Euro. Jnl of Applied Mathematics (2002), *vol.* 13, *pp.* 371–383. © 2002 Cambridge University Press
DOI: 10.1017/S0956792502004850 Printed in the United Kingdom

A stochastic model of sedimentation: probabilities and multifractality

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(*Received 11 February 2000; revised 5 November 2001*)

We consider a one-dimensional stochastic model of sediment deposition in which the complete time history of sedimentation is the sum of a linear trend and a fractional Brownian motion $w_H(t)$ with self-similarity parameter $H \in (0, 1)$. The thickness of the sedimentary layer as a function of time, $d(t)$, looks like the Cantor staircase. The Hausdorff dimension of the points of growth of $d(t)$ is found. We obtain the statistical distribution of gaps in the sedimentary record, periods of time during which the sediments have been eroded. These gaps define sedimentary unconformities. In the case $H = 1/2$ we obtain the statistical distribution of layer thicknesses between unconformities and investigate the multifractality of $d(t)$. We show that the multifractal structures of $d(t)$ and the local time function of Brownian motion are identical; hence $d(t)$ is not a standard multifractal object. It follows that natural statistics based on local estimates of the sedimentation rate produce contradictory estimates of the range of local dimension for $d(t)$. The physical object $d(t)$ is interesting in that it involves the above anomalies, and also in its mechanism of fractality generation, which is different from the traditional multiplicative process.

1 Introduction

A large fraction of the surface of the Earth is covered by a layer of sediments. These sediments were usually deposited in shallow seas as tectonic subsidence was occurring. Subsidence occurs over a long time scale so that we may assume that it occurs at a constant velocity u. Sediment deposition is also affected by variations in sea level which we will consider as a random process $w(t)$ with zero mean value. It is standard practice to assume that the sediments follow sea level. Therefore, the thickness of the sedimentary layer as a function of time is given by $\xi(t) = ut + w(t)$. There are periods when sediments exceed the sea level. Then erosion occurs and all sediments above sea level are lost. There are thus gaps in the sedimentary record and they occur on a world-wide basis. The major gaps define the geological epochs, for example, the beginning and the end of the Cretaceous period. These gaps also define sedimentary unconformities. At the end of an erosional period the older rocks beneath the unconformity have very different physical properties from the younger rocks above the unconformity. Sharp changes in the properties are easily seen in road cuts and are the interfaces that reflect seismic energy in seismic oil exploration. The result of the deposition-erosion process is that the local stratigraphic section of a sedimentary layer will consist of sediments not eroded during the whole time interval T_0. We assume that sediment deposition and erosion are fast processes compared with the time scale of sea level variation. Then the sedimentary record can be described in terms of the complete history of accumulation $\xi(t)$ as follows:

$$d(t) = \inf\{\xi(s) : s \geqslant t\}. \tag{1.1}$$

Any point of growth t_0 of d is a time episode of deposition represented in the stratigraphic section and the value $d(t_0)$ is the thickness of the sedimentary layer at the time t_0. The functions $d(t)$ and $\xi(t)$ are schematically shown in Figure 1. Flat segments of $d(t)$, $d(t)$ = constant, correspond to *time gaps* in the sedimentary record. Time gaps alternate with periods during which sediments were not eroded. They will be called *beds* or *ε-beds* in what follows, if they supplement gaps of size $\geqslant \varepsilon$ on the t-axis. In that case ε is treated as the resolution threshold of time gaps. The total duration of beds measured in fractions of T_0 has been called *stratigraphic completeness*, C. One important observation (Sadler, 1981) is that C is a function of the chosen time scale, ε. The same holds for the parameter R which controls the average rate of deposition. Sadler showed that $C \propto \varepsilon^{1-D}$, $R \propto \varepsilon^{D-1}$ when $\varepsilon \ll T_0$, where the exponent D varies with the environment of deposition in the range $(0.3, 0.7)$. In other words, episodes of deposition represented in the sedimentary record form a fractal set of box-dimension $D \in (0.3, 0.7)$ (Turcotte, 1997). One can also ask the more difficult question about the multifractality (Mandelbrot, 1989) of a sedimentary record. Since geological data are quite noisy, the analysis of $d(\tau)$ is preferably first based on suitable models because the traditional basis for multifractals like stochastic cascade measures (Mandelbrot, 1989) seems to be unsuitable for sedimentation processes.

Since the work of Kolmogorov (1949), the simplest probabilistic models of bed formation were random walk models with discrete time (see Mizutani & Hattori, 1972; Schwarzacher, 1975; Dacey, 1979; Tipper, 1983; Thompson, 1984). Strauss Sadler (1989) extended this class of models to the case of continuous time assuming $\xi(t) = \max(0, ut + \sigma w(t))$, $t \geqslant 0$, where $w(t)$ is a standard Wiener process and the diffusion parameter σ is interpreted as the unsteadiness of the sedimentation process. Strauss & Sadler (1989) found for this model the distribution of the sedimentation rate,

$$R(s) = [d(t+s) - d(t)]/s,$$

and showed that the conditional mean of $R(s)$ given $R(s) > 0$ is proportional to $s^{-0.5}$, $s \ll T_0$, corresponding to the dimension $D = 1/2$.

