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Connection formulae for spectral functions associated with singular Dirac equations

S. M. Riehl
University of Northern Iowa, Cedar Falls, IA 50614-0506, USA

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We consider the Dirac equation given by

$$y' = \begin{pmatrix} p & \lambda + c + v_1 \\ -(\lambda - c + v_2) & -p \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{on } [0, \infty),$$

with initial condition $y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0$, $\alpha \in [0, \pi)$ and suppose the equation is in the limit-point case at infinity. Using $\rho'_\alpha(\mu)$ to denote the derivative of the corresponding spectral function, a formula for $\rho'_\beta(\mu)$ is given when $\rho'_\alpha(\mu)$ is known and positive for three distinct values of α. In general, if $\rho'_\alpha(\mu)$ is known and positive for only two distinct values of α, then $\rho'_\beta(\mu)$ is shown to be one of two possibilities. However, in special cases of the Dirac equation, $\rho'_\beta(\mu)$ can be uniquely determined given $\rho'_\alpha(\mu)$ for only two values of α.

1. Introduction

We consider the spectral derivative functions $\rho'_\alpha(\mu)$, $\mu \in \boldsymbol{R}$, associated with the Dirac equation given by

$$y' = \begin{pmatrix} p & \lambda + c + v_1 \\ -(\lambda - c + v_2) & -p \end{pmatrix} y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{on } [0, \infty), \tag{1.1}$$

together with the initial condition

$$y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \tag{1.2}$$

where $\alpha \in [0, \pi)$. In this notation, $c \geqslant 0$ is a constant, $\lambda = \mu + \mathrm{i}\epsilon$ is the complex spectral parameter and v_1, v_2 and p are real-valued members of $L^1[0, \infty)$. The purpose of this paper is to show how the spectral derivatives $\rho'_\alpha(\mu)$ of (1.1), (1.2) for distinct initial conditions are related. The assumptions are minimal: the equation must be in the limit-point case at infinity and μ must be such that $0 < \rho'_\alpha(\mu) < \infty$. Hinton and Shaw [6] prove these assumptions are met when, for example, p, v_1 and v_2 are integrable and $|\mu| > c$.

To each parameter $\alpha \in [0, \pi)$, we define θ_α and φ_α as solutions of (1.1) that satisfy, for all λ,

$$\theta_\alpha(0, \lambda) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \qquad \varphi_\alpha(0, \lambda) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}. \tag{1.3}$$

Then the Titchmarsh–Weyl $m_\alpha(\lambda)$ function is defined by

$$\psi_\alpha(x, \lambda) = \theta_\alpha(x, \lambda) + m_\alpha(\lambda)\varphi_\alpha(x, \lambda) \in L^2[0, \infty). \tag{1.4}$$

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Since we assume that (1.1) is in the limit-point case at infinity, $m_\alpha(\lambda)$ is well defined and unique for $\mathrm{Im}\{\lambda\} > 0$. Also, in the limit-point case, the $L^2[0,\infty)$ solution of (1.1) is unique up to constant multiples. Thus it follows that $\psi_\alpha(x,\lambda)$ and $\psi_\beta(x,\lambda)$ are linearly dependent. Following Hille [5], the Wronskian of $\psi_\alpha(0,\lambda)$ and $\psi_\beta(0,\lambda)$ is 0, that is,

$$\begin{vmatrix} \cos\alpha - m_\alpha(\lambda)\sin\alpha & \cos\beta - m_\beta(\lambda)\sin\beta \\ \sin\alpha + m_\alpha(\lambda)\cos\alpha & \sin\beta + m_\beta(\lambda)\cos\beta \end{vmatrix} = 0. \tag{1.5}$$

This gives the m connection formula

$$m_\beta(\lambda) = \frac{m_\alpha(\lambda)\cos(\beta-\alpha) - \sin(\beta-\alpha)}{m_\alpha(\lambda)\sin(\beta-\alpha) + \cos(\beta-\alpha)}. \tag{1.6}$$

The spectral derivative functions may then be defined in terms of these $m_\alpha(\lambda)$ functions through the Titchmarsh–Kodaira formula

$$\rho'_\alpha(\mu) = \frac{1}{\pi}\lim_{\epsilon\to 0^+} \mathrm{Im}\{m_\alpha(\mu + \mathrm{i}\epsilon)\}, \tag{1.7}$$

where the limit exists [1, 5, 9].

