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Blow-up for a parabolic system coupled in an equation and a boundary condition

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In this paper, we consider non-negative solutions of

$$\begin{aligned} &u_t = \Delta u + v^p, && v_t = \Delta v && \text{in } \Omega \times (0,\infty),\\ &\frac{\partial u}{\partial \nu} = 0, && \frac{\partial v}{\partial \nu} = u^q && \text{on } \partial\Omega \times (0,\infty),\\ &u(x,0) = u_0(x), && v(x,0) = v_0(x) && \text{in } \Omega. \end{aligned}$$

We prove that if $pq \leqslant 1$, every solution is global while if $pq > 1$, all solutions blow up in finite time. We also show that if $p, q \geqslant 1$, then blow-up can occur only on the boundary.

1. Introduction

This paper is concerned with a parabolic system coupled in an equation and a boundary condition,

$$\left.\begin{aligned} &u_t = \Delta u + v^p, && v_t = \Delta v, && x \in \Omega, && t > 0,\\ &\frac{\partial u}{\partial \nu} = 0, && \frac{\partial v}{\partial \nu} = u^q, && x \in \partial\Omega, && t > 0,\\ &u(x,0) = u_0(x), && v(x,0) = v_0(x), && x \in \Omega, && \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega \subset C^{1+\mu}(0 < \mu < 1)$, ν is the outward normal, $p, q > 0$ and $u_0(x)$ and $v_0(x)$ are non-negative functions such that

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{and} \quad \frac{\partial v_0}{\partial \nu} = u_0^q \quad \text{for } x \in \partial\Omega. \tag{1.2}$$

Problem (1.1) is a special case of the following general system:

$$\left.\begin{aligned} &u_t = \nabla(a(u)\nabla u) f(u,v), && v_t = \nabla(b(v)\nabla v) + g(u,v), && &&\\ & && && x \in \Omega, && t > 0,\\ &\frac{\partial u}{\partial \nu} = \phi(u,v), && \frac{\partial v}{\partial \nu} = \psi(u,v), && x \in \partial\Omega, && t > 0,\\ &u(x,0) = u_0(x), && v(x,0) = v_0(x), && x \in \Omega. && \end{aligned}\right\} \tag{1.3}$$

As is well known, system (1.3) has been formulated from physical models arising in various fields of the applied sciences (cf. [13]). In [1], Acosta and Rossi obtained certain results on the existence and non-existence of global solutions of (1.3).

The direct motivation of considering problem (1.1) comes from [7], wherein the same problem has been studied, with Ω replaced by

$$\mathbb{R}^n_+ = \{(x_1, x') \mid x' \in \mathbb{R}^{n-1},\ x_1 > 0\}.$$

In [7], Fila and Levine showed that for $pq \leqslant 1$, every non-negative solution is global. Furthermore, for $pq > 1$, if $\max((p+2)/(pq-1), (2q+1)/(pq-1)) \geqslant n$, then all non-trivial non-negative solutions are non-global, whereas if $\max((p+2)/(pq-1), (2q+1)/(pq-1)) < n$, there exist both global and non-global non-negative solutions. Their result is interesting because it is 'intermediate' between the result for a system coupled in the equations [6] and the result for a system coupled in the boundary conditions [5]. However, due to the different nature of solutions on the half space, their arguments cannot apply to problem (1.1).

Problem (1.1) is also closely related to other two problems,

$$\left.\begin{array}{llll} u_t = \Delta u + v^p, & v_t = \Delta v + u^q, & x \in \Omega, & t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, & t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \Omega & \end{array}\right\} \tag{1.4}$$

and

$$\left.\begin{array}{llll} u_t = \Delta u, & v_t = \Delta v, & x \in \Omega, & t > 0, \\ \dfrac{\partial u}{\partial \nu} = v^p, & \dfrac{\partial v}{\partial \nu} = u^q, & x \in \partial\Omega, & t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \Omega. & \end{array}\right\} \tag{1.5}$$

