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A Survey in Mathematics for Industry
On inverse problems in secondary oil recovery

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We review simple models of oil reservoirs and suggest some ideas for theoretical and numerical study of this important inverse problem. These models are formed by a system of an elliptic and a parabolic (or first-order hyperbolic) quasilinear partial differential equations. There are and probably there will be serious theoretical and computational difficulties mainly due to the degeneracy of the system. The practical value of the problem justifies efforts to improve the methods for its solution. We formulate 'history matching' as a problem of identification of two coefficients of this system. We consider global and local versions of this inverse problem and propose some approaches, including the use of the inverse conductivity problem and the structure of fundamental solutions. The global approach looks for properties of the ground in the whole domain, while the local one is aimed at recovery of these properties near wells. We discuss the use of the model proposed by Muskat which is a difficult free boundary problem. The inverse Muskat problem combines features of inverse elliptic and hyperbolic problems. We analyse its linearisation about a simple solution and show uniqueness and exponential instability for the linearisation.

1 Introduction

In oil production, it is crucial to find out properties of the ground from various measurements of geophysical fields. In secondary oil recovery, oil is extracted by pumping in water (through injecting wells) and creating pressure which pumps out oil through production wells. The water/oil pressure at wells is monitored, providing information to determine two important characteristics of the medium: permeability and porosity. These characteristics can indicate the location of oil, and hence give valuable recommendations for drilling new wells and finding pumping regimes to optimise oil recovery. Looking for the permeability and porosity can be viewed as an inverse problem for (system of) quasilinear partial differential equations describing the fluid flow in porous media. This inverse problem is called 'history matching'. Despite its obvious applied importance, this inverse problem has been only studied numerically, in most cases by using routine least-squares minimization [14, 16, 30], sometimes with the use of statistical methods [12]. Due to the large size of the problem and its non-convexity, these methods are not efficient and not reliable. In particular, there is no uniqueness and stability analysis.

We review the simplest models of oil reservoirs and suggest some ideas for theoretical and numerical study of this important inverse problem. These models are formed by a system of an elliptic and a parabolic (or first-order hyperbolic) quasilinear partial differential equations. There are and probably there will be serious theoretical and computational difficulties with both direct and inverse problems for this system mainly due to its degeneracy. However, the practical value of the problem justifies efforts producing any improvement in its solution. At present, in inverse problems for elliptic and parabolic equations, there is theoretical and numerical progress [18]. We believe that this progress can generate new efficient mathematical methods in oil recovery.

In Section 2, we recall the systems of quasilinear partial differential equations for pressure and saturation of water and oil and some known results about their solvability. Then we formulate 'history matching' as a problem of identification of two coefficients of this system. We consider global and local versions of this inverse problem. The global one seeks for properties of the ground in the whole domain, while the local one is aimed at recovery of these properties near wells. The difficulties with direct and inverse problems justify linearisation of the inverse problem, which can be viewed as the inverse conductivity problem. We recall known results about this problem and suggest new methods and directions. The local version makes use of various approximations and of the almost explicit algebraic structure of fundamental solutions of elliptic equations. As a result, we obtain a simple linear integral equation for the unknown permeability.

In Section 3, we use a simplification of the original model proposed by Muskat, which is a difficult free boundary problem. The Muskat model seems to be most suitable for local inverse problem. The inverse Muskat problem combines features of inverse elliptic and hyperbolic problems, due to finite speed of propagation of the wet area. We analyse its simplified linearisation at a basic solution and show uniqueness and exponential instability for the linearised inverse problem for two different types of free boundary conditions discussed in the literature.

In Section 4, we consider compressible fluid modelled by a second-order parabolic equation. We adopt to this problem known results on identification of parabolic equations by using hyperbolic or elliptic equations.

We emphasise that all the inverse problems under consideration seem to be exponentially ill-posed. This fact suggests a relatively low resolution in the inverse problems due to their intrinsic nature and not to the numerical methods employed.

2 A general two-phase model

One of the accepted mathematical models of filtration of oil and water through a porous medium [13] (see also [1, 6, 10]) consists of two partial differential equations,

$$-\mathrm{div}(k(\alpha_1\nabla u - \alpha_2\gamma\nabla J(S) - \alpha_3 g\nabla h)) = f \text{ in } \Omega \tag{2.1}$$

and

$$\phi\partial_t S - \mathrm{div}(k\alpha_4(\nabla u - 0.5\gamma\nabla J(S) - \rho_w g\nabla h)) = 0 \text{ in } \Omega\times(0,T), \tag{2.2}$$

where u is the pressure in the medium, $k=k(x)$ is the permeability, $\alpha_1 = k_w\mu_w^{-1} + k_o\mu_o^{-1}$, $k_w = k_w(S), \mu_w$ are the relative permeability and viscosity of water, S is the saturation of water, $k_o(S), \mu_o$ are the relative permeability and viscosity of oil. Next, $\alpha_2 = \frac{1}{2}(k_w\mu_w^{-1} - k_o\mu_o^{-1})$, γ is a known function (proportional to surface tension at the water/oil interface),

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$\phi = \phi(x)$ is the porosity, J is a known capillary pressure (the Leverett function). Moreover, $\alpha_3 = k_w \rho_w \mu_w^{-1} + k_o \rho_o \mu_o^{-1}$, where ρ_w, ρ_o are the densities of water and oil, and $\alpha_4 = k_w \mu_w^{-1}$. Observe that k_w, k_o are known functions of S, and hence $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are known functions of S. Finally, g is the acceleration due to gravity, h is the so-called height function ($h(x) = x_3$ when $n = 3$) and f represents sources and sinks. We will impose the following practically feasible conditions: Ω is a bounded domain in $\mathbb{R}^n, n = 2, 3$, its boundary $\partial\Omega$ is a C^2-surface, $\gamma, k, \phi \in L_\infty(\Omega), \delta_1 < k, \delta_2 < \phi, \delta_3 < \gamma$ on Ω, where $\delta_1, \delta_2, \delta_3$ are some positive constants, $k_o, k_w, \gamma \in C^1(\mathbb{R})$, k_w is non-increasing and k_o non-decreasing, $k_w(0) = 0, k_o(0) = 1$, J is increasing and $J(s_0) = 0$ for some $s_0 \in (0, 1)$.