As mentioned above, the range of observed D is broad enough, $(0.3, 0.7)$. For this reason Pelletier & Turcotte (1997) suggested using the same model where $w(t)$ is to be replaced with a fractional Brownian motion $w_H(t)$ or, in other words,

$$\xi(t) = \max(0, ut + \sigma w_H(t)), \qquad t \geqslant 0, \tag{1.2}$$

where $w_H(t)$ is a Gaussian process with zero mean and the structure function

$$E|w_H(t+s) - w_H(t)|^2 = |s|^{2H}, \qquad 0 < H < 1.$$

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The quantity H is the self-similarity parameter for $w_H(t)$. This means that the scale transformation $|a|^{-H}w_H(at)$ does not affect the stochastic nature of $w_H(t)$. When $H = 1/2$, the fractional Brownian motion becomes identical with the Wiener process. Pelletier & Turcotte (1997) concluded that the value $H = 1/4$ is the most suitable for fluvial depositional environments. These authors did some numerical calculations to show for the case $H = 1/4$ that

(a) $R(\varepsilon) \propto \varepsilon^{-(1-H)}$;
(b) the bed-thickness distribution is of the exponential type;
(c) the time gap distribution falls off for large values faster than the exponential distribution.

Below we prove that the points of growth of $d(t)$ have the dimension H and we refine statement (c) in quantitative terms. In the case $H = 1/2$ for $d(t)$ we discuss statistics of the multifractal type, which are reasonable for applications. The results essentially rely on the work of the first author concerned with the study of zeroes of Brownian motion (Molchan, 1994, 1995) and of the maximum of fractional Brownian motion (Molchan, 1999).

2 Fractality of deposition episodes

We now discuss the fractal properties of the stratigraphic section $d(t)$ for the model (1.1, 1.2), assuming that the history of the sedimentary record is long, $T_0 \to \infty$. Our first statement shows that curve $d(t)$ looks like the Cantor staircase, i.e. $d(\tau)$ is a continuous nondecreasing function with a fractal set of points of growth. The proof is based on nontrivial asymptotics for the distribution of the position of the maximum of fractional Brownian motion in the interval [0,1], (Molchan, 1999).

Statement 1 *For the model* (1.1, 1.2) *the episodes of deposition represented in the sedimentary record form a fractal set* S *of Hausdorff dimension* $\dim S = H$.

Proof For $t_i \in S$, $d(t_1) - d(t_2) = \xi(t_1) - \xi(t_2)$. Taking into account that $\xi(t) = ut + \sigma w_H(t)$ is a Gaussian process and $E|w_H(t) - w_H(s)|^2 = |t-s|^{2H}$, it is easy to show that a.s. $|\Delta\xi(t)| < C|\Delta t|^{H-\varepsilon}$ for any $\varepsilon > 0$ and $t \in [0, T]$ where $C = C(\varepsilon, T, \omega)$ is a finite random constant. Hence $|\Delta d(t)| < C|\Delta t|^{H-\varepsilon}$ a.s.. With the help of Frostman's lemma (e.g. see Falconer, 1990) we obtain that $\dim S \geqslant H - \varepsilon$, $\varepsilon > 0$, i.e. $\dim S \geqslant H$.

Let us prove the converse. Since $w_H(t)$ is self-similar, it is sufficient to show that $\dim S^* \leqslant H$ for $S^* = S \cap [0, 1]$.

Let $\{B_i\}$ be the covering of [0,1] by means of δ-intervals of type $(a, a + \delta)$ with overlapping of size $\delta/2$. Select such intervals B_i' which have non-empty intersection with S. Then $\cup B_i'$ is the covering of S^* and

$$E\sum |B_i'|^{H+\varepsilon} \leqslant \delta^{H+\varepsilon} \cdot \mathbf{p} \cdot 2/\delta,$$

where $\mathbf{p} = \max E\,\mathbf{1}_{B_i \cap S}$. If $\mathbf{p} < \delta^{1-H-\rho}$, $\rho < \varepsilon$, then

$$E\sum |B_i'|^{H+\varepsilon} \leqslant \delta^{\varepsilon'}, \quad \delta \downarrow 0, \quad \varepsilon' = \varepsilon - \rho$$

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and by the Chebyshev inequality we get $P(\sum |B_i'|^{H+\varepsilon} > \delta^{\varepsilon'/2}) \leqslant \delta^{\varepsilon'/2}$. Setting $\delta = 2^{-n}$ and using the Borel–Cantelli lemma we have a.s.

$$\sum |B_i'|^{H+\varepsilon} < \delta^{\varepsilon/2}, \quad n > N(\omega),$$

where N depends upon the realization of w_H. Hence $\dim S \leqslant H+\varepsilon$ or $\dim S \leqslant H$, because ε is an arbitrary positive constant.