On the one hand, for problem (1.5) on $B_R = \{|x| < R\}$ in the radially symmetric case, Deng [2] showed that if $pq \leqslant 1$, all non-negative solutions are global, while if $pq > 1$, there are no non-trivial non-negative global solutions. Later, among other things, Deng [3] and Hu and Yin [11] independently extended this result to an arbitrary bounded domain Ω. On the other hand, arguing analogously as in [3], one can easily see that the result for (1.5) holds for problem (1.4). Because problem (1.1) is 'intermediate' between problems (1.4) and (1.5), we should expect the same result for (1.1). However, there are significant differences at the technical level due to the following facts. Problems (1.4) and (1.5) are symmetric in the sense that one may always assume $p \leqslant q$, which cannot be assumed for (1.1). Also, the representation formulae for both components u, v of solutions of (1.4) and (1.5) have the same form, but solution components of (1.1) take different forms.

The plan of the paper is as follows. In § 2 we establish global existence and global nonexistence results, and in § 3 we localize the blow-up points in the Lipschitz case.

2. Global existence and finite time blow-up

Let $G_N(x, y, t, \tau)$ be Green's function for the heat equation with a homogeneous Neumann boundary condition. Then we have the following representation formulae for the solution components of (1.1):

$$u(x,t)=\int_{\Omega}G_N(x,y,t,0)u_0(y)\,\mathrm{d}y+\int_0^t\int_{\Omega}G_N(x,y,t,\tau)v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau, \tag{2.1}$$

$$v(x,t)=\int_{\Omega}G_N(x,y,t,0)v_0(y)\,\mathrm{d}y+\int_0^t\int_{\partial\Omega}G_N(x,y,t,\tau)u^q(y,\tau)\,\mathrm{d}S_y\mathrm{d}\tau. \tag{2.2}$$

Using the above representation formulae and the contraction mapping principle, as in [13], we can establish the local existence for solutions of (1.1). The argument is rather standard and is therefore omitted here.

We first show the global existence for solutions of (1.1).

THEOREM 2.1. *Assume $pq \leqslant 1$. Then every solution of (1.1) is global, that is, for any $T>0$,*

$$u(x,t)\leqslant C \quad \text{and} \quad v(x,t)\leqslant C \quad \text{in } \bar{\Omega}\times[0,T],$$

with some positive $C=C(T)<\infty$.

Proof. We seek a global supersolution $(\bar{u},\bar{v})$ of (1.1). From [4], we know that there exists a function $\varphi(x)\in C^2(\bar{\Omega})$ satisfying

$$0<\varphi(x)\leqslant 1 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial\varphi}{\partial\nu}\geqslant 1 \quad \text{on } \partial\Omega. \tag{2.3}$$

Let $m_1=\max_{\bar{\Omega}}|\nabla\varphi|$ and $m_2=\max_{\bar{\Omega}}|\Delta\varphi|$. We define

$$\bar{u}(x,t)=M\mathrm{e}^{\sigma t} \quad \text{and} \quad \bar{v}(x,t)=M\mathrm{e}^{\sigma q t+\gamma\varphi},$$

where

$$\begin{aligned} M&=\max(\|u_0\|_{L^\infty(\Omega)},\|v_0\|_{L^\infty(\Omega)}),\\ \gamma&=M^{q-1},\\ \sigma&=\max\{(\gamma m_2+\gamma^2 m_1^2)/q, M^{p-1}\mathrm{e}^{p\gamma}\}.\end{aligned}$$

We then find that $(\bar{u},\bar{v})$ satisfies

$$\left.\begin{array}{llll} \bar{u}_t\geqslant\Delta\bar{u}+\bar{v}^p, & \bar{v}_t\geqslant\Delta\bar{v}, & x\in\Omega, & t>0,\\ \dfrac{\partial\bar{u}}{\partial\nu}=0, & \dfrac{\partial\bar{v}}{\partial\nu}\geqslant\bar{u}^q, & x\in\partial\Omega, & t>0,\\ \bar{u}(x,0)\geqslant u_0(x), & \bar{v}(x,0)\geqslant v_0(x), & x\in\Omega, & \end{array}\right\} \tag{2.4}$$

and hence $(\bar{u},\bar{v})$ is a desired supersolution. □

To establish the blow-up result, we need the following relationship between solution components u and v.

