*Proceedings of the Royal Society of Edinburgh*, **130A**, 335–373, *2000*

# Phragmen–Lindelöf theorems and the asymptotic behaviour of solutions of quasilinear elliptic equations in slabs

**Z. Jin** and **K. Lancaster**
Department of Mathematics and Statistics, Wichita State University, Wichita, KS 67260-0033, USA

(MS received 17 February 1998; accepted 19 January 1999)

The asymptotic behaviour of solutions of second-order quasilinear elliptic partial differential equations defined on unbounded domains in $\mathbb{R}^n$ contained in strips (when $n = 2$) or slabs (when $n > 2$) is investigated when such solutions satisfy Dirichlet boundary conditions and the Dirichlet boundary data have appropriate asymptotic behaviour at infinity. We prove Phragmèn–Lindelöf theorems for large classes of elliptic operators, including uniformly elliptic operators and operators with well-defined *genre*, establish exponential decay estimates for uniformly elliptic operators when the Dirichlet boundary data vanish outside a compact set, establish the uniqueness of solutions, and give examples of solutions for non-uniformly elliptic operators which decay but do not decay exponentially. Our principal theorems are proven using special barrier functions; these barriers are constructed by considering an operator associated to our original operator.

## 1. Introduction

Phragmèn–Lindelöf theorems 'at infinity' for an open set $\Omega \subset \mathbb{R}^n$ and an elliptic partial differential operator $Q$ on $\mathbb{R}^n$ are concerned with the behaviour of $f(X)$ as the norm of $X \in \Omega$ goes to infinity, where $f$ is a solution of a Dirichlet boundary value problem for $Q$ in $\Omega$. The cases in which $\Omega$ is (or is contained in) a strip in $\mathbb{R}^2$ or a cylinder in $\mathbb{R}^3$ have generated particular interest, in part because of applications to problems in continuum mechanics. Decay theorems (alternatively, spatial decay theorems) in such domains are concerned with the rate at which (appropriate) solutions converge to their asymptotic limits, especially when the Dirichlet boundary data vanish outside a compact set. These decay estimates, which have connections with, for example, (linear and nonlinear) heat equations (e.g. [6, 21, 23, 29, 32, 39]), fluid mechanics (e.g. [1, 3]), extensible films (e.g. [17]) and Saint–Venant's principle in elasticity theory (e.g. [12–15, 18, 35]), often begin by assuming that the limiting behaviour at infinity of the solution is known. We might view this interaction as a two-step process, in which a Phragmèn–Lindelöf theorem yields the limiting behaviour of a solution and a decay theorem yields estimates of the rate at which the solution approaches its limiting values. In this paper we will focus primarily on Phragmèn–Lindelöf theorems.

Previous results on Phragmèn–Lindelöf theorems at infinity have generally concerned limited classes of operators, such as the Laplace operator or other linear


https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

uniformly elliptic operators (e.g. [2,4,8,33,40]), nonlinear uniformly elliptic operators (e.g. [22]), the minimal surface operator or other divergence structure operators (e.g. [19, 20, 24, 26, 27, 30, 37]) or operators whose principal part has one of these forms (e.g. [6,16,17,25,31,36]). (Two exceptions, however, are [10] and [28].) Decay estimates were usually obtained for particular classes of operators in special geometries, including strips (e.g. [18,19,22,30,34,37]) and cylinders (e.g. [6,11,15,31]).

Throughout this paper (except in § 5), we will assume $\Omega$ is an unbounded open subset of $\mathbb{R}^n$ such that, for some fixed $M > 0$,

$$\Omega \subset \{X = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_n| < M\}.$$

We will consider elliptic operators of the form

$$Qu(X) = \sum_{i,j=1}^{n} a_{ij}(X, u(X), Du(X)) D_{ij} u(X), \tag{1.1}$$

where $(a_{ij}(X, t, P))$ is a positive definite matrix in which each entry is a $C^1$ function on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. For $\phi \in C^0(\mathbb{R}^n)$, we will consider the Dirichlet problem

$$Qf = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$f = \phi \quad \text{on } \partial\Omega. \tag{1.3}$$

Our goal is to determine the behaviour of $f(X)$ for $X \in \Omega$ as $|X| \to \infty$. We will prove theorems of Phragmèn–Lindelöf type for solutions $f$ of (1.2)–(1.3) when $Q$ satisfies one of two general conditions, $\phi$ has (uniform) limits 'at infinity', and, in some cases, $f(X)$ does not grow too quickly as $|X|$ goes to infinity. The class of elliptic operators satisfying one of these general conditions includes, for example, uniformly elliptic operators, minimal surface operators and operators whose coefficients depend only on the gradient of the solution when these operators are of the form (1.1). Based, for example, on [6] and [22], one might expect that when $Q$ is uniformly elliptic, a solution of (1.2)–(1.3) which converges to zero at infinity does so exponentially when the boundary data vanish outside of some compact set; while spatial decay estimates are not the focus of this paper, we do prove this conjecture in corollary 3.4. We observe that when the operator is not uniformly elliptic, there are solutions of (1.2)–(1.3) which converge to zero but do not do so exponential, as illustrated in § 3 by examples 3.5 and 3.6.

To illustrate some of our results, we will consider two special cases. In the first case, we let

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : |z| < 1, x^2 + y^2 > 1\}$$

be an 'infinite washer', $Q$ be either a uniformly elliptic operator on $\Omega$ or the minimal surface operator on $\mathbb{R}^3$, and $\phi \in C^0(\mathbb{R}^3)$ such that there exist continuous functions $h_1(\theta)$ and $h_2(\theta)$ with

$$\lim_{r\to\infty} \phi(r\cos(\theta), r\sin(\theta), 1) = h_1(\theta), \tag{1.4}$$

$$\lim_{r\to\infty} \phi(r\cos(\theta), r\sin(\theta), -1) = h_2(\theta) \tag{1.5}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

uniformly for $\theta \in [0, 2\pi]$. Then, if $Q$ is the minimal surface operator on $\mathbb{R}^3$ and $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is a solution of (1.2)–(1.3), corollary 4.5 implies

$$\lim_{r\to\infty} f(r\cos(\theta), r\sin(\theta), z) = \tfrac{1}{2}(1+z)h_1(\theta) + \tfrac{1}{2}(1-z)h_2(\theta)$$

uniformly for $0 \leqslant \theta \leqslant 2\pi$ and $-1 \leqslant z \leqslant 1$. Corollary 3.8 implies that this conclusion continues to hold if $Q$ is a uniformly elliptic operator and $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is a solution of (1.2)–(1.3) which satisfies the condition that, for some $C \geqslant 0$, $|f(x, y, z)| \leqslant C\sqrt{x^2 + y^2}$ when $(x, y, z) \in \Omega$ and $x^2 + y^2 \geqslant 1$.

From the conclusion above, we can easily see that the solution to (1.2)–(1.3) is unique when $Q$ is the minimal surface operator and $\phi$ satisfies (1.4)–(1.5). It also easily follows that a solution of (1.2)–(1.3) which grows at most linearly is unique in this class of functions when $Q$ is a uniformly elliptic operator whose coefficients $a_{i,j}(x, y, z, u, Du)$ are independent of $u$, and $\phi$ satisfies (1.4)–(1.5). Since a fully nonlinear uniformly elliptic operator of the form $F(X, Du, D^2u)$ which satisfies $F(X, P, 0) \equiv 0$ can be reduced (by 'linearization') to a uniformly elliptic operator of the form (1.1), we see that our conclusions above can be applied to appropriate solutions of $F(X, Du, D^2u) = 0$.

In the second case, we let $Q$ be a uniformly elliptic operator and $\phi$ vanish outside of a compact set in $\mathbb{R}^n$. Then a solution of (1.2)–(1.3) in the class of functions with (at most) linear growth must actually decay exponentially. For example, suppose $Q$ is the Laplace operator on $\mathbb{R}^3$, $\Omega$ is as before (i.e. an 'infinite washer') and $\phi \in C^0(\mathbb{R}^3)$ satisfies $\phi(x, y, \pm 1) = 0$ if $x^2 + y^2 > 1$ and

$$\phi(\cos(\theta), \sin(\theta), z) = (1 - z^2)(\sin(\theta) - \cos(2\theta)), \quad 0 \leqslant \theta \leqslant 2\pi, \quad -1 \leqslant z \leqslant 1.$$

If $f$ is a solution of (1.2)–(1.3) and, for some $C \geqslant 0$, $|f(x, y, z)| \leqslant C\sqrt{x^2 + y^2}$ for $x^2 + y^2 \geqslant 1$, then corollary 3.2 implies $f(x, y, z) \to 0$ as $x^2 + y^2 \to \infty$ for $|z| \leqslant 1$, and corollary 3.4 implies that, for $r \geqslant 1$ and $0 \leqslant \theta \leqslant 2\pi$,

$$|f(r\cos(\theta), r\sin(\theta), z)| \leqslant \frac{8}{\pi} \mathrm{e}^{-\pi(r-1)/2} \cos(\tfrac{1}{2}\pi z).$$

For another example, let $Qu = 0$ be the equation of gas dynamics for a perfect gas (e.g. [9, eqn (10.8)]); here,

$$Qu = \Delta u - \frac{2D_i u D_j u}{2 - (\gamma - 1)|Du|^2} D_{ij} u$$

for a constant $\gamma > 1$. Suppose $f$ is a solution of (1.2)–(1.3) which satisfies, for some $C \geqslant 0$ and $m < \sqrt{2/(\gamma + 1)}$, $|f(x, y, z)| \leqslant Cr$ when $r \geqslant 1$ and $|Df| \leqslant m$ on $\Omega$; these imply the flow is subsonic and $Qf$ is uniformly elliptic. Then corollary 3.2 implies $f(x, y, z) \to 0$ as $x^2 + y^2 \to \infty$ for $|z| \leqslant 1$, and corollary 3.4 implies

$$|f(r\cos(\theta), r\sin(\theta), z)| \leqslant \frac{4}{k} \mathrm{e}^{-k(r-1)} \mathrm{e}^{k\alpha|z|} \sin(k(1 - |z|)),$$

where $\alpha$ and $k$ are constants depending only on $\gamma$ and $m$.

The aspect of this work which is most crucial to its success (and might also be of independent interest) is the construction of new barrier functions; for ease of presentation, we will limit our discussion here to $\mathbb{R}^3$ and elliptic operators on $\mathbb{R}^2$.

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

The following geometric idea, used, for example, in [20] (see also [27]), is the genesis of our construction. Consider the catenoid $x^2 + y^2 = \cosh^2(z)$, which is a minimal surface with the $z$-axis as an axis of symmetry. Define

$$w(x, y) = \sqrt{\cosh^2(y) - x^2} \quad \text{for } (x, y) \in D = \{(x, y) \in \mathbb{R}^2 : |x| < \cosh(y)\}.$$

The graph of $w$ is a portion of the catenoid obtained by interchanging $y$ and $z$ in the catenoid above. Notice that $w$ is a solution of the minimal surface equation, $w > 0$ in $D$, and

$$\frac{\partial w}{\partial \boldsymbol{n}} = +\infty \quad \text{on } \partial D,$$

where $\boldsymbol{n}$ is the exterior normal to $D$ on $\partial D$. Let $S = \{(x, y) \in \mathbb{R}^2 : |y| < 1\}$ and suppose $f \in C^2(S) \cap C^0(\bar{S})$ is any solution of the minimal surface equation in $S$ satisfying $f(x, \pm 1) \leqslant 0$ for $|x| \leqslant \cosh(1)$. Using well-known comparison principles, we see that $f \leqslant w$ on $S \cap D$. In particular, $f(0, y) \leqslant w(0, y)$ for $|y| \leqslant 1$. In [38] (see also [28]), upper and lower barriers of the form $g(x, y) = h(\sqrt{x^2 + y^2})$ are used; for upper barriers, $g$ is a supersolution of the elliptic equation under consideration. Since the surfaces $x^2 + y^2 = \cosh^2(z)$ and $z = h(\sqrt{x^2 + y^2})$ have some similarities, one might wish to mimic the process above. Given a suitable operator $Q$ of the form (1.1), our construction is based on finding supersolutions and subsolutions for an operator $Q^{\#}$ corresponding to $Q$, defining $w$ so that

$$z = w(x, y) \quad \Longrightarrow \quad y = g(x, z),$$

and concluding that $w$ is a supersolution or subsolution for $Q$. However, certain technical differences between minimal surfaces and solutions of $Qf = 0$ should be apparent; for example, $Q^{\#} = Q$ when $Q$ is the minimal surface operator, while $Q^{\#} \neq Q$ in general. One unexpected aspect of this construction is that upper (lower) barriers for $Q$ will come from subsolutions (supersolutions) for $Q^{\#}$.

The remainder of the paper is organized as follows. In § 2, we state our principal theorems. In §§ 3–6, we apply our theorems to various kinds of elliptic equations by verifying the hypotheses of our theorems. Our theorems are proven in §§ 8, 10 and 11, where we apply the barriers constructed in §§ 7 and 9. In § 12 we prove corollaries 3.8, 4.5 and 4.6.

## 2. Main results

We will assume from now on that the coefficients of $Q$ have been normalized so that

$$\sum_{i=1}^{n} a_{ii}(X, z, P) = 1 \quad \text{for } (X, z, P) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n. \tag{2.1}$$

We will write elements $X = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ as $(\boldsymbol{x}, y)$, where $\boldsymbol{x} = (x_1, \ldots, x_{n-1})$ and $y = x_n$. Corresponding to the operator $Q$, we define an operator $Q^{\#}$ by

$$Q^{\#} v(\boldsymbol{x}, z) = \sum_{i,j=1}^{n} A_{ij}(\boldsymbol{x}, z, v, Dv) D_{ij} v \tag{2.2}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

for $v = v(\boldsymbol{x}, z)$ in $C^2(R^n)$ with $\partial v/\partial z \neq 0$, where

$$A_{ij}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = q^2 a_{ij}, \quad 1 \leqslant i, \quad j \leqslant n-1, \tag{2.3}$$

$$A_{in}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = q a_{in} - \sum_{j=1}^{n-1} p_j q a_{ij}, \quad 1 \leqslant i \leqslant n-1, \tag{2.4}$$

$$A_{nn}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = a_{nn} - 2\sum_{j=1}^{n-1} p_j a_{jn} + \sum_{i,j=1}^{n-1} p_i p_j a_{ij}. \tag{2.5}$$

Here $a_{ij}$ means $a_{ij}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q)$ for $1 \leqslant i$, $j \leqslant n$, $\boldsymbol{p} = (p_1, \dots, p_{n-1}) \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, $q \neq 0$, $D_i = \partial/\partial x_i$ for $1 \leqslant i \leqslant n-1$, $D_n = \partial/\partial z$, $Dv = (D_1 v, \dots, D_n v)$ and $D_{ij} = D_i D_j$ for $1 \leqslant i$, $j \leqslant n$.

The operators $Q$ and $Q^{\#}$ are related in the following way. If $w = w(\boldsymbol{x}, y)$ is in $C^2(\mathbb{R}^n)$, $g = g(\boldsymbol{x}, z)$ is in $C^2(\mathbb{R}^n)$, $g_z \neq 0$, and $g(\boldsymbol{x}, w(\boldsymbol{x}, y)) = y$, then

$$Qw(\boldsymbol{x}, y) = \frac{-1}{g_z^3(\boldsymbol{x}, w(\boldsymbol{x}, y))} Q^{\#} g(\boldsymbol{x}, w(\boldsymbol{x}, y)). \tag{2.6}$$

In particular, if $g_z > 0$ and $Q^{\#} g \geqslant 0$, then $Qw \leqslant 0$. This is the crucial observation in the paper, which enables us to construct supersolutions and subsolutions for Dirichlet problems related to $Q$ on $\Omega$.

We shall assume the following hypothesis on the behaviour of the boundary data $\phi$.

Assumption 2.1. There is a function $\Phi \in C^0(S^{n-2})$ such that

$$\phi(r\omega, y) \to \Phi(\omega) \quad \text{as } r \to \infty$$

uniformly for $\omega \in S^{n-2}$ and $|y| \leqslant M$.

The assumptions on the operator $Q$ will be described by the behaviour of the following functions.

Definition 2.2. For an operator $Q$ in (1.1) satisfying (2.1), let

$$\varepsilon(X, z, P) = \varepsilon(\boldsymbol{x}, y, z, P) = \sum_{i,j=1}^{n} a_{ij}(X, z, P) P_i P_j \tag{2.7}$$

for $X$, $P \in \mathbb{R}^n$, $z \in \mathbb{R}$, and

$$\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = \left[\sum_{i,j=1}^{n-1} A_{ij} p_i p_j + 2\sum_{i=1}^{n-1} A_{in} p_i q + A_{nn} q^2\right] \Big/ \sum_{i=1}^{n} A_{ii} \tag{2.8}$$

for $q \neq 0$, where $A_{ij} = A_{ij}(\boldsymbol{x}, z, t, \boldsymbol{p}, q)$ for $1 \leqslant i$, $j \leqslant n$ are given by (2.3)–(2.5), and $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$.

Remark 2.3. A brief computation shows that (using (2.1))

$$\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) = \frac{a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q)}{1 + \varepsilon(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q) - a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q)}. \tag{2.9}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Our first theorem follows.

THEOREM 2.4. *Suppose we have the following.*

(1) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2)–(1.3).*

(2) $Q$ *satisfies (2.1) and there exist* $L \geqslant 0$ *and a positive continuous function* $\sigma$ *on* $[1, \infty)$ *such that*

$$\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2) \tag{2.10}$$

*whenever* $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$, $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ *and* $q \neq 0$.