The partial differential equations (2.1), (2.2) for u, S are supplemented by the initial conditions

$$S = S(\,;0) \text{ on } \Omega \times \{0\} \tag{2.3}$$

and the boundary conditions

$$\alpha_1 \partial_\nu u - \alpha_2 \gamma \partial_\nu J(S) - \alpha_3 g \partial_\nu h = 0 \text{ on } \partial\Omega \times (0, T),\ S = S(\,;0) \text{ on } \partial\Omega \times (0, T). \tag{2.4}$$

It is natural to assume that

$$\int_\Omega f = 0, \tag{2.5}$$

and to normalise pressure as follows:

$$\int_\Omega u = 0. \tag{2.6}$$

Observe that equation (2.1) for u is of elliptic type, while equation (2.2) is a partial differential equation for S which is of parabolic type. The initial boundary value problem (2.1), (2.2), (2.3), (2.4) is a system of quasilinear equations with natural boundary conditions. For some results on (weak) solvability of this problem, we refer to [1, 6], and more recently to [7, 8]. A feature of the filtration system is the degeneracy at $S = 0$, which occurs near the domain filled by oil. This degeneracy accounts for several unusual phenomena, including the finite speed of propagation, which is one of the basic properties of solutions of hyperbolic equations and which is not possible for non-generate (linear or non-linear) elliptic and parabolic equations. As shown below, in some cases equation (2.2) can be replaced by a quasilinear first-order equation for the saturation S. This equation is hyperbolic and it exhibits shock solutions which collapse in finite time. In filtration theory, this causes a phenomenon called 'fingering' (developing of wet zones recalling the shape of a hand with long fingers) and causing instability and blowup of solutions. 'Fingering' does not always occur, but only under some conditions (e.g. when water is less viscous than oil and displaces it). These reasons lead to serious difficulties in the theory of solvability of the boundary value problem (2.1), (2.2), (2.3), (2.4): currently, only existence of weak solutions is established. We will briefly describe a typical result.

The pair (u, S) is a weak solution of the initial boundary value problem (2.1), (2.2), (2.3), (2.4) if

$$u \in L_\infty(0, T; H^1(\Omega)),$$

$$S \in L_\infty(\Omega \times (0, T)),\ \nabla J(S) \in L_2(\Omega \times (0, T)),\ 0 \leqslant S \leqslant 1 \text{ on } \Omega \times (0, T)$$

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and

$$\int_{\Omega} k(\alpha_1 \nabla u - \alpha_2 \gamma \nabla J(S) - \alpha_3 g \nabla h)) \cdot \nabla \varphi_1 = \int_{\Omega} f \varphi_1,$$

$$\int_{\Omega \times (0,t)} \left(-\phi S \partial_t \varphi_2 + \left(k\alpha_4 \left(\nabla u - \frac{1}{2} \gamma \nabla J(S) - \rho_w g \nabla h \right) \right) \cdot \nabla \varphi_2 \right)$$
$$= - \int_{\partial(\Omega \times (0,t))} \phi S \varphi_2 v_t$$

for all test functions $\varphi_1 \in H^1(\Omega), \varphi_2 \in H^1(\Omega \times (0,T)), t < T$. Let us assume that $f \in L_\infty((0,T), L_2(\Omega)), S(;0) \in C([0,T], H^1(\Omega), 0 \leqslant S(;0) \leqslant 1, S(;0) = 0$ on $\partial\Omega \times (0,T)$ (no water on the outer boundary) or $S(;0) = 1$ on $\partial\Omega$, $k_o(1) = 0$ (no oil on the outer boundary). Following the arguments in [1, 6, 8], one can show that there is a weak solution (u, S) to the initial boundary value problem (2.1), (2.2), (2.3), (2.4). As in case of the Navier–Stokes system, uniqueness of a weak solution remains unknown, as well as existence of regular solutions. It seems that the main obstacle to completing theory of existence and uniqueness of solution of the initial boundary value problem (2.1), (2.2), (2.3), (2.4) is its degeneration (of the parabolic equation (2.2) for S) at $S = 0$. In cases when the data exclude degeneracy (when one can show, say by using maximum principles, that for all solutions with this data $\varepsilon_0 < S$ for some positive ε_0), global existence and uniqueness of solutions most likely can be derived from known theory of quasilinear elliptic and parabolic equations and systems [23, 24]. Maximum principles for the system of filtration theory can be found in [1].

In oil production, the source function f can be modelled as the sum of point sources at $x(m) \in \omega$, where ω is a subdomain of Ω, with intensities q_m, so

$$f(x) = \sum_{m=1}^{M} q_m \delta(x - x(m)). \tag{2.7}$$

The system of equations (2.1), (2.2) is too complicated, and satisfactory mathematical results for the direct initial boundary value problem are not available. However, in practical situations, simplifying assumptions are possible. In particular, one neglects capillarity and gravity, to arrive at a version of Buckley–Leverett system

$$\operatorname{div}(k\alpha_1 \nabla u) = f, \tag{2.8}$$

$$\phi \partial_t S - \operatorname{div}(k\alpha_4 \nabla u) = 0 \text{ in } \Omega \times (0,T). \tag{2.9}$$

Expressing $\operatorname{div}(k\nabla u)$ from equation (2.8), one transforms equation (2.9) into

$$\phi \partial_t S + \alpha_1 \left(\frac{\alpha_4}{\alpha_1} \right)' k \nabla u \cdot \nabla S = \frac{\alpha_4}{\alpha_1} f \tag{2.10}$$

with the initial and boundary value conditions

$$\partial_\nu u = 0 \text{ on } \partial\Omega, S = S(;0) \text{ on } \Omega \times (0,T). \tag{2.11}$$

Even for the simplified system (2.8), (2.10), it is too hard to obtain theoretical results in direct and inverse problems. We write the system (2.8), (2.10) in the most convenient form possible,

$$\mathrm{div}(k\alpha_1(S)\nabla u) = \mu_w f, \tag{2.12}$$

$$\phi\partial_t S + \alpha_5(S)k\nabla u \cdot \nabla S = \alpha_6 f, \tag{2.13}$$

where

$$\alpha_5(S) = \frac{k_w' k_o - k_w k_o'}{\mu_o k_w + \mu_w k_o}(S), \quad \alpha_6(S) = \frac{\mu_o k_w}{\mu_o k_w + \mu_w k_o}(S).$$

The functions $k_w(S), k_o(S)$ are known from numerous experiments. In agreement with these experiments, a good form of these functions is

$$k_w(S) = \left(\frac{S-0.1}{0.9}\right)^{\lambda}, \quad \text{when } 0.1 < S, \quad k_w(S) = 0, \quad \text{when } 0 < S \leqslant 0.1,$$

$$k_o(S) = k_w(1-S), \tag{2.14}$$

where $\lambda = \frac{log10 - log3}{log9 - log4}$. In many cases, $\mu_o = 0.6$ while $\mu_w = 1$ (see [3]).

Since typically there are hundreds of wells located in some region ω, we can assume that f is a function in $L^2(\Omega)$ supported in a subdomain $\omega \subset \Omega$ $\partial\omega \in C^2$. In practice, pressure is measured at wells for a single given f or for many of them, provided one can change the rates of pumping. It is of great importance to know two characteristics of the medium Ω, the permeability k and porosity ϕ. These functions can be assumed to be piecewise constant, with constants reflecting the properties of 20–30 typical materials forming the rock. Complying with equation (2.5) and physical reality, we partition ω into two open subsets ω_+ and ω_- with C^2-boundaries, so that $0 < f$ on ω_+ and $f < 0$ (and constant) on ω_-.

Hence we formulate the following.

Inverse Problem 1

Find k, ϕ entering the initial boundary value problem (2.1), (2.2), (2.3), (2.4) (or (2.8), (2.10), (2.11)) from the additional data

$$u = g_0 \text{ on } \omega \times (0, T) \tag{2.15}$$

given for one or many f on ω_+.

Even the direct problem for the system (2.1), (2.2) is not well understood. So not surprisingly, there are no theoretical results about uniqueness and stability of k, ϕ from the data (2.15). We will outline some linearisation approach to the inverse problem (2.8), (2.10), (2.11), (2.15) where one can at least claim uniqueness of $k(x)$ for the data from many f.