To prove the desired upper bound, $\mathbf{p} < \delta^{1-H-\rho}$, we have to show that

$$P(A_\delta(a)) < \delta^{1-H-\rho}, \quad 0 < a < 1, \quad \rho < \varepsilon,$$

where $A_\delta(a) = \{\exists\, t_0 \in B = (a, a+\delta) : \xi(t_0) < \xi(t), \forall t > a+\delta\}$. Taking into account that $w_H(t) - w_H(a) \stackrel{\mathrm{d}}{=} w_H(t-a)$, where $\stackrel{\mathrm{d}}{=}$ denotes equality for finite-dimensional distributions, we have

$$P(A_\delta(a)) = P(A_\delta(0)) < P(A_\delta^\xi).$$

Here $A_\delta^\xi = \{\exists\, t_0 \in (0,\delta) : \xi(t_0) < \xi(t),\ \delta < t < 1\}$. The Gaussian measures $\mathscr{P}_\xi$ and $\mathscr{P}_w$ corresponding to the processes $\xi(t) = ut + \sigma w_H(t)$ and $w = \sigma w_H(t)$, $\quad 0 < t < 1$, are mutually absolutely continuous (Molchan & Golosov, 1969). Using the Cameron–Martin formula for the Radon–Nikodim derivative $\mathscr{P}_\xi(dw)/\mathscr{P}_w(dw) = l(w)$ we have

$$P(A_\delta^\xi) = E\,\mathbf{1}_{A_\delta^w} l(w)$$

where the functional $\ln l(w)$ has a Gaussian distribution with mean $m = -\|\varphi\|^2/2$ and variance $\|\varphi\|^2$, $\varphi = ut$ and $\|\cdot\|$ is the norm of the Hilbert space with the reproducing kernel $B(t,s) = \sigma^2 E w_H(t) w_H(s)$ that is

$$B(t,s) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), 0 < t, s < 1.$$

Using the Hölder inequality with indices $p = (1-\rho)^{-1}$ and $q = \rho^{-1}$ we get

$$P(A_\delta^\xi) < c_\rho [P(A_\delta^w)]^{1-\rho}$$

where $c_\rho = (E[l(w)]^{1/\rho})^\rho = \exp\{\frac{1}{2}(\rho^{-1} - 1)\|\varphi\|^2\}$. Note that

$$P(A_\delta^w) = P(\exists\, t_0 \in (0,\delta) : w_H(t_0) < w_H(t),\ \delta < t < 1) = P(\tau_{\min} < \delta)$$

where τ_{min} is the position of the minimum of $w_H(t)$ on [0,1].

According to Molchan (1999), we have $P(\tau_{min} < \delta) < \delta^{1-H}\mathscr{L}_\delta$ where $\mathscr{L}_\delta$ is slowly varying at the origin in the sense of Karamata. Therefore, $\mathbf{p} < c_\rho[\delta^{1-H}\mathscr{L}_\delta]^{1-\rho} < c_\rho \delta^{1-H-\rho'}$ for any $\rho' > (1-H)\rho$. Choosing a small enough ρ we can make $\rho' < \varepsilon$. □

3 The distribution of time gaps

The time scale ε is a parameter in what follows. Time gaps of size $\geqslant \varepsilon$ will be denoted Δ_ε ($|\Delta_\varepsilon|$ is the length of Δ_ε). The complement of ε-gaps on the t-axis consists of intervals δ_ε, to be called ε-beds. The increments $d(t)$ in δ_ε, or ε-*bed thickness*, will be denoted as d_ε (see Figure 1).

The numerical calculations of time gaps due to Pelletier & Turcotte (1997) furnish no clue as to the distribution of $|\Delta_\varepsilon|$ for large values. The next statement shows that the tail

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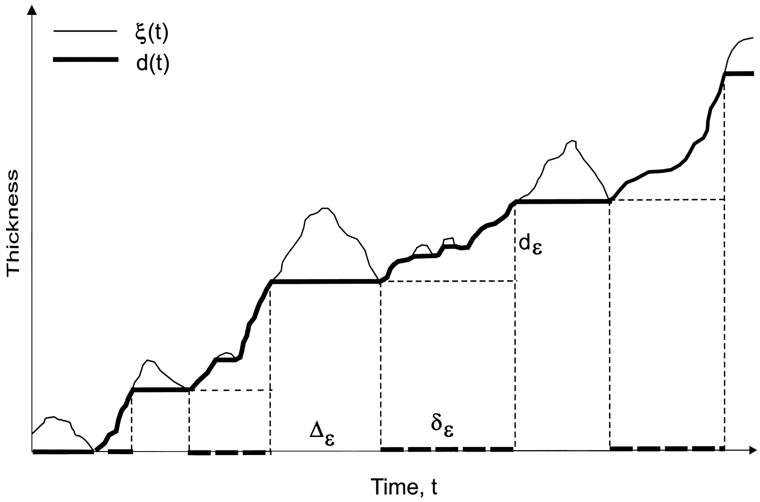

FIGURE 1. A sketch of the sedimentary process: the complete history of accumulation $\xi(t)$ (*fine* and *bold* lines) and the sedimentary record $d(t)$ (*bold* line). For the time resolution scale ε we show the time gaps Δ_ε, ε-beds δ_ε and ε-bed thicknesses d_ε.

of the distribution of $|\Delta_\varepsilon|$ may fall off slower or faster as the case may be than does the exponential distribution depending on H.