(3) $\phi$ *satisfies assumption 2.1.*

*Then*

$$\lim_{j\to\infty} f(\boldsymbol{x}_j, y_j) = \Phi(\omega) \tag{2.11}$$

*uniformly for* $\omega \in S^{n-2}$ *and sequences* $\{(\boldsymbol{x}_j, y_j)\}$ *in* $\bar{\Omega}$ *with* $|\boldsymbol{x}_j| \to \infty$ *and* $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ *as* $j \to \infty$.

If condition (2) in theorem 2.4 holds only for $|q| \geqslant \delta_0 > 0$, we need to add more restrictions on the solution $f$.

THEOREM 2.5. *Suppose we have the following.*

(1) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2)–(1.3), and* $f(\boldsymbol{x}, y)$ *has at most linear growth in its variables; that is, there is a constant* $C > 0$ *such that*

$$|f(\boldsymbol{x}, y)| \leqslant C|\boldsymbol{x}| \quad \text{for } |\boldsymbol{x}| \text{ large}, \quad (\boldsymbol{x}, y) \in \Omega. \tag{2.12}$$

(2) $Q$ *satisfies (2.1) and there exist* $L \geqslant 0$, $\delta_0 > 0$, *and a positive continuous function* $\sigma$ *on* $[1, \infty)$, *such that*

$$\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2) \tag{2.13}$$

*whenever* $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ *with* $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ *and* $|q| \geqslant \delta_0$.

(3) $\phi$ *satisfies assumption 2.1.*

*Then*

$$\lim_{j\to\infty} f(\boldsymbol{x}_j, y_j) = \Phi(\omega) \tag{2.14}$$

*uniformly for* $\omega \in S^{n-2}$ *and sequences* $\{(\boldsymbol{x}_j, y_j)\}$ *in* $\bar{\Omega}$ *with* $|\boldsymbol{x}_j| \to \infty$ *and* $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ *as* $j \to \infty$.

If condition (3) in theorem 2.4 is not assumed, we can still obtain a bound on a solution given by the bound on the boundary data.

THEOREM 2.6. *Suppose we have the following.*

(1) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2).*

(2) *$Q$ satisfies (2.1) and there exist $L \geqslant 0$ and a positive continuous function $\sigma$ on $[1, \infty)$ such that*

$$\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2) \tag{2.15}$$

*whenever $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ and $q \neq 0$.*

*Then*

$$\lim_{R\to\infty} \sup\{|f(\boldsymbol{x}, y)| \mid |x| \geqslant R, (\boldsymbol{x}, y) \in \Omega\} \leqslant \lim_{R\to\infty} \sup\{|f(\boldsymbol{x}, y)| \mid |x| \geqslant R, (\boldsymbol{x}, y) \in \partial\Omega\}.$$

*In particular, if $|f(\boldsymbol{x}, y)| \leqslant K$ on $\partial\Omega$, then $||f||_{L^\infty(\Omega)} \leqslant K$.*

## 3. Uniformly elliptic and other equations

In this section, we consider the application of theorem 2.5 to particular classes of operators. In order to do this, we need to verify that assumption (2) of theorem 2.5 is satisfied. Let us consider the following four conditions.

(1) $Q$ is a uniformly elliptic operator.

(ii) $a_{nn}(\boldsymbol{x}, t, z, \boldsymbol{p}, q)$ is independent of $\boldsymbol{x}$, $t$ and $z$.

(iii) There exist $L \geqslant 0$ and a positive continuous function $\sigma$ on $[1, \infty)$ such that

$$a_{nn}(\boldsymbol{x}, t, z, \boldsymbol{p}, q) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2)$$

whenever $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ and $|q| > 0$.

(iv) There exist $L \geqslant 0$, $\delta_0 > 0$ and a positive continuous function $\sigma$ on $[1, \infty)$ such that

$$a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2) \tag{3.1}$$

whenever $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ and $|q| \geqslant \delta_0$.

LEMMA 3.1. *Suppose $Q$ satisfies (2.1) and one of conditions* (i), (ii) *or* (iii). *Then $Q$ satisfies condition* (iv).

*Proof.* Suppose first that $Q$ satisfies (i). Then there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{n} a_{i,j}(\boldsymbol{x}, y, z, \boldsymbol{p}, q)\xi_i\xi_j \geqslant \mu|\xi|^2$$

for all $\xi \in \mathbb{R}^n$, $(\boldsymbol{x}, y) \in \Omega$, $z$, $q \in \mathbb{R}$ and $\boldsymbol{p} \in \mathbb{R}^{n-1}$. If we set $\xi = (0, \dots, 0, 1)$, we get $a_{nn}(\boldsymbol{x}, t, z, \boldsymbol{p}, q) \geqslant \mu$. Thus we see that

$$a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q) \geqslant \mu,$$

and so (iv) is satisfied if we let $\sigma(\rho)$ be the constant function which equals $\mu$.

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Suppose next that $Q$ satisfies (ii). Let $\delta_0 \in (0,1]$ be fixed. Set

$$k(\rho) = \min\left\{ a_{nn}(\boldsymbol{p}, q) : \boldsymbol{p} \in \mathbb{R}^{n-1}, q \in \mathbb{R}, \frac{1}{\delta_0^2} \leqslant |\boldsymbol{p}|^2 + q^2 \leqslant \rho \right\}.$$

Then $k(\cdot)$ is a positive decreasing continuous function on $[\delta_0^2, \infty)$. For $\boldsymbol{p} \in \mathbb{R}^{n-1}$ and $q \in \mathbb{R}$ with $|q| \geqslant \delta_0$ and $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, we have

$$a_{nn}\left(-\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \geqslant k\left(\frac{|\boldsymbol{p}|^2}{q^2} + \frac{1}{q^2}\right) \geqslant k\left(\frac{2}{\delta_0^2}(|\boldsymbol{p}|^2 + q^2)\right),$$

since

$$\frac{|\boldsymbol{p}|^2}{q^2} + \frac{1}{q^2} \leqslant \frac{|\boldsymbol{p}|^2 + q^2}{q^2} + \frac{|\boldsymbol{p}|^2 + q^2}{q^2} \leqslant \frac{2}{\delta_0^2}(|\boldsymbol{p}|^2 + q^2).$$

If we define $\sigma$ by $\sigma(\rho) = k(2\delta_0^{-2}\rho)$ and set $L = 0$, then (iv) is satisfied.

Suppose finally that (iii) is true. Let $\delta_0 \in (0,1]$ be fixed. Set

$$k(\rho) = \min\left\{ \sigma(|\boldsymbol{p}|^2 + q^2) : \boldsymbol{p} \in \mathbb{R}^{n-1}, q \in \mathbb{R}, \frac{1}{\delta_0^2} \leqslant |\boldsymbol{p}|^2 + q^2 \leqslant \rho \right\}.$$

Then $k(\cdot)$ is a positive decreasing continuous function on $[\delta_0^2, \infty)$. For $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$ and $y$, $t$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $(\boldsymbol{x}, q) \in \Omega$, $|q| \geqslant \delta_0$ and $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, we have

$$a_{nn}\left(\boldsymbol{x}, y, t, -\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \geqslant \sigma\left(\frac{|\boldsymbol{p}|^2}{q^2} + \frac{1}{q^2}\right) \geqslant k\left(\frac{|\boldsymbol{p}|^2}{q^2} + \frac{1}{q^2}\right) \geqslant k\left(\frac{2}{\delta_0^2}(|\boldsymbol{p}|^2 + q^2)\right),$$

and so (iv) holds. □

COROLLARY 3.2. *Suppose we have the following.*

(1) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2)–(1.3) and* $f(\boldsymbol{x}, y)$ *has at most linear growth in its variables.*

(2) $Q$ *satisfies (2.1) and one of conditions (i)–(iv).*

(3) $\phi$ *satisfies assumption 2.1.*

*Then*

$$\lim_{j \to \infty} f(\boldsymbol{x}_j, y_j) = \Phi(\omega) \tag{3.2}$$

*uniformly for* $\omega \in S^{n-2}$ *and sequences* $\{(\boldsymbol{x}_j, y_j)\}$ *in* $\bar{\Omega}$ *with* $|\boldsymbol{x}_j| \to \infty$ *and* $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ *as* $j \to \infty$.

*Proof.* We shall assume condition (iv) is satisfied. From (2.1), we see the largest eigenvalue of $(a_{ij})$ is bounded by one. Hence

$$\varepsilon(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \leqslant |\boldsymbol{p}|^2 + |q|^2.$$

Then, for any fixed $\delta_0 > 0$, if $|q| \geqslant \delta_0$ and $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, we have

$$\varepsilon\left(\boldsymbol{x}, z, t, -\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \leqslant (|\boldsymbol{p}|^2 + 1)|q|^{-2} \leqslant \frac{2}{\delta_0^2}(|\boldsymbol{p}|^2 + q^2).$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Combining this with (2.9) and (3.1) implies

$$\begin{aligned}\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) &\geqslant \frac{a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q)}{1 - a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q) + 2\delta_0^{-2}(|\boldsymbol{p}|^2 + q^2)} \\ &\geqslant \frac{\delta_0^2 \sigma(|\boldsymbol{p}|^2 + q^2)}{\delta_0^2 + 2(|\boldsymbol{p}|^2 + q^2)}\end{aligned}$$

for $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ and $|q| \geqslant \delta_0$. The corollary now follows from theorem 2.5. □

REMARK 3.3. If $F = F(X, z, P, R)$ is a $C^2$ function of $X$, $P \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $R \in \mathbb{R}^{n \times n}$ (i.e. real symmetric $n \times n$ matrices) and $f \in C^2(\Omega)$ is a solution of the fully nonlinear equation

$$F(X, u(X), Du(X), D^2u(X)) = 0,$$

then, as in [9, p. 444], $f$ can be considered as the solution of a quasilinear elliptic equation. If $F(X, z, P, 0) = 0$, then this quasilinear is of the form (1.1); if additionally $F(X, f(X), Df(X), D^2u(X))$ is a uniformly elliptic operator, then corollary 3.2 (and corollary 3.4) can be applied to $f$.

Suppose $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (1.2)–(1.3), $f(\boldsymbol{x}, y) = 0$ for $(\boldsymbol{x}, y) \in \partial\Omega$ when $|\boldsymbol{x}|$ is large, $f(\boldsymbol{x}, y)$ grows at most linearly in $|\boldsymbol{x}|$ and the linear operator

$$\mathcal{L}u(X) = \sum_{i,j=1}^{n} a_{i,j}(X, f(X), Df(X)) D_{ij} u(X) \tag{3.3}$$

is uniformly elliptic. Then $f(\boldsymbol{x}, y)$ decays exponentially, as indicated in the following corollary. We also observe that the decay rate obtained in corollary 3.4 (i.e. $k$) may not be optimal. If the operator $L$ is not uniformly elliptic, then $f(\boldsymbol{x}, y)$ need not decay exponentially, even if $f(\boldsymbol{x}, y) \to 0$ as $|\boldsymbol{x}| \to \infty$, as examples 3.5 and 3.6 demonstrate.

COROLLARY 3.4. *Suppose we have the following.*

(1) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2)–(1.3) and* $f(\boldsymbol{x}, y)$ *has at most linear growth in its variables.*

(2) $Q$ *satisfies (2.1) and is uniformly elliptic with ellipticity constants* $\mu_1$ *and* $\mu_2$, $0 < \mu_1 \leqslant \mu_2 \leqslant 1$; *that is,*

$$\mu_1 |\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{i,j}(X, z, P) \xi_i \xi_j \leqslant \mu_2 |\xi|^2$$

*for* $X \in \Omega$, $P, \xi \in \mathbb{R}^n$, $z \in \mathbb{R}$.

(3) $\alpha = \max\{1, (\mu_2 - \mu_1)/\mu_1\}$ *and* $k = (1/M)\tan^{-1}(1/\alpha)$ *if* $\mu_2 > \mu_1$, *while* $\alpha = 0$ *and* $k = \pi/2M$ *if* $\mu_2 = \mu_1$.

(4) *There exists* $L \geqslant 0$ *such that* $\phi(\boldsymbol{x}, y) = 0$ *if* $|\boldsymbol{x}| > L$ *and* $|y| \leqslant M$.

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

(5) *There exists* $K \geqslant 0$ *such that* $|f(\boldsymbol{x}, y)| \leqslant K \sin(k(M - |y|))\mathrm{e}^{-kL}$ *if* $|\boldsymbol{x}| = L$ *and* $(\boldsymbol{x}, y) \in \Omega$.

*Then*

$$|f(\boldsymbol{x}, y)| \leqslant K\mathrm{e}^{-k|\boldsymbol{x}|}\mathrm{e}^{k\alpha|y|} \sin(k(M - |y|)) \tag{3.4}$$

*for* $(\boldsymbol{x}, y) \in \Omega$ *with* $|\boldsymbol{x}| \geqslant L$.

*Proof.* Let $\mathcal{L}$ be the (linear) operator defined by

$$\mathcal{L}u(X) = \sum_{i,j=1}^{n} a_{i,j}(X, f(X), Df(X))D_{ij}u(X).$$

As in [22], define $\zeta(\boldsymbol{x}, y) = \mathrm{e}^{-k|\boldsymbol{x}|}\psi(y)$ for a positive function $\psi$ (with $\psi'' \leqslant 0$) to be determined. Set

$$\eta(\boldsymbol{\xi}) = \eta(X, \xi) = \sum_{i,j=1}^{n} a_{i,j}(X, f(X), Df(X))\xi_i\xi_j$$

for $X \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{R}^n$. Let $\tilde{\mu}_1 = \min\{\mu_1, \frac{1}{2}\mu_2\}$ if $\mu_2 > \mu_1$ and $\tilde{\mu}_1 = \mu_1 = \mu_2$ otherwise; then $\alpha = (\mu_2 - \tilde{\mu}_1)/\tilde{\mu}_1$. Notice that

$$\tilde{\mu}_1|\boldsymbol{\xi}|^2 \leqslant \eta(\boldsymbol{\xi}) \leqslant \mu_2|\boldsymbol{\xi}|^2,$$
$$\tilde{\mu}_1 \leqslant a_{n,n} \leqslant \mu_2,$$
$$\left|2\sum_{i=1}^{n-1} \frac{x_i}{|\boldsymbol{x}|}a_{i,n}\right| = \left|\eta\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}, 1\right) - \eta\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}, 0\right) - a_{n,n}\right| \leqslant 2(\mu_2 - \tilde{\mu}_1).$$

For each $X = (\boldsymbol{x}, y) \in \Omega$,

$$\begin{aligned}
\mathrm{e}^{k|\boldsymbol{x}|}\mathcal{L}\zeta = \psi(y)\left(\sum_{i,j=1}^{n-1} \frac{k^2x_ix_j}{|\boldsymbol{x}|^2}a_{i,j} + \sum_{i,j=1}^{n-1} \frac{kx_ix_j}{|\boldsymbol{x}|^3}a_{i,j} - \frac{k}{|\boldsymbol{x}|}\sum_{i=1}^{n-1} a_{i,i}\right)& \\
- 2\psi'(y)\sum_{i=1}^{n-1} \frac{kx_i}{|\boldsymbol{x}|}a_{i,n} + \psi''(y)a_{n,n}& \\
= \eta\left(\frac{k\boldsymbol{x}}{|\boldsymbol{x}|}, 0\right)\psi(y) - k\left(\eta\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}, 1\right) - \eta\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}, 0\right) - a_{n,n}\right)\psi'(y)& \\
+ a_{n,n}\psi''(y) - \frac{k}{|\boldsymbol{x}|}\left(\sum_{i,j=1}^{n-1}\left(\delta_{i,j} - \frac{x_ix_j}{|\boldsymbol{x}|^2}\right)a_{i,j}\right)\psi(y).&
\end{aligned}$$

Notice that

$$\sum_{i,j=1}^{n-1}\left(\delta_{i,j} - \frac{x_ix_j}{|\boldsymbol{x}|^2}\right)a_{i,j} = \frac{1}{|\boldsymbol{x}|^2}\sum_{i=2}^{n-1}\sum_{j=1}^{i-1} \eta(x_j\boldsymbol{e}_i - x_i\boldsymbol{e}_j) \geqslant 0,$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

and so

$$\mathrm{e}^{k|\boldsymbol{x}|}\mathcal{L}\zeta \leqslant a_{n,n}\biggl(\psi''(y) - \frac{k}{a_{n,n}}\biggl(\eta\biggl(\frac{\boldsymbol{x}}{|\boldsymbol{x}|},1\biggr) - \eta\biggl(\frac{\boldsymbol{x}}{|\boldsymbol{x}|},0\biggr) - a_{n,n}\biggr)\psi'(y) + \frac{1}{a_{n,n}}\eta\biggl(\frac{k\boldsymbol{x}}{|\boldsymbol{x}|},0\biggr)\psi(y)\biggr)$$

$$\leqslant a_{n,n}\biggl(\psi''(y) + 2\frac{k}{\tilde{\mu}_1}(\mu_2 - \tilde{\mu}_1)|\psi'(y)| + \frac{\mu_2 k^2}{\tilde{\mu}_1}\psi(y)\biggr).$$

Let us define $\psi \in C^2(\mathbb{R})$ by

$$\psi(y) = C_1 \mathrm{e}^{k\alpha|y|}\sin(k(M - |y|))$$

for some $C_1 \geqslant 0$; notice that $\psi'(0) = 0$ because of our choice of $k$. Now

$$\psi'(y) = C_1\mathrm{e}^{k\alpha|y|}\frac{y}{|y|}(k\alpha\sin(k(M-|y|)) - k\cos(k(M-|y|))),$$

$$|\psi'(y)| = C_1\mathrm{e}^{k\alpha|y|}k(-\alpha\sin(k(M-|y|)) + \cos(k(M-|y|)))$$

and

$$\psi''(y) = C_1\mathrm{e}^{k\alpha|y|}k^2((\alpha^2-1)\sin(k(M-|y|)) - 2\alpha\cos(k(M-|y|))),$$

hence

$$\begin{aligned}\psi''(y) &+ 2\frac{k}{\tilde{\mu}_1}(\mu_2 - \tilde{\mu}_1)|\psi'(y)| + \frac{\mu_2 k^2}{\tilde{\mu}_1}\psi(y)\\ &= C_1\mathrm{e}^{k\alpha|y|}k^2\sin(k(M-|y|))\biggl(\alpha^2 - 1 - \frac{2\alpha(\mu_2-\tilde{\mu}_1)}{\tilde{\mu}_1} + \frac{\mu_2}{\tilde{\mu}_1}\biggr)\\ &\quad + C_1\mathrm{e}^{k\alpha|y|}k^2\cos(k(M-|y|))\biggl(-2\alpha + 2\frac{\mu_2-\tilde{\mu}_1}{\tilde{\mu}_1}\biggr).\end{aligned}$$

Since $\mu_2 \geqslant 2\tilde{\mu}_1$ and $\alpha = (\mu_2 - \tilde{\mu}_1)/\tilde{\mu}_1$ (or $\mu_2 = \tilde{\mu}_1$ and $\alpha = 0$), we have

$$\alpha^2 - 1 - \frac{2\alpha(\mu_2-\tilde{\mu}_1)}{\tilde{\mu}_1} + \frac{\mu_2}{\tilde{\mu}_1} = \frac{1}{\tilde{\mu}_1}(\mu_2-\tilde{\mu}_1)(2\tilde{\mu}_1 - \mu_2) \leqslant 0$$

and

$$-2\alpha + 2\frac{\mu_2 - \tilde{\mu}_1}{\tilde{\mu}_1} = 0.$$

Hence, for any $C_1 \geqslant 0$, $\mathcal{L}\zeta \leqslant 0$. Notice further that

$$\mathrm{e}^{k|\boldsymbol{x}|}\mathcal{L}(-\zeta) \geqslant -a_{n,n}\biggl(\psi''(y) + 2\frac{k}{\tilde{\mu}_1}(\mu_2-\tilde{\mu}_1)|\psi'(y)| + \frac{\mu_2 k^2}{\tilde{\mu}_1}\psi(y)\biggr) \geqslant 0$$

if $C_1 \geqslant 0$.

