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EXTREMAL PROBLEMS FOR THE CLASSES $S_{\mathbf{R}}^{-p}$ AND $T_{\mathbf{R}}^{-p}$

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1. Introduction. Let $H(D)$ be the linear space of analytic functions on a domain D of $\mathbb{C}$ endowed with the topology of locally uniform convergence and let $H'(D)$ be the topological dual space of $H(D)$. For domains D which are symmetric with respect to the real axis we use the notation $H_{\mathbf{R}}(D) = \{f \in H(D) : f(D \cap \mathbb{R}) \subset \mathbb{R}\}$. Furthermore, denote by S the set of all univalent mappings f defined on the unit disk Δ which are normalized by $f(0) = 0$ and $f'(0) = 1$. A well studied subclass of $H(\Delta)$ is the set $T_{\mathbf{R}}$ of typically real functions f which have the following properties:

$$(1) \qquad f(0) = f'(0) - 1 = 0$$

$$(2) \qquad \operatorname{Im}\{z\}.\operatorname{Im}\{f(z)\} \geqq 0 \quad \text{for all } z \in \Delta.$$

There is a one-to-one correspondence between $T_{\mathbf{R}}$ and the set $\mathbb{P}_{[-1,1]}$ of all probability measure μ on the Borel σ-algebra over $[-1, 1]$. Indeed, if $\mu \in \mathbb{P}_{[-1,1]}$, then

$$(1.1) \qquad f(z) = \int_{[-1,1]} z/(1 - 2tz + z^2)d\mu(t)$$

belongs to the class $T_{\mathbf{R}}$. Conversely, for each $f \in T_{\mathbf{R}}$ there is a unique $\mu \in \mathbb{P}_{[-1,1]}$ such that (1.1) holds. It follows from there that $T_{\mathbf{R}}$ is convex and compact. For simplicity, we shall use the notation

$$(1.2) \qquad q_t(z) = z/(1 - 2tz + z^2), -1 \leqq t \leqq 1.$$

Observe that the mappings q_t are univalent on Δ and that

$$q_t(\Delta) = \mathbb{C}\backslash\{(-\infty, -1/(2 + 2t)] \cup [1/(2 - 2t), \infty)\}.$$

The set of all univalent mappings in $T_{\mathbf{R}}$ we shall denote by $S_{\mathbf{R}}$.

If $f \in T_{\mathbf{R}}$, then f is strictly monotone increasing on the interval $(-1, 1)$. For simplicity, we shall denote the radial limit of f at $z = -1$ by

$$(1.3) \qquad f(-1) = \lim_{x \to -1} f(x).$$

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Let A be a compact subset of $H(\Delta)$.

Definition 1.1.

(a) A function $f \in A$ is called to be a *support point* of A, $f \in \sigma(A)$, if there is an $L \in H'(\Delta)$ such that $\mathrm{Re}\{L(f)\} = \max\{\mathrm{Re}\{L(g)\} : g \in A\}$ and that $\mathrm{Re}\{L\}$ is not constant on A.

(b) A function $f \in A$ is called to be an *extreme point* of A, $f \in \mathcal{E}(A)$, if f is not a proper convex combination of two other functions in A.

The set of all finite convex combination of functions in A we denote by $co(A)$ and its closure by $\overline{co}(A)$. For example, $\mathcal{E}(T_{\mathbf{R}}) = \{q_t : -1 \leqq t \leqq 1\}$ and $\sigma(T_{\mathbf{R}}) = co(\mathcal{E}(T_{\mathcal{R}}))$ and

$$(1.4) \qquad \overline{co}(S_{\mathbf{R}}) = T_{\mathbf{R}}.$$

Lately, W. Koepf [**6**] has shown that $\mathcal{E}(S_{\mathbf{R}}) = \sigma(S_{\mathbf{R}}) = \mathcal{E}(T_{\mathbf{R}})$.