Statement 2 *For the model* (1.1, 1.2) *the distribution of the gap* Δ_ε containing a point $a > 0$ *has the following form:*

$$\ln P(|\Delta_\varepsilon| > t | \Delta_\varepsilon \ni a) = -0.5\,[u\sigma^{-1}\, t^{1-H}]^2 c(t), \tag{3.1}$$

where $a_H < c(t) < b_H$ for $t \gg 1$; *the thresholds* a_H *and* b_H *are given by the table*

H	0.3	0.4	0.6	0.7
a_H	0.08	0.14	0.35	0.53
b_H	4.24	4.08	4.12	4.66

In the case $H = 1/2$ *the unconditional distribution* $|\Delta_\varepsilon|$ *can be found exactly:*

$$\begin{aligned} P(|\Delta_\varepsilon| > t) &= c_\varepsilon \int_t^\infty \exp(-0.5(u/\sigma)^2 x)\, x^{-3/2}\, dx, \quad t > \varepsilon \\ &\simeq 2c_\varepsilon(\sigma/u)^2 \exp(-0.5(u/\sigma)^2 t)\, t^{-3/2}, \quad t \to \infty, \end{aligned} \tag{3.2}$$

where c_ε *is a normalizing constant such as to make* $P(\Delta_\varepsilon \geqslant \varepsilon) = 1$.

Proof We are going to prove (3). The relation (4) is more conveniently discussed in the next section. Let us consider the ε-gap $\Delta_\varepsilon \ni a$ on the semiaxis $t > 0$ of the length Δ. We are interested in the event $A = \{\Delta > T\}$, where $T > a$.

The lower bound for $P(A)$
Let us fix the point $t_0 = a + T(1+\varepsilon)$, $\varepsilon > 0$, and define the following event:

$$\begin{aligned} B &= \{\xi(t) > \xi(t_0),\ t \in (a, T+a)\} = \\ &= \{w_H(t) - w_H(t_0) > b(t_0 - t),\ t \in (a, T+a)\}, \end{aligned}$$

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where $b = u/\sigma$. Then $B \subset A$ and $P(A) > P(B)$. Now note that

$$w_H(t) - w_H(t_0) \overset{d}{=} T^H w_H\left(\frac{t_0 - t}{T}\right).$$

Therefore, putting $\theta = T^{1-H}$, we get

$$\begin{aligned} P(B) &= P\{w_H(x) > b\theta x, \quad x \in (\varepsilon, 1+\varepsilon)\} \\ &= \{w_H(x) - w_H(-\varepsilon) \geqslant b\theta(x+\varepsilon), \quad x \in (0,1)\}. \end{aligned}$$

Here we have used the stationary property of inrements of $w_H(t)$.

Decompose the Gaussian variable $w_H(-\varepsilon)$ in a sum of independent terms $w_\perp + w_\wedge$, where $w_\wedge = E(w_H(-\varepsilon) \,|\, w_H,\, t \in (0,\infty))$ is the best forecast of $w_H(-\varepsilon)$ based on $\{w_H(\tau),\, \tau > 0\}$. Therefore, $w_\perp$ is independent of $\{w_\wedge, w_H(\tau), \tau \in (0,1)\}$, so that for any $\varphi_T > 0$

$$\begin{aligned} P(B) &> P(w_\perp > \theta b(1+\varepsilon) + \varphi_T, \quad w_\wedge + w_H(x) \geqslant -\varphi_T, x \in (0,1)) \\ &= \Psi(\frac{\theta b(1+\varepsilon) + \varphi_T}{\sigma(\varepsilon)}) q_T. \end{aligned} \tag{3.3}$$

The first factor above is $P(w_\perp > \theta b(1+\varepsilon) + \varphi_T)$, and therefore can be expressed in terms of the error function:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du \sim \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x^{-1}, \quad x \to \infty, \tag{3.4}$$

and of the variance of $w_\perp$: $\sigma_\varepsilon^2 = E w_\perp^2 = \varepsilon^{2H} s_H^2$. Here, s_H^2 is variance of the error of the best forecast of $w_H(-1)$ based on $\{w_H(t),\, t \in (0,\infty)\}$:

$$s_H^2 = \Gamma(1.5 - H)/[\Gamma(0.5+H)\Gamma(2-2H)], \tag{3.5}$$

where Γ is the gamma-function (e.g. see Molchan, 1997).

The second factor q_T is

$$q_T = P\{w_\wedge + w_H(\tau) \geqslant -\varphi_T, \quad \tau \in (0,1)\} \to 1, \; T \to \infty,$$

provided $\varphi_T \to \infty$. Choose $\varphi_T = T^{(1-H)}$ then the desired bound follows from (5,6):

$$P(A) > C_T \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} T^{2-2H} b^2 \rho_{H,T}\right) [T^{1-H} b \, \rho_{H,T}^{1/2}]^{-1}, \tag{3.6}$$

where $C_T \sim 1$, $T \to \infty$ and $\rho_{H,T} \sim (1+\varepsilon)^2 \varepsilon^{-2H}/s_H^2 = \rho_H$, $\quad T \to \infty$.

Now let us minimize ρ_H over the values of ε. We then get $\varepsilon = H/(1-H)$ and

$$\rho_H = [(1-H)^{(1-H)} H^H \; s_H]^{-2}. \tag{3.7}$$

The upper bound of $P(A)$

If a gap Δ_ε is such that $|\Delta_\varepsilon| > T$ and $a \in \Delta_\varepsilon \subset (0,\infty)$ then $\Delta_\varepsilon \supset (a, T)$. Thus, for any $\delta > 0$

$$\begin{aligned} P(A) &\leqslant P(\min_{(a,T)}(\xi(t) > \min_{(T,\infty)} \xi(t)) = \\ &= P(\min_{(a-T,0)}(w_H(t) + bt) > \min_{(0,\infty)}(w_H(t) + bt)) \leqslant \\ &\leqslant P(\min_{(a-T,0)}(w_H(t) + bt) > -bT\delta) + P(\min_{(0,\infty)}(w_H(t) + bt) < -bT\delta) = \\ &:= p_1 + p_2. \end{aligned}$$

