < 0)$ depend on the value of the safety loading θ and on the distribution of the X_k. Note also that the second assertion of Theorem 1 does not hold for the Laplace transform of the time to ruin or more general Gerber–Shiu functions.

If the X_k have a light-tailed distribution, that is, there is some $\delta > 0$ that satisfies $\mathbb{E}[\exp(\delta X_k)] < \infty$, then we also have the following.

Corollary 1. *If the* X_k *have a light-tailed distribution and a positive R exists that satisfies* $\mathbb{E}[\exp(RX_k)] - 1 - (1+\theta)\mu R = 0$*, then this is the Lundberg adjustment coefficient of our generalized Cramér–Lundberg model, and the ruin probability is written as*

$$\mathbb{P}\Big(\inf_{t\in[0,\infty)} U_t < 0\Big) = \frac{\exp(-Ru)}{\mathbb{E}[\,\exp(-RU_\tau)\mid \tau < \infty]}.$$

The following lemma is the key to the proof of our theorem.

Lemma 1. *For each* $k \in \{0, 1, \dots\}$*, the random variable* I_{k+1} *is independent of* $\mathcal{F}(\tau_k)$ *and is exponentially distributed with mean 1.*

The next proposition addresses the dependence of $\{U(\tau_k)\}_{k\in\{1,2,\dots\}}$ and the Λ_i.

Proposition 1. *Suppose that at least one* Λ_i *is non-constant. Then, the discrete-time process* $\{U(\tau_k)\}_{k\in\{0,1,\dots\}}$ *is not independent of* $(\Lambda_1, \dots, \Lambda_n)$*:* $\{U(\tau_k)\}_{k\in\{0,1,\dots\}} \mid (\Lambda_1, \dots, \Lambda_n) \overset{\mathrm{d}}{\neq} \{U(\tau_k)\}_{k\in\{0,1,\dots\}}$.

Remark 5. Note that the aggregated claim number process N is also a mixed Poisson process with mixing variable $\sum_{i=1}^n \Lambda_i$. A natural question that arises is whether our instantaneous premium rate $c(t)$ remains unchanged if we replace $\sum_{i=1}^n \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(t)]$ with $\mathbb{E}\big[\sum_{i=1}^n \Lambda_i \mid \mathcal{F}^N(t)\big]$ in its definition, where $\mathcal{F}^N$ is the filtration generated by N. We provide two examples that concern the comparison.

- Suppose that $\Lambda_i \sim \Gamma(\alpha_i, \beta)$ for each i, where the second parameter β is common for all Λ_i. It then follows from Remark 2 that

$$\sum_{i=1}^n \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(t)] = \sum_{i=1}^n \frac{N_i(t)+\alpha_i}{\beta+t} = \frac{N(t)+\sum_{i=1}^n \alpha_i}{\beta+t} \quad \text{a.s.}$$

  Moreover, because $\sum_{i=1}^n \Lambda_i \sim \Gamma\big(\sum_{i=1}^n \alpha_i, \beta\big)$,

$$\mathbb{E}\Bigg[\sum_{i=1}^n \Lambda_i \mid \mathcal{F}^N(t)\Bigg] = \frac{N(t)+\sum_{i=1}^n \alpha_i}{\beta+t} \quad \text{a.s.}$$

  Therefore, for this example, $\sum_{i=1}^n \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(t)] = \mathbb{E}\big[\sum_{i=1}^n \Lambda_i \mid \mathcal{F}^N(t)\big]$ a.s.

- Suppose that each Λ_i is uniformly distributed in the interval $(0, a_i)$. Because

$$\int_0^{a_i} \lambda^y \mathrm{e}^{-\lambda t}\,\mathrm{d}\lambda = \frac{y!}{t^{y+1}}\left\{1 - \mathrm{e}^{-a_i t}\sum_{k=0}^{y}\frac{(a_i t)^k}{k!}\right\} = \frac{y!\mathrm{e}^{-a_i t}}{t^{y+1}}\sum_{k=y+1}^{\infty}\frac{(a_i t)^k}{k!}$$

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TABLE 1. Ruin probability for finite time