The class of univalent mappings $f \in H(\Delta)$ with fixed value $f(0)$ and fixed omitted values was examined by G. M. Goluzin and others (see e.g. [**3**], [**5**]). Recently, P. Duren and G. Schober [**4**] gave some geometric properties of extreme points and support points of the class S_o of univalent nonvanishing functions f on Δ with $f(0) = 1$. The corresponding case, when f has real coefficients, was studied by W. Koepf [**6**]. In this paper we consider the classes

$$(1.5) \qquad T_{\mathbf{R}}^{-p} = \{f = z + \sum_{k \geqq 2} a_k(f) z^k \in T_{\mathbf{R}} : f \text{ omits a given point } -p\},\ p > 0$$

and

$$(1.6) \qquad S_{\mathbf{R}}^{-p} = T_{\mathbf{R}}^{-p} \cap S.$$

Since we require that $f'(0) = 1$, the choice of p is important. For instance, $T_{\mathbf{R}}^{-p}$ and $S_{\mathbf{R}}^{-p}$ are empty, if $0 < p < 1/4$, and contain only the Koebe mapping q_1, if $p = 1/4$. Furthermore, for $1/4 < s < t$, we have the strict inclusions $T_{\mathbf{R}}^{-s} \subset T_{\mathbf{R}}^{-t}$ and $S_{\mathbf{R}}^{-s} \subset S_{\mathbf{R}}^{-t}$ and $T_{\mathbf{R}}^{-\infty}(S_{\mathbf{R}}^{-\infty}$ resp.) is the usual class $T_{\mathbf{R}}(S_{\mathbf{R}}$ resp.). Hence, the solutions of most of the optimization problems will depend on the omitted value $-p$.

There is a close relation between $S_{\mathbf{R}}^{-p}$ and the class $S_{\mathbf{R}}(M)$ of all univalent typically real functions which are bounded by M. Indeed, if $g \in S_{\mathbf{R}}(M)$, then $f = M . q_1(g/M) \in S_{\mathbf{R}}^{-M/4}$ and, vice versa, if $f \in S_{\mathbf{R}}^{-p}$, then $g = 4pq_1^{-1}(f/(4p)) \in S_{\mathbf{R}}(4p)$. The class $S_{\mathbf{R}}(M)$ has been studied extensively by O. Tammi [**12**].

Extremal functions in $S_{\mathbf{R}}^{-p}$, $p > 1/4$, (i.e. extreme points and support points) are slit mappings (Proposition 3.7 and Corollary 3.10) but they can split at several finite points or at infinity (Theorem 3.5). However, if $-p$ is not an endpoint of the slit on the negative real axis, then no splitting can occur at infinity (Proposition 3.8). The main result of section 3 is a kind of a Schiffer-Goluzin differential equation (Theorem 3.9).

In section 4 we complete a result of W. Koepf in determining explicitly the set of all support points of the class $(S_o)_{\mathbf{R}}$ which consists of all univalent nonvanishing mappings f, $f(0) = 1$, which have real coefficients.

Section 5 deals with the class $T_{\mathbf{R}}^{-p}$, $p > 1/4$. Evidently $T_{\mathbf{R}}^{-p}$ is compact and convex but, in contrast to (1.4), $T_{\mathbf{R}}^{-p}$, $p > 1/4$, is not the closed convex hull of $S_{\mathbf{R}}^{-p}$ (Proposition 5.1). There is an interesting difference between the extreme points different from q_t, $1/(2p) - 1 \leqq t \leqq 1$, in $S_{\mathbf{R}}^{-p}$ and $T_{\mathbf{R}}^{-p}$. While in the first class splitting occurs, they are two-valent in the second class and all its boundary values lie in $\mathbb{R} \cup \infty$ (Theorem 5.2). Indeed, the set of extreme points for the class $T_{\mathbf{R}}^{-p}$ is exactly the set of all mappings

$$(1.7) \qquad f = q_s q_t / q_{2p(1+s)(1+t)-1}, \; -1 \leqq s \leqq 1/(2p) - 1 \leqq t \leqq 1.$$

Note that any f of the form (1.7) can be expressed as a convex combination of q_s and q_t. First we give sharp lower bounds and upper bounds for $f(x)$, $f'(x)$, $a_2(f)$, $a_3(f)$, and $a_4(f)$ (Proposition 5.4, Theorem 5.6, Proposition 5.8, and Theorem 5.10) and we determine in Theorem 5.5 the set of values of $f(z)$ for a given nonreal z in Δ. Next (Lemma 5.11), we give a sufficient condition for $L \in H'(\Delta)$ in order to get a univalent extremal function. In theorem 5.12 we apply the above Lemma to the odd coefficients of f. The last Theorem is surprising. We show that for each $L \in H'(\Delta)$ there is a $p_L > 0$ such that, if $p > p_L$, there is a univalent mapping $f \in T_{\mathbf{R}}^{-p}$ such that $\operatorname{Re}\{L(f)\} = \max \operatorname{Re}\{L(T_{\mathbf{R}}^{-p})\}$.