Clearly, spectral functions are related to $m_\alpha(\lambda)$ functions and, for different initial conditions, these m functions are connected by (1.6). Furthermore, for special cases of the Dirac equation, it is possible to relate the spectral functions to solutions of a Riccati equation (see [11]). As in the work of Gilbert and Harris [3], the motivation to seek connection formulae for $\rho'_\alpha(\mu)$ arose since it is known that the cross ratio of four solutions of a Riccati equation is constant [8].

The connection formulae presented here are analogous to those previously established for limit-point Sturm–Liouville problems (see [2, 3, 10]). That the formulae are similar is perhaps not surprising, since the Sturm–Liouville problem can be considered a special case of the Dirac problem. (Set $p = 0$, $\lambda + c + v_1 = 1$ and $-(\lambda - c + v_2) = q - \lambda$ in (1.1).) On the other hand, one connection formula in each problem is proved using the asymptotic behaviour of the respective spectral derivative functions—and this is markedly different for the two problems.

The results are given in § 2 and proved in § 3. Examples are given in § 4.

2. The connection formulae

THEOREM 2.1. *Let $\rho'_\alpha(\mu)$ denote spectral derivatives associated with (1.1), (1.2). For almost all μ, if there is an $\alpha \in [0,\pi)$ such that $\rho'_\alpha(\mu)$ exists with $0 < \rho'_\alpha(\mu) < \infty$, then the following hold.*

(i) *$\rho'_\beta(\mu)$ exists with $0 < \rho'_\beta(\mu) < \infty$ for all $\beta \in [0,\pi)$.*

(ii) *For distinct $\alpha, \beta, \gamma, \delta \in [0,\pi)$, the spectral derivatives satisfy*

$$\begin{aligned} 0 = \frac{\sin(\beta-\gamma)\sin(\gamma-\delta)\sin(\delta-\beta)}{\rho'_\alpha(\mu)} - \frac{\sin(\gamma-\delta)\sin(\delta-\alpha)\sin(\alpha-\gamma)}{\rho'_\beta(\mu)} \\ + \frac{\sin(\delta-\alpha)\sin(\alpha-\beta)\sin(\beta-\delta)}{\rho'_\gamma(\mu)} - \frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{\rho'_\delta(\mu)}. \end{aligned} \tag{2.1}$$

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As in the Sturm–Liouville equation case, if $\rho'_\alpha(\mu)$ and $\rho'_\beta(\mu)$ exist and satisfy $0 < \rho'_\alpha(\mu) < \infty$, $0 < \rho'_\beta(\mu) < \infty$ for distinct $\alpha, \beta \in [0, \pi)$, then $\rho'_\gamma(\mu)$ exists and satisfies $0 < \rho'_\gamma(\mu) < \infty$ for all $\gamma \in [0, \pi)$ (see [3, remark 5.2]).

THEOREM 2.2. *If there are distinct $\alpha, \beta \in [0, \pi)$ such that $\rho'_\alpha(\mu), \rho'_\beta(\mu)$ exist with $0 < \rho'_\alpha(\mu) < \infty, 0 < \rho'_\beta(\mu) < \infty$, then, for any $\gamma \in [0, \pi)$, the spectral derivatives associated with (1.1) satisfy*

$$\left[\frac{\sin^2(\beta-\alpha)}{\rho'_\gamma(\mu)} - \frac{\sin^2(\beta-\gamma)}{\rho'_\alpha(\mu)} - \frac{\sin^2(\gamma-\alpha)}{\rho'_\beta(\mu)}\right]^2 = 4\sin^2(\beta-\gamma)\sin^2(\gamma-\alpha)\left(\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2\sin^2(\beta-\alpha)\right). \quad (2.2)$$

Thus $\sin^2(\beta-\alpha)/\rho'_\gamma(\mu)$ *must be one of the two choices*

$$\frac{\sin^2(\beta-\gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma-\alpha)}{\rho'_\beta(\mu)} \pm 2\sin(\beta-\gamma)\sin(\gamma-\alpha)\sqrt{\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2\sin^2(\beta-\alpha)}. \quad (2.3)$$