	$t=1$	5	10	50	100	1000
$n=1$	0.02	0.100	0.155	0.274	0.309	0.356
10	0.162	0.312	0.344	0.362	0.362	0.362
100	0.345	0.361	0.361	0.361	0.361	0.361

for each nonnegative integer y, we see from Remark 2 that

$$\sum_{i=1}^{n} \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(t)] = \sum_{i=1}^{n} \frac{\{N_i(t)+1\} \sum_{k=N_i(t)+2}^{\infty} \frac{(a_i t)^k}{k!}}{t \sum_{k=N_i(t)+1}^{\infty} \frac{(a_i t)^k}{k!}} \quad \text{a.s.}$$

Because this is not $\mathcal{F}^N(t)$-measurable, for this example

$$\mathbb{P}\left[\sum_{i=1}^{n} \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(t)] \neq \mathbb{E}\left[\sum_{i=1}^{n} \Lambda_i \mid \mathcal{F}^N(t)\right]\right] > 0.$$

We now present a numerical example of the ruin probability in finite time using a Monte Carlo simulation. We assume the following: the claim size X_k is exponentially distributed with parameter 1, Λ_i is exponentially distributed with parameter 1, the initial surplus $u_0 = 5$, and $\theta = 0.2$.

We consider the three cases of $n = 1$, 10, and 100 insureds. We generate 100 000 random scenarios and calculate the ruin probability numerically at time $t = 1$, 5, 10, 50, 100, and 1000.

In all cases, the ruin probability for infinite time equals $\frac{1}{1.2} \exp\left(-\frac{0.2 \cdot 5}{1.2}\right) \simeq 0.362$. The results for finite time are shown in Table 1. The following two points are observed:

1. For finite time, the smaller the number of insureds, the lower the probability of ruin. This result is easy to understand because the smaller the number of insureds, the larger the variance of the surplus in finite time.
2. In all cases, the ruin probability approaches a certain level as t approaches infinity. As expected from Theorem 1, the ruin probability for infinite time is independent of the number of insureds.

3. Proofs

We first provide a lemma that is a consequence of Bayes' theorem.

Lemma 2. *For each nonnegative integer k, the posterior distribution of* $(\Lambda_1, \ldots, \Lambda_n) \mid \mathcal{F}(\tau_k)$ *satisfies*

$$\mathbb{P}\big(\Lambda_i \leq \lambda_i, i = 1, \ldots, n \mid \mathcal{F}(\tau_k)\big) = \prod_{i=1}^{n} \frac{\int_0^{\lambda_i} \nu_i^{N_i(\tau_k)} \mathrm{e}^{-\nu_i \tau_k} \, \mathrm{d}F_i(\nu_i)}{\int_0^{\infty} \nu_i^{N_i(\tau_k)} \mathrm{e}^{-\nu_i \tau_k} \, \mathrm{d}F_i(\nu_i)}$$

for $\lambda_i > 0$, $i = 1, \ldots, n$.

Proof of Lemma 2. Let $\{t_\ell\}_{\ell \in \{0,1,\ldots,k\}}$ be a strictly increasing sequence of nonnegative real numbers with $t_0 = 0$, and let $\{y_{i,\ell}\}$, where $i \in \{1, \ldots, n\}$ and $\ell \in \{0, 1, \ldots, k\}$, be a double sequence of nonnegative integers that satisfy the following two properties:

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- for all i, the sequence $\{y_{i,\ell}\}_{\ell\in\{0,1,\ldots,k\}}$ is non-decreasing;
- for all ℓ, $\sum_{i=1}^{n} y_{i,\ell}=\ell$.