Let the initial data ε_0 be a small positive constant, $f = f_0 + \tau f_1$ where τ is a small parameter. First we assume that $f_0 = 0, S(;0) = 0$. By a standard perturbation argument,

$$u = u_0 + \tau u_1 + \cdots, \; S = \tau S_1 + \cdots,$$

where $\cdots$ are terms bounded (in certain standard norms) by $C\tau^2$. Observe that this linearisation can be most likely rigorously justified, however we do not do it in this review paper.

If $f_0 = 0$, we have $u_0 = 0$ and

$$\operatorname{div}(k\alpha_1(0)\nabla u_1) = \mu_w f_1, \tag{2.16}$$

with the boundary condition (2.4) for u_1. One can assume that k is known on ω_+. Then the additional data (2.15) uniquely determine

$$k\alpha_1(0)\partial_\nu u_1 = g_1 \text{ on } \partial\omega_+. \tag{2.17}$$

We will call a function k piecewise Lipschitz constant on Ω if there is a partition of Ω into finitely many Lipschitz subdomains, such that k is constant on each subdomain. In inverse problem 1, these subdomains and constants on these domains are not known and are to be found.

Theorem 2.1. *The map $f_1 \to u$ on ω_+ ($0 < f_1$ on ω_+, $f_1 < 0$ on ω_-, $f \in L_2(\omega)$ and satisfies equation (2.5)) uniquely determines piecewise Lipschitz constant k on Ω.*

We observe that Theorem 2.1 guarantees uniqueness of domains where k is constant and constants on these domains.

Outline of proof. By using approximations, one can show that for the solution $u(\cdot; y^+, y^-)$ to the Neumann problem

$$-\operatorname{div}(k\nabla u) = \delta(\cdot - y^+) - \delta(\cdot - y^-) \text{ on } \Omega, \quad \partial_\nu u = 0 \text{ on } \partial\Omega,$$

the function $u(\cdot\ ; y^+, y^-)$ on ω_+ is uniquely determined. From the definition of a weak solution,

$$\int_\Omega k\nabla u(\cdot\ ; y^+, y^-) \cdot \nabla v = v(y^+) - v(y^-)$$

for any $v \in C^1(\bar{\Omega})$. We can assume that k is a known constant k_0 in a connected neighbourhood $\Omega(0)$ of $\bar{\omega}_+$. Then by uniqueness of the continuation for elliptic equations, $u(\cdot\ ; y^+, y^-)$ is given on $\Omega(0)$. Let $\Omega(1)$ be a connected component of Ω where $k = k_1$, $k_1 \neq k_0$ and which has a common piece of boundary $\Gamma(1)$ with $\Omega(0)$. If v is given, then we are given

$$\int_{\Omega\setminus\omega_+} k\nabla u(\cdot\ ; y^+, y^-) \cdot \nabla v.$$

Now let $v(x) = |x - y^+|^{-1}$. Let γ be a (piecewise) analytic curve in $\Omega(0)$. Integrating by parts and using harmonicity of v outside ω^+, we conclude that we are given

$$\int_{\partial\omega_+} k_0 u(\cdot\ ; y^+, y^-)\partial_\nu v + \int_{\Gamma(1)} (k_1 - k_0) u(\cdot\ ; y^+, y^-)\partial_\nu v + \cdots,$$

where $\cdots$ denotes terms (integrals over surfaces of discontinuity of k away from $\Omega \cup \Gamma(1)$, which are bounded when $y^+ \in \gamma$. Observe that the first term in the preceding equality is known when $y^+ \in \omega_+$. Hence

$$I(y^+) = \int_{\Gamma(1)} (k_1 - k_0) u(\cdot\ ; y^+, y^-)\partial_\nu v + \cdots$$

is given when $y^+ \in \omega_+$ and is obviously analytic with respect to $y^+ \in \Omega(0)$. Since $k_0 \neq k_1$, as in ([18], Section 5.7), the first integral behaves as $C \log d(y^+)$, where $d(y^+)$ is the distance from y^+ to $\Gamma(1)$. Hence varying curves γ with starting points inside ω_+ and continuing $I(y^+)$ along γ, we can uniquely identify $\Gamma(1)$ and k_1.

We can repeat the same step and, moving subsequently from known discontinuity surfaces to adjacent ones, identify all discontinuity surfaces and piecewise constant k.

Remark. Slightly changing this proof one can show that for uniqueness it is sufficient to have the data of Theorem 2.1 for one f_1 on ω_-.

Theorem 2.1 is new, but the idea of using singular solutions of partial differential equations for identification of domains was proposed in [17]. Efficient numerical algorithms based on this idea are given by Potthast [27].

For smooth k, uniqueness can be shown if $\Omega = \mathbf{R}^3$, by using the Kelvin transform with the pole inside ω_+ and known results of Sylvester and Uhlmann for the inverse conductivity problem in bounded domains. If Ω is a ball in $\mathbf{R}^3$, then we expect that uniqueness can be shown by the methods of the paper [19]. When $\Omega = \mathbf{R}^2$ by using inversion with respect to a point of ω_+ and recent results of Astala and Päivärinta [2], one obtains uniqueness of $k \in L_\infty(\Omega)$. Currently, there is progress in the inverse conductivity problem with partial boundary data [5], [21]; however, complete results with data at a part of the boundary of a general domain Ω are still not available.

For general f_0, S_0 and $k = k_0 + \tau k_1 + ..., \phi = \phi_0 + \tau\phi_1 + \cdots$, one similarly obtains

$$
\begin{aligned}
\operatorname{div}(k_0\alpha_1(S_0)\nabla u_1) + \operatorname{div}(k_0\alpha_1'(S_0)S_1\nabla u_0) &= \mu_w f_1 - \operatorname{div}(k_1\alpha_1(S_0)\nabla u_0), \\
\phi_0\partial_t S_1 + \alpha_5(S_0)k_0\nabla S_0 \cdot \nabla u_1 + \alpha_5(S_0)k_0\nabla u_0 \cdot \nabla S_1 & \\
+ (\alpha_5'(S_0)k_0\nabla u_0 \cdot \nabla S_0 - \alpha_6'(S_0)f_0)S_1 & \\
= -(\partial_t S_0)\phi_1 - \alpha_5(S_0)\nabla u_0 \cdot \nabla S_0 k_1 + \alpha_6(S_0)f_1. & \qquad (2.18)
\end{aligned}
$$

These equations are augmented by zero initial and boundary value conditions (2.11) for u_1, S_1. The data for the linear inverse problem for first-order corrections k_1, ϕ_1 are similar to equation (2.15). The term f_1 can be interpreted as a new well, and solution of the linear inverse problem as updating old solution k_0, ϕ_0.

Practical needs stimulate numerical solution of the inverse problem (in general and simplified formulations). Currently this is done mostly by regularised least-squares matching [14, 16, 30], without analytic justification. Non-convexity of the minimisation problems and absence of analytic theory result in poor resolution and low reliability of these numerical methods. For this reason, simplifications of the inverse problem which preserve its essential features are of obvious interest.