COROLLARY 2.3. *If $0 < \rho'_\alpha(\mu) < \infty$, $0 < \rho'_\beta(\mu) < \infty$ for distinct $\alpha, \beta \in [0, \pi)$, then*

$$\rho'_\alpha(\mu)\rho'_\beta(\mu) \leqslant \frac{1}{\pi^2\sin^2(\beta-\alpha)}. \quad (2.4)$$

Theorem 2.1 is the analogue of theorem 2.1 in [3] in the form given by Eastham [2]. Similarly, theorem 2.2 is comparable with theorem 2.2 in [3], where there need be no special relationship among the three initial conditions. The analogue to the corollary is corollary 1.2 of [10].

In [2], Eastham obtains several relationships among the spectral derivatives associated with Sturm–Liouville equation for special values of α, β, γ, δ. Analogous corollaries are valid for the Dirac equation (1.1) and are listed here.

COROLLARY 2.4. *Suppose that $\rho'_\alpha(\mu)$ exists and satisfies $0 < \rho'_\alpha(\mu) < \infty$ for all $\alpha \in [0, \pi)$. Then*

$$\left(\frac{1}{\rho'_\alpha(\mu)} - \frac{1}{\rho'_{\alpha+\pi/2}(\mu)}\right)$$

does not depend on α.

COROLLARY 2.5. *Suppose that $\rho'_\alpha(\mu)$ exists and satisfies $0 < \rho'_\alpha(\mu) < \infty$ for all $\alpha \in [0, \pi)$. Then, for any fixed η,*

$$\left(\frac{1}{\rho'_{\eta+\alpha}(\mu)} - \frac{1}{\rho'_{\eta-\alpha}(\mu)}\right)\csc 2\alpha$$

does not depend on α ($\alpha \neq 0, \frac{1}{2}\pi$).

We adopt the convention of using mod π values if the parameter falls outside the interval $[0, \pi)$. As in [2], the proofs of these corollaries follow quickly from (2.1) with $\alpha + \beta = \gamma + \delta$.

Theorem 2.2 states that, given $\rho'_\alpha(\mu)$ for fixed μ and two distinct values of α, a third derivative must be one of two choices. The next theorem refines this result for cases when additional information is known about the asymptotic behaviour of the spectral derivative. Additional hypotheses are required.

Hypothesis 2.6. *There is a $\Lambda_0 \in \boldsymbol{R}$ such that, for all $|\mu| \geqslant \Lambda_0$, $\rho'_\alpha(\mu)$ is continuous and $0 < \rho'_\alpha(\mu) < \infty$ for all $\alpha \in [0, \pi)$.*

Hypothesis 2.7. *There exist real-valued functions $S(\mu)$ and $T(\mu)$ such that, for $|\mu| \geqslant \Lambda_0$,*

$$\rho'_\alpha(\mu) = \frac{1}{\pi} \frac{T(\mu)}{(S(\mu)^2 + T(\mu)^2)\sin^2\alpha + S(\mu)\sin 2\alpha + \cos^2\alpha}, \tag{2.5}$$

with $S(\mu) \to 0$ and $T(\mu) \to \sqrt{(\mu - c)/(\mu + c)}$ as $|\mu| \to \infty$.

These hypotheses are met, for example, by Dirac equations where there exists a decreasing $L^1[0, \infty)$ function $a(x)$ such that

$$\left| \int_x^\infty \mathrm{e}^{2\mathrm{i}\sqrt{\lambda^2 - c^2}(t-x)} \left(\frac{\lambda - c}{\lambda + c} v_1(t) - v_2(t) - 2\mathrm{i}\sqrt{\frac{\lambda - c}{\lambda + c}} p(t) \right) \mathrm{d}t \right| \leqslant \frac{a(x)}{|\sqrt{\lambda^2 - c^2}|}, \quad 0 \leqslant x < \infty,$$

and the coefficient functions p, v_1, v_2 are small enough (see [11] for details).