Arguing as in [30], we will show that $|f| \leqslant \zeta$ on $\Omega$. Set $C_1 = K$, so that

$$\zeta(\boldsymbol{x}, y) = K\mathrm{e}^{-k|\boldsymbol{x}|}\mathrm{e}^{k\alpha|y|}\sin(k(M-|y|)).$$

Let $\epsilon > 0$. Corollary 3.2 implies $|f(\boldsymbol{x}, y)| \to 0$ as $|\boldsymbol{x}| \to \infty$ and so there exists $l(\epsilon) > 0$ such that

$$|f(\boldsymbol{x}, y)| \leqslant \tfrac{1}{2}\epsilon \quad \text{and} \quad 0 \leqslant \zeta(\boldsymbol{x}, y) < \tfrac{1}{2}\epsilon$$

if $(\boldsymbol{x}, y) \in \bar{\Omega}$ and $|\boldsymbol{x}| \geqslant l(\epsilon)$. Therefore,

$$f(\boldsymbol{x}, y) \leqslant \zeta(\boldsymbol{x}, y) + \epsilon \quad \text{for } (\boldsymbol{x}, y) \in \bar{\Omega}, \quad |\boldsymbol{x}| \geqslant l(\epsilon).$$

Now condition (4) implies $f(\boldsymbol{x}, y) = 0$ if $(\boldsymbol{x}, y) \in \partial\Omega$, $|\boldsymbol{x}| \geqslant L$, and so

$$f(\boldsymbol{x}, y) \leqslant \zeta(\boldsymbol{x}, y) < \zeta(\boldsymbol{x}, y) + \epsilon \quad \text{for } (\boldsymbol{x}, y) \in \partial\Omega, \quad |\boldsymbol{x}| \geqslant L.$$

Finally, condition (5) implies $|f(\boldsymbol{x}, y)| \leqslant \zeta(\boldsymbol{x}, y)$ if $(\boldsymbol{x}, y) \in \bar{\Omega}$ and $|\boldsymbol{x}| = L$. Hence

$$f \leqslant \zeta + \epsilon \quad \text{on } \partial\{(\boldsymbol{x}, y) \in \Omega : L < |\boldsymbol{x}| < l\}$$

for any $l \geqslant l(\epsilon)$. Using the comparison principle (see [9]), we see that

$$f(\boldsymbol{x}, y) \leqslant \zeta(\boldsymbol{x}, y) + \epsilon \quad \text{if } (\boldsymbol{x}, y) \in \bar{\Omega}, \quad |\boldsymbol{x}| \geqslant L.$$

Since $-\zeta$ is a subsolution, a similar argument shows that

$$-\zeta(\boldsymbol{x}, y) - \epsilon \leqslant f(\boldsymbol{x}, y) \quad \text{if } (\boldsymbol{x}, y) \in \bar{\Omega}, \quad |\boldsymbol{x}| \geqslant L.$$

Since $\epsilon > 0$ is arbitrary, we see that

$$|f(\boldsymbol{x}, y)| \leqslant \zeta(\boldsymbol{x}, y) \quad \text{for } (\boldsymbol{x}, y) \in \partial\Omega, \quad |\boldsymbol{x}| \geqslant L.$$

□

Some operators of the form (1.1), such as the minimal surface operator, are not uniformly elliptic but become uniformly elliptic for functions $f$ with bounded gradient; that is, the linear operator $\mathcal{L}$ given in (3.3) is uniformly elliptic when $|\nabla f|$ is bounded. For such operators, corollary 3.4 is applicable. For other operators, however, obtaining a bound on $|\nabla f|$ does not imply that the linear operator $\mathcal{L}$ is uniformly elliptic; as the following examples show, solutions of (1.2)–(1.3) may decay to zero without decaying exponentially, for such operators.

Example 3.5. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, |y| < \pi\}$ and define $Q$ by

$$Qu = \frac{1}{1 + u^2 + u_y^2} u_{xx} + \frac{u^2 + u_y^2}{1 + u^2 + u_y^2} u_{yy}.$$

Notice that

$$a_{2,2}\left(x, y, z, -\frac{p}{q}, \frac{1}{q}\right) = \frac{z^2 q^2 + 1}{q^2 + z^2 q^2 + 1} \geqslant \frac{1}{q^2 + 1} \geqslant \frac{1}{2(p^2 + q^2)}$$

when $p^2 + q^2 \geqslant 1$ and $q \neq 0$. Let $\phi(x, y) = \sin(y)$. Then one solution of the Dirichlet problem $Qu = 0$ in $\Omega$, $u = \phi$ on $\partial\Omega$ is

$$f(x, y) = \frac{\sqrt{2}}{x + \sqrt{2}} \sin(y).$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Notice that the hypotheses of corollary 3.2 are satisfied and $f(x, y) \to 0$ as $x \to \infty$; however, $f$ does not decay exponentially. (The fact that $a_{ij}(x, y, z, p, q)$ is only positive definite on $\{(x, y, z, p, q) : z^2 + q^2 > 0\}$ is not important, since the linear operator $L$ obtained by replacing the coefficients $a_{ij}(z, q)$ of $Q$ by $b_{ij}(x, y) = a_{ij}(f, f_y)$ (so $Lu$ is proportional to $u_{xx} + (f^2 + f_y^2)u_{yy}$) is elliptic for all $u$.)

Example 3.6. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, |y| < \pi\}$ and define $Q$ by

$$Qu = u_{xx} + \alpha(u^2 + u_y^2)^k u_{yy}$$

for $\alpha$, $k > 0$. Notice that

$$f(x, y) = c(x + a)^{-1/k} \sin(y)$$

is a solution of $Qu = 0$ if $k + 1 = \alpha k^2 c^{2k}$ and $a \in \mathbb{R}$; further, $f(x, \pm\pi) = 0$ for $x > 0$. For example,

$$f(x, y) = \tfrac{1}{2}(x + \tfrac{1}{4})^{-1/2} \sin(y)$$

is a solution of

$$u_{xx} + 3(u^2 + u_y^2)^2 u_{yy} = 0 \quad \text{in } \Omega$$

with $f(0, y) = \sin(y)$ and $f(x, \pm\pi) = 0$ for $x > 0$.

Let us consider an example for which assumption (2) in theorems 2.4 or 2.5 is not satisfied.

Example 3.7. Suppose $\Omega \subset \{(x, y) \in \mathbb{R}^2 : |y| \leqslant M\}$. Let $Q$ be the operator on $\mathbb{R}^2$ defined by

$$Qu(x, y) = u_{xx}(x, y) + \frac{1}{x^2 + 1} u_{yy}(x, y).$$

Then

$$\varepsilon(x, t, z, p, q) = \frac{x^2 + 1}{x^2 + 2} p^2 + \frac{1}{x^2 + 2} q^2,$$

$$\varepsilon^{\#}(x, z, t, p, q) = \frac{q^2}{1 + (x^2 + 1)(p^2 + q^2)}.$$

Notice that hypotheses (2) in theorems 2.4 or 2.5 cannot be satisfied since $\varepsilon^{\#}(x, z, t, p, q) \to 0$ as $|x| \to \infty$.

Let us next consider a case in which $\boldsymbol{x}_j \to \infty$, $(\boldsymbol{x}_j, y_j) \in \partial\Omega$, $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$, and yet the limit $\phi(\boldsymbol{x}_j, y_j)$ may not exist.

Corollary 3.8. *Suppose we have the following.*

(1) *The domain $\Omega$ is given by*

$$\Omega = \{(x_1, \dots, x_{n-1}, y) \in \mathbb{R}^n \mid -M < y < M\}.$$

(2) *$Q$ satisfies (2.1) and one of conditions* (i)–(iii).

(3) *$f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (1.2)–(1.3) and there exists $C \geqslant 0$ such that*

$$|f(\boldsymbol{x}, y)| \leqslant C|\boldsymbol{x}| \quad \textit{for } |\boldsymbol{x}| \geqslant 1, \quad (\boldsymbol{x}, y) \in \Omega.$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

(4) *There are two functions* $\Phi_1(\omega) \in C^0(S^{n-2})$ *and* $\Phi_2(\omega) \in C^0(S^{n-2})$ *such that*

$$\lim_{r\to\infty} \phi(r\omega, M) = \Phi_1(\omega), \qquad \lim_{r\to\infty} \phi(r\omega, -M) = \Phi_2(\omega),$$

*uniformly for* $\omega \in S^{n-1}$.

*Then*

$$\lim_{j\to\infty} f(\boldsymbol{x}_j, y) = \frac{1}{2M}(\Phi_1(\omega) - \Phi_2(\omega))(y + M) + \Phi_2(\omega)$$

*uniformly for* $\omega \in S^{n-2}$ *and sequences* $\{(\boldsymbol{x}_j, y_j)\}$ *in* $\bar{\Omega}$ *with* $|\boldsymbol{x}_j| \to \infty$ *and* $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ *as* $j \to \infty$.

The proof will be given after the proof of theorem 2.6.

## 4. Equations with well-defined genre

Now we consider equations to which we can apply theorems 2.4 and 2.5. The class of equations to which we shall apply these theorems is the class of elliptic operators $Q$ with a well-defined *genre* $\lambda$.

Definition 4.1 (see [5], [38, p. 425]). Equation (1.1) has *genre* $\lambda$ if and only if it satisfies (2.1) and there are positive constants $\mu_1$ and $\mu_2$ such that for $|P| \geqslant 1$, $X \in \Omega$, $t \in \mathbb{R}$, $P \in \mathbb{R}^n$,

$$\mu_1 |P|^{2-\lambda} \leqslant \varepsilon(X, t, P) \leqslant \mu_2 |P|^{2-\lambda}. \tag{4.1}$$

As examples, we see the minimal surface operator and operators of minimal surface type (e.g. [7,9]) have *genre* two, while uniformly elliptic operators, such as Laplace's equation, have *genre* zero.

Corollary 4.2. *Suppose we have the following.*

(1) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2)–(1.3).*

(2) $Q$ *has genre* $\lambda \geqslant 2$, *satisfies (2.1) and there exist* $L \geqslant 0$ *and a positive continuous function* $\sigma$ *on* $[1, \infty)$ *such that*

$$a_{nn}\left(\boldsymbol{x}, t, z, -\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2) \tag{4.2}$$

*whenever* $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ *with* $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ *and* $|q| > 0$.

(3) $\phi$ *satisfies assumption 2.1.*

*Then*

$$\lim_{j\to\infty} f(\boldsymbol{x}_j, y_j) = \Phi(\omega) \tag{4.3}$$

*uniformly for* $\omega \in S^{n-2}$ *and sequences* $\{(\boldsymbol{x}_j, y_j)\}$ *in* $\bar{\Omega}$ *with* $|\boldsymbol{x}_j| \to \infty$ *and* $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ *as* $j \to \infty$.

*Proof.* We will apply theorem 2.4. We only need to verify condition (2) in theorem 2.4. Since $Q$ has a well-defined *genre* $\lambda \geqslant 2$, there exists $\mu_2 > 0$ such that, if $|\boldsymbol{p}|^2/q^2 + 1/q^2 \geqslant 1$,

$$\varepsilon\left(\boldsymbol{x}, t, z, -\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \leqslant \mu_2\left(\frac{|\boldsymbol{p}|^2 + 1}{q^2}\right)^{1-\lambda/2} \leqslant \mu_2|q|^{\lambda-2},$$

and so from (2.9) and (4.2) we have (notice that $\lambda \geqslant 2$)

$$\begin{aligned}\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) &\geqslant \frac{a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q)}{1 - a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q) + \mu_2|q|^{\lambda-2}} \\ &\geqslant \frac{a_{nn}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q)}{1 + \mu_2|q|^{\lambda-2}} \\ &\geqslant \frac{\sigma(|\boldsymbol{p}|^2 + q^2)}{1 + \mu_2(\boldsymbol{p}|^2 + q^2)^{(\lambda-2)/2}}.\end{aligned}$$

If $|\boldsymbol{p}|^2/q^2 + 1/q^2 < 1$, then (2.1) implies

$$\varepsilon\left(\boldsymbol{x}, t, z, -\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \leqslant \frac{|\boldsymbol{p}|^2 + 1}{q^2} \leqslant 1.$$

Hence from (2.9) and (4.2), we have

$$\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \geqslant \tfrac{1}{2}a_{nn}\left(\boldsymbol{x}, t, z, -\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \geqslant \frac{\sigma(|\boldsymbol{p}|^2 + q^2)}{2 + 2\mu_2(|\boldsymbol{p}|^2 + q^2)^{(\lambda-2)/2}}.$$

□

Remark 4.3. Let $Q$ be the minimal surface operator on $\mathbb{R}^3$. We wish to compare one result in [24] with this corollary. If $\Omega$ is an open subset of the wedge domain $W = \{x, y, z) \in \mathbb{R}^3 : z >, |y| < z, -\infty < x < \infty\}$, $\phi \equiv 0$ and $f$ is a solution of (1.2)–(1.3), then the corollary to theorem 6 in [24] implies $f \equiv 0$. On the other hand, if $\Omega$ is an open subset of the slab $S = \{(x, y, z) \in \mathbb{R}^3 : |z| < 1, -\infty < x, y < \infty\}$, $\phi \equiv 0$ and $f$ is a solution of (1.2)–(1.3), corollary 3.8 implies $f \equiv 0$. We observe that these results are independent and complementary.

One of the differences between theorems 2.4 and 2.5 is that for theorem 2.5 we need to assume the solution grows at most linearly in its variables, while for theorem 2.4 we do not. On the other hand, if $Q$ has a well-defined *genre* $\lambda > 0$ and $\Omega$ is a domain inside a (translate of a) *salient cone* (that is, a cone with vertex at the origin whose closure contains no subspaces other than $\{0\}$), then we can use [28, theorem 6] to conclude that $f(\boldsymbol{x}, y)$ grows at most linearly in its variables; for convenience of notation, we will refer to any translation of a salient cone as a salient cone. Thus we have the following.

Corollary 4.4. *Suppose we have the following.*

(1) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2)–(1.3).*

(2) $Q$ *satisfies one of the conditions* (ii), (iii) *or* (iv).

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

(3) *$\phi$ satisfies assumption 2.1.*

(4) *$Q$ has genre $\lambda > 0$ and satisfies (2.1).*

(5) *$\Omega \subset \mathbb{R}^2$ or there is a bounded set $D \subset \Omega$ such that $\Omega \setminus D$ has finitely many components and each component of $\Omega \setminus D$ is contained inside a salient cone.*

*Then*

$$\lim_{j\to\infty} f(\boldsymbol{x}_j, y_j) = \Phi(\omega) \tag{4.4}$$

*uniformly for $\omega \in S^{n-2}$ and sequences $\{(\boldsymbol{x}_j, y_j)\}$ in $\bar{\Omega}$ with $|\boldsymbol{x}_j| \to \infty$ and $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ as $j \to \infty$.*

*Proof.* Since $\phi$ satisfies assumption 2.1, there is a constant $C$ such that

$$|f(\boldsymbol{x}, y)| \leqslant C \quad \text{on } \partial\Omega.$$

If $\Omega \subset \mathbb{R}^2$, then since $\Omega$ is inside a strip, we can choose a bounded set $D$ such that $\Omega \setminus D$ contains two components and each component is contained in a salient cone. If $n \geqslant 3$, condition (5) implies there is a bounded set $D \subset \Omega$ such that $\Omega \setminus D$ has finitely many components and each component is contained inside a salient cone. Now we apply [28, theorem 6] to conclude that $f(\boldsymbol{x}, y)$ grows at most linearly in its variables on each component. Thus the solution grows at most linearly in its variables. We apply corollary 3.2 to complete the proof. □

Next we consider a situation in which $\boldsymbol{x}_j \to \infty$, $(\boldsymbol{x}_j, y_j) \in \partial\Omega$, $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ and the limit $\phi(\boldsymbol{x}_j, y_j)$ may not exist.