Let $0 < \varepsilon < 1 - a/T$. Then

$$p_1 < P(w_H(-\varepsilon T) - b\varepsilon T > -bT\delta) = \Psi(bT^{1-H}(\varepsilon - \delta)\varepsilon^{-H}).$$

Let us estimate p_2, using the property $\{w_H(t)\} \stackrel{\mathrm{d}}{=} \{-w_H(t)\}$:

$$p_2 < \sum_{k=0}^{\infty} P\left(\min_{(0,T)}[w_H(kT + \tau) + b\tau] < -bT(k + \delta)\right) \leqslant$$
$$\leqslant \sum_{k=0}^{\infty} P\left(\max_{(0,T)} w_H(Tk + \tau) > bT(k + \delta)\right).$$

Now we use the Fernique (1975) estimate for the distribution of the maximum of a Gaussian process. In application to w_H we have for all $x \geqslant \sqrt{5}$:

$$P\left(\max_{(0,T)} w_H(Tk + \tau) > xT^H[(k + 1)^H + c_H]\right) \leqslant 10 \int_x^{\infty} e^{-u^2/2} du,$$

where

$$c_H = (2 + \sqrt{2}) \int_1^{\infty} 2^{-u^2 H}\, du. \tag{3.8}$$

Hence

$$p_2 < 10\sqrt{2\pi} \sum_{k\geqslant 0} \Psi(bT^{1-H}(k + \delta)((k + 1)^H + c_H)^{-1}) <$$
$$< 10\sqrt{2\pi} \sum_{k\geqslant 0} \Psi\left(bT^{1-H} \frac{(k + \delta)(k + 1)^{-H}}{1 + c_H}\right).$$

As $T \to \infty$, the sum has the same order as the first term, i.e.

$$p_2 \leqslant 10\sqrt{2\pi}\Psi\left(bT^{1-H} \frac{\delta}{1 + c_H}\right)(1 + o(1)), \qquad T \to \infty.$$

To make the orders of p_1 and p_2 equal we require that $(\varepsilon - \delta)\varepsilon^{-H} = \delta/(1 + c_H)$, and that this quantity have the maximum value in $\varepsilon \in (0,\, 1 - a/T)$. The result is

$$\frac{\delta}{1 + c_H} = \frac{\varepsilon}{1 + c_H + \varepsilon^H} |_{\varepsilon=1-\frac{a}{T}} = \frac{1}{c_H}\left(1 - O\left(\frac{a}{T}\right)\right).$$

Consequently,

$$P(A) \leqslant C\Psi\left(bT^{1-H} \frac{1 - \varepsilon_T}{c_H}\right), \tag{3.9}$$

where $\varepsilon_T \to 0$ as $T \to \infty$.

So, taking into account (3.6), (3.9) and (3.5), (3.7) and (3.8), one has

$$a_H(1 + o(1)) < \ln P(A) / [-0.5\,(T^{1-H}\mu/\sigma)^2] <$$
$$< b_H(1 + o(1)), \qquad T \to \infty$$

where $a_H = c_H^{-2}$ (see (3.8)) and $b_H = \rho_H$ (see (3.7)). The particular values of a_H and b_H for $H = 0.3 - 0.7$ are given by the table in the statement 2. □

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4 Distributions of Δ_ε and d_ε: the case $H = 1/2$

The stochastic analysis of sediments is mostly concerned with the process $d(t)$. However, when $H = 1/2$, the inverse (continuous from the right) process $\tau(d)$ is a simpler object. Let τ_0 be the first moment of time that has remained in the sedimentary record. Speaking formally, $\xi(t) > \xi(\tau_0)$ for all $t > \tau_0$. In that case, *the process* $\tilde{\tau}(d) = \tau(d) - \tau_0$, $d \geqslant 0$, *has homogeneous independent increments whose distribution is given by the Laplace transform*

$$E \exp\{-\theta(\tilde{\tau}(d + x) - \tilde{\tau}(x))\} = \exp\{-du\sigma^{-2}\sqrt{1 + 2\theta(\sigma/u)^2} - 1\}, \tag{4.1}$$

(see, for instance, Exercise VII.3 in Bertoin, 1996).

From this we can derive the probability density for increments of $\tilde{\tau}(d)$:

$$P(\tilde{\tau}(d + x) - \tilde{\tau}(x) \in dt)/dt = (2\pi)^{-1/2}(d/\sigma)t^{-3/2}\exp\{-(ut - d)^2/(2\sigma^2 t)\}. \tag{4.2}$$

This is the so-called 'inverse Gaussian law'. Note that $\tilde{\tau}(d+x)-\tilde{\tau}(x)$ is the period required for generating a sedimentary layer d thick.