From (2.5), we note that $\rho'_\alpha(\mu) \to 1/\pi$ for all $\alpha \in [0, \pi)$. This is key in being able to distinguish the correct value of $\rho'_\alpha(\mu)$ from the two possibilities given by theorem 2.2. The result is as follows.

Theorem 2.8. *For the Dirac problem (1.1), (1.2) satisfying hypotheses 2.6, 2.7, distinct $\alpha, \beta, \gamma \in [0, \pi)$ and $|\mu| \geqslant \Lambda_0 > c$, we have the following.*

(i) *If* $\cos(\beta - \alpha) > 0$,

$$\frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} = \frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)} + 2\sin(\beta - \gamma)\sin(\gamma - \alpha)\sqrt{\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2 \sin^2(\beta - \alpha)}. \tag{2.6}$$

(ii) *If* $\cos(\beta - \alpha) < 0$,

$$\frac{\sin^2(\beta - \alpha)}{\rho'_\gamma(\mu)} = \frac{\sin^2(\beta - \gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma - \alpha)}{\rho'_\beta(\mu)} - 2\sin(\beta - \gamma)\sin(\gamma - \alpha)\sqrt{\frac{1}{\rho'_\alpha(\mu)\rho'_\beta(\mu)} - \pi^2 \sin^2(\beta - \alpha)}. \tag{2.7}$$

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Theorem 2.8 omits the case where α and β differ by $\frac{1}{2}\pi$, for then the two choices are asymptotically indistinguishable. However, if additional information is available on $S(\mu)$, the theorem can be extended to this case.

Theorem 2.8 has an analogue in the Sturm–Liouville case. In [4], the spectral derivative for certain Sturm–Liouville problems is written in a form comparable with (2.5). But there $\rho'_\alpha(\mu)$ tends to ∞ or 0, depending on whether α is zero or non-zero, and this behaviour is used to establish theorem 1.3 in [10].

3. Proofs

Proof of theorem 2.1. (i) The proof of this is the same as the proof of the corresponding statement in the Sturm–Liouville equation case (see [3, theorem 2.1i]). The properties of Herglotz functions and (1.7) imply that, for almost all $\mu \in \boldsymbol{R}$, if $\rho'_\alpha(\mu)$ exists with $0 < \rho'_\alpha(\mu) < \infty$, then $m_\alpha(\mu + \mathrm{i}\epsilon)$ converges to a finite non-real limit as $\epsilon \to 0^+$, in which case, by (1.6), $m_\beta(\mu+\mathrm{i}\epsilon)$ also converges to a finite non-real limit as $\epsilon \to 0^+$. It then follows from (1.7) that $\rho'_\beta(\mu)$ exists and has the required properties, for all $\beta \in [0, \pi)$.

(ii) $m_\alpha(\mu + \mathrm{i}\epsilon)$ converges to a finite non-real limit as $\epsilon \to 0^+$ for almost all μ for which $\rho'_\alpha(\mu)$ exists with $0 < \rho'_\alpha(\mu) < \infty$ for some $\alpha \in [0, \pi)$. For such a μ, we denote

$$m_\alpha(\mu) = \lim_{\epsilon\to 0^+} m_\alpha(\mu + \mathrm{i}\epsilon) = X_\alpha(\mu) + \mathrm{i}\pi\rho'_\alpha(\mu), \tag{3.1}$$

where X_α is real valued and the imaginary part follows from the Titchmarsh–Kodaira formula. Then, by (1.7) and (1.6),

$$\begin{aligned}\pi\rho'_\beta(\mu) &= \mathrm{Im}\left\{\frac{(X_\alpha(\mu) + \mathrm{i}\pi\rho'_\alpha(\mu))\cos(\beta-\alpha) - \sin(\beta-\alpha)}{(X_\alpha(\mu) + \mathrm{i}\pi\rho'_\alpha(\mu))\sin(\beta-\alpha) + \cos(\beta-\alpha)}\right\} \\ &= \frac{\pi\rho'_\alpha(\mu)}{|X_\alpha(\mu)\sin(\beta-\alpha) + \cos(\beta-\alpha) + \mathrm{i}\pi\rho'_\alpha(\mu)\sin(\beta-\alpha)|^2} \\ &= \frac{\pi\rho'_\alpha(\mu)}{(X_\alpha^2(\mu) + \pi^2\rho'^2_\alpha(\mu))\sin^2(\beta-\alpha) + X_\alpha(\mu)\sin 2(\beta-\alpha) + \cos^2(\beta-\alpha)}.\end{aligned} \tag{3.2}$$