LEMMA 2.2. *Let (u,v) be a non-negative solution of (1.1). Then there exists a positive constant $c=c(p,q,\Omega)$ such that*

(i) *If $p,q\geqslant 1$,*

$$\int_{\partial\Omega}u^q(x,t)\,\mathrm{d}S_x\geqslant c\biggl(\int_0^t\int_{\Omega}v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\biggr)^q \quad \textit{for } t>0. \tag{2.5}$$

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(ii) *If $p > 1 > q$, with $pq > 1$,*

$$\int_{\partial\Omega} u^q(x,t)\,\mathrm{d}S_x \geqslant ct^{q(1-p)}\biggl(\int_0^t\int_\Omega v(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\biggr)^{pq} \quad \textit{for } t>0. \tag{2.6}$$

(iii) *If $q > 1 > p$, with $pq > 1$,*

$$\int_{\Omega} v^p(x,t)\,\mathrm{d}x \geqslant ct^{p(1-q)}\biggl(\int_0^t\int_{\partial\Omega} u(y,\tau)\,\mathrm{d}S_y\mathrm{d}\tau\biggr)^{pq} \quad \textit{for } t\geqslant 1. \tag{2.7}$$

Proof. We first prove (2.5). Note that, as in [11], one can show

$$\int_{\partial\Omega} G_N(x,y,t,\tau)\,\mathrm{d}S_x \geqslant c_0 \quad \text{for } y\in\bar{\Omega}, \quad t>\tau\geqslant 0, \tag{2.8}$$

with c_0 a positive constant. By means of (2.1), (2.8) and applying Jensen's inequality, we have

$$\begin{aligned}\int_{\partial\Omega} u^q(x,t)\,\mathrm{d}S_x &\geqslant \int_{\partial\Omega}\biggl(\int_0^t\int_\Omega G_N(x,y,t,\tau)v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\biggr)^q\mathrm{d}S_x\\ &\geqslant |\partial\Omega|^{1-q}\biggl(\int_{\partial\Omega}\int_0^t\int_\Omega G_N(x,y,t,\tau)v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\mathrm{d}S_x\biggr)^q\\ &= |\partial\Omega|^{1-q}\biggl[\int_0^t\int_\Omega v^p(y,\tau)\biggl(\int_{\partial\Omega}G_N(x,y,t,\tau)\,\mathrm{d}S_x\biggr)\mathrm{d}y\mathrm{d}\tau\biggr]^q\\ &\geqslant c_0^q|\partial\Omega|^{1-q}\biggl(\int_0^t\int_\Omega v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\biggr)^q. \end{aligned} \tag{2.9}$$

We next prove (2.6). As is well known, Green's function G_N satisfies

$$c_1 \leqslant \int_\Omega G_N(x,y,t,\tau)\,\mathrm{d}x \leqslant c_2 \quad \text{for } y\in\bar{\Omega}, \quad t>\tau\geqslant 0, \tag{2.10}$$

where c_1 and c_2 are positive constants. By (2.10) and Jensen's inequality, we find

$$\begin{aligned}\int_0^t\int_\Omega G_N(x,y,t,\tau)v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau &\geqslant \frac{\bigl(\int_0^t\int_\Omega G_N(x,y,t,\tau)v(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\bigr)^p}{\bigl(\int_0^t\int_\Omega G_N(x,y,t,\tau)\,\mathrm{d}y\mathrm{d}\tau\bigr)^{p-1}}\\ &\geqslant c_2^{1-p}t^{1-p}\biggl(\int_0^t\int_\Omega G_N(x,y,t,\tau)v(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\biggr)^p. \end{aligned} \tag{2.11}$$