Conditional on $\sigma(\Lambda_1,\ldots,\Lambda_n)$, the processes N_i are independent Poisson processes, and

$$\mathbb{P}\big(\tau_\ell\in[t_\ell,t_\ell+dt_\ell),\ \ell=1,\ldots,k\mid\Lambda_i=\lambda_i,\ i=1,\ldots,n\big)$$
$$=\left\{\prod_{\ell=1}^{k}\left(\sum_{i=1}^{n}\lambda_i\right)\mathrm{e}^{-(\sum_{i=1}^{n}\lambda_i)(t_\ell-t_{\ell-1})}\right\}\mathrm{d}t_1\ldots\mathrm{d}t_k=\left(\sum_{i=1}^{n}\lambda_i\right)^{k}\mathrm{e}^{-(\sum_{i=1}^{n}\lambda_i)t_k}\,\mathrm{d}t_1\ldots\mathrm{d}t_k.$$

Moreover,

$$\mathbb{P}\big(N_i(\tau_\ell)=y_{i,\ell},\ i=1,\ldots,n,\ \ell=1,\ldots,k\mid\Lambda_i=\lambda_i,\ \tau_\ell=t_\ell,\ i=1,\ldots,n,\ \ell=1,\ldots,k\big)$$
$$=\prod_{\ell=1}^{k}\frac{\prod_{i=1}^{n}\lambda_i^{y_{i,\ell}-y_{i,\ell-1}}}{\sum_{i=1}^{n}\lambda_i}=\frac{\prod_{i=1}^{n}\lambda_i^{y_{i,k}}}{\big(\sum_{i=1}^{n}\lambda_i\big)^{k}}.$$

It thus follows that

$$\mathbb{P}\big(\tau_\ell\in[t_\ell,t_\ell+dt_\ell),\ N_i(\tau_\ell)=y_{i,\ell},\ i=1,\ldots,n,\ \ell=1,\ldots,k\mid\Lambda_i=\lambda_i,\ i=1,\ldots,n\big)$$
$$=\left(\prod_{i=1}^{n}\lambda_i^{y_{i,k}}\right)\mathrm{e}^{-(\sum_{i=1}^{n}\lambda_i)t_k}\,\mathrm{d}t_1\ldots\mathrm{d}t_k=\left(\prod_{i=1}^{n}\lambda_i^{y_{i,k}}\mathrm{e}^{-\lambda_i t_k}\right)\mathrm{d}t_1\ldots\mathrm{d}t_k,$$

which, together with Bayes' theorem, completes the proof. □

Proof of Lemma 1. It suffices to show that

$$\mathbb{E}\big[\mathrm{e}^{-aI_{k+1}}\mid\mathcal{F}(\tau_k)\big]=\frac{1}{a+1}\qquad\text{for all } a\in[0,\infty).\tag{1}$$

It follows from Remark 2 that

$$I_{k+1}=\sum_{i=1}^{n}\int_{\tau_k}^{\tau_{k+1}}\frac{\int_0^\infty\lambda^{1+N_i(t)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty\lambda^{N_i(t)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t=\sum_{i=1}^{n}\int_{\tau_k}^{\tau_{k+1}}\frac{\int_0^\infty\lambda^{1+N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty\lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t,$$

which implies that the left-hand side of (1) is equal to

$$\mathbb{E}\left[\exp\left\{-a\sum_{i=1}^{n}\int_{\tau_k}^{\tau_{k+1}}\frac{\int_0^\infty\lambda^{1+N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty\lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t\right\}\mid\mathcal{F}(\tau_k)\right]$$
$$=\mathbb{E}\left[\mathbb{E}\left[\exp\left\{-a\sum_{i=1}^{n}\int_{\tau_k}^{\tau_{k+1}}\frac{\int_0^\infty\lambda^{1+N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty\lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t\right\}\mid\sigma(\Lambda_1,\ldots,\Lambda_n)\vee\mathcal{F}(\tau_k)\right]\mid\mathcal{F}(\tau_k)\right]\tag{2}$$

according to the tower property of conditional expectations. Because the conditional distribution of the inter-arrival time $\tau_{k+1}-\tau_k$ given $\sigma(\Lambda_1,\ldots,\Lambda_n)\vee\mathcal{F}(\tau_k)$ is exponential with mean