As known [9], the inverse conductivity problem with complete boundary data is exponentially unstable. Hence, the simplest inverse problem for k is exponentially ill-posed. We expect this type of instability for more realistic problems with data at a part of the boundary and for more complicated original quasilinear equations.

Due to theoretical and numerical difficulties with the quasilinear degenerate system (2.12), (2.13), one uses a different linear elliptic partial differential equation for the pressure u

$$-\operatorname{div}(a(x)\nabla(u - \rho gh)) = f \text{ in } \Omega \qquad (2.19)$$

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with the natural boundary condition

$$\partial_\nu(u - \rho g h) = 0 \text{ on } \partial\Omega. \tag{2.20}$$

Here $a(x) = k(x)\mu(x)^{-1}$, k is the permeability and μ is some average viscosity. Of course, this average viscosity is not known. But if in some cases we assume μ to be known, then we arrive at the inverse conductivity problem where there has been some theoretical and numerical progress (as described above and in [18]). The model (2.19) is used in some cases when one can assume that compressibility of the fluids is small.

If f is given by equation (2.7) and $n = 3$, then, as known from the theory of elliptic equations,

$$u(x) = \sum_{m=1}^{M} q_m(K_1(x, x(m)) + K_2(x, x(m)) + \cdots,$$

where

$$K_1(x, x(m)) = \frac{1}{4\pi a(x)|x - x(m)|},$$

$$K_2(x, x(m)) = \frac{1}{4\pi a(y)} \int_{\Omega^*} K_1(x, v)\left(-\nabla_v a(v) \cdot \nabla_v \frac{1}{|v - x(m)|}\right) dv,$$

and $\cdots$ are bounded as well as their gradients with respect to x. Here Ω^* is some domain containing Ω. Integrating by parts, it is not hard to show that

$$K_2(x, x(m)) = -\frac{1}{16\pi^2 a(x(m))}\left(\int_{\partial\Omega^*} \frac{\log a(v) - \log a(y)}{|x - v|} \partial_{\nu(y)} \frac{1}{|v - x(m)|} d\Gamma(y) \right.$$
$$\left. + \int_{\Omega^*} (\log a(x(m) - \log a(v))\nabla_v \frac{1}{|x - v|} \cdot \nabla_v \frac{1}{|v - x(m)|} dv \right). \tag{2.21}$$

When $u(x)$ is given for $x \in \partial\omega$ and ω is a small neighbourhood of $x(m)$ (typically, a sphere), the most singular terms $K_1, \nabla K_1, \nabla K_2$ are uniquely determined by $u, \nabla u$. One can use the formula for K_1 to find a near $x(m)$. Moreover, one can use the second (less explicit) term K_2 of the expansion to get more detailed information about a near sources and sinks by solving linear integral equation with respect to $a(v)$. The first term on the right-hand side of equation (2.21) can be assumed to be known. Since we can take Ω^* to be a sufficiently large ball, we can neglect the first term on the right-hand side of equation (2.21) which will be small. Letting

$$b(v) = -\frac{1}{4\pi a(x(m))}(\log a(v) - \log a(y)),$$

we obtain for b the following integral equation:

$$\frac{1}{4\pi} \int_{\Omega^*} b(v)\nabla_v |x - v|^{-1} \cdot \nabla_v |v - x(m)|^{-1} dv = U(x), \qquad x \in \omega_+, \tag{2.22}$$

where $U(x) = K_2(x, x(m))$. It is not hard to show (integrating by parts when b is C^1 smooth and compactly supported in Ω^*) that

$$\Delta U = \mathrm{div}(b\nabla| - x(m)|^{-1}) \text{ in } \Omega^*.$$

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Hence the integral equation (2.22) can be viewed as a linearisation of the inverse conductivity problem ([18], Section 10.1). Also it has similarities with the inverse gravimetrical problem ([18], Section 4.1). Uniqueness of the solution b of the integral equation (2.22) is not known (and it is not anticipated for general b), and stability is expected to be of logarithmic type (i.e. the corresponding linear inverse problem is exponentially ill-posed).

3 Muskat free boundary model

In this model suggested in the 1930s, Muskat assumed that oil and water do not mix and occupy the domain $\Omega \subset \mathbf{R}^n, n = 2, 3$, so that the domain occupied by water is Ω_w and by oil is Ω_o. As known, [6, 25]

$$
\begin{aligned}
-\mathrm{div}(a_w \nabla(u_w - \rho_w g h)) &= f_w \text{ in } \Omega_w, \\
-\mathrm{div}(a_o \nabla(u_o - \rho_o g h)) &= f_o \text{ in } \Omega_o,
\end{aligned} \tag{3.1}
$$

where $a_j = \sigma k k_j \mu_j^{-1}, j = o, w$. The function $\sigma(x)$ is the so-called section of field at x, $\sigma = 1$ when $n = 3$, but for $n = 2$ it depends on the reduction of a 3-dimensional problem to the 2-dimensional one. Here ρ_j is density of the jth fluid. According to [15, 25], the boundary conditions at the oil–water interface $\Gamma = \partial\Omega_o \cap \partial\Omega_w$ are

$$
u_o - u_w = 0 \text{ on } \Gamma, \quad a_w \partial_\nu (u_w - \rho_w g h) = a_o \partial_\nu (u_o - \rho_o g h) = \phi V_\nu \text{ on } \Gamma, \tag{3.2}
$$

where V_ν is the normal velocity (with respect to t) of Γ and the standard outer boundary conditions are

$$
\partial_\nu u_j = 0 \text{ on } \partial\Omega \cap \partial\Omega_j. \tag{3.3}
$$

Given all the coefficients and source terms, we have an elliptic free boundary problem (3.1), (3.2), (3.3). There are only few partial analytic results on this problem [6, 15, 29]. Observe that f_w (rate of water injection) is given, while f_o (rate of oil production) may be assumed to be constant determined from solvability condition (2.7).

As above in the inverse problem, one is looking for the coefficient $k(x)$ from the additional data

$$
u_w = g \text{ in } \omega, \tag{3.4}
$$

where g is a given function for one or several given f_w. An advantage of the Muskat model for inverse problem is a possibility to use moving, due to pumping in water, interfaces Γ to introduce into the inverse problem an additional parameter θ, and hence some evolution with respect to this parameter.

Inverse Problem 2

Find k from the data (3.4) for the solution to (3.1), (3.2), (3.3) given for a one-parametric family of domains Ω_w and one f_w.

Now we will show that in a simple realistic case, one has uniqueness in this inverse problem with the reduced (and most likely minimal) data.