Hence

$$\frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} - 1 = (X_\alpha^2(\mu) + \pi^2\rho'^2_\alpha(\mu) - 1)\sin^2(\beta-\alpha) + X_\alpha(\mu)\sin 2(\beta-\alpha). \tag{3.3}$$

The coefficients of $\sin^2(\beta-\alpha)$ and $\sin 2(\beta-\alpha)$ depend on α and μ, not β. So replacing β in turn by γ and δ in (3.3) gives three equations in two variables, which must therefore be linearly dependent. Thus

$$\det\begin{pmatrix} \dfrac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)} - 1 & \sin^2(\beta-\alpha) & \sin 2(\beta-\alpha) \\ \dfrac{\rho'_\alpha(\mu)}{\rho'_\gamma(\mu)} - 1 & \sin^2(\gamma-\alpha) & \sin 2(\gamma-\alpha) \\ \dfrac{\rho'_\alpha(\mu)}{\rho'_\delta(\mu)} - 1 & \sin^2(\delta-\alpha) & \sin 2(\delta-\alpha) \end{pmatrix} = 0. \tag{3.4}$$

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Formula (2.1) follows upon expanding about the first column and using trigonometric identities. This completes the proof of theorem 2.1. □

Proof of theorem 2.2. As in the proof of theorem 2.1, $\rho'_\alpha(\mu)$, $\rho'_\beta(\mu)$ and $X_\alpha(\mu) = \mathrm{Re}\{m_\alpha(\mu)\}$ are related by equation (3.2). Rearranging yields an equation quadratic in X_α,

$$0 = X_\alpha^2(\mu)\sin^2(\beta-\alpha) + X_\alpha(\mu)\sin(2\beta-2\alpha) + \cos^2(\beta-\alpha) + \pi^2\rho'^2_\alpha(\mu)\sin^2(\beta-\alpha) - \frac{\rho'_\alpha(\mu)}{\rho'_\beta(\mu)}.$$

So $X_\alpha(\mu)$ is one of

$$\frac{-\cos(\beta-\alpha) \pm \sqrt{\rho'_\alpha(\mu)/\rho'_\beta(\mu) - \pi^2\rho'^2_\alpha(\mu)\sin^2(\beta-\alpha)}}{\sin(\beta-\alpha)}. \tag{3.5}$$

Equation (3.2) remains valid when β is replaced by γ. Replacing $X_\alpha(\mu)$ by (3.5) in this expression and rearranging gives formulae (2.2) and (2.3). □

Proof of corollary 2.3. If $0 < \rho'_\alpha(\mu) < \infty$, $0 < \rho'_\beta(\mu) < \infty$, then, by theorem 2.2, $\rho'_\gamma(\mu)$ exists for all $\gamma \in [0,\pi)$. So the radicand in (2.3) must be non-negative and the result follows. □

Proof of theorem 2.8. The two possible values for $\rho'_\alpha(\mu)$ are given in theorem 2.2. These expressions may be written in terms of $S(\mu)$, $T(\mu)$ using (2.5). That is, the two choices for $\sin^2(\beta-\alpha)/\rho'_\gamma(\mu)$ are

$$\begin{aligned}\frac{\pi}{T}\{&((S^2+T^2)\sin^2\alpha + S\sin 2\alpha + \cos^2\alpha)\sin^2(\beta-\gamma)\\ &+ ((S^2+T^2)\sin^2\beta + S\sin 2\beta + \cos^2\beta)\sin^2(\gamma-\alpha)\\ &\pm 2\sin(\beta-\gamma)\sin(\gamma-\alpha)|(S^2+T^2)\sin\alpha\sin\beta\\ &+ S\sin(\alpha+\beta) + \cos\alpha\cos\beta|\}.\end{aligned} \tag{3.6}$$