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$(\sum_{i=1}^{n}\Lambda_i)^{-1}$, expression (2) is further equal to

$$\mathbb{E}\left[\int_0^\infty \exp\left\{-a\sum_{i=1}^{n}\int_{\tau_k}^{\tau_k+x}\frac{\int_0^\infty \lambda^{1+N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t\right\}\left(\sum_{i=1}^{n}\Lambda_i\right)\mathrm{e}^{-x\sum_{i=1}^{n}\Lambda_i}\,\mathrm{d}x \,\Big|\, \mathcal{F}(\tau_k)\right]$$

$$=\int_0^\infty \exp\left\{-a\sum_{i=1}^{n}\int_{\tau_k}^{\tau_k+x}\frac{\int_0^\infty \lambda^{1+N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t\right\}\mathbb{E}\left[\left(\sum_{i=1}^{n}\Lambda_i\right)\mathrm{e}^{-x\sum_{i=1}^{n}\Lambda_i} \,\Big|\, \mathcal{F}(\tau_k)\right]\mathrm{d}x$$

$$=-\int_0^\infty \exp\left\{-a\sum_{i=1}^{n}\int_{\tau_k}^{\tau_k+x}\frac{\int_0^\infty \lambda^{1+N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t\right\}\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\left[\mathrm{e}^{-x\sum_{i=1}^{n}\Lambda_i} \,\Big|\, \mathcal{F}(\tau_k)\right]\right)\mathrm{d}x$$

$$=-\int_0^\infty \exp\left\{-a\sum_{i=1}^{n}\int_{\tau_k}^{\tau_k+x}\frac{\int_0^\infty \lambda^{1+N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t\right\}$$
$$\times\left(\frac{\mathrm{d}}{\mathrm{d}x}\prod_{i=1}^{n}\frac{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda(\tau_k+x)}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda\tau_k}\,\mathrm{d}F_i(\lambda)}\right)\mathrm{d}x \tag{3}$$

by Lemma 2.

In the following, we show that (3) equals $1/(a+1)$. The idea is to observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \lambda^y\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)=\int_0^\infty \lambda^y\frac{\mathrm{d}\mathrm{e}^{-\lambda t}}{\mathrm{d}t}\,\mathrm{d}F_i(\lambda)=-\int_0^\infty \lambda^{1+y}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)$$

for each i and $y\geq 0$, where the interchangeability of integration and differentiation is guaranteed by Lebesgue's dominated convergence theorem and our assumption that $\mathbb{E}[\Lambda_i]<\infty$. Consequently, the integral with respect to t in (3) is, via integration by substitution, rewritten as follows:

$$\int_{\tau_k}^{\tau_k+x}\frac{\int_0^\infty \lambda^{1+N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)}\,\mathrm{d}t=\left[-\log\left(\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda t}\,\mathrm{d}F_i(\lambda)\right)\right]_{t=\tau_k}^{t=\tau_k+x}$$
$$=-\log\left(\frac{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda(\tau_k+x)}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda\tau_k}\,\mathrm{d}F_i(\lambda)}\right).$$

Expression (3) is therefore equal to

$$-\int_0^\infty\left\{\prod_{i=1}^{n}\frac{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda(\tau_k+x)}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda\tau_k}\,\mathrm{d}F_i(\lambda)}\right\}^a\left(\frac{\mathrm{d}}{\mathrm{d}x}\prod_{i=1}^{n}\frac{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda(\tau_k+x)}\,\mathrm{d}F_i(\lambda)}{\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda\tau_k}\,\mathrm{d}F_i(\lambda)}\right)\mathrm{d}x$$

$$=\frac{-\int_0^\infty\left\{\prod_{i=1}^{n}\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda(\tau_k+x)}\,\mathrm{d}F_i(\lambda)\right\}^a\left\{\frac{\mathrm{d}}{\mathrm{d}x}\prod_{i=1}^{n}\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda(\tau_k+x)}\,\mathrm{d}F_i(\lambda)\right\}\mathrm{d}x}{\left\{\prod_{i=1}^{n}\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda\tau_k}\,\mathrm{d}F_i(\lambda)\right\}^{a+1}}$$