Corollary 4.5. *Suppose*

(1) *The domain $\Omega$ is given by*

$$\Omega = \{(x_1, \ldots, x_{n-1}, y) \in \mathbb{R}^n \mid -M < y < M\}.$$

(2) *$f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (1.2)–(1.3).*

(3) *$Q$ has a well-defined genre $\lambda \geqslant 2$.*

(4) *There exist $L \geqslant 0$ and a positive continuous function $\sigma$ on $[1, \infty)$ such that*

$$a_{nn}\left(\boldsymbol{x}, t, z, -\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2) \tag{4.5}$$

*whenever $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 > 0$, $|t| \leqslant M$ and $|q| > 0$.*

(5) *There are two functions $\Phi_1(\omega) \in C^0(S^{n-2})$ and $\Phi_2(\omega) \in C^0(S^{n-2})$ such that*

$$\lim_{r\to\infty} \phi(r\omega, M) = \Phi_1(\omega), \qquad \lim_{r\to\infty} \phi(r\omega, -M) = \Phi_2(\omega),$$

*uniformly for $\omega \in S^{n-1}$.*

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

*Then*

$$\lim_{j\to\infty} f(\boldsymbol{x}_j, y) = \frac{1}{2M}(\Phi_1(\omega) - \Phi_2(\omega))(y+M) + \Phi_2(\omega)$$

*uniformly for* $\omega \in S^{n-2}$ *and sequences* $\{(\boldsymbol{x}_j, y_j)\}$ *in* $\bar{\Omega}$ *with* $|\boldsymbol{x}_j| \to \infty$ *and* $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ *as* $j \to \infty$.

The proof will be given after the proof of theorem 2.6.

As we will see from the proof, assumptions (3), (4) and (5) in corollary 4.5 are primarily used to conclude that the solution has at most linearly growth in its variables. Since we can use [28, theorem 6] to conclude that solutions grow at most linearly if $Q$ has a well-defined *genre* $\lambda > 0$ and the domain is inside a salient cone, we obtain the following result.

Corollary 4.6. *Suppose we have the following.*

(1) *The domain* $\Omega$ *is given by*

$$\Omega = \{(x_1, \dots, x_{n-1}, y) \in \mathbb{R}^n \mid -M < y < M\}.$$

(2) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2)–(1.3).*

(3) $Q$ *has a well-defined genre* $\lambda$, $0 < \lambda < 2$.

(4) $\Omega \subset \mathbb{R}^2$ *or there is a bounded set* $D \subset \Omega$, *such that* $\Omega \setminus D$ *has finitely many components and each component of* $\Omega \setminus D$ *is contained inside a salient cone.*

(5) *There exist* $L \geqslant 0$ *and a positive continuous function* $\sigma$ *on* $[1, \infty)$ *such that*

$$a_{nn}(\boldsymbol{x}, t, z, \boldsymbol{p}, q) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2) \tag{4.6}$$

*whenever* $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ *with* $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 > 0$, $|t| \leqslant M$, *and* $|q| > 0$.

(6) *There are two functions* $\Phi_1(\omega) \in C^0(S^{n-2})$ *and* $\Phi_2(\omega) \in C^0(S^{n-2})$ *such that*

$$\lim_{r\to\infty} \phi(r\omega, M) = \Phi_1(\omega), \qquad \lim_{r\to\infty} \phi(r\omega, -M) = \Phi_2(\omega),$$

*uniformly for* $\omega \in S^{n-1}$.

*Then*

$$\lim_{j\to\infty} f(\boldsymbol{x}_j, y) = \frac{1}{2M}(\Phi_1(\omega) - \Phi_2(\omega))(y+M) + \Phi_2(\omega)$$

*uniformly for* $\omega \in S^{n-2}$ *and sequences* $\{(\boldsymbol{x}_j, y_j)\}$ *in* $\bar{\Omega}$ *with* $|\boldsymbol{x}_j| \to \infty$ *and* $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ *as* $j \to \infty$.

The proof will be given after the proof of theorem 2.6.

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

## 5. Domains with 'finitely many ends'

In this section, we will consider domains $\Omega$ which are not contained in a slab of finite width. We will say that a domain $\Omega$ has 'finitely many ends' if there is a compact subset $D$ of $\mathbb{R}^n$ such that $\Omega \setminus D$ is a disjoint union of components $\Omega_k$, $1 \leqslant k \leqslant J$, for some $J > 1$, and

$$\Omega_k \subset S_k = \{X \in \mathbb{R}^n : |X \cdot \nu^k| \leqslant M_k\}$$

for some $M_k$ and some $\nu^k = (\nu_1^k, \dots, \nu_n^k)$ with $|\nu^k| = 1$ for each $k \in \{1, \dots, J\}$. For a domain $\Omega$ with 'finitely many ends' and a function $\phi$ defined on $\mathbb{R}^n$, we say $\phi$ converges uniformly on each end if, for each $k \in \{1, \dots, J\}$, there is a function $\Phi_k \in C^0(S^{n-1})$ such that

$$\lim_{i \to \infty} \phi(r_i \omega + t\nu^k) = \Phi_k(\omega)$$

uniformly for $\omega \in S^{n-1}$ with $\omega \cdot \nu^k = 0$, $|t| \leqslant M_k$, and sequences $\{r_i\}$ such that $r_i \to \infty$ as $i \to \infty$.

COROLLARY 5.1. *Suppose we have the following.*

(1) *The domain $\Omega$ has 'finitely many ends'.*

(2) *$f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (1.2)–(1.3).*

(3) *$Q$ satisfies (2.1) and has a well-defined genre $\lambda \geqslant 2$.*

(4) *For each $k \in \{1, \dots, J\}$ there exist $L \geqslant 0$ and a positive continuous function $\sigma$ on $[1, \infty)$ such that*

$$\sum_{i,j=1}^{n} \nu_i^k \nu_j^k a_{i,j}(X, z, P) \geqslant \sigma(|P|^2) \tag{5.1}$$

*whenever $X, P \in \mathbb{R}^n$, $z \in \mathbb{R}$ with $|X \cdot \nu^k| \leqslant M_k$, $|X - (X \cdot \nu^k)\nu^k| \geqslant L$, $|P|^2 \geqslant 1$ and $P \cdot \nu^k \neq 0$.*

(5) *The boundary data $\phi$ converge uniformly on each end.*

*Then on each end $\Omega_k$, $1 \leqslant k \leqslant J$,*

$$\lim_{i \to \infty} f(X_i) = \Phi_k(\omega)$$

*uniformly for $\omega \in S^{n-1}$ with $\omega \cdot \nu^k = 0$ and sequences $\{X_i\}$ in $\bar{\Omega}_k$ with $|X_i| \to \infty$ and $X_i/|X_i| \to \omega$ as $i \to \infty$.*

*Proof.* Let us fix $k \in \{1, \dots, J\}$, which we may assume is 1. Let us now make a linear transformation of coordinates which maps $\nu^1$ to $\boldsymbol{e}_n = (0, \dots, 0, 1)$. Let us denote the coefficients of $Q$, in these new coordinates, as $\tilde{a}_{i,j}$. Then

$$\tilde{a}_{n,n} = \sum_{i,j=1}^{n} \nu_i^1 \nu_j^1 a_{i,j}.$$

The proof that $f$ behaves as claimed on the end $\Omega_1$ now follows from corollary 3.2. If we repeat the same procedure for each $\Omega_k$, the proof follows. □

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Similar to the way in which corollaries 4.5 and 4.6 are related, we obtain the following from corollary 4.6.

COROLLARY 5.2. *Suppose we have the following.*

(1) *The domain $\Omega$ has 'finitely many ends'.*

(2) $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies (1.2)–(1.3).*

(3) $Q$ *satisfies (2.1) and has a well-defined genre* $\lambda$, $0 < \lambda < 2$.

(4) $\Omega \subset \mathbb{R}^2$ *or there is a bounded set* $D \subset \Omega$ *such that* $\Omega \setminus D$ *has finitely many components and each component of* $\Omega \setminus D$ *is contained inside a salient cone.*

(5) *For each* $k \in \{1, \dots, J\}$, *there exist* $L \geqslant 0$ *and a positive continuous function* $\sigma$ *on* $[1, \infty)$ *such that*

$$\sum_{i,j=1}^{n} \nu_i^k \nu_j^k a_{i,j}(X, z, P) \geqslant \sigma(|P|^2)$$

*whenever* $X, P \in \mathbb{R}^n$, $z \in \mathbb{R}$ *with* $|X \cdot \nu^k| \leqslant M_k$, $|X - (X \cdot \nu^k)\nu^k| \geqslant L$, $|P|^2 \geqslant 1$ *and* $P \cdot \nu^k \neq 0$.

(6) *The boundary data* $\phi$ *converge uniformly on each end.*

*Then on each end* $\Omega_k$, $1 \leqslant k \leqslant J$,

$$\lim_{i \to \infty} f(X_i) = \Phi_k(\omega)$$

*uniformly for* $\omega \in S^{n-1}$ *with* $\omega \cdot \nu_k = 0$ *and sequences* $\{X_i\}$ *in* $\bar{\Omega}_k$ *with* $|X_i| \to \infty$ *and* $X_i/|X_i| \to \omega$ *as* $i \to \infty$.

*Proof.* The proof is almost the same as that for corollary 4.6, except now we need to apply [28, theorem 6] to conclude that the solution has at most linear growth in its variables, and then apply corollary 3.2. □

## 6. Uniqueness

Another application of theorems 2.4 and 2.5 is the uniqueness of solutions to the Dirichlet problem (1.2)–(1.3).

COROLLARY 6.1. *Suppose we have the following.*

(1) *All assumptions in one of the corollaries 3.2, 3.8, 4.2, 4.4, 4.5, 4.6 or 5.1 are satisfied.*

(2) *The coefficients* $Q$ *are independent of* $z$.

*Then the solutions to the Dirichlet problem*

$$\begin{aligned} Qf &= 0 \quad \text{in } \Omega, \\ f &= \phi \quad \text{on } \partial\Omega \end{aligned}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

*are unique in the class $C^2(\Omega)\cap C^0(\bar{\Omega})$ in the cases of corollaries 3.8, 4.2, 4.4, 4.5, 4.6 and 5.1, or in the class $C^2(\Omega)\cap C^0(\bar{\Omega})$ with at most linear growth in the case of corollary 3.2.*

*Proof.* Let $f_1(\boldsymbol{x},y)$ and $f_2(\boldsymbol{x},y)$ be two such solutions. Consider

$$w(\boldsymbol{x},y)=f_1(\boldsymbol{x},y)-f_1(\boldsymbol{x},y).$$

Then $w(\boldsymbol{x},y)=0$ on $\partial\Omega$. By one of the corollaries, we have

$$w(\boldsymbol{x},y)\to 0 \quad \text{as } |\boldsymbol{x}|\to\infty, \quad (\boldsymbol{x},y)\in\Omega.$$

Thus if $\sup_\Omega w>0$, there is a point $(\boldsymbol{x}_0,y_0)\in\Omega$ such that

$$w(\boldsymbol{x}_0,y_0)=\sup_\Omega w.$$

Since the coefficients of $Q$ do not depend on $z$, from $Qf_1=0$ and $Qf_2=0$, we easily get $Q^{**}w=0$ in $\Omega$ for some linear elliptic operator $Q^{**}$ without zero order term in $w$. Thus, from the maximum principle for solutions of linear elliptic equations, $w$ can not achieve an interior local maximum. Thus $\sup w>0$ is impossible. Similarly, $\inf_\Omega w<0$ is impossible. Hence $w=0$ in $\Omega$; that is, $f_1=f_2$ in $\Omega$. □

## 7. Barrier functions 1

In this section we assume there exist $L\geqslant 0$ and a positive continuous function $\sigma$ on $[1,\infty)$ such that

$$\varepsilon^{\#}(\boldsymbol{x},z,t,\boldsymbol{p},q)\geqslant\sigma(|\boldsymbol{p}|^2+q^2) \tag{7.1}$$

whenever $\boldsymbol{x}$, $\boldsymbol{p}\in\mathbb{R}^{n-1}$, $z$, $t$, $q\in\mathbb{R}$ with $|\boldsymbol{x}|\geqslant L$, $|\boldsymbol{p}|^2+q^2\geqslant 1$, $|t|\leqslant M$ and $q\neq 0$. We will first construct 'upper barriers' for the Dirichlet problem (1.2)–(1.3); this construction uses the geometric idea in [20] applied to barriers from [38] and [28]. Specifically, we will prove that there exist functions $A(t)>0$, $\chi(t)>0$ such that for $H>1$, $a\geqslant A(H)$, $\boldsymbol{x}_0\in\mathbb{R}^{n-1}$ with $|\boldsymbol{x}_0|\geqslant L+a\mathrm{e}^{\chi(H)}$ and $\gamma\in\mathbb{R}$, there exists $w=w_{a,\boldsymbol{x}_0,\gamma,H}\in C^2(\Omega_{a,\boldsymbol{x}_0,H})$ such that, for any constant $b$,

$$Q(w+b)\leqslant 0 \qquad \text{in } \Omega_{a,\boldsymbol{x}_0,H}, \tag{7.2}$$

$$w\geqslant\gamma \qquad \text{on } \bar{\Omega}_{a,\boldsymbol{x}_0,H}, \tag{7.3}$$

$$\frac{\partial w}{\partial\boldsymbol{n}}=+\infty \qquad \text{on } \Omega\cap\partial\Omega_{a,\boldsymbol{x}_0,H}, \tag{7.4}$$

$$w(\boldsymbol{x}_0,y)\leqslant\gamma+\frac{2M}{H} \qquad \text{for } |y|\leqslant M, \tag{7.5}$$

where $\boldsymbol{n}$ is the exterior unit normal to $\partial\Omega_{a,\boldsymbol{x}_0,H}$ and $\Omega_{a,\boldsymbol{x}_0,H}$ is an open subset of $\mathbb{R}^n$ (the definition of $\Omega_{a,\boldsymbol{x}_0,H}$ is given in (7.12)).

### 7.1. The construction

Define $\Psi\in C^0([1,\infty))$ by

$$\Psi(\rho)=\frac{1}{\min\{1,\sigma(\rho^2)\}}.$$

Then

$$\int_1^\infty \frac{1}{\rho^3 \Psi(\rho)}\,\mathrm{d}\rho < \infty$$

and

$$\varepsilon^\#(\boldsymbol{x}, z, t, \boldsymbol{p}, q)\Psi(\sqrt{|\boldsymbol{p}|^2 + q^2}) \geqslant 1 \tag{7.6}$$

for $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ and $q \neq 0$. Define $\Psi_1$ by setting $\Psi_1(\rho) = \rho^{-2}$ if $0 < \rho < 1$ and $\Psi_1(\rho) = \Psi(\rho)$ if $1 \leqslant \rho$. Define $\chi$ by

$$\chi(\alpha) = \int_\alpha^\infty \frac{\mathrm{d}\rho}{\rho^3 \Psi_1(\rho)} \quad \text{for } \alpha > 0.$$

Then it is clear that $\chi(\alpha)$ is a decreasing function with range $(0, \infty)$. Let $\eta$ be the inverse of $\chi$. Then $\eta$ is a positive decreasing function with range $(0, \infty)$.

Let $H \geqslant 1$. Since $\eta(\chi(H)) = H$ and $\eta$ is decreasing, we have

$$\eta(\beta) > H \quad \text{for } 0 < \beta < \chi(H).$$

For each $a > 0$, define $h_a = h_{a,H}$ by

$$h_a(r) = \int_r^{a\mathrm{e}^{\chi(H)}} \eta\left(\ln \frac{t}{a}\right) \mathrm{d}t \quad \text{for } a \leqslant r \leqslant a\mathrm{e}^{\chi(H)}. \tag{7.7}$$

Then

$$\begin{aligned} h_a(a\mathrm{e}^{\chi(H)}) &= 0, \\ h_a(a) &= \int_a^{a\mathrm{e}^{\chi(H)}} \eta\left(\ln \frac{t}{a}\right) \mathrm{d}t \\ &= a \int_1^{\mathrm{e}^{\chi(H)}} \eta(\ln t)\,\mathrm{d}t = a h_1(1). \end{aligned}$$

Recall that $2M$ is the width of the strip which contains the domain $\Omega$; we define a function $A(H)$ by

$$A(H) = 2M \left( \int_1^{\mathrm{e}^{\chi(H)}} \eta(\ln t)\,\mathrm{d}t \right)^{-1}. \tag{7.8}$$

Then, for $a \geqslant A(H)$, $h_a(a\mathrm{e}^{\chi(H)}) \geqslant 2M$. Furthermore, for $a < r \leqslant a\mathrm{e}^{\chi(H)}$,

$$h_a'(r) = -\eta\left(\ln \frac{r}{a}\right) < 0, \qquad |h_a'(r)| > H$$

and

$$h_a''(r) = \frac{1}{r}\left(\eta\left(\ln \frac{r}{a}\right)\right)^3 \Psi_1\left(\eta\left(\ln \frac{r}{a}\right)\right) > 0.$$

Thus, for $a < r \leqslant a\mathrm{e}^{\chi(H)}$,

$$\frac{h_a''(r)}{(h_a'(r))^2} = -\frac{h_a'(r)}{r}\Psi_1(-h_a'(r)). \tag{7.9}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

For $\boldsymbol{x}_0 \in \mathbb{R}^{n-1}$ with $|\boldsymbol{x}_0| \geqslant L + a\mathrm{e}^{\chi(H)}$, $a \geqslant A(H)$ and a constant $\gamma \in \mathbb{R}$, let $\Gamma = \gamma + a\mathrm{e}^{\chi(H)}$, we define a function $g = g_{a,\boldsymbol{x}_0,\Gamma,M,H}$ by

$$g_{a,\boldsymbol{x}_0,\Gamma,M,H}(\boldsymbol{x}, z) = h_a(\sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}) - M \tag{7.10}$$

for $a^2 < |\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2 < a^2\mathrm{e}^{2\chi(H)}$.