We assume, for the present, that the expression within absolute value is non-negative and consider the choice using the positive sign. Then we have

$$\begin{aligned}\frac{\pi}{T}\{&(S^2+T^2)(\sin\alpha\sin(\beta-\gamma) + \sin\beta\sin(\gamma-\alpha))^2\\ &+ S(\sin 2\alpha\sin^2(\beta-\gamma) + \sin 2\beta\sin^2(\gamma-\alpha)\\ &+ 2\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha+\beta))\\ &+ (\cos\alpha\sin(\beta-\gamma) + \cos\beta\sin(\gamma-\alpha))^2\},\end{aligned}$$

which, upon applying trigonometric identities, simplifies to

$$\frac{\pi\sin^2(\beta-\alpha)}{T}\{(S^2+T^2)\sin^2\gamma + S\sin 2\gamma + \cos^2\gamma\}. \tag{3.7}$$

This is identically $\sin^2(\beta-\alpha)/\rho'_\gamma(\mu)$. If the expression within absolute value is negative, then the choice using the negative sign is the one that simplifies as above. By hypothesis 2.6, the spectral derivatives $\rho'_\gamma(\mu)$ are continuous on $\mu \geqslant \Lambda_0$ and

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on $\mu \leqslant -\Lambda_0$. Therefore, $\rho'_\gamma(\mu)$ for two μ on the same half-line are both given by either the choice using the positive sign or the choice using the negative sign. We can thus use the asymptotic behaviour of $S(\mu)$ and $T(\mu)$ to choose the correct sign for μ with large absolute value and be assured the choice is correct for all μ on the same half-line with $|\mu| \geqslant \Lambda_0$. In fact, since

$$|(S^2(\mu)+T^2(\mu))\sin\alpha\sin\beta + S(\mu)\sin(\alpha+\beta)+\cos\alpha\cos\beta| \to |\cos(\beta-\alpha)|,$$

formulae (2.6) and (2.7) are established. But, if $|\beta-\alpha| = \frac{1}{2}\pi$, then the quantity within absolute value tends to 0 and more information on the size and sign of $S(\mu)$ is required to determine whether the approach is from above or below. In general, we may only conclude that

$$\frac{1}{\rho'_\gamma(\mu)} \to \frac{\sin^2(\alpha+\frac{1}{2}\pi-\gamma)}{\rho'_\alpha(\mu)} + \frac{\sin^2(\gamma-\alpha)}{\rho'_{\alpha+\pi/2}(\mu)} \quad \text{as } |\mu| \to \infty.$$

□

4. Examples

If the coefficient functions p, v_1 and $v_2 \in L^1[0,\infty)$, then $0 < \rho'_\alpha(\mu) < \infty$ for all $|\mu| > c$ and $\alpha \in [0,\pi)$ (see [6,7]). Hence the connection formulae of this paper hold. As a particular example, we take $p \equiv v_1 \equiv v_2 \equiv 0$. Then the spectral derivatives can be computed directly from (1.7) and (1.4) and

$$\rho'_\alpha(\mu) = \begin{cases} \dfrac{\sqrt{\mu^2-c^2}}{\pi(\mu+c\cos 2\alpha)}, & \mu > c, \\[2ex] \dfrac{-\sqrt{\mu^2-c^2}}{\pi(\mu+c\cos 2\alpha)}, & \mu < -c, \end{cases} \tag{4.1}$$

for $\alpha \in [0,\pi)$. Further, hypotheses 2.6, 2.7 are satisfied and $S(\mu)$, $T(\mu)$ are given by

$$S(\mu) \equiv 0, \qquad T(\mu) = \sqrt{\frac{\mu-c}{\mu+c}}.$$

To illustrate theorem 2.2, we take $\alpha = \frac{1}{4}\pi$, $\beta = \frac{1}{2}\pi$ and compute the choices for $\rho'_0(\mu)$. Since, for $\mu > c$,

$$\rho'_{\pi/4}(\mu) = \frac{1}{\pi}\frac{\sqrt{\mu^2-c^2}}{\mu} \quad \text{and} \quad \rho'_{\pi/2}(\mu) = \frac{1}{\pi}\frac{\sqrt{\mu^2-c^2}}{\mu-c},$$

theorem 2.2 yields

$$\left[\frac{1}{2\rho'_0(\mu)} - \frac{\pi}{2}\frac{3\mu-c}{\sqrt{\mu^2-c^2}}\right]^2 = \pi^2\left(\frac{\mu-c}{\mu+c}\right)$$

and the choices for $\rho'_0(\mu)$, $\mu > c$, are

$$\frac{1}{\pi}\frac{\sqrt{\mu^2-c^2}}{5\mu-3c} \quad \text{and} \quad \frac{1}{\pi}\frac{\sqrt{\mu^2-c^2}}{\mu+c}.$$