Following P. Levy (see Bertoin, 1996), we will describe a process with independent increments in terms of time gaps to be observed in samples of $\tau(d)$. Let Δ_i be a sequence of time gaps of size $\geqslant \varepsilon$ where $d(t) = d_i$ with $0 < d_i < d$ and $v_\varepsilon(d)$ the number of the gaps. Then the quantity $v_\varepsilon(d)$ will obey the Poisson distribution with parameter $d\Lambda_\varepsilon$, where

$$\Lambda_\varepsilon = \sigma^{-1}\int_\varepsilon^\infty (2\pi)^{-1/2}t^{-3/2}\exp\{-0.5\,(u/\sigma)^2 t\}\,dt. \tag{4.3}$$

Besides, the amplitudes d_i are independent and uniformly distributed in the interval $(0, d)$, the $|\Delta_i|$ are independent and obey the distribution (3.2) or, which amounts to the same thing, $P(|\Delta| > t) = \Lambda_t/\Lambda_\varepsilon$. The sets of random variables $\{|\Delta_i|\}$, $\{\tau_i\}$ are independent as well.

Suppose $\tilde{\tau}_\varepsilon(d)$ is the total length of the time gaps Δ_i for which $d_i < d$. In that case $\tilde{\tau}_\varepsilon(d)$ converges to $\tilde{\tau}(d)$ as $\varepsilon \to 0$ in the sense of finite dimensional distributions.

The above description of $\tilde{\tau}(d)$ explains the result (4) concerning the distribution of time-gap size Δ_ε. The description will also be used to derive the distribution of bed-thickness d_ε:

For the model (1.1, 1.2), $H = 1/2$, the increment of $d(t)$ in ε-bed, d_ε, has the exponential distribution

$$P(d_\varepsilon > x) = \exp(-\Lambda_\varepsilon x), \qquad x > 0. \tag{4.4}$$

The proof is as follows. The increments of $\tilde{\tau}(d)$ are homogeneous and $\tilde{\tau}(d)$ has the strict Markov property. Therefore, the ε-bed δ_ε can be considered to begin at 0. However, the event $\{d_\varepsilon > x\}$ then means that the path of $\tilde{\tau}(d)$ has no discontinuities of size $\geqslant \varepsilon$ in $(0, x)$. Hence $P(d_\varepsilon > x) = P(v_\varepsilon(x) = 0) = \exp(-x\,\Lambda_\varepsilon)$. The last equality is true, because $v_\varepsilon(x)$ has a Poisson distribution with parameter $x\,\Lambda_\varepsilon$.

Strauss & Sadler (1989) studied the distribution of increments of $d(t)$ in a fixed time interval. The distribution is significantly different from the exponential, because it falls off according to the Gaussian law. The exponential type was to be expected for the distribution of d_ε, considering that it arises in a discrete random-walk model for bed formation (Dacey, 1979). The nontrivial thing for applications, which follows from (4.3), is the particular functional form of the parameter of the exponential distribution in terms of time scale ε and model parameters (u, σ).

The structure of the distribution of d_ε in the general case of H can be inferred only from the computations of Pelletier & Turcotte (1997) for $H = 1/4$. The appearance of an exponential distribution for d_ε would be a surprise in the case $H \neq 1/2$.

5 The multifractility of $d(t)$: the case $H = 1/2$

The complete sedimentation history $\xi(t)$ in the above models is described by a curve whose increments $\Delta\xi$ have order Δ^H. It would be natural to expect that $\Delta d(t)$ has the same property for times $\{t\}$ that have been preserved in the record and which form a fractal set of Hausdorff dimension H. As a matter of fact, the structure of $d(t)$ for the model (1.2) is more complex and calls for a multifractal description. We recall that a multifractal measure $d\mu(x)$ is by definition such that there is a set of fractal subsets S_α, $\alpha \in (\alpha_1, \alpha_2)$ for which $\Delta\mu(x) \sim \Delta^\alpha$ when $x \in S_\alpha$. The dimensions of the S_α, which are $\dim S_\alpha = f(\alpha)$, $\alpha \in (\alpha_1, \alpha_2)$, determine the multifractal spectrum of $d\mu(x)$. Here and below the notation $a \sim b$ means $\ln|a| = \ln|b|(1 + o(1))$.

According to Genyuk (1997), "a typical measure μ typically has no local dimension":

$$\alpha = \lim_{\varepsilon\to 0} \log[\mu(x+\varepsilon) - [\mu(x-\varepsilon)]/\log 2\varepsilon. \tag{5.1}$$

Therefore, replacing the limit in (5.1) with lim sup or lim inf we arrive at a multifractal description of the upper and lower local dimensions α; suppose the sets $S_\alpha^\pm$ and $\dim S_\alpha^\pm = f^\pm(\alpha)$, respectively, correspond to these. The theoretical data $d(t)$ with $H = 1/2$ concern the case in which $S_\alpha^+ \neq S_\alpha^-$. Below we will show how that feature affects the natural (from the applications standpoint) statistics of the multifractal type.

Now we formulate the key statement for multifractal analysis of $d(t)$. Let $\tilde{\tau}_0(d)$ be a Levy process whose increments are described by the Laplace transform (4.1) with $u = 0$, while $d_0(t)$ is the inverse process. It is a well known fact that $d_0(t)$ has the same probability structure as the process of local time for $\sigma w(t)$, where $w(t)$ is the standard Wiener process (see Bertoin, 1996).

Lemma *The probability measures corresponding to the processes* $d_0(t)$ *and* $d(t)$, $d(0) = 0$ *are mutually absolutely continuous on any finite interval* $[0, T]$.

Proof The Levy processes $\tilde{\tau}(d)$ defined by (4.2) have jump-like increasing paths. They can be completely described by the spectral measure density of the jumps $\lambda_u(t) = \sigma^{-1}(2\pi)^{-1/2}t^{-3/2}\exp\{-0.5(u/\sigma)^2 t\}$.