Then, for $r = \sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}$, $a < r \leqslant a\mathrm{e}^{\chi(H)}$, $1 \leqslant i, j \leqslant n-1$,

$$\left.\begin{aligned}
\frac{\partial g}{\partial x_j} &= \frac{x_j - x_{0j}}{r} h_a'(r), \\
\frac{\partial g}{\partial z} &= \frac{z-\Gamma}{r} h_a'(r), \\
\frac{\partial^2 g}{\partial z^2} &= h_a''(r)\frac{(z-\Gamma)^2}{r^2} - h_a'(r)\frac{(z-\Gamma)^2}{r^3} + h_a'(r)\frac{1}{r}, \\
\frac{\partial^2 g}{\partial x_i \partial z} &= h_a''(r)\frac{(x_i - x_{0i})(z-\Gamma)}{r^2} - h_a'(r)\frac{(x_i - x_{0i})(z-\Gamma)}{r^3}, \\
\frac{\partial^2 g}{\partial x_i \partial x_j} &= h_a''(r)\frac{(x_i - x_{0i})(x_j - x_{0j})}{r^2} \\
&\qquad - h_a'(r)\frac{(x_i - x_{0i})(x_j - x_{0j})}{r^3} + \delta_{ij} h_a'(r)\frac{1}{r},
\end{aligned}\right\} \tag{7.11}$$

where $\delta_{ij}$ is the Kronecker delta.

Then it is clear that for any number $b$ (with $Q_1 g(\boldsymbol{x}, z) = Q^{\#} g(\boldsymbol{x}, z+b)$)

$$\begin{aligned}
Q_1 g(\boldsymbol{x}, z) &= \sum_{i,j=1}^{n} A_{ij}(\boldsymbol{x}, z+b, g, Dg) D_{ij} g \\
&= \left(\frac{h_a''(r)}{(h_a'(r))^2}\varepsilon^{\#}(\boldsymbol{x}, z+b, g, Dg) + \frac{1}{r}h_a'(r) - \frac{1}{r h_a'(r)}\varepsilon^{\#}(\boldsymbol{x}, z+b, g, Dg)\right) \\
&\qquad \times \sum_{i=1}^{n} A_{ii}(\boldsymbol{x}, z, g, Dg) \\
&\geqslant \left(\sum_{i=1}^{n} A_{ii}(\boldsymbol{x}, z+b, g, Dg)\right)\left(\varepsilon^{\#}(\boldsymbol{x}, z+b, g, Dg)\frac{h_a''(r)}{(h_a'(r))^2} + \frac{h_a'(r)}{r}\right) \\
&= \left(\sum_{i=1}^{n} A_{ii}(\boldsymbol{x}, z+b, g, Dg)\right) \\
&\qquad \times \left(-\frac{h_a'(r)}{r}\Psi_1(-h_a'(r))\varepsilon^{\#}(\boldsymbol{x}, z+b, g, Dg) + \frac{h_a'(r)}{r}\right) \\
&= \left(\sum_{i=1}^{n} A_{ii}(\boldsymbol{x}, z+b, g, Dg)\right)\frac{h_a'(r)}{r}\left(1 - \Psi_1(|Dg|)\varepsilon^{\#}(\boldsymbol{x}, z+b, g, Dg)\right) \\
&\geqslant 0.
\end{aligned}$$

In the above equation, we have used the definition of $\varepsilon^{\#}(\boldsymbol{x}, z+b, g, Dg)$ and the fact that if $a < r < a\mathrm{e}^{\chi(H)}$, then $h_a'(r) < 0$, $|Dg| = |h_a'(r)| \geqslant H \geqslant 1$ and

$$\Psi(|Dg|)\varepsilon^{\#}(\boldsymbol{x}, z+b, g, Dg) \geqslant 1 \quad \text{for } |Dg| \geqslant 1.$$

Now we define a domain $\Omega_{a,\boldsymbol{x}_0,H}$ in $(\boldsymbol{x}, y)$ space by

$$\Omega_{a,\boldsymbol{x}_0,H} = \{(\boldsymbol{x}, y) : |y| < M, |\boldsymbol{x} - \boldsymbol{x}_0| < h_a^{-1}(y + M)\}. \tag{7.12}$$

The domain $\Omega_{a,\boldsymbol{x}_0,H}$ is obtained by projecting the portion of the graph of the function

$$y = g(\boldsymbol{x}, z), \qquad a < \sqrt{|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (z - \Gamma)^2} < a\mathrm{e}^{\chi(H)}$$

which satisfies $|y| < M$ and $z < \Gamma$ onto the plane $z = 0$. Since $a \geqslant A(H)$, from the definition of $A(H)$ we see that $(\boldsymbol{x}_0, y) \in \Omega_{a,\boldsymbol{x}_0,H}$ for all $-M < y < M$.

Since $\partial g/\partial z < 0$ for $z < \Gamma$, we see that there is a function $z = w_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, y)$ defined on $\Omega_{a,\boldsymbol{x}_0,H}$ such that

$$g(\boldsymbol{x}, w_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, y)) = y \quad \text{for } (\boldsymbol{x}, y) \in \Omega_{a,\boldsymbol{x}_0,H}. \tag{7.13}$$

In fact, the function $z = w_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, y)$ can be easily solved from the formula for $g$ to get

$$w_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, y) = \Gamma - \sqrt{(h_a^{-1}(y + M))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}. \tag{7.14}$$

Then from

$$Q_1 g(\boldsymbol{x}, z) \geqslant 0, \qquad \frac{\partial g}{\partial z} < 0$$

and the relationship between $Q^{\#}$ and $Q$ we see that for $w = w_{a,\boldsymbol{x}_0,\gamma,H}$,

$$Q(w + b) \leqslant 0 \quad \text{in } \Omega_{a,\boldsymbol{x}_0,H}$$

for any constant $b$. This is (7.2).

Since

$$Dw(\boldsymbol{x}, y) = \frac{1}{\sqrt{(h_a^{-1}(y + M))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}} \left(\boldsymbol{x} - \boldsymbol{x}_0, \frac{-h_a^{-1}(y + M)}{h_a'(h_a^{-1}(y + M))}\right)$$

and if $(\boldsymbol{x}, y) \in \Omega \cap \partial\Omega_{a,\boldsymbol{x}_0,H}$, then $|y| < M$ and $|\boldsymbol{x} - \boldsymbol{x}_0| = h_a^{-1}(y + M)$. Thus $h_a(|\boldsymbol{x} - \boldsymbol{x}_0|) - M = y$ and the outer unit normal along $\partial\Omega_{a,\boldsymbol{x}_0,H}$ is

$$\boldsymbol{n} = \frac{1}{\sqrt{1 + (h_a'(h_a^{-1}(y + M)))^2}} \left(\frac{\boldsymbol{x} - \boldsymbol{x}_0}{|\boldsymbol{x} - \boldsymbol{x}_0|}, h_a'(h_a^{-1}(y + M))\right).$$

Thus from the fact that $h_a' < 0$, we have

$$\frac{\partial w}{\partial \boldsymbol{n}} \geqslant \frac{|\boldsymbol{x} - \boldsymbol{x}_0|}{\left(\sqrt{(h_a^{-1}(y + M))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}\right)\left(\sqrt{1 + (h_a'(h_a^{-1}(y + M)))^2}\right)}.$$

Let $|\boldsymbol{x} - \boldsymbol{x}_0| \to h_a^{-1}(y + M)$, we see that (7.4) holds. To verify (7.3) and (7.5) we see that $w(\boldsymbol{x}, y) \geqslant \Gamma - h_a^{-1}(y + M)$. Since $h_a(r)$ is a decreasing function, $h_a^{-1}$ is also a decreasing function. Thus $h_a^{-1}(y + M) \leqslant h_a^{-1}(0) = a\mathrm{e}^{\chi(H)}$ for $y \geqslant -M$. Thus

$$w(\boldsymbol{x}, y) \geqslant \Gamma - h_a^{-1}(0) = \Gamma - a\mathrm{e}^{\chi(H)}.$$

This is exactly (7.3) since $\Gamma = \gamma + a\mathrm{e}^{\chi(H)}$.

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

For (7.5), since $w(\boldsymbol{x}_0, y) = \Gamma - h_a^{-1}(y+M)$,

$$\frac{\partial w}{\partial y}(\boldsymbol{x}_0, y) = \frac{-1}{h_a'(h_a^{-1}(y+M))} = \frac{1}{\eta(\ln((1/a)h_a^{-1}(y+M)))}.$$

Using the fact that $h_a^{-1}(r)$ is a decreasing function again, we have

$$\ln\frac{1}{a}h_a^{-1}(y+M) \leqslant \ln \mathrm{e}^{\chi(H)} = \chi(H) \quad \text{for } |y| \leqslant M.$$

Since $\eta$ is also decreasing, we have

$$\frac{\partial w}{\partial y}(\boldsymbol{x}_0, y) \leqslant \frac{1}{\eta(\chi(H))} = \frac{1}{H} \quad \text{for } |y| \leqslant -M.$$

Then (7.5) follows from this and

$$w(\boldsymbol{x}_0, -M) = \Gamma - h_a^{-1}(0) = \Gamma - a\mathrm{e}^{\chi(H)} = \gamma.$$

By an argument similar to the one given above, we can construct 'lower barriers' for the Dirichlet problem (1.2)–(1.3) to get a function $l = l_{a,\boldsymbol{x}_0,\gamma,H} \in C^2(\Omega_{a,\boldsymbol{x}_0,H})$ by

$$l_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, y) = \gamma - a\mathrm{e}^{\chi(H)} + \sqrt{(h_a^{-1}(y+M))^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}$$

for $(\boldsymbol{x}, y) \in \Omega_{a,\boldsymbol{x}_0,H}$ such that for any number $b$

$$Q(l+b) \geqslant 0 \qquad \text{in } \Omega_{a,\boldsymbol{x}_0,H}, \tag{7.15}$$

$$w \leqslant \gamma \qquad \text{on } \bar{\Omega}_{a,\boldsymbol{x}_0,H}, \tag{7.16}$$

$$\frac{\partial w}{\partial \boldsymbol{n}} = +\infty \qquad \text{on } \Omega \cap \partial\Omega_{a,\boldsymbol{x}_0,H}, \tag{7.17}$$

$$w(\boldsymbol{x}_0, y) \geqslant \gamma - \frac{2M}{H} \qquad \text{for } |y| \leqslant M, \tag{7.18}$$

where $a > a(H)$ and $|\boldsymbol{x}_0| \geqslant L + a\mathrm{e}^{\chi(H)}$. We omit the details here.

## 8. Proof of theorem 2.4

Let $\omega \in S^{n-2}$. For any $\epsilon > 0$, by the assumption on $\phi(\boldsymbol{x}, y)$ and the continuity of $\Phi(\omega)$, there exist $\delta > 0$ and $R > 0$ such that if $(\boldsymbol{x}, y) \in \partial\Omega$, $|\boldsymbol{x}| \geqslant R$, $|y| \leqslant M$ and $|(\boldsymbol{x}/|\boldsymbol{x}|) - \omega| < \delta$, we have

$$|\phi(\boldsymbol{x}, y) - \Phi(\omega)| < \epsilon. \tag{8.1}$$

We choose $H \geqslant 1$ such that $2M/H < \epsilon$. Let $A(H)$ be the number given in (7.2)–(7.5). We choose a large number $R_1 > R + L + A(H)\mathrm{e}^{\chi(H)}$ and a small number $0 < \delta_1 < \delta$ such that if $|\boldsymbol{x}| > R_1$, $|(\boldsymbol{x}/|\boldsymbol{x}|) - \omega| < \delta_1$, we have

$$\left|\frac{\boldsymbol{v}}{|\boldsymbol{v}|} - \omega\right| < \delta \quad \text{for all } \boldsymbol{v} \text{ with } |\boldsymbol{v} - \boldsymbol{x}| \leqslant A(H)\mathrm{e}^{\chi(H)}.$$

Set

$$W = \left\{\boldsymbol{x} \mid |\boldsymbol{x}| > R_1, \left|\frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \omega\right| < \delta_1\right\}.$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

We claim that if $(\boldsymbol{x}_0, y) \in \bar{\Omega}$ and $\boldsymbol{x}_0 \in W$, then

$$f(\boldsymbol{x}_0, y) < \Phi(\omega) + 3\epsilon.$$

In fact, let $w(\boldsymbol{x}, y) = w_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, y)$ be the upper barrier given by (7.2)–(7.5) with $\gamma = \Phi(\omega) + 2\epsilon$ and $a = A(H)$. $w$ is defined on the domain $\Omega_{a,\boldsymbol{x}_0,H}$. We compare the functions $f(\boldsymbol{x}, y)$ and $w(\boldsymbol{x}, y)$ on the domain $\Omega_1 \equiv \Omega_{a,\boldsymbol{x}_0,H} \cap \Omega$. If $(\boldsymbol{x}, y) \in \partial\Omega \cap \partial\Omega_1$, from the definition of $W$ and (7.3), (8.1),

$$f(\boldsymbol{x}, y) = \phi(\boldsymbol{x}, y) < \Phi(\omega) + 2\epsilon = \gamma \leqslant w(\boldsymbol{x}, y). \tag{8.2}$$

Thus

$$f(\boldsymbol{x}, y) - w(\boldsymbol{x}, y) < 0 \quad \text{on } \partial\Omega \cap \partial\Omega_1. \tag{8.3}$$

$f \in C^2(\Omega)$ and (7.4) imply that

$$f(\boldsymbol{x}, y) - w(\boldsymbol{x}, y) < 0 \quad \text{on } \Omega \cap \partial\Omega_1. \tag{8.4}$$

Now we claim that $f \in C^2(\Omega)$ and (7.4) imply that

$$f(\boldsymbol{x}, y) - w(\boldsymbol{x}, y) < 0 \quad \text{on } \Omega_1. \tag{8.5}$$

Indeed, from (7.2) we have

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}, y, w(\boldsymbol{x}, y) + b, Dw(\boldsymbol{x}, y))D_{ij}w(\boldsymbol{x}, y)) \leqslant 0 \quad \text{on } \Omega_1$$

for any constant $b$. In particular, for any $(\boldsymbol{x}_1, y_1) \in \Omega_1$, let $b = f(\boldsymbol{x}_1, y_1) - w(\boldsymbol{x}_1, y_1)$ in above formula; then we have

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}_1, y_1, f(\boldsymbol{x}_1, y_1), Dw(\boldsymbol{x}_1, y_1))D_{ij}w(\boldsymbol{x}_1, y_1) \leqslant 0.$$

Since $(\boldsymbol{x}_1, y_1) \in \Omega_1$ can be arbitrary,

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}, y, f(\boldsymbol{x}, y) + b, Dw(\boldsymbol{x}, y))D_{ij}w(\boldsymbol{x}, y)) \leqslant 0 \quad \text{in } \Omega_1.$$

Now (8.5) follows from a standard argument, along with the fact (8.3) and (8.4). In particular, from (8.5) we have

$$f(\boldsymbol{x}_0, y) \leqslant w(\boldsymbol{x}_0, y) \quad \text{for } (\boldsymbol{x}_0, y) \text{ in } \Omega_1.$$

Thus (7.5) and the choices of $\gamma$ and $H$ yield

$$f(\boldsymbol{x}_0, y) \leqslant \gamma + \frac{2M}{H} \leqslant \Phi(\omega) + 3\epsilon \quad \text{for } (\boldsymbol{x}_0, y) \text{ in } \Omega_1.$$

Since $(\boldsymbol{x}_0, y) \in \Omega_1$ is the same as $(\boldsymbol{x}_0, y) \in \Omega$ (this follows from the definition of $\Omega_{a,\boldsymbol{x}_0,H}$), this proves the claim.

Similarly, using the *lower barriers* given by (7.15)–(7.18), we can conclude that

$$f(\boldsymbol{x}_0, y) \geqslant \Phi(\omega) - 3\epsilon \quad \text{for } (\boldsymbol{x}_0, y) \text{ in } \Omega.$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Thus

$$|f(\boldsymbol{x}_0, y) - \Phi(\omega)| \leqslant 3\epsilon \quad \text{for } (\boldsymbol{x}_0, y) \in \Omega.$$

Since $\boldsymbol{x}_0 \in W$ is arbitrary, we finally have

$$|f(\boldsymbol{x}, y) - \Phi(\omega)| \leqslant 3\epsilon \quad \text{for } (\boldsymbol{x}, y) \in \Omega \text{ with } \boldsymbol{x} \in W. \tag{8.6}$$

Now if $\boldsymbol{x}_j/|\boldsymbol{x}_j| \to \omega$ as $j \to \infty$, there exists $N > 0$ such that $\boldsymbol{x}_j \in W$. Then from (8.6), for $(\boldsymbol{x}_j, y_j) \in \Omega$, we have

$$|f(\boldsymbol{x}_j, y_j) - \Phi(\omega)| \leqslant 3\epsilon \quad \text{if } j \geqslant N.$$

Since $\epsilon > 0$ is arbitrary, the conclusion of theorem 2.4 follows.