Then (2.1), together with (2.8) and (2.11), yields

$$\begin{aligned}&\int_{\partial\Omega} u^q(x,t)\,\mathrm{d}S_x\\ &\quad\geqslant c_2^{q(1-p)}t^{q(1-p)}\int_{\partial\Omega}\biggl(\int_0^t\int_\Omega G_N(x,y,t,\tau)v(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\biggr)^{pq}\mathrm{d}S_x\end{aligned}$$

$$
\begin{aligned}
&\geqslant c_2^{q(1-p)}|\partial\Omega|^{1-pq}t^{q(1-p)}\bigg[\int_0^t\int_\Omega v(y,\tau)\bigg(\int_{\partial\Omega}G_N(x,y,t,\tau)\,\mathrm{d}S_x\bigg)\,\mathrm{d}y\mathrm{d}\tau\bigg]^{pq}\\
&\geqslant c_0^{pq}c_2^{q(1-p)}|\partial\Omega|^{1-pq}t^{q(1-p)}\bigg(\int_0^t\int_\Omega v(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\bigg)^{pq}.
\end{aligned}
$$

We then prove (2.7). Using the estimate of [8], as in [11], we can see

$$
\int_0^t\int_{\partial\Omega}G_N(x,y,t,\tau)\,\mathrm{d}S_y\mathrm{d}\tau\leqslant c_3t\quad\text{for } x\in\bar{\Omega},\quad t\geqslant 1, \tag{2.12}
$$

with c_3 a positive constant. By (2.2), (2.10) and (2.12), we have that

$$
\begin{aligned}
\int_\Omega v^p(x,t)\,\mathrm{d}x&\\
&\geqslant\int_\Omega\bigg(\int_0^t\int_{\partial\Omega}G_N(x,y,t,\tau)u^q(y,\tau)\,\mathrm{d}S_y\mathrm{d}\tau\bigg)^p\,\mathrm{d}x\\
&\geqslant\frac{\int_\Omega\big(\int_0^t\int_{\partial\Omega}G_N(x,y,t,\tau)u(y,\tau)\,\mathrm{d}S_y\mathrm{d}\tau\big)^{pq}\,\mathrm{d}x}{\big(\int_0^t\int_{\partial\Omega}G_N(x,y,t,\tau)\,\mathrm{d}S_y\mathrm{d}\tau\big)^{p(q-1)}}\\
&\geqslant c_3^{p(1-q)}|\Omega|^{1-pq}t^{p(1-q)}\bigg[\int_0^t\int_{\partial\Omega}u(y,\tau)\bigg(\int_\Omega G_N(x,y,t,\tau)\,\mathrm{d}x\bigg)\,\mathrm{d}S_y\mathrm{d}\tau\bigg]^{pq}\\
&\geqslant c_1^{pq}c_3^{p(1-q)}|\Omega|^{1-pq}t^{p(1-q)}\bigg(\int_0^t\int_{\partial\Omega}u(y,\tau)\,\mathrm{d}S_y\mathrm{d}\tau\bigg)^{pq}.
\end{aligned}
$$

□

We are now in a position to present the following result.

THEOREM 2.3. *Suppose $pq>1$. Then all non-negative solutions of (1.1) blow up in finite time.*

Proof. In view of the preceding lemma, we consider three cases.

CASE 1 ($p,q\geqslant 1$). Introduce the function

$$
F(t)=\int_0^t\int_\Omega v(x,\tau)\,\mathrm{d}x\mathrm{d}\tau.
$$

By (2.2) and (2.10), $F(t)$ satisfies

$$
\begin{aligned}
F(t)&\geqslant\int_0^t\int_\Omega\bigg(\int_\Omega G_N(x,y,\tau,0)v_0(y)\,\mathrm{d}y\bigg)\,\mathrm{d}x\mathrm{d}\tau\\
&\geqslant\int_0^t\int_\Omega v_0(y)\bigg(\int_\Omega G_N(x,y,\tau,0)\,\mathrm{d}x\bigg)\,\mathrm{d}y\mathrm{d}\tau\\
&\geqslant c_1\int_0^t\int_\Omega v_0(y)\,\mathrm{d}y\mathrm{d}\tau=c_4t.
\end{aligned} \tag{2.13}
$$