2. Some auxiliary Lemmas. For $A \subset H(\Delta)$ let $\mathcal{E}(A)$, $\sigma(A)$, $co(A)$ and $\overline{co}(A)$ denote the set of all extreme points of A, the set of all (proper) support points of A, the convex hull of A and the closed convex hull of A respectively. Let T be a compact metrizable space and $\mathbb{P}_T$ the set of all probability measures μ on the σ-algebra of Borel subsets of T. The support of μ we denote by $\operatorname{supp}(\mu)$. Furthermore, $\mathcal{E}(\mathbb{P}_T)$ consists of all Dirac measures δ_t concentrated at the points $t \in T$. The Krein Milman Theorem states that $\mathbb{P}_T$ is the closed convex hull of $\mathcal{E}(\mathbb{P}_T)$ with respect to the weak*-topology of the dual space of $C(T)$. Finally, if $\mu \in \mathbb{P}_T$ and A is a Borel set of T, we shall use the notation

$$(2.1) \qquad \mu_A(B) = \mu(A \cap B) \quad \text{for all Borel sets } B \text{ in } T.$$

Our first Lemma characterizes compact and convex sets in $H(\Delta)$.

LEMMA 2.1. *A set $A \subset H(\Delta)$ is compact and convex if and only if there exists T described as above and a continuous function $Q : \Delta \times T \longrightarrow \mathbb{C}$ such that $Q(\cdot, t) \in H(\Delta)$ for all $t \in T$ and*

$$(2.2) \qquad A = \left\{ f_\mu = \int_T Q(\cdot, t) d\mu(t) : \mu \in \mathbb{P}_T \right\}.$$

Furthermore, we have

$$(2.3) \qquad \mathcal{E}(A) \subset \{Q(\cdot, t) : t \in T\}.$$

Equality holds if the mapping $\mu \longrightarrow f_\mu$ is injective on $\mathbb{P}_T$, (i.e. if the linear space spanned by $f \equiv 1$, the real parts and imaginary parts of the Taylor coefficients (as functions of t) of the kernel function is dense in $C(T)$).

The necessity of the existence of Q was shown in [**8**] and the sufficiency can be found in [**2**]. The case of equality is discussed in [**9**]. For example, the simplest realisation of the Lemma is the case

$$T = \overline{\mathcal{E}(A)} \quad \text{and} \quad Q(z, f) \equiv f(z).$$

The next Lemma considers a special case of Lemma 2.1.

LEMMA 2.2. *Let T and Q be as in Lemma* (2. 1). *If T is a line segment or a circle and if Q is analytic at each point of $\Delta \times T$, then we have*

$$(2.4) \qquad \sigma(A) \subset co(\mathcal{E}(A)).$$

Proof. Let $L \in H'(\Delta)$ such that $m = \max\{\mathrm{Re}\{L(f) : f \in A\} > \min\{\mathrm{Re}\{Lf)\} : f \in A\}$. Consider the hyperplane $M = \{f \in H(\Delta) : \mathrm{Re}\{L(f)\} = m\}$ which supports the set A. Denote by t^* the reflected point of t with respect to T and let $\Phi(t)$ be defined by

$$\Phi(t) = L(Q(\cdot, t)) + \overline{L(Q(\cdot, t^*))}.$$

By the above assumptions, the function Φ is analytic on a domain containing T and Φ is not constant on T. Therefore there are only finitely many functions $Q(\cdot, t)$ which belong to $A \cap M$. From (2.3) we conclude that

$$\sigma(A) \cap M = A \cap M = co(\mathcal{E}(A \cap M)) \subset co(\mathcal{E}(A)). \qquad \square$$

The following Lemma is a particular case of a more general result given in [**8**, see also **9–11**]. For the convenience of the reader we shall give a proof of it.