Let $n = 2$. We will neglect gravity, arriving at

$$
\mathrm{div}(k \nabla u_w) = f_0 \text{ in } \Omega_w. \tag{3.5}
$$

Let ω be the unit disc $\{x : |x| < 1$, $\Omega(\theta) = \{|x| < 1 + \theta\}$, $\Gamma_\theta = \partial\Omega(\theta)$. We assume that

$$k = 1 + f, \tag{3.6}$$

where f is 'small' and equal to 0 in ω. We will assume that the data (3.4) are given for $\Omega_w = \Omega(\theta), 0 < \theta < T$. By standard perturbation argument, $u = u_0 + u_1 + \cdots$ where u_0 corresponds to $f = 0$, u_1 to first-order perturbations (with respect to f), etc. Then

$$\Delta u_1 = -\mathrm{div}(f\nabla u_0) \text{ in } \Omega(\theta). \tag{3.7}$$

We have

$$\partial_\nu u_1 = 0 \text{ on } \Gamma(\theta),\ \partial_\nu u_1 = 0 \text{ on } \Gamma(0), \tag{3.8}$$

and we are given

$$u_1 = g_1(;\theta) \text{ on } \Gamma(0),\ g = g_0 + g_1 + \cdots. \tag{3.9}$$

In order to simplify the exposition, we assume the second boundary condition in (3.8) to be homogeneous. A much more delicate analysis is needed to completely justify this condition or to replace it with a similar condition. We plan to do this analysis in the near future. We assume constant water pumping letting

$$u_0(x;\theta) = \log \frac{r}{1+\theta}, \tag{3.10}$$

where $r = |x|$.

Theorem 3.1. *The solution $f \in L^2(\Omega(T))$ to the linearised inverse problem (3.7), (3.8), (3.9), (3.10) is unique. This inverse problem is severely (exponentially) ill-posed.*

Proof. To find f we make use of its angular Fourier series

$$f(x) = \sum_{-\infty}^{\infty} f_m(r)e^{im\sigma}, \quad x = r(\cos\sigma, \sin\sigma)$$

and of the solutions

$$u_m^*(x;\theta) = \left(\left(\frac{r}{1+\theta}\right)^m + \left(\frac{r}{1+\theta}\right)^{-m}\right)e^{-im\sigma}$$

of an 'adjoint' problem. From the definition of a weak solution to equation (3.7), we have

$$-\int_{\Omega(\theta)} \nabla u_1 \cdot \nabla u_m^* = -\int_{\Gamma(\theta)} f\partial_\nu u_0 u_m^* + \int_{\Omega(\theta)} f\nabla u_0 \cdot \nabla u_m^*.$$

Using that u_m^* are harmonic in $\Omega(\theta)\setminus\omega$ and $\partial_\nu u_m^* = 0$ on $\Gamma(\theta)$ to replace the left side by the integral over $\gamma(\theta)$ and orthogonality of exponents, polar coordinates, and the equality $\partial_\nu u_0 = r^{-1}$ to transform the right-hand side, we get

$$-\int_{\Gamma(0)} u_1\partial_\nu u_m^* = -\int_{\Gamma(\theta)} f(1+\theta)^{-1}2e^{-im\sigma}d\Gamma(\sigma)$$
$$+\int_{\Gamma(0)} f((1+\theta)^{-m} + (1+\theta)^m)e^{-im\sigma}d\Gamma(\sigma) + \int_{\Omega(\theta)} f\partial_r u_0\partial_r u_m^*.$$

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Using polar coordinates again we obtain

$$m\int_0^{2\pi} g_1(\sigma;\theta)((1+\theta)^{-m}-(1+\theta)^{m})e^{-im\sigma}d\sigma = -2\int_0^{2\pi} f(1+\theta,\sigma)e^{-im\sigma}d\sigma$$
$$+m\int_0^{1+\theta}\left(\int_0^{2\pi} f(r,\sigma)((1+\theta)^{-m}r^{m-1}-(1+\theta)^{m}r^{-m-1})e^{-im\sigma}d\sigma\right)dr.$$

Introducing the angular Fourier coefficients

$$f_m(1+\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(1+\theta,\sigma)e^{-im\sigma}d\sigma,$$
$$g_{1m}(1+\theta)=\frac{1}{2\pi}\int_0^{2\pi} g_1(\cos\sigma,\sin\sigma;\theta)e^{-im\sigma}d\sigma,$$

we obtain the following Volterra integral equation:

$$f_m(1+\theta)+\frac{m}{2}\int_1^{1+\theta}((1+\theta)^{m}r^{-m-1}-(1+\theta)^{-m}r^{m-1})f_m(r)dr$$
$$=\frac{m}{2}((1+\theta)^{m}-(1+\theta)^{-m})g_{1m}(1+\theta). \tag{3.11}$$

Of course, the solution f_m to this Volterra equation and hence to the linearised inverse problem 2 is unique.

To show exponential instability, we solve the integral equation (3.11) explicitly. We managed to do it by using elementary but not standard arguments given in the proof of the following key lemma. □

Lemma 3.2. *Let*

$$F_m(1+\theta)=\frac{m}{2}((1+\theta)^{m}-(1+\theta)^{-m})g_{1m}(1+\theta).$$

Then the function

$$f_m(1+\theta)=F_m(1+\theta)+m^2\int_1^{1+\theta}\left(log\frac{r}{1+\theta}\right)r^{-1}F_m(r)dr \tag{3.12}$$

solves the integral equation (3.11).

Although this statement can be checked by elementary direct calculations, we prefer to give a proof explaining how to obtain equation (3.12).

Proof of Lemma 3.2.
Letting

$$v_m(1+\theta)=(1+\theta)^{m}v_m(1+\theta), \tag{3.13}$$

and multiplying both sides by $(1+\theta)^m$, we transform the integral equation (3.11) for f_m into

$$v_m(1+\theta)+\frac{m}{2}\int_1^{1+\theta}((1+\theta)^{2m}r^{-2m-1}-r^{-1})v_m(r)dr=(1+\theta)^{m}F_m(1+\theta).$$

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Differentiating we replace this equation by a new homogeneous integro-differential equation with an initial condition

$$v_m'(1+\theta) + m^2 \int_1^{1+\theta} (1+\theta)^{2m-1} r^{-2m-1} v_m(r)dr = ((1+\theta)^m F_m(1+\theta))', \quad v_m(1) = 0,$$

where we used that $F_m(1) = 0$. Finally letting

$$w_m(1+\theta) = \int_1^{1+\theta} r^{-2m-1} v_m(r)dr, \tag{3.14}$$

we obtain the Euler differential equation with the initial conditions

$$(1+\theta)^2 w_m''(1+\theta) + (2m+1)(1+\theta)w_m'(1+\theta) + m^2 w_m(1+\theta) = ((1+\theta)^m F_m(1+\theta))',$$
$$w_m(1) = 0, \quad w_m'(1) = 0 \tag{3.15}$$

that are equivalent to the integro-differential equation for v_m.