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Moreover, integrating by parts and recalling (2.5), we have

$$\begin{aligned} F''(t) &= \int_{\Omega} v_t(x,t)\,\mathrm{d}x \\ &= \int_{\partial\Omega} u^q(x,t)\,\mathrm{d}S_x \\ &\geqslant c\biggl(\int_0^t \int_{\Omega} v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\biggr)^q \\ &\geqslant c|\Omega|^{q(1-p)} t^{q(1-p)} F^{pq}(t). \end{aligned}$$

Since $F' > 0$, we may multiply this last inequality by F' and integrate by parts on the right to obtain

$$(F'(t))^2 \geqslant \frac{2c|\Omega|^{q(1-p)}}{pq+1} t^{q(1-p)} F^{pq+1}(t)$$

or, equivalently,

$$F'(t) \geqslant c_5 t^{q(1-p)/2} F^{(pq+1)/2}(t), \tag{2.14}$$

where $c_5 = [2c|\Omega|^{q(1-p)}/(pq+1)]^{1/2}$. Choose a positive number δ such that

$$1 < \delta < \min\{\tfrac{1}{2}(pq+1), \tfrac{1}{2}(q+3)\}.$$

We then use the lower bound in (2.13) for $F^{(pq+1)/2-\delta}(t)$ to get

$$F'(t) \geqslant c_4^{(pq+1)/2-\delta} c_5 t^{(q+1)/2-\delta} F^{\delta}(t), \tag{2.15}$$

that is,

$$\frac{F'(t)}{F^{\delta}(t)} \geqslant c_6 t^{(q+1)/2-\delta}.$$

Integrating over (η, t) yields

$$\frac{1}{F^{\delta-1}(t)} - \frac{1}{F^{\delta-1}(\eta)} \leqslant c_7(\eta^{(q+3)/2-\delta} - t^{(q+3)/2-\delta}), \tag{2.16}$$

with a positive fixed η.

By virtue of (2.13), we find

$$F^{\delta-1}(t) \geqslant (c_4^{1-\delta}\eta^{1-\delta} + c_7\eta^{(q+3)/2-\delta} - c_7 t^{(q+3)/2-\delta})^{-1},$$

which implies that F and hence the solution, cannot be global.

Case 2 ($p > 1 > q$). By letting

$$F(t) = \int_0^t \int_{\Omega} v(x,\tau)\,\mathrm{d}x\mathrm{d}\tau$$

and recalling (2.6), the proof is essentially the same as that in case 1, and hence is omitted.

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CASE 3 ($q > 1 > p$). On the one hand, by (2.1) and (2.8), we can see that for $t > 0$,

$$\begin{aligned}\int_{\partial\Omega} u(x,t)\,\mathrm{d}S_x &\geqslant \int_{\partial\Omega}\int_{\Omega} G_N(x,y,t,0)u_0(y)\,\mathrm{d}y\mathrm{d}S_x \\ &\geqslant \int_{\Omega} u_0(y)\Big(\int_{\partial\Omega} G_N(x,y,t,0)\,\mathrm{d}S_x\Big)\,\mathrm{d}y \\ &\geqslant c_0\int_{\Omega} u_0(y)\,\mathrm{d}y \\ &= \tilde{c}_0 > 0. \end{aligned}\tag{2.17}$$

On the other hand, again by (2.1) and (2.8), we have

$$\begin{aligned}\int_{\partial\Omega} u(x,t)\,\mathrm{d}S_x &\geqslant \int_{\partial\Omega}\int_0^t\int_{\Omega} G_N(x,y,t,\tau)v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau\mathrm{d}S_x \\ &= \int_0^t\int_{\Omega} v^p(y,\tau)\Big(\int_{\partial\Omega} G_N(x,y,t,\tau)\,\mathrm{d}S_x\Big)\,\mathrm{d}y\mathrm{d}\tau \\ &\geqslant c_0\int_0^t\int_{\Omega} v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau, \end{aligned}$$

which, combined with (2.7), leads to

$$\int_{\partial\Omega} u(x,t)\,\mathrm{d}S_x \geqslant cc_0\int_0^t \tau^{p(1-q)}\Big(\int_0^\tau\int_{\partial\Omega} u(y,\zeta)\,\mathrm{d}S_y\mathrm{d}\zeta\Big)^{pq}\,\mathrm{d}\tau \quad \text{for } t\geqslant 1. \tag{2.18}$$