## 9. Barrier functions 2

In this section we will construct 'upper barriers' for the Dirichlet problem (1.2)–(1.3) under the assumptions that there exist $L \geqslant 0$, $\delta_0 > 0$ and a positive continuous function $\sigma$ on $[1, \infty)$ such that

$$\varepsilon^{\#}(\boldsymbol{x}, z, t, \boldsymbol{p}, q) \geqslant \sigma(|\boldsymbol{p}|^2 + q^2) \tag{9.1}$$

whenever $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^{n-1}$, $z, t, q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ and $|q| \geqslant \delta_0$. ('Lower barriers' can be constructed similarly; we leave the details to the reader.)

We will use the same upper barriers $w_{a,\boldsymbol{x}_0,\gamma,H}$ defined in (7.13)–(7.14). However, since (9.1) need not hold for all $q \neq 0$, there will be a different domain on which $Q^{\#}g \geqslant 0$. (In fact, we can not have $Q^{\#}g \geqslant 0$ on $\Omega_{a,\boldsymbol{x}_0,H}$ in general if (9.1) only holds for $|q| \geqslant \delta_0$.)

We will prove that for each $H \geqslant 1$, $\gamma \in \mathbb{R}$ and $K > 0$, there exist a number $B(H, K, \gamma) \geqslant A(H)$ and a domain $\Lambda_{a,\boldsymbol{x}_0,H,K-\gamma} \subset \Omega_{a,\boldsymbol{x}_0,H}$ such that for $a \geqslant B(H, K, \gamma)$, $\boldsymbol{x}_0 \in \mathbb{R}^{n-1}$ with $|\boldsymbol{x}_0| \geqslant L + a\mathrm{e}^{\chi(H)}$, the function $w = w_{a,\boldsymbol{x}_0,\gamma,H}$ (given by (7.13)–(7.14) and restricted to $\Lambda_{a,\boldsymbol{x}_0,H,K-\gamma}$) satisfies

$$Qw \leqslant 0 \qquad \text{in } \Lambda_{a,\boldsymbol{x}_0,H,K-\gamma}, \tag{9.2}$$

$$w \geqslant \gamma \qquad \text{on } \bar{\Lambda}_{a,\boldsymbol{x}_0,H,K-\gamma}, \tag{9.3}$$

$$w \geqslant K \qquad \text{on } \Omega \cap \partial\Lambda_{a,\boldsymbol{x}_0,H,K-\gamma}, \tag{9.4}$$

$$w(\boldsymbol{x}_0, y) \leqslant \gamma + \frac{4M}{H} \qquad \text{for } |y| \leqslant M. \tag{9.5}$$

We use the same notation as in the construction of (7.2)–(7.5). Now let $H$ be a number such that $H \geqslant \sqrt{2}\delta_0 + 1$ for the $\delta_0$ defined in (9.1).

For each $\boldsymbol{x}_0 \in \mathbb{R}^{n-1}$, $|\boldsymbol{x}_0| \geqslant L + a\mathrm{e}^{\chi(H)}$ and $a \geqslant A(H)$, let $\Lambda_{a,\boldsymbol{x}_0,H}$ be the domain in $(\boldsymbol{x}, y)$ space defined by

$$\Lambda_{a,\boldsymbol{x}_0,H} = \{(\boldsymbol{x}, y) : |y| < M, \sqrt{2}|\boldsymbol{x} - \boldsymbol{x}_0| < h_a^{-1}(y + M)\}. \tag{9.6}$$

The domain $\Lambda_{a,\boldsymbol{x}_0,H}$ is obtained by projecting the portion of the graph of the function

$$y = g(\boldsymbol{x}, z), \qquad a < \sqrt{|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (z - \Gamma)^2} < a\mathrm{e}^{\chi(H)}$$

which satisfies $|y| < M$ and $z - \Gamma < -\frac{1}{2}h_a^{-1}(y + M)$ onto the plane $z = 0$. Once again, since $a \geqslant A(H)$, $(\boldsymbol{x}_0, y) \in \Lambda_{a,\boldsymbol{x}_0,H}$ for all $-M < y < M$.

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

If $(\boldsymbol{x}, y) \in \Lambda_{a,\boldsymbol{x}_0,H}$, then $|y| < M$ and $\sqrt{2}|\boldsymbol{x}-\boldsymbol{x}_0| < h_a^{-1}(y+M)$. Since $y = h_a(\sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}) - M$, we have

$$\sqrt{2}|\boldsymbol{x}-\boldsymbol{x}_0| < \sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}.$$

Thus

$$|\boldsymbol{x}-\boldsymbol{x}_0|^2 < (z-\Gamma)^2.$$

Since $z \leqslant \Gamma$, we have

$$\frac{z-\Gamma}{\sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}} \leqslant -\frac{1}{\sqrt{2}}.$$

Thus for $r = \sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}$, $a < r \leqslant a\mathrm{e}^{\chi(H)}$, since $h_a'(r) < 0$ we have, from (7.11),

$$\frac{\partial g}{\partial z} = \frac{z-\Gamma}{r} h_a'(r) \geqslant \frac{1}{\sqrt{2}}|h_a'(r)| = \frac{1}{\sqrt{2}}\eta\left(\ln\frac{r}{a}\right) \geqslant \frac{1}{\sqrt{2}}H > \delta_0. \tag{9.7}$$

Then using (9.1), in the same way as we verified (7.2), for any number $b$ we have

$$Q_1 g(\boldsymbol{x}, z) = \sum_{i,j=1}^{n} A_{ij}(\boldsymbol{x}, z+b, g, Dg) D_{ij} g \geqslant 0 \quad \text{for } a < r < a\mathrm{e}^{\chi(H)}.$$

Since $\partial g/\partial z < 0$ for $z < \Gamma$, we see that there is a function $z = w_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, y)$ (the same function given by (7.13)–(7.14)) defined on $\Lambda_{a,\boldsymbol{x}_0,H}$ such that

$$g(\boldsymbol{x}, w_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, y)) = y \quad \text{for } (\boldsymbol{x}, y) \in \Lambda_{a,\boldsymbol{x}_0,H}.$$

Also,

$$w_{a,\boldsymbol{x}_0,\Gamma,H}(\boldsymbol{x}, y) = \Gamma - \sqrt{(h_a^{-1}(y+M))^2 - |\boldsymbol{x}-\boldsymbol{x}_0|^2}.$$

Then from

$$Q_1 g(\boldsymbol{x}, z) \geqslant 0, \qquad \frac{\partial g}{\partial z} < 0,$$

and the relationship between $Q^{\#}$ and $Q$, we see that for $w = w_{a,\boldsymbol{x}_0,\gamma,H}$,

$$Q(w+b) \leqslant 0 \quad \text{on } \Lambda_{a,\boldsymbol{x}_0,H}$$

for any constant $b$. This is (9.2). Equations (9.3) and (9.5) are verified in exactly in the same way as (7.3) and (7.5). For (9.4), we see that if $(\boldsymbol{x}, y) \in \partial\Lambda_{a,\boldsymbol{x}_0,H} \cap \Omega$, then

$$\sqrt{2}|\boldsymbol{x}-\boldsymbol{x}_0| = h_a^{-1}(y+M). \tag{9.8}$$

Thus

$$y + M = h_a(\sqrt{2}|\boldsymbol{x}-\boldsymbol{x}_0|).$$

However, since $y = h_a(\sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}) - M$, we have

$$\sqrt{2}|\boldsymbol{x}-\boldsymbol{x}_0| = \sqrt{|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (z-\Gamma)^2}.$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

That is (using (9.8)),

$$|\boldsymbol{x}-\boldsymbol{x}_0| = |z-\varGamma| = \frac{1}{\sqrt{2}}h_a^{-1}(y+M).$$

Solving for $z$, we have (recall that $z<\varGamma$)

$$w(\boldsymbol{x},y) = z = \varGamma - \frac{1}{\sqrt{2}}h_a^{-1}(y+M) \quad \text{on } \partial\varLambda_{a,\boldsymbol{x}_0,H}\cap\varOmega.$$

As we has seen before, using the fact that $h_a^{-1}$ is decreasing, we have

$$w(\boldsymbol{x},y) \geqslant \varGamma - \frac{1}{\sqrt{2}}h_a^{-1}(0) = \varGamma - \frac{1}{\sqrt{2}}a\mathrm{e}^{\chi(H)}.$$

Since $\varGamma = \gamma + a\mathrm{e}^{\chi(H)}$, we have

$$w(\boldsymbol{x},y) \geqslant \gamma + \left(1-\frac{1}{\sqrt{2}}\right)a\mathrm{e}^{\chi(H)}.$$

Then if we choose

$$B(H,K,\gamma) = \max\left\{\frac{\sqrt{2}(K-\gamma)\mathrm{e}^{-\chi(H)}}{\sqrt{2}-1}, A(H)\right\}, \tag{9.9}$$

we have

$$w(\boldsymbol{x},y) \geqslant K \quad \text{on } \partial\varLambda_{a,\boldsymbol{x}_0,H}\cap\varOmega.$$

This is (9.4).

## 10. Proof of theorem 2.5

The proof of theorem 2.5 is very similar to that of theorem 2.4 except we shall use the barriers given by (9.2)–(9.5) instead of (7.2)–(7.5). Thus we shall refer back to the proof of theorem 2.4 whenever possible.

From (2.12) we may assume that

$$|f(\boldsymbol{x},y)| \leqslant C|\boldsymbol{x}| \quad \text{for } |\boldsymbol{x}|\geqslant 1, \quad (\boldsymbol{x},y)\in\varOmega. \tag{10.1}$$

For each $\epsilon>0$, from assumption 2.1 and the continuity of $\varPhi(\omega)$, we see that there are numbers $\delta_1>0$, $R_1>0$ such that if $r>R_1$, $r_1>R_1$, $r_2>R_1$,

$$|\phi(r\omega,y)-\varPhi(\omega)|<\epsilon \quad \text{for all } \omega\in S^{n-2} \text{ and } (r\omega,y)\in\partial\varOmega \tag{10.2}$$

and

$$|\phi(r_1\omega_1,y_1)-\phi(r_2\omega_2,y_2)| \leqslant \epsilon \tag{10.3}$$

for all $\omega_1$, $\omega_2\in S^{n-2}$, with $|\omega_1-\omega_2|\leqslant\delta_1$, and $(r_1\omega_1,y_1)$, $(r_2\omega_2,y_2)\in\partial\varOmega$.

Then we can choose a number $\delta_2>0$ (independent of $\boldsymbol{x}_0$) such that if $|\boldsymbol{x}_0|\geqslant R_1$ (choose $R_1$ larger if it is necessary),

$$\left|\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}-\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right| \leqslant \delta_1 \quad \text{if } |\boldsymbol{x}-\boldsymbol{x}_0|\leqslant\delta_2|\boldsymbol{x}_0|. \tag{10.4}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Now we choose $\beta = \frac{1}{16}(1+\frac{1}{4}\delta_2)^{-1}\delta_2$ and consider the function $f_1 = (\beta/C)f(\boldsymbol{x}, y)$, where $C$ is the number defined in (10.1). Then

$$|f_1(\boldsymbol{x}, y)| \leqslant \beta|\boldsymbol{x}| \quad \text{for } |\boldsymbol{x}| \geqslant 1, \quad (\boldsymbol{x}, y) \in \Omega. \tag{10.5}$$

$f_1(\boldsymbol{x}, y)$ satisfies (from (1.2))

$$Q_2 f_1(\boldsymbol{x}, y) \equiv \sum_{i,j=1}^{n} a_{ij}\biggl(\boldsymbol{x}, y, \frac{C}{\beta}f_1(\boldsymbol{x}, y), \frac{C}{\beta}Df_1(\boldsymbol{x}, y)\biggr)D_{ij}f_1(\boldsymbol{x}, y) = 0 \tag{10.6}$$

and $f_1 = (\beta/C)\phi$ on $\partial\Omega$. Thus, from (10.2),

$$|f_1(r\omega, y) - \frac{\beta}{C}\Phi(\omega)| < \frac{\beta}{C}\epsilon \quad \text{for all } \omega \in S^{n-2} \text{ and } (r\omega, y) \in \partial\Omega. \tag{10.7}$$

From (10.3),

$$|f_1(r_1\omega_1, y_1) - f_1(r_2\omega_2, y_2)| \leqslant \frac{\beta}{C}\epsilon \tag{10.8}$$

for all $\omega_1$, $\omega_2 \in S^{n-2}$, with $|\omega_1 - \omega_2| \leqslant \delta_1$, and $(r_1\omega_1, y_1)$, $(r_2\omega_2, y_2) \in \partial\Omega$.

Now for the $Q_2$ given in (10.6), we compute the $\varepsilon_2^{\#}$ corresponding to $Q_2^{\#}$ and $Q_2$. We have (by (2.13))

$$\varepsilon_1^{\#}(\boldsymbol{x}, y, z, \boldsymbol{p}, q) = \frac{\beta^2}{C^2}\varepsilon^{\#}\biggl(\boldsymbol{x}, y, \frac{C}{\beta}z, \frac{C}{\beta}\boldsymbol{p}, \frac{C}{\beta}q\biggr) \geqslant \frac{\beta^2}{C^2}\sigma\biggl(\frac{C^2}{\beta^2}(\boldsymbol{p}^2 + q^2)\biggr).$$

Thus the construction of *barrier functions 2* (i.e. § 9) applies to the equation (10.6), and the functions $A(H)$, $\chi(H)$ do not depend on $\omega$ and $\boldsymbol{x}$.

We set $H = (2MC)(\beta)^{-1}\epsilon^{-1}$ and choose a number $R_3 > R + L$ such that if $|\boldsymbol{x}_0| \geqslant R_3$,

$$\delta_2|\boldsymbol{x}_0| \geqslant 16\biggl(\frac{(\sqrt{2}-1)\beta}{\sqrt{2}C}A(H) + \frac{\beta}{C}\max|\Phi(\omega)| + \frac{\beta}{C}\biggr). \tag{10.9}$$

Set

$$W = \{\boldsymbol{x} \mid |\boldsymbol{x}| > R_3\}.$$

We claim that if $(\boldsymbol{x}_0, y) \in \bar{\Omega}$ and $\boldsymbol{x}_0 \in W$, then

$$f_1(\boldsymbol{x}_0, y) < \frac{\beta}{C}\Phi\biggl(\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}\biggr) + 3\frac{\beta}{C}\epsilon.$$

In fact, let $w(\boldsymbol{x}, y) = w_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, y)$ be the upper barrier given by (9.2)–(9.5) with

$$K = \tfrac{1}{16}\delta_2|\boldsymbol{x}_0|, \qquad \gamma = \frac{\beta}{C}\biggl(\Phi\biggl(\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}\biggr) + 2\epsilon\biggr), \qquad a = B(H, K, \gamma),$$

with the number $H$ as chosen above. $w$ is defined on the domain $\Lambda_{a,\boldsymbol{x}_0,H}$. We compare the functions $f_1(\boldsymbol{x}, y)$ and $w(\boldsymbol{x}, y)$ on the domain $\Omega_2 \equiv \Lambda_{a,\boldsymbol{x}_0,H} \cap \Omega$.