It is easy to see that the Kakutani criterion is fulfilled:

$$\int_0^\infty \left(\sqrt{\lambda_0(t)} - \sqrt{\lambda_u(t)}\right)^2 dt < \infty,$$

which guarantees that the measures corresponding to $\tilde{\tau}(d)$ and $\tilde{\tau}_0(d)$ are equivalent on any finite interval $0 \leqslant d \leqslant D$ (see Jacod & Shiryaev, 1987). We stop the processes $\tau(d)$ and $\tau_0(d)$ at the times d^* and d_0^* when the level T was first reached. Obviously, $d^* < \infty$, since from (4.2) it follows that

$$P(d^* = \infty) = \lim_{x\to\infty} P(d^* > x) = \lim_{d\to\infty} P(\tilde{\tau}(d) < T) = 0.$$

Hence, the measures corresponding to the inverse processes $d(t)$ and $d_0(t)$ in the interval $(0, T)$ are equivalent (see Jacod & Shiryaev, 1987). □

The lemma yields one corollary that is important for what follows: all statements (events) that have probability 0 or 1 for the process $d(t)$ with $u = 0$ automatically have the same probabilities for the process $d(t)$ with $u \neq 0$ as well. In other words, the '0-1' properties of local Brownian motion (see Molchan, 1994, 1995; Dolgopyat & Sidorov, 1995; and Hu & Taylor, 1997) are automatically transferred to the process $d(t)$.

According to (Hu & Taylor, 1997), $d_0(t)$ has the a.s. single local dimension $\alpha = 1/2$ for which $f(\alpha) = 1/2$. On the other hand, there is the following nontrivial spectrum of upper local dimensions:

$$f^+(\alpha) = \frac{1-\alpha}{2\alpha}, \quad 1/2 \leqslant \alpha < 1. \tag{5.2}$$

Molchan (1998) found for $d_0(t)$ the so-called Renyi τ-function which is traditionally used in multifractal analysis. This is defined as the limit

$$\tau(q) = \lim_{N\to\infty} \frac{\log \sum' d^q(\Delta_i)}{\log |\Delta|}, \tag{5.3}$$

where $\{\Delta_i\}$ is the partition of I into equal intervals of size $\Delta = |I|/N$. The sum $\sum'$ means that terms with $d(\Delta_i) \neq 0$ are incorporated, $d(\Delta_i)$ being the increment of $d(t)$ over the interval Δ_i. As is well known for the *regular* situation, the multifractal spectrum $f(\alpha)$ and $\tau(q)$ form a Legendre transform pair:

$$\tau(q) = \min_\alpha(\alpha q - f(\alpha)) := \mathscr{L} f \tag{5.4}$$

and $\mathscr{L}\tau = f$.

According to Molchan (1998), *the limit* (5.3) *exists with probability one when* $N = 2^n$, $n \to \infty$ *and equals*

$$\tau(q) = 0.5 \min(q-1, 2q). \tag{5.5}$$

Obviously, $\tau = \mathscr{L} f^+$, but not vice versa, because the Legendre transform produces concave functions. It follows that in our case the Renyi function provides little relevant information. Using $\tau(q) = 1/2(q-1)$ on the half-axis $q > 0$, that is typical for applications, we see from the slope of $\tau(q)$ the typical local dimension $\alpha = 1/2$ only.

We will deal with statistics that are more natural for our object of study. Let us use ε-beds for local estimation of sedimentation rate: $R(\delta_\varepsilon) = d_\varepsilon/|\delta_\varepsilon|$. Then *for the model* (1.2), *with* $H = 1/2$ *the number of* ε*-beds with* $R(\delta) \sim |\delta|^{\alpha-1}$ *is increasing with* ε^{-1} *as follows*:

$$\#\{\delta_\varepsilon \subset I : |\delta|^{\alpha-1}/|\ln\delta| < R(\delta) < c|\delta|^{\alpha-1}\} \sim \varepsilon^{-f^*(\alpha)} \tag{5.6}$$

where $f^*(\alpha) = 3/2 - 2\alpha$, $\alpha \in [1/2, 3/4]$ *and* $f^*(\alpha) = -\infty$ *for the other values of* α (here and below $\varepsilon^\infty = 0$ is understood to hold). *The asymptotic expression* (5.6) *is true with probability one, when* $\varepsilon = 2^{-n}$, $n \to \infty$.

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Molchan (1995) proves (5.6) in probability as $\varepsilon \to 0$. A trivial use of the Borel–Cantelli lemma yields a.s. convergence, when the choice $\varepsilon = 2^{-n}$, $n \to \infty$ has been made.

Relation (5.6) is similar to a multifractal description where the sedimentation rate index α plays the part of the order of singularities, while the exponent $f^*(\alpha)$, which characterizes the number of ε-beds with the relevant sedimentation rate index, plays the part of multifractal spectrum. However, the dimension functions $f^*(\alpha)$ and $f^+(\alpha)$ differ, not only in form, but also in their respective singularity supports: [1/2, 3/4] and [1/2, 1], respectively. The Legendre transform of the former is

$$\tau^*(q) = \mathscr{L}f^* = \frac{1}{2}\min\,(q-1,\, 3/2\; q) \tag{5.7}$$

and can be derived independently on the lines of Halsey *et al.* (1986) as follows.