Set

$$H(t) = \int_0^t\int_{\partial\Omega} u(x,\tau)\,\mathrm{d}S_x\mathrm{d}\tau \quad \text{for } t\geqslant 1.$$

From (2.17) and (2.18), it follows that $H(t)$ satisfies

$$H'(t) \geqslant \tilde{c}_1 + \tilde{c}_2\int_0^t \tau^{p(1-q)}H^{pq}(\tau)\,\mathrm{d}\tau \quad \text{for } t\geqslant 1. \tag{2.19}$$

Integration of the above inequality over $(1,t)$ then yields

$$\begin{aligned} H(t) &\geqslant \tilde{c}_3 t + \tilde{c}_2\int_1^t\int_1^\tau \zeta^{p(1-q)}H^{pq}(\zeta)\,\mathrm{d}\zeta\mathrm{d}\tau \\ &= \tilde{c}_3 t + \tilde{c}_2\int_1^t (t-\zeta)\zeta^{p(1-q)}H^{pq}(\zeta)\,\mathrm{d}\zeta \\ &\geqslant \tilde{c}_3 t + \tilde{c}_2 t^{p(1-q)}\int_1^t (t-\zeta)H^{pq}(\zeta)\,\mathrm{d}\zeta \end{aligned}\tag{2.20}$$

for $t\geqslant 2$. Assume on the contrary that (1.1) has a global solution (u,v). Then for any positive number T (greater than or equal to 2), we have

$$H(t) \geqslant \tilde{c}_3 T + \tilde{c}_4 T^{p(1-q)}\int_T^t (t-\zeta)H^{pq}(\zeta)\,\mathrm{d}\zeta \quad \text{for } T\leqslant t\leqslant 2T. \tag{2.21}$$

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Thus, by comparison, $H(t) \geqslant h(t)$ on $[T, 2T]$, where

$$h(t) = \tilde{c}_3 T + \tilde{c}_4 T^{p(1-q)} \int_T^t (t-\zeta) h^{pq}(\zeta)\, \mathrm{d}\zeta \quad \text{for } T \leqslant t \leqslant 2T. \tag{2.22}$$

Clearly, $h(t)$ satisfies

$$\left.\begin{aligned} h''(t) &= \tilde{c}_4 T^{p(1-q)} h^{pq}(t), \quad T < t < 2T, \\ h(T) &= \tilde{c}_3 T, \\ h'(T) &= 0. \end{aligned}\right\} \tag{2.23}$$

Multiplying the equation in (2.23) by $h'(t)$ and integrating from T to t, we obtain

$$h'(t) = \tilde{c}_5 T^{p(1-q)/2} (h^{pq+1}(t) - h^{pq+1}(T))^{1/2}. \tag{2.24}$$

Integration of this relation over $(T, 2T)$ then leads to

$$\begin{aligned} \tilde{c}_5 T^{(p-pq+2)/2} &= \int_{h(T)}^{h(2T)} (z^{pq+1} - h^{pq+1}(T))^{-1/2}\, \mathrm{d}z \\ &\leqslant (pq+1)^{-1/2} h^{-pq/2}(T) \int_{h(T)}^{2h(T)} (z - h(T))^{-1/2}\, \mathrm{d}z \\ &\qquad + 2^{(pq+1)/2} \int_{2h(T)}^{\infty} z^{-(pq+1)/2}\, \mathrm{d}z \\ &= 2[(pq+1)^{-1/2} + 2(pq-1)^{-1}] \tilde{c}_3^{(1-pq)/2} T^{(1-pq)/2}, \end{aligned} \tag{2.25}$$

which is equivalent to

$$T^{(p+1)/2} \leqslant 2[(pq+1)^{-1/2} + 2(pq-1)^{-1}] \tilde{c}_3^{(1-pq)/2} / \tilde{c}_5. \tag{2.26}$$