From (10.9) and (9.9), we have

$$B(H, K, \gamma) = \frac{\sqrt{2}(K-\gamma)\mathrm{e}^{-\chi(H)}}{\sqrt{2}-1}.$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Thus

$$B(H,K,\gamma)\mathrm{e}^{\chi(H)} = \frac{\sqrt{2}(K-\gamma)}{\sqrt{2}-1} \leqslant 4K \leqslant \tfrac{1}{4}\delta_2|\boldsymbol{x}_0|.$$

Then if $(\boldsymbol{x},y) \in \varLambda_{a,\boldsymbol{x}_0,H}$, from the definition of $\varLambda_{a,\boldsymbol{x}_0,H}$ and the monotonicity of $h_a^{-1}$, we have

$$|\boldsymbol{x}-\boldsymbol{x}_0| \leqslant h_a^{-1}(0) = a\mathrm{e}^{\chi(H)} = B(H,K,\gamma)\mathrm{e}^{\chi(H)} \leqslant \tfrac{1}{4}\delta_2|\boldsymbol{x}_0|. \tag{10.10}$$

Now if $(\boldsymbol{x},y) \in \partial\varOmega \cap \partial\varOmega_2$, from (10.4), (10.7), (10.8) and (10.10), we have

$$f_1(\boldsymbol{x},y) = \frac{\beta}{C}\phi(\boldsymbol{x},y) < \frac{\beta}{C}\biggl(\varPhi\biggl(\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}\biggr) + 2\epsilon\biggr) = \gamma \leqslant w(\boldsymbol{x},y).$$

Thus

$$f_1(\boldsymbol{x},y) - w(\boldsymbol{x},y) < 0 \quad \text{on } \partial\varOmega \cap \partial\varOmega_2. \tag{10.11}$$

From (10.5) and (10.10), we have that if $(\boldsymbol{x},y) \in \varOmega_2$,

$$|f_1(\boldsymbol{x},y)| \leqslant \beta|\boldsymbol{x}| \leqslant \beta(|\boldsymbol{x}_0| + \tfrac{1}{4}\delta_2|\boldsymbol{x}_0|) \leqslant \tfrac{1}{16}\delta_2|\boldsymbol{x}_0| = K$$

by the choices of $\beta$ and $K$. Thus

$$f_1(\boldsymbol{x},y) - w(\boldsymbol{x},y) < 0 \quad \text{on } \varOmega \cap \partial\varOmega_2. \tag{10.12}$$

Then, similar to the proof of theorem 2.4, we conclude that

$$f_1(\boldsymbol{x},y) - w(\boldsymbol{x},y) < 0 \quad \text{in } \varOmega_2.$$

In particular, from the choices of $\gamma$ and $H$, we have

$$f_1(\boldsymbol{x}_0,y) \leqslant \gamma + \frac{2M}{H} = \frac{\beta}{C}\biggl(\varPhi\biggl(\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}\biggr) + 3\epsilon\biggr).$$

This proves the claim. Since $f_1(\boldsymbol{x}_0,y) = (\beta/C)f_1(\boldsymbol{x}_0,y)$, we have

$$f(\boldsymbol{x}_0,y) \leqslant \varPhi\biggl(\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}\biggr) + 3\epsilon$$

for all $\boldsymbol{x}_0 \in W$. Similarly, we can obtain

$$f(\boldsymbol{x}_0,y) \geqslant \varPhi\biggl(\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}\biggr) - 3\epsilon$$

for all $\boldsymbol{x}_0 \in W$. Thus

$$\biggl|f(\boldsymbol{x}_0,y) - \varPhi\biggl(\frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|}\biggr)\biggr| \leqslant 3\epsilon$$

for all $\boldsymbol{x}_0 \in W$. Now the conclusion of theorem 2.5 follows easily.

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

## 11. Proof of theorem 2.6

Without loss of generality, we may assume

$$D = \lim_{R\to\infty} \sup\{|f(\boldsymbol{x}, y)| \mid |x| \geqslant R, (\boldsymbol{x}, y) \in \partial\Omega\} < +\infty.$$

Then for any $\epsilon > 0$, there is a number $R_3 > 0$ such that if $(\boldsymbol{x}, y) \in \partial\Omega$, $|\boldsymbol{x}| \geqslant R_3$, we have

$$f(\boldsymbol{x}, y) \leqslant D + \epsilon. \tag{11.1}$$

We choose $H \geqslant 1$ such that $2M/H < \epsilon$. Let $A(H)$ be the number given in (7.2)–(7.5). We choose a large number $R_4 > R_3 + L + A(H)\mathrm{e}^{\chi(H)}$ and set

$$W = \{\boldsymbol{x} \mid |\boldsymbol{x}| > R_4\}.$$

We claim that if $(\boldsymbol{x}_0, y) \in \bar{\Omega}$ and $\boldsymbol{x}_0 \in W$, then

$$f(\boldsymbol{x}_0, y) < D + 2\epsilon.$$

In fact, let $w(\boldsymbol{x}, y) = w_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, y)$ be the upper barrier given by (7.2)–(7.5) with $\gamma = D + \epsilon$ and $a = A(H)$. $w$ is defined on the domain $\Omega_{a,\boldsymbol{x}_0,H}$. We compare the functions $f(\boldsymbol{x}, y)$ and $w(\boldsymbol{x}, y)$ on the domain $\Omega_0 = \Omega_{a,\boldsymbol{x}_0,H} \cap \Omega$. If $(\boldsymbol{x}, y) \in \partial\Omega \cap \partial\Omega_0$, from the definition of $W$ and (7.3), (11.1),

$$f(\boldsymbol{x}, y) = \phi(\boldsymbol{x}, y) < D + \epsilon = \gamma \leqslant w(\boldsymbol{x}, y). \tag{11.2}$$

Thus

$$f(\boldsymbol{x}, y) - w(\boldsymbol{x}, y) < 0 \quad \text{on } \partial\Omega \cap \partial\Omega_0. \tag{11.3}$$

$f \in C^2(\Omega)$ and (7.4) imply that

$$f(\boldsymbol{x}, y) - w(\boldsymbol{x}, y) < 0 \quad \text{on } \Omega \cap \partial\Omega_0.$$

Then, similar to the proof of theorem 2.4, we can conclude that $f(\boldsymbol{x}, y) - w(\boldsymbol{x}, y) < 0$ on $\Omega_0$ and

$$f(\boldsymbol{x}_0, y) \leqslant \gamma + \frac{2M}{H} \leqslant D + 2\epsilon \quad \text{for } (\boldsymbol{x}_0, y) \text{ in } \Omega.$$

This proves the claim.

Now let $\boldsymbol{x}_0$ take all possible value such that $|\boldsymbol{x}_0| \geqslant R_4$ and $(\boldsymbol{x}_0, y) \in \Omega$ for some $y$; then we have

$$f(\boldsymbol{x}, y) \leqslant \gamma + \frac{2M}{H} \leqslant D + 2\epsilon$$

for all $\boldsymbol{x}$ with $|\boldsymbol{x}| \geqslant R_4$ and $(\boldsymbol{x}, y) \in \Omega$ for some $y$. Thus

$$\sup\{f(\boldsymbol{x}, y) \mid |\boldsymbol{x}| \geqslant R_4, (\boldsymbol{x}, y) \in \Omega\} \leqslant D + 2\epsilon.$$

Letting $\epsilon \to 0$, we obtain

$$\lim_{R\to+\infty} \sup\{f(\boldsymbol{x}, y) \mid |\boldsymbol{x}| \geqslant R, (\boldsymbol{x}, y) \in \Omega\} \leqslant D.$$

Similarly, we can prove

$$\lim_{R\to+\infty} \sup\{-f(\boldsymbol{x},y) \mid |\boldsymbol{x}| \geqslant R, (\boldsymbol{x},y)\in\Omega\} \leqslant \lim_{R\to+\infty} \sup\{-f(\boldsymbol{x},y) \mid |\boldsymbol{x}| \geqslant R, (\boldsymbol{x},y)\in\partial\Omega\}.$$

Thus we have

$$\lim_{R\to+\infty} \sup\{|f(\boldsymbol{x},y)| \mid |\boldsymbol{x}| \geqslant R, (\boldsymbol{x},y)\in\Omega\} \leqslant \lim_{R\to+\infty} \sup\{|f(\boldsymbol{x},y)| \mid |\boldsymbol{x}| \geqslant R, (\boldsymbol{x},y)\in\partial\Omega\}.$$

In particular, if $|f(\boldsymbol{x},y)| \leqslant K$ on $\partial\Omega$, we have $|f(\boldsymbol{x},y)| \leqslant K$ on $\Omega$ by combining the conclusion above and the maximum principle for elliptic equations.

## 12. Proofs of corollaries 3.8, 4.5 and 4.6

First of all, for corollary 4.5, we can apply theorem 2.6 to conclude that the solution $f(\boldsymbol{x},y)$ is bounded; that is, using conditions (3) and (4), we can verify condition (2) in theorem 2.6 in the same way as that in the proof of corollary 3.8. For corollary 4.6, as in the proof of corollary 3.8, using conditions (3) and (4), we can apply [28, theorem 6] to conclude the solution has at most linear growth in its variables.

To prove the conclusions in corollaries 3.8, 4.5 and 4.6, we only have to show that for every $\epsilon > 0$, there is a number $R_5 > 0$ such that for all $\omega \in S^{n-2}$, $-M < y < M$,

$$|f(r\omega,y) - G(\omega,y)| \leqslant 7\epsilon \quad \text{for } r > R_5, \quad -M < y < M, \tag{12.1}$$

where

$$G(\omega,y) = \frac{1}{2M}(\Phi_1(\omega) - \Phi_2(\omega))(y+M) + \Phi_2(\omega), \quad \omega\in S^{n-2}, \quad -M<y<M.$$

By a covering argument, it is clear that (12.1) follows from the following claim.

For each fixed $\omega_1 \in S^{n-2}$, there exist $\delta > 0$, $R_6 > 0$ ($\delta$ may depend on $\omega_1$, $R_6$ may depend on $\delta$) such that

$$|f(r\omega,y) - G(\omega,y)| \leqslant 7\epsilon \quad \text{for } r > R_6, \quad -M < y < M, \quad |\omega - \omega_1| \leqslant \delta. \tag{12.2}$$

As with the proof of theorem 2.5, for any $\epsilon > 0$, from the assumption on $\phi(\boldsymbol{x},y)$ and the continuity of $\Phi_1(\omega)$, $\Phi_2(\omega)$, there exist $\delta_1 > 0$ and $R_1 > 0$ ($\delta_1$ and $R_1$ are independent of $\omega$ and depend on $\epsilon$) such that if $|\boldsymbol{x}| \geqslant R_1$, $\omega \in S^{n-2}$ and $|(\boldsymbol{x}/|\boldsymbol{x}|) - \omega| < \delta_1$, we have

$$|\phi(\boldsymbol{x},M) - \Phi_1(\omega)| < \epsilon, \qquad |\phi(\boldsymbol{x},-M) - \Phi_2(\omega)| < \epsilon. \tag{12.3}$$

And if $|\omega - \omega_1| < \delta_1$, we have

$$|\Phi_1(\omega) - \Phi_1(\omega_1)| \leqslant \epsilon \quad \text{and} \quad |\Phi_2(\omega) - \Phi_2(\omega_1)| \leqslant \epsilon. \tag{12.4}$$

We prove (12.2) by considering two cases.

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

**Case 1 ($\boldsymbol{\Phi_1(\omega_1) = \Phi_2(\omega_1)}$).**

In this case, since the solution is either bounded or has at most linear growth in its variables, from condition (2) in corollary 3.8, condition (4) in corollary 4.5 or condition (5) in corollary 4.6 and the proof of corollary 3.2, we see that condition (2) in theorem 2.5 is satisfied. Then we basically will go through the proof of theorem 2.5 again (with the same notation unless stated otherwise) and indicate the necessary changes along the way. Hence one may wish to refer back to the proof of theorem 2.5 to understand the proof of this part. As before, we choose a number $\delta_2$ by (10.4). Set

$$\beta = \tfrac{1}{16}(1 + \tfrac{1}{4}\delta_2)^{-1}\delta_2, \qquad f_1 = \frac{\beta}{C} f.$$

Consider the equation $Q_2$ satisfied by $f_1$. Now we understand that if $(\boldsymbol{x}, y) \in \partial\Omega$ and appears in one formula, we need to assume $y$ is either always $M$ or $-M$, and $\Phi$ is defined by two functions $\Phi_1$ and $\Phi_2$.

We set

$$W_1 = \left\{ \boldsymbol{x} \mid |\boldsymbol{x}| > R_3, \left| \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \omega_1 \right| < \delta_1 \right\}.$$

Then we claim that if $r > R_3$ and $|\omega - \omega_1| < \delta_1$, (that is, $r\omega \in W_1$), then

$$|f(r\omega, y) - G(\omega, y)| \leqslant 3\epsilon \quad \text{for } -M < y < M.$$

In fact, let $w(\boldsymbol{x}, y) = w_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, y)$ be the upper barrier given by (9.2)–(9.5) with

$$\begin{gathered} \boldsymbol{x}_0 = r\omega, \qquad K = \tfrac{1}{16}\delta_2|\boldsymbol{x}_0|, \\ \gamma = \frac{\beta}{C}(\Phi_1(\omega_1) + 3\epsilon) = \frac{\beta}{C}(\Phi_2(\omega_1) + 3\epsilon), \\ H = 2MC\beta^{-1}\epsilon^{-1}, \qquad a = B(H, K, \gamma). \end{gathered}$$

$w$ is defined on the domain $\Lambda_{a,\boldsymbol{x}_0,H}$. We compare the functions $f_1(\boldsymbol{x}, y)$ and $w(\boldsymbol{x}, y)$ on the domain $\Omega_2 \equiv \Lambda_{a,\boldsymbol{x}_0,H} \cap \Omega$. As we have seen in the proof of theorem 2.5, if $(\boldsymbol{x}, y) \in \Lambda_{a,\boldsymbol{x}_0,H}$, then

$$|\boldsymbol{x} - \boldsymbol{x}_0| \leqslant \tfrac{1}{4}\delta_2|\boldsymbol{x}_0|. \tag{12.5}$$

Then if $(\boldsymbol{x}, y) \in \partial\Omega \cap \partial\Omega_2$, from (10.4), (10.7), (10.8), (12.5) and (12.4), we have

$$\begin{aligned} f_1(\boldsymbol{x}, y) &= \frac{\beta}{C}\phi(\boldsymbol{x}, y) \\ &< \frac{\beta}{C}\left( \max\left\{ \Phi_1\left( \frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|} \right), \Phi_2\left( \frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|} \right) \right\} + 2\epsilon \right) \\ &= \frac{\beta}{C}(\max\{\Phi_1(\omega), \Phi_2(\omega)\} + 2\epsilon) \\ &\leqslant \frac{\beta}{C}(\Phi_1(\omega_1) + 3\epsilon) \\ &= \gamma \\ &\leqslant w(\boldsymbol{x}, y). \end{aligned}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

Thus, as in the proof in theorem 2.5, we have

$$f_1(\boldsymbol{x}_0, y) \leqslant \gamma + \frac{2M}{H} = \frac{\beta}{C}(\Phi_1(\omega_1) + 4\epsilon).$$

Since $f_1(\boldsymbol{x}_0, y) = (\beta/C)f_1(\boldsymbol{x}_0, y)$, we have

$$f(\boldsymbol{x}_0, y) \leqslant \Phi_1(\omega_1) + 4\epsilon, \quad -M < y < M$$

for all $\boldsymbol{x}_0 = r\omega \in W_1$. Similarly, we can get

$$f(\boldsymbol{x}_0, y) \geqslant \Phi_1(\omega_1) - 4\epsilon, \quad -M < y < M$$

for all $\boldsymbol{x}_0 = r\omega \in W_1$. Thus

$$|f(r\omega, y) - \Phi_1(\omega_1)| \leqslant 4\epsilon, \quad -M < y < M$$

for all $r\omega \in W_1$. However, if $|\omega - \omega_1| < \delta_1$, from the definition of $G(\omega, y)$, $\Phi_1(\omega_1) = \Phi_2(\omega_2)$ and (12.4), we have that for $-M < y < M$,

$$\begin{aligned} |G(\omega, y) - \Phi_1(\omega_1)| &\leqslant |\Phi_1(\omega) - \Phi_2(\omega)| + |\Phi_2(\omega) - \Phi_1(\omega_1)| \\ &\leqslant |\Phi_1(\omega) - \Phi_1(\omega_1)| + 2|\Phi_2(\omega) - \Phi_2(\omega_1)| \\ &\leqslant 3\epsilon. \end{aligned}$$

Thus

$$|f(r\omega, y) - G(\omega, y)| \leqslant 7\epsilon$$

for all $r\omega \in W_1$, or $r \geqslant R_3$ and $|\omega - \omega_1| < \delta_1$. This proves the claim and completes the proof of case 1.

**Case 2 ($\boldsymbol{\Phi_1(\omega_1) \neq \Phi_2(\omega_1)}$).**

Since $\Phi_1(\omega_1) \neq \Phi_2(\omega_1)$, there are numbers $a_1 > 0$ and $\delta_4 > 0$ such that, if $|\omega - \omega_1| \leqslant \delta_4$,

$$|\Phi_1(\omega) - \Phi_2(\omega)| \geqslant a_1 > 0. \tag{12.6}$$

Set

$$V = \{\omega \mid \omega \in S^{n-2}, |\omega - \omega_1| < \delta_4\}.$$

We want to show that there is a number $R_7 > R_1$ (which may depend on $\delta_4$) such that

$$|f(r\omega, y) - G(\omega, y)| \leqslant 3\epsilon \quad \text{for } -M < y < M, \quad \omega \in V, \quad r > R_7. \tag{12.7}$$

To prove (12.7), we fix a $\omega_0 \in V$ and consider the function

$$f_0(\boldsymbol{x}, y) = f(\boldsymbol{x}, y) - \left(\frac{1}{2M}(\Phi_1(\omega_0) - \Phi_2(\omega_0))(y + M) + \Phi_2(\omega_0)\right).$$

Then, from (12.3), we see that if $|\boldsymbol{x}| \geqslant R_1$, and $|(\boldsymbol{x}/|\boldsymbol{x}|) - \omega_0| < \delta_1$, then

$$|f_0(\boldsymbol{x}, M)| = |\phi(\boldsymbol{x}, M) - \Phi_1(\omega_0)| < \epsilon \tag{12.8}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

and

$$|f_0(\boldsymbol{x},-M)| = |\phi(\boldsymbol{x},-M) - \Phi_2(\omega_0)| < \epsilon. \tag{12.9}$$

Furthermore, it is straightforward to check that in $\Omega$, $f_0(\boldsymbol{x}_0, y)$ satisfies the elliptic equation

$$\begin{aligned} &Q_0 f_0(\boldsymbol{x}_0, y) \\ &\quad \equiv \sum_{i,j=1}^{n} a_{ij}\left(\boldsymbol{x}, y, f_0 + G(\omega_0, y), D_{\boldsymbol{x}} f_0, D_y f_0 + \frac{1}{2M}(\Phi_1(\omega_0) - \Phi_2(\omega_0))\right) D_{ij} f_0 \\ &\quad = 0. \end{aligned} \tag{12.10}$$

Let $\varepsilon_0^{\#}(\boldsymbol{x}, t, z, -\boldsymbol{p}/q, 1/q)$ be the function defined in (2.8) corresponding to $Q_0$ and $Q_0^{\#}$. We claim that there exist $L_1 \geqslant 0$, $\delta_5 > 0$ and a positive continuous function $\sigma_2$ on $[1, \infty)$ ($L_1$, $\delta_5$, $\sigma_2$ may depend on the set $V_1$, but do not depend on the specific choice of $\omega_0 \in V_1$) such that

$$\varepsilon_0^{\#}\left(\boldsymbol{x}, t, z, -\frac{\boldsymbol{p}}{q}, \frac{1}{q}\right) \geqslant \sigma_2(|\boldsymbol{p}|^2 + q^2) \tag{12.11}$$

for all $\omega_0 \in \bar{V}$, $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L_1$, $|\boldsymbol{p}|^2 + q^2 \geqslant 1$, $|t| \leqslant M$ and $|q| \geqslant \delta_5$.

