Notes

106.17 An interesting spin-off

Recently, we came across the following problem which was posed in a mathematical contest for undergraduates.

Determine all polynomials p with real coefficients that satisfy

$$(p(x))^2 = 1 + xp(x + 1)$$

for all real x.

As both sides of the functional equation are polynomials and they agree for all real x, they must have the same degree. If m is the degree of p then

$$2m = m + 1$$

implying $m = 1$. Thus, if $p(x) = ax + b$ for some real numbers a and b then

$$(ax + b)^2 = 1 + x(ax + a + b)$$

for all real x and this implies $(a, b) = (1, 1)$. So $p(x) = x + 1$ is the solution and it can be easily checked that it satisfies the given equation.

One may now ask the following question:

Does there exist a polynomial p with real coefficients that satisfies

$$(p(x))^2 = 1 + x^k p(x + 1)$$

for all real x where $k \geqslant 2$ is an integer?

Evidently, if such a polynomial exists, its degree has to be k. Let

$$p(x) = a_k x^k + a_{k-1}x^{k-1} + \ldots + a_1 x + a_0$$

where a_j is a real number for $0 \leqslant j \leqslant k$ and $a_k \neq 0$. Observe that $a_0^2 = (p(0))^2 = 1$ and if

$$(p(x))^2 = 1 + \sum_{j=1}^{2k} b_j x^j$$

then $b_j = 0$ for $1 \leqslant j \leqslant k - 1$. But $b_1 = 2a_0 a_1 = 0$ implies $a_1 = 0$, $b_2 = 2a_0 a_2 + a_1^2 = 0$ implies $a_2 = 0$ and by an induction argument we see that $a_j = 0$ for $1 \leqslant j \leqslant k - 1$. Hence $p(x) = a_k x^k \pm 1$ is a possible candidate for a solution of the equation. Substituting in the equation we obtain, for all real x,

$$a_k^2 x^{2k} \pm 2a_k x^k + 1 = 1 + a_k x^k (x + 1)^k \pm x^k$$

which leads to

$$a_k (x + 1)^k - a_k^2 x^k = \pm(2a_k - 1)$$

for all x. Since the coefficient of x^k is zero on the right-hand side of the equality, we obtain $a_k - a_k^2 = 0$, whence $a_k = 1$, as we are given that $a_k \neq 0$. This leads to

$$(x + 1)^k - x^k = \pm 1$$

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for all x and a given integer $k \geqslant 2$. This is absurd. Thus no such polynomial exists. However, if we replace the monomial x^k by $p(x-1)$ in the previous question then, as we shall see shortly, we do get polynomials satisfying the new equation

$$(p(x))^2 = 1 + p(x-1)p(x+1) \quad (1)$$

for all real x. In fact the perceptive reader would immediately spot that $p(x) = \pm x$ are solutions. Are these all? Let us answer this question.

Observe that on replacing x by $x + 1$ in (1) we obtain

$$(p(x+1))^2 = 1 + p(x)p(x+2). \quad (2)$$

On subtracting (2) from (1) and rearranging terms we are led to

$$p(x)q(x+1) = p(x+1)q(x), \quad (3)$$

where $q(x) = p(x-1) + p(x+1)$.

We now show that if arbitrary non-zero complex polynomials p, q satisfy (3), then one must be a scalar multiple of the other by induction on $\deg p$. Indeed, the base case where $\deg p = 0$ is clear. Now, for the general case, take the root α of p with the largest possible real part. If such a choice is not unique, take any one such root as α. Then from (3), $p(\alpha+1)q(\alpha) = 0$. But $\alpha + 1$ is not a root of p, so $q(\alpha) = 0$. Thus we may write $p(x) = (x-\alpha)p_1(x)$ and $q(x) = (x-\alpha)q_1(x)$. Substituting these into (3) gives

$$(x-\alpha)p_1(x)(x+1-\alpha)q_1(x+1) = (x+1-\alpha)p_1(x+1)(x-\alpha)q_1(x)$$

which becomes $p_1(x)q_1(x+1) = p_1(x+1)q_1(x)$, and by the induction hypothesis, $\lambda p_1(x) = q_1(x)$, for some non-zero scalar λ, which means

$$\lambda p(x) = \lambda(x-\alpha)p_1(x) = (x-\alpha)q_1(x) = q(x),$$

and we are done.

We now return to the special case where $q(x) = p(x-1) + p(x+1)$. The coefficient of x^k in $q(x)$ is $2a_k$ and that in $\lambda p(x)$ is λa_k. Therefore $\lambda a_k = 2a_k$ implying $\lambda = 2$ since $a_k \neq 0$ and we have

$$2p(x) = p(x-1) + p(x+1).$$

This equation can be written as

$$p(x+1) - p(x) = p(x) - p(x-1).$$

If $r(x) = p(x+1) - p(x)$ then $r(x) = r(x-1)$ for all real x. Suppose $r(x)$ is a non-constant polynomial. Then, if γ is a root of $r(x)$ then so is $\gamma \pm t$ for any integer t, which is absurd. Thus $r(x) = c$ for some constant c. But for $\alpha \in \{\beta : p(\beta) = 0\}$,

$$p(\alpha+1) - p(\alpha) = c = p(\alpha) - p(\alpha-1),$$

implying $p(\alpha+1) = c = -p(\alpha-1)$ whence $c = \pm 1$ by (1). Hence we have

$$p(x+1) - p(x) = \pm 1.$$

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But

$$p(x + 1) - p(x) = \sum_{j=0}^{k} a_j \{(x + 1)^j - x^j\},$$

and one readily observes that $p(x + 1) - p(x) = \pm 1$ holds only if $a_1 = \pm 1$ and $a_j = 0$ for $2 \leqslant j \leqslant k$. Therefore $p(x) = x + a$ or $p(x) = -x + a$, for some real constant a and it can be checked that these polynomials indeed satisfy the given functional equation.

We leave the reader with the following question to settle.

Determine all polynomials p with real coefficients that satisfy

$$p(x^2) = 1 + p(x - 1)p(x + 1)$$

for all real x.

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106.18 Impossibility of solving the quintic using Cardano's solution

In [1], the author formulated a modified version of Cardano's solution of the cubic using the binomial theorem of the third power and presented an alternative solution of the quartic by the modified method using the trinomial theorem of the fourth power. In the present paper, it is shown that solving the quintic and higher degree polynomial equations by the modified method using the quadrinomial theorem of the fifth power and the $(n - 1)$th multinomial theorem of the nth power for $n > 5$, respectively, is impossible.

Cardano's solution of the cubic $x^3 + p_1x + p_2 = 0$ was found by del Ferro and Tartaglia, and published by Cardano in 1545 in his book *Ars Magna*. We show the solution in the following (e.g. [2]). In the cubic $x^3 + p_1x + p_2 = 0$, let $x = u_1 + u_2$. Then we have

$$u_1^3 + u_2^3 + (3u_1u_2 + p_1)(u_1 + u_2) + p_2 = 0.$$

Finding u_1 and u_2 such that $3u_1u_2 + p_1 = 0$ and $u_1^3 + u_2^3 + p_2 = 0$, we obtain Cardano's formulas:

$$x = \omega^k \sqrt[3]{-\frac{p_2}{2} + \sqrt{\frac{p_2^2}{4} + \frac{p_1^3}{27}}} + \omega^{3-k} \sqrt[3]{-\frac{p_2}{2} - \sqrt{\frac{p_2^2}{4} + \frac{p_1^3}{27}}}$$

with $k = 0, 1, 2$ where ω is a primitive cube root of unity and the product of the two cube roots is equal to $-\frac{p_1}{3}$.

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