Consider the following function:

$$\Phi_\varepsilon(q,\tau) = \sum \tilde{d}_\varepsilon^q(i)\,|\delta_\varepsilon(i)|^{-\tau}, \quad \delta_\varepsilon(\cdot) \subset [0,1],$$

where $\tilde{d}_\varepsilon(i) = d_\varepsilon(i)/\sum d_\varepsilon(i)$ *are normalized* ε*-bed thicknesses. Then the following limit exists a.s. when* $\varepsilon = 2^{-n} \downarrow 0$:

$$\lim_{\varepsilon\downarrow 0} \Phi_\varepsilon(q,\tau) = \begin{cases} \infty & \tau > \tau^*(q) \\ 0 & \tau < \tau^*(q) \end{cases}$$

The function $\tau^*(q)$ *coincides with* (5.7) (Molchan, 1995).

Thus, parallel with (5.2) and (5.5), we have again the dual pair of statements (5.6), (5.7) as the multifractal formalism prescribes.

The result (5.6) may be due to the fact that ε-beds have strongly varying lengths. The variability of $|\delta_\varepsilon|$ is indeed present and can be described in a multifractal manner (see Molchan, 1995):

$$\#\{\delta_\varepsilon \subset I \,:\, \varepsilon^\alpha/|\ln\varepsilon| < |\delta_\varepsilon| < c\,\varepsilon^\alpha\} \sim \varepsilon^{-f_\delta(\alpha)}, \text{a.s.} \tag{5.8}$$

when $\varepsilon = 2^{-n}$ *and* $n \to \infty$. *Here,* $f_\delta(\alpha) = 1 - \alpha/2$, $\alpha \in [1, 2]$ *and* $f_\delta(\alpha) = -\infty$ *for the other values of* α.

Thus, a typical ε-bed has a size of order ε, while nontypical ones are of order ε^α, $1 < \alpha < 2$. Treating ε as a time-resolution scale, all lengths of ε-beds should be made equal to ε. One then gets a rough estimate of sedimentation rate based on the ε-beds: $\hat{R}(\delta_\varepsilon) = d_\varepsilon/\varepsilon$. The multifractal description like (5.6) holds for the new estimate as well. The variable factors are the range of index α and the exponent $f(\alpha)$. To be more specific,

$$\#\{\delta_\varepsilon \subset I \,:\, \varepsilon^{\alpha-1}/|\ln\varepsilon| < \tilde{R}(\delta_\varepsilon) < c\,\varepsilon^{\alpha-1}\} \sim \varepsilon^{-f_*(\alpha)} \tag{5.9}$$

with probability one, when ε *goes through the sequence* 2^{-n}, $n \to \infty$. *Here,* $f_*(\alpha) = 1 - \alpha$, $\alpha \in [1/2, 1]$ *and* $f_*(\alpha) = -\infty$ *for the other values of* α.

To attach a fractal geometrical meaning to (5.9), we write the right-hand side of (5.9) as

$$\varepsilon^{-f_*(\alpha)} = (\varDelta_\alpha)^{-f_*(\alpha)/(2\alpha)} = (\varDelta_\alpha)^{-f^+(\alpha)}$$

where $\varDelta_\alpha = \varepsilon^{2\alpha}$ is a typical size of ε-beds that have the property $d_\varepsilon \sim \varepsilon^\alpha$ (Dolgopyat &

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Sidorov, 1995). We see that (5.2) and (5.9) are consistent, but statistics (5.5), (5.9) and (5.7), (5.6) give different information on the range of possible local dimensions α.

6 Conclusion

We have considered a model for the deposition of sediments. The model assumes constant tectonic subsidence and a random variation in sea level. An important aspect of sedimentary sequences are periods of erosion that produce time gaps in the record and generate well defined sedimentary beds. We have presented analytical results on the statistics of the time gaps, bed thicknesses, and sedimentation rates. The object under study $d(t)$ is interesting in several respects. First, typical multifractal objects in applications have their origin in multiplicative processes. Examples are turbulent cascades and their associated local energy dissipation fields (Frisch, 1995; Mandelbrot, 1974). The dynamics of an evolving section $d(t)$ has an essentially different origin and, as we have seen, a different type of multifractal spectrum (a convex instead of a concave function). Also, the object $d(t)$ is not a multifractal in the strict sense of the term, $f(\alpha) \neq f^{+}(\alpha)$. Consequently, any inferences as to multifractality are nearly meaningless here without a relevant theoretical basis. Lastly, the model case $H = 1/2$ provides a nearly unique opportunity to follow all complexities of the multifractal analysis of $d(t)$. A series of statistics that are natural from the practical point of view allows an analysis that demonstrates the multifractal nature of $d(t)$. However, the statistics can lead us to different conclusions about the multifractal spectra of local and upper local dimensions.

Acknowledgements

Remarks by an anonymous reviewer and by the Co-Editor-in-Chief, J. R. Ockendon, have been very helpful in revising this paper. This research was supported by James S. McDonnell Foundation within the framework of the 21st Century Collaborative Activity Award for Studying Complex Systems (project 'Understanding and Prediction of Critical Transitions in Complex Systems') and in part by the International Science and Technology Center (project code 1293). We also acknowledge support under NFS Grant EAR 9804859.

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