By standard integration method of ordinary differential equations, functions $(1+\theta)^{-m}$, $(1+\theta)^{-m}\log(1+\theta)$ are two linearly independent solutions to the homogeneous differential equation (3.15). According to the variation of the parameters technique, we look for general solution to equation (3.15) in the form

$$w_m(1+\theta) = u_1(1+\theta)(1+\theta)^{-m} + u_2(1+\theta)(1+\theta)^{-m}\log(1+\theta).$$

Using also the initial conditions, we obtain

$$\begin{aligned} u_1(1+\theta) &= -\int_1^{1+\theta} r^{-m}\log r (r^m F_m(r))' dr \\ &= -\log(1+\theta)F_m(1+\theta) + \int_1^{1+\theta} (-m\log r + 1) r^{-1} F_m(r)dr, \\ u_2(1+\theta) &= \int_1^{1+\theta} r^{-m}(r^m F_m(r))' dr = F_m(1+\theta) + m\int_1^{1+\theta} r^{-1}F_m(r)dr. \end{aligned}$$

Due to our choice of w_m and u_1, u_2,

$$\begin{aligned} w_m'(1+\theta) &= m(1+\theta)^{-m-1}\log(1+\theta)F_m(1+\theta) + m(1+\theta)^{-m-1}\int_1^{1+\theta}(m\log r - 1)r^{-1}F_m(r)dr \\ &+ \left(F_m(1+\theta) + m\int_1^{1+\theta} r^{-1}F_m(r)dr\right)\left(-m(1+\theta)^{-m-1}\log(1+\theta) + (1+\theta)^{-m-1}\right) \\ &= (1+\theta)^{-m-1}F_m(1+\theta) + m^2(1+\theta)^{-m-1}\int_1^{1+\theta}\log\frac{r}{1+\theta} r^{-1}F_m(r)dr, \end{aligned}$$

so using that

$$f_m(1+\theta) = (1+\theta)^{-m} v_m(1+\theta) = (1+\theta)^{m+1} w_m'(1+\theta),$$

we obtain the needed solution (3.12). The proof is complete.

Now we complete the proof of Theorem 3.1 by showing exponential instability of solution to equation (3.11). Indeed, the terms on the right-hand side of equation (3.12)

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containing $(1+\theta)^{-m}$ form the stable part of f_m, so to exhibit instability we look at the remaining part of $f_m(1+\theta)$ which, after integration by parts, is

$$\begin{aligned}
&\frac{m}{2}(1+\theta)^m g_{1m}(1+\theta) + \frac{m^3}{2}\int_1^{1+\theta} \log\frac{r}{1+\theta} r^{-1+m} g_{1m}(r)dr \\
&= \frac{m}{2}(1+\theta)^m g_{1m}(1+\theta) + \frac{m^2}{2}\int_1^{1+\theta}\left(r^m \log\frac{r}{1+\theta}r^{-1+m} - \frac{r^m}{m}\right)' g_{1m}(r)dr \\
&= \left(\frac{m^2}{2}\log(1+\theta) + \frac{m}{2}\right) g_{1m}(1) - \frac{m^2}{2}\int_1^{1+\theta} r^m\left(\log\frac{r}{1+\theta} - \frac{1}{m}\right) g'_{1m}(r)dr.
\end{aligned}$$

Due to the presence of powers r^m, the last formula shows exponential magnifying of the Fourier coefficients of g_{1m} with growing m, and hence exponential instability for the Volterra equation (3.11) and for the linearised inverse Muskat problem. The proof is complete.

According to [6], the boundary conditions at the oil–water interface $\Gamma = \partial\Omega_o \cap \partial\Omega_w$ are

$$u_o - u_w = 0 \text{ on } \Gamma, \quad a_w \nabla(u_w - \rho_w g h) = a_o \nabla(u_o - \rho_o g h) \text{ on } \Gamma. \tag{3.16}$$

Given all the coefficients and source terms, we again have an elliptic free boundary problem (3.1), (3.16), (3.3). This free boundary problem is not overdetermined in $\mathbf{R}^2$ and it is overdetermined in $\mathbf{R}^3$.

Letting ∇_τ be the tangential component of ∇ we have, from (3.2), $\nabla_\tau u_o = \nabla_\tau u_w$ on Γ and hence

$$\left(\frac{k_o}{\mu_o} - \frac{k_w}{\mu_w}\right)\nabla_\tau u_w = \left(\frac{k_o\rho_o}{\mu_o} - \frac{k_w\rho_w}{\mu_w}\right) g\nabla_\tau h \text{ on } \Gamma,$$

so given Γ, one can uniquely (modulo constant) determine u_w on Γ. Neglecting gravity (i.e. letting $\nabla_\tau h = 0$) we conclude that u_w is constant on each connected component of Γ.

Inverse Problem 2(D)

Find k from the data (3.4) for the direct problem (3.1), (3.16), (3.3), for a one-parametric family of domains Ω_w and one f_w.

Now we will demonstrate uniqueness for a linearisation of this inverse problem and analyse its stability.

Let $n=2$. As above, from condition (3.16), we have $u_w = const$ on Γ. We can assume that this constant is zero. We have

$$u_1 = 0 \text{ on } \Gamma(\theta), \quad \partial_\nu u_1 = 0 \text{ on } \Gamma(0), \tag{3.17}$$

and we are given equation (3.9).

Theorem 3.3.

A solution $f \in L^2(\Omega(T))$ to the linearised inverse problem (3.7), (3.17), (3.9), (3.10) is unique. This inverse problem is severely (exponentially) ill-posed.

Proof. To find f we again make use of its angular Fourier series

$$f(x)=\sum_{-\infty}^{\infty} f_m(r)e^{im\sigma}, \quad x=r(\cos\sigma,\sin\sigma)$$

and of solutions u_m^* of an 'adjoint' problem

$$u_m^*(x;\theta)=\left(\left(\frac{r}{1+\theta}\right)^m-\left(\frac{r}{1+\theta}\right)^{-m}\right)e^{-im\sigma}.$$

Using that u_m^* are harmonic in $\Omega(\theta)\setminus\omega$ and $u_m^*=0$ on $\Gamma(\theta)$ from the definition of a weak solution to equation (3.7), we have

$$\int_{\Gamma(0)} f\partial_\nu u_0 u_m^* + \int_{\Omega(\theta)} f\nabla u_0\cdot\nabla u_m^* = -\int_{\Gamma(0)} g_1\partial_\nu u.$$

As the expansion is orthogonal, we get

$$\begin{aligned}\int_{\Omega(\theta)} f\nabla u_0\cdot\nabla u_m^* &= \int_{\Omega(\theta)} f\partial_r u_0\partial_r u_m^*\\ &=2\pi\int_1^{1+\theta} f_m(r)r^{-1}\frac{m}{1+\theta}\left(\left(\frac{r}{1+\theta}\right)^{m-1}+\left(\frac{r}{1+\theta}\right)^{-m-1}\right)rdr\\ &=2\pi\int_1^{1+\theta} kf_m(r)(r^{m-1}(1+\theta)^{-m}+r^{-m-1}(1+\theta)^m)dr.\end{aligned}$$

Hence, we have the following Volterra integral equation:

$$\int_1^{1+\theta} f_m(r)(r^{m-1}(1+\theta)^{-m}+r^{-m-1}(1+\theta)^m)dr = F_m(\theta), \tag{3.18}$$

where

$$F_m(\theta)=\frac{1}{2\pi}((1+\theta)^m+(1+\theta)^{-m})\int_0^{2\pi} g_1(\cos\sigma,\sin\sigma;\theta)e^{-im\sigma}d\sigma.$$

To solve equation (3.18) for f_m we introduce the two functions

$$\Phi_{1m}(\theta)=\int_1^{1+\theta} f_m(r)r^{m-1}dr,\ \Phi_{2m}(\theta)=\int_1^{1+\theta} f_m(r)r^{-m-1}dr. \tag{3.19}$$

From equations (3.18), (3.19), we have

$$\begin{gathered}\Phi'_{1m}(\theta)=(1+\theta)^{m-1}f_m(1+\theta),\ \Phi'_{2m}(\theta)=(1+\theta)^{-m-1}f_m(1+\theta),\\ (1+\theta)^{-m}\Phi_{1m}(\theta)+(1+\theta)^m\Phi_{2m}(\theta)=F_m(\theta).\end{gathered} \tag{3.20}$$

Expressing Φ_{2m} from the third equation (3.20), differentiating with respect to θ and using the second relation (3.20) we arrive at the following linear ordinary differential equation with respect to Φ:

$$(\theta+1)^{-2k}\Phi'_{1m}(\theta)=-(1+\theta)^{-2m}\Phi'_{1m}(\theta)+2m(1+\theta)^{-2m-1}\Phi_{1m}(\theta)+((1+\theta)^{-m}F_m(\theta))' \tag{3.21}$$

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with the initial condition

$$\Phi_{1m}(0) = 0. \tag{3.22}$$

The initial condition (3.22) follows from equation (3.19). On the other hand, it is not hard to see that equations (3.21), (3.22), and the first relation (3.20) are equivalent to the Volterra equation (3.18).