For sufficiently large T, inequality (2.26) yields a contradiction, which completes the proof. □

3. Blow-up on the boundary

In [10], with a monotonicity assumption, Hu and Yin showed that for the heat equation $u_t = \Delta u$ under the nonlinear boundary condition $\partial u/\partial \nu = u^p$ $(p > 1)$, blow-up can occur only on the boundary. Later, Hu [9] proved the same result without the assumption. Based on their general ideas, in this section we show that for problem (1.1) in the Lipschitz case, blow-up cannot occur at the interior of the domain. For definiteness, we may assume that T is the blow-up time.

THEOREM 3.1. *If $p, q \geqslant 1$, then blow-up can occur only on the boundary.*

Proof. We first establish another relationship between the solution components u and v of (1.1). By (2.2) and (2.10), we have

$$\begin{aligned}\int_\Omega v^p(x,t)\,\mathrm{d}x &\geqslant \int_\Omega\biggl(\int_0^t\int_{\partial\Omega} G_N(x,y,t,\tau)u^q(y,\tau)\,\mathrm{d}S_y\mathrm{d}\tau\biggr)^p\mathrm{d}x\\ &\geqslant |\Omega|^{1-p}\biggl[\int_0^t\int_{\partial\Omega} u^q(y,\tau)\biggl(\int_\Omega G_N(x,y,t,\tau)\,\mathrm{d}x\biggr)\mathrm{d}S_y\mathrm{d}\tau\biggr]^p\\ &\geqslant c\biggl(\int_0^t\int_{\partial\Omega} u^q(y,\tau)\,\mathrm{d}S_y\mathrm{d}\tau\biggr)^p. \end{aligned} \tag{3.1}$$

Let $A(t)=\|u(\cdot,t)\|_{L^q(\partial\Omega)}$ and $B(t)=\|v(\cdot,t)\|_{L^p(\Omega)}$. By virtue of (2.5) and (3.1), we find

$$A(t)\geqslant c^{1/q}\int_0^t B^p(\tau)\,\mathrm{d}\tau\equiv c^{1/q}J(t) \tag{3.2}$$

and

$$B(t)\geqslant c^{1/p}\int_0^t A^q(\tau)\,\mathrm{d}\tau\equiv c^{1/p}K(t). \tag{3.3}$$

As a consequence, we obtain

$$J'(t)\geqslant cK^p(t)\quad\text{and}\quad K'(t)\geqslant cJ^q(t). \tag{3.4}$$

From theorem 2.3 of [12], it follows that

$$J(t)\leqslant C(T-t)^{-\alpha}\quad\text{and}\quad K(t)\leqslant C(T-t)^{-\beta}\quad\text{for } t\in[0,T), \tag{3.5}$$

where $\alpha=(p+1)/(pq-1)$ and $\beta=(q+1)/(pq-1)$.

We now take an arbitrary $\Omega'\subset\subset\Omega$ with $\mathrm{dist}(\partial\Omega,\Omega')=\epsilon>0$. For this Ω', we can further take $\Omega''\subset\subset\Omega$ such that $\Omega'\subset\subset\Omega''$, $\mathrm{dist}(\partial\Omega'',\Omega')\geqslant\frac{1}{3}\epsilon$ and $\mathrm{dist}(\partial\Omega,\Omega'')\geqslant\frac{1}{3}\epsilon$. It is well known that for any $\epsilon>0$,

$$0\leqslant G_N(x,y,t,\tau)\leqslant C_\epsilon\quad\text{for } |x-y|\geqslant\tfrac{1}{3}\epsilon,\quad x,y\in\bar\Omega,\quad 0<\tau<t<T. \tag{3.6}$$

Then, by (2.2) and (3.6), we can see that

$$\max_{\bar\Omega''}v(x,t)\leqslant C_0+C_\epsilon\int_0^t A^q(\tau)\,\mathrm{d}\tau\leqslant C_1(T-t)^{-\beta}. \tag{3.7}$$