$$=\frac{-\frac{1}{a+1}\left[\left\{\prod_{i=1}^{n}\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda(\tau_k+x)}\,\mathrm{d}F_i(\lambda)\right\}^{a+1}\right]_{x=0}^{x=\infty}}{\left\{\prod_{i=1}^{n}\int_0^\infty \lambda^{N_i(\tau_k)}\mathrm{e}^{-\lambda\tau_k}\,\mathrm{d}F_i(\lambda)\right\}^{a+1}}=\frac{1}{a+1},$$

which verifies our desired equality (1). □

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Proof of Theorem 1(i). Each increment of the discrete-time process $\{U(\tau_k)\}_{k\in\{1,2,\ldots\}}$ satisfies

$$U(\tau_k) - U(\tau_{k-1}) = (1+\theta)\mu I_k - X_k \tag{4}$$

for $k = 1, 2, \ldots$. By Lemma 1, it also follows that $\mathcal{F}(\tau_{k-1}) \vee \sigma(X_1, \ldots, X_{k-1})$, I_k, and X_k are independent, and thus the right-hand side of (4) is independent of $\mathcal{F}(\tau_{k-1}) \vee \sigma(X_1, \ldots, X_{k-1})$. This implies that the discrete-time process $\{U(\tau_k)\}_{k\in\{0,1,\ldots\}}$ is a random walk. Moreover, each increment of the discrete-time process, that is, the right-hand side of (4), has a distribution that does not depend on either n or the distributions of the Λ_i, because I_k is independent of X_k and is exponentially distributed with mean 1. This completes the proof of the first assertion in Theorem 1. □

Proof of Theorem 1(ii). Process U decreases only in jumps, and this implies that $\inf_{t\in[0,\infty)} U(t) = \inf_{k\in\{0,1,\ldots\}} U(\tau_k)$ a.s. This, together with Theorem 1(i), completes the proof. □

Proof of Corollary 1. Theorem 1(ii) yields $\mathbb{P}\big(\inf_{t\in[0,\infty)} U_t < 0\big) = \psi^0(u)$, and Theorem 1(i) implies that the distribution of $U^0(\tau^0)$ is equal to $U(\tau)$.

See Lundberg (1903) for the proof on the existence of a positive R and $\psi^0(u) = \exp(-Ru)/\mathbb{E}[\exp(-RU^0_{\tau^0}) \mid \tau^0 < \infty]$. □

Proof of Proposition 1. The conditional distribution of τ_1 given $\sigma(\Lambda_1, \ldots, \Lambda_n)$ is exponential with mean $\big(\sum_{i=1}^n \Lambda_i\big)^{-1}$, and thus

$$\begin{aligned}\mathbb{E}[I_1 \mid \sigma(\Lambda_1, \ldots, \Lambda_n)] &= \int_0^\infty \left\{\sum_{i=1}^n \int_0^t \frac{\int_0^\infty \lambda e^{-\lambda u}\, dF_i(\lambda)}{\int_0^\infty e^{-\lambda u}\, dF_i(\lambda)}\, du\right\}\left(\sum_{i=1}^n \Lambda_i\right) e^{-t\sum_{i=1}^n \Lambda_i}\, dt \\ &= \int_0^\infty \left\{\sum_{i=1}^n \int_0^{s(\sum_{i=1}^n \Lambda_i)^{-1}} \frac{\int_0^\infty \lambda e^{-\lambda u}\, dF_i(\lambda)}{\int_0^\infty e^{-\lambda u}\, dF_i(\lambda)}\, du\right\} e^{-s}\, ds \quad \text{a.s.,}\end{aligned}$$

where $s := t\sum_{i=1}^n \Lambda_i$. Moreover, it follows readily by definition that

$$\mathbb{E}[U(\tau_1) \mid \sigma(\Lambda_1, \ldots, \Lambda_n)] = u_0 + (1+\theta)\mu\mathbb{E}[I_1 \mid \sigma(\Lambda_1, \ldots, \Lambda_n)] - \mathbb{E}[X_1],$$

and $\mathbb{E}[U(\tau_1) \mid \sigma(\Lambda_1, \ldots, \Lambda_n)]$ is therefore almost surely equal to a strictly decreasing function of $\sum_{i=1}^n \Lambda_i$. Because $\sum_{i=1}^n \Lambda_i$ is non-constant by our assumption, the random variable $U(\tau_1)$ is not independent of the Λ_i. □

4. Second proof via stochastic calculus

In this section we use stochastic calculus to provide another proof of our key lemma.