The solution to equations (3.21), (3.22) is

$$\Phi_{1m}(\theta) = \frac{1}{2}(\theta+1)^m \int_0^\theta (s+1)^m((s+1)^{-m}F_m(s))'ds.$$

Hence from equations (3.20) and (3.18) by elementary calculations,

$$\begin{aligned} f_m(1+\theta) &= (1+\theta)^{1-m}\Phi'_{1m}(\theta) \\ &= \frac{m}{2}\int_0^\theta (s+1)^m(((s+1)^{-m}F_m(s))'ds)' + \frac{1}{2}(\theta+1)^{2m+1}((\theta+1)^{-m}F_m(\theta))' \\ &= \frac{\theta+1}{2}F'_m(\theta) - \frac{m^2}{2}\int_0^\theta (s+1)^{-1}F_m(s)ds. \end{aligned} \tag{3.23}$$

Due to equation (3.18), we have

$$\begin{aligned} F'_m(\theta) \; &= m((1+\theta)^{m-1} - (1+\theta)^{-m-1})\frac{1}{2\pi}\int_0^{2\pi} g_1(\cos\sigma,\sin\sigma;\theta)e^{-im\sigma}d\sigma \\ &\quad + ((1+\theta)^m + (1+\theta)^{-m})\frac{1}{2\pi}\int_0^{2\pi} \partial_\theta g_1(\cos\sigma,\sin\sigma;\theta)e^{-im\sigma}d\sigma. \end{aligned}$$

This formula combined with equations (3.18) and (3.23) implies exponential ill-conditioning of the linearised inverse problem 2D.

We analysed the simplest 2-dimensional version of the linearised inverse problem which preserves some features of the complete non-linear 3-dimensional problem. There are no results on the full (non-linear) inverse problem 2 when 1-dimensional family of domains growing with pumping time θ depends on the permeability. It will be interesting to study complete non-linear case analytically and numerically and to adjust it to some realistic situations of oil recovery. □

4 Compressible fluids

For compressible fluids a simple model [26] is the linear parabolic partial differential equation for the pressure

$$a_0(x)\partial_t u - \operatorname{div}(a(x)\nabla u) = f \text{ in } \Omega \times (0, T) \tag{4.1}$$

with the initial data

$$u = u_0 \text{ on } \Omega \times \{0\}$$

and the natural boundary condition

$$\partial_\nu u = 0. \tag{4.2}$$

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Here $a_0(x) = \phi(x)c(x)$, where ϕ is the porosity of the medium and $c(x)$ is the compressibility of the fluid, and $a(x) = \mu^{-1}k(x)$, where $\mu(x)$ is some average viscosity of fluids which is assumed to be known. We assume that $\phi, c \in L_\infty(\Omega)$ and $\delta_0 < \phi, \delta_0 < c$ on Ω for some positive number δ_0. As above, we can assume that f is a function in $L^2(\Omega)$ supported in a subdomain $\omega \subset \Omega$. We denote $\Omega_\omega = \Omega \setminus \bar{\omega}$.

Inverse Problem 3

Find a_0, a in Ω_ω from solution u to the problem given on $\omega \times (0, T)$ for any function $f \in L_2(\Omega \times (0, T))$ which is zero on $\Omega_\omega \times (0, T)$.

Theorem 4.1. *The solution a_0, a to inverse problem 3 is unique, if either (a) $a_0, a \in C^\infty(\bar{\Omega})$ or (b) a_0, a are Lipschitz piecewise constant functions.*

We outline a proof of part (a) based on the heat equation transform of parabolic problems to hyperbolic ones and application of results obtained by the boundary control method, and a proof of part (b) based on reduction to elliptic equations via stabilisation method.

First we will show that the data of inverse problem 3 uniquely determine the parabolic Dirichlet-to-Neumann map $\Lambda_p(\Omega, \omega)$. To define it we consider the parabolic initial boundary value problem

$$a_0(x)\partial_t w - \operatorname{div}(a(x)\nabla w) = 0 \text{ in } \Omega_\omega \times (0, T), \tag{4.3}$$

with the initial data

$$w = 0 \text{ on } \Omega_\omega \times \{0\} \tag{4.4}$$

and the boundary conditions

$$w = g_0 \text{ on } \partial\omega \times (0, T), \quad \partial_\nu w = 0 \text{ on } \partial\Omega \times (0, T). \tag{4.5}$$

One can show that any $g_0 \in C^1(\partial\omega \times [0, T])$ can be approximated in $L_\infty(0, T; H^{\frac{1}{2}}(\partial\omega))$ by solutions u of the problem (with various $f \in L_\infty(0, T; L_2(\omega))$). By the conditions u on $\omega \times (0, T)$ is given for any f. Hence $\partial_\nu u$ on $\partial\omega \times (0, T)$ is given as well. From standard estimates for solutions to parabolic initial boundary value problems, it follows that convergence of u in $L_\infty(0, T; H^{\frac{1}{2}}(\partial\omega \times (0, T))$ implies convergence of $\partial_\nu u$ in $L_\infty(0, T; H^{-\frac{1}{2}}(\partial\omega \times (0, T))$. Hence g_0 uniquely determines $\partial_\nu w$ on $\partial\omega \times (0, T)$, so we are given the parabolic Dirichlet-to-Neumann map $\Lambda_p(\Omega, \omega)$.