If a third derivative is known, say, $\rho'_{\pi/3}(\mu) = \sqrt{\mu^2 - c^2}/(\mu - \frac{1}{2}c)$, the ambiguity is resolved and

$$\rho'_0(\mu) = \frac{1}{\pi}\frac{\sqrt{\mu^2 - c^2}}{\mu + c}$$

for $\mu > c$ by theorem 2.1. Note that the chosen value of $\rho'_0(\mu)$ tends to $1/\pi$ as $\mu \to \infty$, but the rejected value does not have the correct asymptotic behaviour. As expected, theorem 2.8 also identifies the correct value of $\rho'_0(\mu)$. The calculation for $\mu < -c$ is similar.

Suppose now that α, β differ by $\frac{1}{2}\pi$, so theorem 2.8 does not apply. Writing $\beta = \alpha + \frac{1}{2}\pi$ and neglecting the absolute value, both expressions in (3.6) simplify and are $\sin^2(\beta - \alpha)/\rho'_\gamma(\mu)$ and $\sin^2(\beta - \alpha)/\rho'_{2\alpha-\gamma}(\mu)$. In other words, both the positive and negative branches of the square root yield valid expressions for spectral derivatives. As a specific example (with $p \equiv v_1 \equiv v_2 \equiv 0$), we set $\alpha = \frac{1}{4}\pi$, $\beta = \frac{3}{4}\pi$. Then, if $\gamma = 0$, the choices for $\sin^2(\beta - \alpha)/\rho'_0(\mu)$ for $\mu > c$ are

$$\pi\left[\frac{\mu}{\sqrt{\mu^2 - c^2}} \pm 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)\sqrt{\frac{c^2}{\mu^2 - c^2}}\right].$$

That is, for $\mu > c$, $\rho'_0(\mu)$ is either

$$\frac{\sqrt{\mu^2 - c^2}}{\pi(\mu - c)} \quad \text{or} \quad \frac{\sqrt{\mu^2 - c^2}}{\pi(\mu + c)}.$$

The expression that is not $\rho'_0(\mu)$ is $\rho'_{2\alpha-\gamma}(\mu) = \rho'_{\pi/2}(\mu)$. Asymptotically, these are indistinguishable. Here, however, since $S(\mu)$ and $T(\mu)$ are explicitly known, the expression within absolute value in (3.6) can be evaluated. In fact,

$$\begin{aligned}|(S^2 + T^2)\sin\alpha\sin(\alpha + \tfrac{1}{2}\pi) + S\sin(2\alpha + \tfrac{1}{2}\pi) + \cos\alpha\cos(\alpha + \tfrac{1}{2}\pi)| \\ = \left|\frac{\mu - c}{\mu + c}\sin\alpha\cos\alpha - \sin\alpha\cos\alpha\right| \\ = \left|\frac{-2c}{\mu + c}\sin\alpha\cos\alpha\right|.\end{aligned}$$

Since c is non-negative and α is in the first quadrant, the quantity within absolute value is non-positive for $\mu > c$ and so (2.7) is used and yields

$$\rho'_0(\mu) = \frac{\sqrt{\mu^2 - c^2}}{\pi(\mu + c)}.$$

For $\mu < -c$, the quantity is non-negative and so (2.6) must be used and

$$\rho'_0(\mu) = \frac{\sqrt{\mu^2 - c^2}}{\pi(-\mu - c)} = \frac{-\sqrt{\mu^2 - c^2}}{\pi(\mu + c)}$$

is easily calculated.

Finally, in this simple example with $p \equiv v_1 \equiv v_2 \equiv 0$, equality in corollary 2.3 is attained when $\alpha = 0$, $\beta = \frac{1}{2}\pi$.

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