Assuming the claim for the moment, for the operator $Q_0$ we can use the *barrier functions 2* (i.e. § 9) and the functions $A(t)$, $\chi(t)$ do not depend on $\omega_0 \in V$ (but may depend on $V$).

Now we will basically go through the proof of theorem 2.5 again. Using what have been proved at the beginning of the proof and the definition of $f_0$, we may assume (the constant $C$ is independent of the choice of $\omega_0 \in S^{n-2}$)

$$|f_0(\boldsymbol{x}, y)| \leqslant C|\boldsymbol{x}| \quad \text{for } |\boldsymbol{x}| \geqslant 1, \quad (\boldsymbol{x}, y) \in \Omega. \tag{12.12}$$

Let $\delta_2$ be the number given in the proof of theorem 2.5 by (10.4). Let $\beta = \frac{1}{16}(1 + \frac{1}{4}\delta_2)^{-1}\delta_2$ and consider the function $f_1 = (\beta/C) f_0(\boldsymbol{x}, y)$, where $C$ is the number defined in (12.12). Then

$$|f_1(\boldsymbol{x}, y)| \leqslant \beta|\boldsymbol{x}| \quad \text{for } |\boldsymbol{x}| \geqslant 1, \quad (\boldsymbol{x}, y) \in \Omega. \tag{12.13}$$

$f_1(\boldsymbol{x}, y)$ satisfies (from (12.10))

$$\begin{aligned} Q_3 f_1(\boldsymbol{x}, y) \equiv \sum_{i,j=1}^{n} a_{ij}\bigg(&\boldsymbol{x}, y, \frac{C}{\beta} f_1 + G(\omega_0, y), \frac{C}{\beta} D_{\boldsymbol{x}} f_1, \\ &\frac{C}{\beta} D_y f_1 + \frac{1}{2M}(\Phi_1(\omega_0) - \Phi_2(\omega_0))\bigg) D_{ij} f_1 = 0 \end{aligned} \tag{12.14}$$

and $f_1 = (\beta/C) f_0$ on $\partial\Omega$. Thus, from (12.8) and (12.9), for $\omega \in S^{n-2}$, $|\omega - \omega_0| < \delta_1$, and $r > R_1$, we have

$$|f_1(r\omega, M)| < \frac{\beta}{C}\epsilon \quad \text{and} \quad |f_1(r\omega, -M)| < \frac{\beta}{C}\epsilon. \tag{12.15}$$

https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

From the claim (12.11), the construction of *barrier functions 2* applies to equation (12.10), and the functions $A(H)$, $\chi(H)$ do not depend on the choice of $\omega_0 \in V$ (but may depend on $V$).

Let $R_3$ be the same number as that given in the proof of theorem 2.5. Set

$$W_2 = \left\{ \boldsymbol{x} \mid |\boldsymbol{x}| > R_3, \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \in V \right\}.$$

We claim that if $r > R_3$, $\omega \in V$, (that is, $r\omega \in W_2$), then

$$f_1(r\omega, y) < \frac{3\beta}{C}\epsilon.$$

In fact, let $w(\boldsymbol{x}, y) = w_{a,\boldsymbol{x}_0,\gamma,H}(\boldsymbol{x}, y)$ be the upper barrier given by (9.2)–(9.5) with

$$\boldsymbol{x}_0 = r\omega, \qquad K = \tfrac{1}{16}\delta_2|\boldsymbol{x}_0|,$$
$$\gamma = \frac{2\beta}{C}\epsilon, \qquad H = 2MC\beta^{-1}\epsilon^{-1}, \qquad a = B(H, K, \gamma).$$

We compare the functions $f_1(\boldsymbol{x}, y)$ and $w(\boldsymbol{x}, y)$ on the domain $\Omega_2 \equiv \Lambda_{a,\boldsymbol{x}_0,H} \cap \Omega$.

In the same way as we have seen in the proof of theorem 2.5, if $(\boldsymbol{x}, y) \in \Lambda_{a,\boldsymbol{x}_0,H}$, we have

$$|\boldsymbol{x} - \boldsymbol{x}_0| \leqslant \tfrac{1}{4}\delta_2|\boldsymbol{x}_0|. \tag{12.16}$$

Now, if $(\boldsymbol{x}, y) \in \partial\Omega \cap \partial\Omega_2$, from (10.4), (12.15) and (12.16), we have

$$f_1(\boldsymbol{x}, y) = \frac{\beta}{C}\phi(\boldsymbol{x}, y) < \frac{2\beta}{C}\epsilon = \gamma \leqslant w(\boldsymbol{x}, y).$$

Thus

$$f_1(\boldsymbol{x}, y) - w(\boldsymbol{x}, y) < 0 \quad \text{on } \partial\Omega \cap \partial\Omega_2.$$

Then, similar to the proof of theorem 2.5, we conclude that

$$f_1(\boldsymbol{x}_0, y) \leqslant \gamma + \frac{2M}{H} = \frac{3\beta}{C}\epsilon.$$

Since $f_1(\boldsymbol{x}_0, y) = (\beta/C)f_0(\boldsymbol{x}_0, y)$, we have

$$f_0(\boldsymbol{x}_0, y) \leqslant 3\epsilon$$

for all $r\omega = \boldsymbol{x}_0 \in W_2$. Similarly, we can get

$$f_0(\boldsymbol{x}_0, y) \geqslant -3\epsilon$$

for all $r\omega = \boldsymbol{x}_0 \in W_2$. Thus

$$|f_0(r\omega_0, y)| \leqslant 3\epsilon$$

for $r \geqslant R_3$, $\omega_0 \in V$. Using the definition of $f_0$, we have proved (12.7).

It still remains to prove the claim (12.11). For corollary 4.5, from assumption (4), we have

$$a_{nn}(\boldsymbol{x}, t, z, \boldsymbol{p}, q) \geqslant \sigma\left(\frac{|\boldsymbol{p}|^2 + 1}{q^2}\right) \tag{12.17}$$

whenever $\boldsymbol{x}$, $\boldsymbol{p} \in \mathbb{R}^{n-1}$, $z$, $t$, $q \in \mathbb{R}$ with $|\boldsymbol{x}| \geqslant L$, $|t| \leqslant M$ and $|q| \neq 0$.

Then, from $Q_0$,

$$a_{nn}^0\left(\boldsymbol{x},t,z,-\frac{\boldsymbol{p}}{q},\frac{1}{q}\right)=a_{nn}\left(\boldsymbol{x},t,z,-\frac{\boldsymbol{p}}{q},\frac{1}{q}+\frac{1}{2M}(\Phi_1(\omega_0)-\Phi_2(\omega_0))\right).$$

By (12.6) we can choose a number $\delta_7>0$ such that, if $|q|\geqslant\delta_7$, $\omega\in\bar{V}$,

$$\left|1+\frac{q}{2M}(\Phi_1(\omega)-\Phi_2(\omega))\right|>0. \tag{12.18}$$

Then the function (with $\sigma$ given in (12.17))

$$\sigma_3(\rho)=\min\left\{\sigma\left(\frac{|\boldsymbol{p}|^2+q^2}{|1+(q/2M)(\Phi_1(\omega)-\Phi_2(\omega))|^2}\right)\mid|\boldsymbol{p}|^2+q^2=\rho,\omega\in\bar{V},|q|\geqslant\delta_7\right\}$$

is a well-defined positive continuous function. Hence we have

$$a_{nn}^0\left(\boldsymbol{x},t,z,-\frac{\boldsymbol{p}}{q},\frac{1}{q}\right)\geqslant\sigma_3(|\boldsymbol{p}|^2+q^2)$$

for all $\omega_0\in V$, $\boldsymbol{x}$, $\boldsymbol{p}\in\mathbb{R}^{n-1}$, $z$, $t$, $q\in\mathbb{R}$ with $|\boldsymbol{x}|\geqslant L$, $|t|\leqslant M$ and $|q|>\delta_7$. Now the claim follows from the proof of corollary 3.2.

For corollary 4.6, from assumption (5) in the corollary, we have, for $|q|\neq 0$,

$$\begin{aligned}a_{nn}^0\left(\boldsymbol{x},t,z,-\frac{\boldsymbol{p}}{q},\frac{1}{q}\right)&=a_{nn}\left(\boldsymbol{x},t,z,-\frac{\boldsymbol{p}}{q},\frac{1}{q}+\frac{1}{2M}(\Phi_1(\omega_0)-\Phi_2(\omega_0))\right)\\&\geqslant\sigma\left(|\boldsymbol{p}|^2+\left|\frac{1}{q}+\frac{1}{2M}(\Phi_1(\omega_0)-\Phi_2(\omega_0))\right|^2\right).\end{aligned}$$

Now using the $\delta_7>0$ in (12.18), we see the function

$$\sigma_4(\rho)=\min\left\{\sigma\left(|\boldsymbol{p}|^2+\left|\frac{1}{q}+\frac{1}{2M}(\Phi_1(\omega)-\Phi_2(\omega))\right|^2\right):|\boldsymbol{p}|^2+q^2=\rho,\omega\in\bar{V},|q|\geqslant\delta_7\right\}.$$

is a well-defined positive continuous function. Then we have

$$a_{nn}^0\left(\boldsymbol{x},t,z,-\frac{\boldsymbol{p}}{q},\frac{1}{q}\right)\geqslant\sigma_4(|\boldsymbol{p}|^2+q^2)$$

for all $\omega_0\in V$, $\boldsymbol{x}$, $\boldsymbol{p}\in\mathbb{R}^{n-1}$, $z$, $t$, $q\in\mathbb{R}$ with $|\boldsymbol{x}|\geqslant L$, $|t|\leqslant M$ and $|q|>\delta_7$. Now the claim follows from the proof of corollary 3.2.

For corollary 3.8, we notice that conditions (i) and (ii) of § 3 each imply (iii). The remainder of the proof now follows as for corollary 4.6.

## References


1 G. A. Ache. An exponential decay estimate for the stationary axisymmetric perturbation of Poiseuille flow in a circular pipe. *ZAMP* **45** (1994), 116–134.

2 L. Ahlfors. On Phragmén–Lindelöf's principle. *Trans. Am. Math. Soc.* **41** (1937), 1–8.

3 K. Ames and L. Payne. Decay estimates in steady pipe flow. *SIAM J. Math. Analysis* **20** (1989), 789–815.

4 H. Bear and G. Hile. Behavior of solutions of elliptic differential inequalities near a point of discontinuous boundary data. *Commun. Partial Diff. Eqns* **8** (1983), 1175–1197.


https://doi.org/10.1017/S0308210500000196 Published online by Cambridge University Press

5 S. Bernstein. Sur les équations du calcul des variations. *Ann. Sci. Ecole Norm Sup* **29**, 431–485.

6 S. Breuer and J. Roseman. Decay theorems for nonlinear Dirichlet problems in semi-infinite cylinders. *Arch. Ration. Mech. Analysis* **94** (1986), 363–371.

7 R. Finn. New Estimates for equations of minimal surface type. *Arch. Ration. Mech. Analysis* **7** (1963), 337–375.

8 A. Friedman. On two theorems of Phragmén–Lindelöf for linear elliptic and parabolic differential equations of the second order. *Pacific J. Math.* **7** (1957), 1563–1575.

9 D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, 2nd edn (Springer, 1983).

10 J. O. Herzog. Phragmén–Lindelöf theorems for second order quasilinear elliptic partial differential equations. *Proc. Am. Math. Soc.* **16** (1964), 721–728.

11 C. Horgan. A note on the spatial decay of a three-dimensional minimal surface over a semi-infinite cylinder. *J. Math. Analysis Appl.* **107** (1985), 285–290.

12 C. Horgan. Recent developments concerning Saint–Venant's principle: an update. *Appl. Mech. Rev.* **42** (1989), 295–303.

13 C. Horgan. Decay estimates for boundary-value problems in linear and nonlinear continuum mechanics. In *Mathematical problems in elasticity* (ed. R. Russo), pp. 47–89 (World Scientific, 1996).

14 C. Horgan. Recent developments concerning Saint–Venant's principle: a second update. *Appl. Mech. Rev.* **49** (1996), 101–111.

15 C. Horgan and L. Payne. Decay estimates for a class of second-order quasilinear equations in three dimensions. *Arch. Ration. Mech. Analysis* **86** (1984), 279–289.

16 C. Horgan and L. Payne. On the asymptotic behavior of solutions of inhomogeneous second-order quasilinear partial differential equations. *Q. Appl. Math.* **47** (1989), 753–771.

17 C. Horgan and L. Payne. Exponential decay estimates for capillary surfaces and extensible films. *Stability Appl. Analysis Continuous Media* **1** (1991), 261–282.

18 C. Horgan and L. Payne. The effect of constitutive law perturbations on finite antiplane shear deformations of a semi-infinite strip. *Q. Appl. Math.* **51** (1993), 441–465.

19 C. Horgan and L. Payne. Spatial decay estimates for a class of second-order quasilinear elliptic partial differential equations arising in anisotropic nonlinear elasticity. *Math. Mech. Solids* **1** (1996), 411–423.

20 C. Horgan and D. Siegel. On the asymptotic behavior of a minimal surface over a semi-infinite strip. *J. Math. Analysis Appl.* **20** (1990), 397–406.

21 C. Horgan and L. Wheeler. Spatial decay estimates for the heat equation via the maximum principle. *ZAMP* **27** (1976), 371–376.

22 C. Horgan and L. Wheeler. Exponential decay estimates for second-order quasi-linear elliptic equations. *J. Math. Analysis Appl.* **59** (1977), 267–277.

23 C. Horgan, L. Payne and L. Wheeler. Spatial decay estimates in transient heat conduction. *Q. Appl. Math.* **42** (1984), 119–127.

24 C. Hsieh, J. Hwang and F. Liang. Phragmén–Lindelöf theorem for minimal surface equations in higher dimensions. Preprint.

25 W. Huang. Boundedness of surfaces of constant sign mean curvature and the Phragmén–Lindelöf problem. In *Advances in geometric analysis and continuum mechanics* (ed. P. Concus and K. Lancaster), pp. 116–123 (Boston, MA: International, 1995).

26 J. Hwang. Phragmén–Lindelöf theorem for the minimal surface equation. *Proc. Am. Math. Soc.* **104** (1988), 825–828.

27 J. Hwang. Growth property for the minimal surface equations in unbounded domains. *Proc. Am. Math. Soc.* **121** (1994), 1027–1037.

28 Z. Jin and K. Lancaster. Theorems of Phragmén–Lindelöf type for quasilinear elliptic equations. *J. Reine Angew. Math.* **514** (1999), 165–197.

29 J. Knowles. On the spatial decay of solutions of the heat equation. *ZAMP* **22** (1971), 1050–1056.

30 J. Knowles. A note on the spatial decay of a minimal surface over a semi-infinite strip. *J. Math. Analysis Appl.* **59** (1977), 29–32.

31 C. Lin. A Phragmén–Lindelöf alternative for a class of second order quasilinear equations in $\mathbb{R}^n$. *Acta Math. Sci.* **16** (1996), 181–191.

32 C. Lin and L. Payne. The influence of domain and diffusivity perturbations on the decay of end effects in heat conduction. *SIAM J. Math. Analysis* **25** (1994), 1241–1258.

33 J. Oddson. Phragmén–Lindelöf theorems for elliptic equations in the plane. *Trans. Am. Math. Soc.* **145** (1969), 347–356.

34 R. Quintanilla. Some theorems of Phragmén–Lindelöf type for nonlinear partial differential equations. *Publ. Mat.* **37** (1993), 443–463.

35 J. Roseman. The principle of Saint–Venant in linear and nonlinear plane elasticity. *Arch. Ration. Mech. Analysis* **26** (1967), 142–162.

36 J. Roseman. Phragmén–Lindelöf theorems for some non-linear elliptic partial differential equations. *J. Math. Analysis Appl.* **43** (1973), 587–602.

37 J. Roseman. The rate of decay of a minimal surface defined over a semifinite strip. *J. Math. Analysis Appl.* **46** (1974), 545–554.

38 J. Serrin. The problem of Dirichlet for quasilinear equations with many independent variables. *Phil. Trans. R. Soc. Lond.* A **264** (1969), 413–496.

39 Y. Shenker and J. Roseman. On the asymptotic uniformization of the temperature in a laterally insulated rod for the steady-state nonlinear model. *Can. Appl. Math. Q.* **2** (1994), 115–149.

40 D. Siegel. The Dirichlet problem in a half-space and a new Phragmén–Lindelöf principle. In *Maximum principles and eigenvalue problems in partial differential equations (Knoxville, TN, 1987)*, pp. 208–217. Pitman Research Notes in Mathematics Series 175 (Longman Science and Technology, Harlow, 1988).

(*Issued 17 March 2000*)