Proceeding similarly as in the proof of theorem 4.1 of [10], we find

$$v(x,t)\leqslant\frac{C_3}{[\psi(x)+(C_2+1)(T-t)]^\beta}\quad\text{in }\bar\Omega'\times[0,T), \tag{3.8}$$

where $\psi(x)\in C^2(\bar\Omega')$ satisfies

$$\left.\begin{aligned}\psi(x)>0\quad\text{in }\Omega',\quad \psi(x)=0\quad\text{on }\partial\Omega',\\ \Delta\psi-\frac{\max(\alpha+1,\beta+1)|\nabla\psi|^2}{\psi}\geqslant -C_2\quad\text{in }\Omega'\end{aligned}\right\} \tag{3.9}$$

for some $C_2>0$. Inequality (3.8) shows that $v(x,t)$ cannot blow up in $\Omega'\times(0,T)$.

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Next we turn our attention to $u(x,t)$. Making use of (2.1), (2.10) and (3.5)–(3.7), we have that in $\bar{\Omega}' \times [0,T)$

$$\begin{aligned} u(x,t) &\leqslant C_0 + \int_0^t \int_{\Omega''} G_N(x,y,t,\tau) v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &\quad + \int_0^t \int_{\Omega\setminus\Omega''} G_N(x,y,t,\tau) v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &\leqslant C_0 + C_1^p \int_0^t (T-\tau)^{-p\beta} \biggl(\int_{\Omega''} G_N(x,y,t,\tau)\,\mathrm{d}y \biggr)\,\mathrm{d}\tau \\ &\quad + C_\epsilon \int_0^t \int_{\Omega\setminus\Omega''} v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &\leqslant C_0 + c_2 C_1^p \int_0^t (T-\tau)^{-p\beta}\,\mathrm{d}\tau + C_\epsilon \int_0^t \int_{\Omega} v^p(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \\ &\leqslant C_0 + \frac{c_2 C_1^p}{\alpha}(T-t)^{-\alpha} + CC_\epsilon (T-t)^{-\alpha} \\ &\leqslant C_4 (T-t)^{-\alpha}. \end{aligned} \tag{3.10}$$

Introduce the function $\Psi(x,t) = C_5[\psi(x) + (C_2+1)(T-t)]^{-\alpha} - u(x,t)$, where C_5 is a positive constant to be determined. Through a routine calculation, we find

$$\begin{aligned} \Psi_t - \Delta\Psi \geqslant{} & \frac{1}{[\psi(x) + (C_2+1)(T-t)]^{\alpha+1}} \\ & \cdot \biggl[\alpha C_5 - C_3^p + \alpha C_5 \biggl(C_2 + \Delta\psi - \frac{(\alpha+1)|\nabla\psi|^2}{\psi + (C_2+1)(T-t)} \biggr) \biggr] \geqslant 0 \end{aligned} \tag{3.11}$$

if $C_5 \geqslant C_3^p/\alpha$.

On the parabolic boundary, $\Psi(x,0) \geqslant 0$ for $x \in \bar{\Omega}'$ if

$$C_5 \geqslant \max_{\bar{\Omega}'} (\psi(x) + (C_2+1)T)^\alpha u_0(x)$$

and $\Psi(x,t) \geqslant 0$ for $(x,t) \in \partial\Omega' \times (0,T)$ if $C_5 \geqslant (C_2+1)^\alpha C_4$. Thus the maximum principle implies that $\Psi(x,t) \geqslant 0$ in $\bar{\Omega}' \times [0,T)$, that is,

$$u(x,t) \leqslant \frac{C_5}{[\psi(x) + (C_2+1)(T-t)]^\alpha} \quad \text{in } \bar{\Omega}' \times [0,T). \tag{3.12}$$

Hence $u(x,t)$ cannot blow up in $\Omega' \times (0,T)$. Since Ω' is an arbitrary compact subset of Ω, the proof is completed. □

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