Proof of Lemma 1. For each $i \in \{1, \ldots, n\}$, the process

$$\left\{N_i(t) - \int_0^t \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(s)]\, ds\right\}_{t\in[0,\infty)} \tag{5}$$

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is an $\mathcal{F}_i$-martingale. Indeed, for $0 \le u < t$,

$$\begin{aligned}
&\mathbb{E}\left[N_i(t) - \int_0^t \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(s)]\,\mathrm{d}s \mid \mathcal{F}_i(u)\right] - \left\{N_i(u) - \int_0^u \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(s)]\,\mathrm{d}s\right\} \\
&\quad = \mathbb{E}[N_i(t) - N_i(u) \mid \mathcal{F}_i(u)] - \mathbb{E}\left[\int_u^t \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(s)]\,\mathrm{d}s \mid \mathcal{F}_i(u)\right] \\
&\quad = \mathbb{E}\big[\mathbb{E}[N_i(t) - N_i(u) \mid \sigma(\Lambda_i) \vee \mathcal{F}_i(u)] \mid \mathcal{F}_i(u)\big] - \int_u^t \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(u)]\,\mathrm{d}s \\
&\quad = \mathbb{E}[(t-u)\Lambda_i \mid \mathcal{F}_i(u)] - (t-u)\mathbb{E}[\Lambda_i \mid \mathcal{F}_i(u)] = 0 \quad \text{a.s.}
\end{aligned}$$

Because every $\mathcal{F}_i$-martingale is also an $\mathcal{F}$-martingale, the process (5) is an $\mathcal{F}$-martingale. By aggregation, we see that the process $\big\{N(t) - \int_0^t \sum_{i=1}^n \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(s)]\,\mathrm{d}s\big\}_{t\in[0,\infty)}$ is also an $\mathcal{F}$-martingale, and hence the $\mathcal{F}$-compensator of the point process N is $A(t) := \int_0^t \sum_{i=1}^n \mathbb{E}[\Lambda_i \mid \mathcal{F}_i(s)]\,\mathrm{d}s$. Because this compensator has continuous paths, it follows from the Grigelionis (1977) Poisson reduction result (see also Kallenberg 2017, Corollary 9.30) that the time-changed process $N(A^{-1}(\cdot))$, where $A^{-1}(s) := \inf\{t \ge 0 \mid A(t) > s\}$, $s \in [0, \infty)$, is an $\mathcal{F}(A^{-1}(\cdot))$-Poisson process with intensity parameter 1. Consequently, for each $k \ge 0$, the random variable $I_{k+1} = A(\tau_{k+1}) - A(\tau_k)$ is independent of $\mathcal{F}(\tau_k)$ and is exponentially distributed with mean 1. □

5. Conclusion

We proposed an extension of Dubey's ruin model (1977) with the Bayesian estimator to multiple insureds. We showed that, for each fixed value of the safety loading, the ruin probability derived from our model is the same as that of the classical Cramér–Lundberg model and does not depend on either the distribution of the mixing variable of the driving mixed Poisson process or the number of insurance contracts; that is, the ruin probability of the model with the Bayesian estimator in infinite time derived by Dubey (1977) also applies to the case of multiple insureds. Our result is also consistent with the result of Dubey's model that the ruin probability of multiple insureds with the premium based on the Bühlmann estimator converges to the ruin probability of the classical model when the number of insureds approaches infinity. We emphasize that our result is not approximate.

We provided two proofs, the first based on Bayes' theorem and the second via stochastic calculus. The second proof suggests that our assumption that the counting process is a mixed Poisson process is not a necessary condition for the validity of our result. A sufficient condition for our proposition is that the compensator of the counting process is continuous and equal to the time integral of the Bayesian estimator. This leads to two tasks for future research. One is to determine somewhat different descriptions of the sufficient condition, and the other is to study whether the distribution of real loss data satisfies the condition.

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