To use results for hyperbolic equations, we define

$$w = a^{-\frac{1}{2}} v \tag{4.6}$$

which as known ([18], Section 5.2), transforms the partial differential equation for w into

$$a_1(x)\partial_t v - \Delta v + c(x)v = 0 \text{ in } \Omega_\omega \times (0, T), a_1 = a^{\frac{1}{2}} a_0, \ c = a^{-\frac{1}{2}} \Delta a^{\frac{1}{2}}, \tag{4.7}$$

$$v = 0 \text{ on } \Omega_\omega \times \{0\}, \tag{4.8}$$

$$u = g_0 \text{ on } \partial\omega \times (0, T), \quad \partial_\nu v = 0 \text{ on } \partial\Omega \times (0, T). \tag{4.9}$$

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We recall the heat equation transform v of a function v^*,

$$v(x,t) = (\pi t)^{-\frac{1}{2}} \int_0^\infty exp(-\frac{\tau^2}{4t})v^*(x,\tau)d\tau.$$

Let v^* solve the initial value mixed hyperbolic problem

$$a_1(x)\partial_t^2 v^* - \Delta v^* + c(x)v^* = 0 \text{ in } \Omega_\omega \times (0, T^*), \tag{4.10}$$

$$v^* = \partial_t v^* = 0 \text{ on } \Omega_\omega \times \{0\}, \tag{4.11}$$

$$v^* = g_0^* \text{ on } \partial\omega \times (0, T), \; \partial_\nu v^* = 0 \text{ on } \partial\Omega \times (0, T), \tag{4.12}$$

with $g_0^* \in C_0^2(\partial\omega \times (0,\infty))$ and g_0 being the heat equation transform of g_0^*. Available theory of the mixed hyperbolic problem (4.10), (4.11), (4.12) guarantees existence and uniqueness of the solution u^* with $e^{-\tau t}v^* \in L_\infty(0,\infty; H^2(\Omega))$, $e^{-\tau t}\partial_t^2 v^* \in L_\infty(0,\infty; L_2(\Omega))$ for some positive τ. For such functions the heat equation transform is well defined. We refer for its properties to ([18], Section 9.2). In particular, v solves the parabolic initial boundary value problem (4.7), (4.8), (4.9). Since for this parabolic equation the Dirichlet-to-Neumann is known, g_0^* uniquely determines $\partial_\nu v$ on $\partial\omega \times (0, T)$. It is clear that $\partial_\nu v$ is the heat transform of $\partial_\nu v^*$. Since the inverse heat transform is unique ([17], Section 9.2), $\partial_\nu v^*$ on $\partial\omega \times (0, T^*)$ is known (for any T^*). From known results on inverse hyperbolic problems [4, 20], we derive that a_1, c are uniquely determined on Ω. Now from definition of c in (4.7), we have the elliptic partial differential equation $\Delta a^{\frac{1}{2}} - ca^{\frac{1}{2}} = 0$ in Ω_ω for $a^{\frac{1}{2}}$. Since a is known on ω, we have Cauchy data for a on $\partial\omega$. By uniqueness in the Cauchy problem for elliptic equations ([17], Section 3.3), a is unique. Since a_1, a are unique, from the definition of a_1 in (4.7), a_0 is unique.

This completes an outline of a proof in case (a). Now we explain ideas of proof in case (b).

The substitution

$$w = ve^{\tau t} \tag{4.13}$$

transforms equation (4.3) into equation

$$a_0\partial_t v - \text{div}(a\nabla v) + \tau a_0 v = 0 \text{ in } \Omega_\omega \times (0, T). \tag{4.14}$$

Let $g_0 = G_0\phi$, where G_0 is any function in $C^2(\bar{\Omega})$ and $\phi(t) \in C^\infty(\mathbf{R})$ satisfies the conditions: $\phi(t) = 0$ on $(-\infty, \frac{T}{4})$ and $\phi(t) = e^{\tau t}$ if $\frac{T}{2} < t$. Since the coefficients of equation (4.14) and the boundary data do not depend on $t > \frac{T}{2}$, the solution to the initial boundary value problem is analytic with respect to $t > \frac{T}{2}$. So it is uniquely determined for all $t > 0$.

Since equation (4.14) satisfies the conditions of the maximum principle and the boundary data are time-independent for large t, by known stabilisation results

$$\|v(,t) - v_0\|_{(1)}(\Omega_\omega) \to 0 \text{ as } t \to \infty, \tag{4.15}$$

where v_0 solves the following mixed boundary value problem

$$-\text{div}(a\nabla v_0) + \tau a_0 v_0 = 0 \text{ in } \Omega_\omega, \; v_0 = G_0 \text{ on } \partial\omega, \; \partial_\nu v_0 = 0 \text{ on } \partial\Omega. \tag{4.16}$$

Since $\partial_\nu v(,t)$ on $\partial\omega$ is given from equation (4.15), it follows that $\partial_\nu v_0$ on $\partial\omega$ is given as well (for any Dirichlet data G_0). Hence we are given a partial Dirichlet-to-Neumann map for all elliptic equations (4.16). Due to analyticity, we can take $\tau = 0$; then uniqueness of piecewise Lipschitz constant a follows from the generalisation of results of Kohn and Vogelius [22] given by Sever [28]. Similarly, one can show uniqueness of a_0.

The proof is complete.

As follows from the proof of Theorem 4.1 and known results on the inverse conductivity problem and on inverse hyperbolic problems, Problem 3 should be also severely (exponentially) ill-conditioned.

Let ε will be the operator norm of the difference of the lateral parabolic Dirichlet-to-Neumann operators corresponding to the coefficients a_{01}, a_1 and a_{02}, a_2 of equation (4.1) (with zero initial data). The operators are from $L_2((0,T); H_{(\frac{1}{2})}(\partial(\Omega \setminus \omega))$ into $L_2((0,T); H_{(-\frac{1}{2})}(\partial(\Omega \setminus \omega))$. The data of the inverse problem 3 can be in a standard and stable way recalculated into the lateral Dirichlet-to-Newmann map as in [11].

Theorem 4.2. *Assume that $\Omega = \mathbb{R}^3$, that a_{0j}, a_j are constants outside some ball B, and that*

$$\|a_{0j}\|_{\infty,1}(\Omega) + \|a_j\|_{\infty,3}(\Omega) \leqslant M, \ j = 1, 2.$$

Then there is a constant C depending on M, B, ω such that

$$\|a_{02} - a_{01}\|(\Omega) + \|a_2 - a_1\|_{\infty}(\Omega) \leqslant C|\log \varepsilon|^{-0.2}.$$

A proof can be obtained by using the Kelvin transform and some modification of the proof of Theorem 9.4.3 in [18]. A linearisation of the inverse problem 3 with reduced data was considered analytically and numerically in [11].

5 Conclusion

We have reviewed analytical results for inverse problems based on some models of filtration of water and oil through soil and outlined some new possible directions of analytical and numerical research on inverse problems. The results on linearised versions of the inverse Muskat problem are new. In most cases we only outlined proofs which can be made complete. We plan to rigorously analyse the inverse problems 1–2 in near future. We hope that methods of proofs (especially formulas for the inverse Muskat model and the heat equation transform) can be used to design more efficient and reliable numerical algorithms. It seems that all possible formulations of inverse problems are severely ill-posed, which indicates that it is hard to expect fine numerical resolution. We observe that similar inverse problems arise in hydraulics [31]. The filtration system is a combination of a conservation–diffusion law (elliptic equation) with a drift (parabolic or first-order equation). The same structure is a feature of partial differential equations models of semiconductors and ion channels. In all our considerations, we decoupled the system into scalar equations. It would be very interesting to find some approach to inverse problems for full drift–diffusion systems.

As mentioned, the direct problem is a serious mathematical challenge which is not likely to be completely addressed in the near future. So as in hydrodynamics, it makes sense to simplify general system of filtration to particular interesting cases and study these cases analytically and numerically. Simultaneously, one can try to look at more difficult but more realistic model of three phases filtration (oil, water, and gas).

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