LEMMA 2.3. *Let $\Phi : \mathbb{P}_{[a,b]} \longrightarrow \mathbb{R}$ be an affine continuous mapping. Then we have for all $\tau \in \Phi(\mathbb{P}_{[a,b]})$*

$$(2.5) \qquad \mathcal{E}\{\mu \in \mathbb{P}_{[a,b]} : \Phi(\mu) = \tau\} = \{\nu = (1-\lambda)\delta_s + \lambda\delta_t : s, t \in [a, b], \; 0 \leqq \lambda \leqq 1, \; \Phi(\nu) = \tau\}.$$

Proof. Let $\nu \in \mathcal{E}\{\mu \in \mathbb{P}_{[a,b]} : \Phi(\mu) = \tau\}$ and assume that $\mathrm{supp}(\mu)$ contains at least three points x_1, x_2, x_3, $a \leqq x_1 < x_2 < x_3 \leqq b$. Choose $c \in (x_1, x_2)$ and $d \in (x_2, x_3)$. Then the intervals $T_1 = [a, c)$, $t_2 = [c, d]$ and $T_3 = (d, b]$ form a partition of $[a, b]$ and we have

$$\nu(T_j) > 0, \sum_{j=1}^{3} \nu(T_j) = 1 \quad \text{and} \quad \nu = \sum_{j=1}^{3} \nu(T_j)\mu_j,$$
$$\text{where } \mu_j = \nu_{T_j}/\nu(T_j) \in \mathbb{P}_{[a,b]}.$$

Next, there are three real numbers s_j such that

$$\sum_{j=1}^{3} s_j = 0, \sum_{j=1}^{3} |s_j| > 0 \quad \text{and} \quad \sum_{j=1}^{3} s_j \Phi(\mu_j) = 0.$$

Consider now the real measure $\nu_0 = \epsilon \sum_{j=1}^{3} s_j \mu_j$, $0 < \epsilon < \min\{\nu(T)\}/\max\{|s_j|\}$. Then $\nu_o(T_j) = \epsilon . s_j$ and ν_0 is not the zero measure. Moreover we have

$$\nu = [\nu - \nu_0) + \{\nu + \nu_0)]/2,$$

$$\nu \pm \nu_0 = \sum_{j=1}^{3} [\nu(T_j) \pm \epsilon s_j]\mu_j \in \mathbb{P}_{[a,b]} \quad \text{and}$$

$$\Phi(\nu \pm \nu_0) = \sum_{j=1}^{3} [\nu(T_j) \pm \epsilon s_j]\Phi(\mu_j) = \Phi(\nu) = \tau.$$

which leads to a contradiction.

The converse inclusion is trivial since $\operatorname{supp}[(1 - s)\nu_1 + s\nu_2] = \operatorname{supp}[\nu_1] \cup \operatorname{supp}[\nu_2]$ for all $\nu_1, \nu_2 \in \mathbb{P}_{[a,b]}$ and all $0 < s < 1$. □

To each $L \in H'(\Delta)$ we associate the linear functional $L^* \in H'(\Delta)$ defined by

(2.6) $\quad L^*(f) = (1/2)[L(f) + \overline{L(f^*)}]$

where

(2.7) $\quad f^*(z) = \overline{f(\bar{z})}$.

The Toeplitz representation for L and L^* is then of the form

$$L(f) = \sum_{n=0}^{\infty} b_n a_n(f) \quad \text{and } L^*(f) = \sum_{n=0}^{\infty} \operatorname{Re}(b_n) a_n(f)$$

where $a_n(f) = f^{(n)}(0)/n!$.

Furthermore, if $f \in H_{\mathbb{R}}(\Delta)$, then $L^*(f) = \operatorname{Re}\{L(f)\}$.

LEMMA 2.4. *Let* $L \in H'(\Delta), c \in \mathbb{R}$ *and suppose that the equation* $L^*(q_t) = c$ ($L^*(q_t/z) = c$ respectively) *has an infinite number of solutions. Then we have* $L^*(f) = \operatorname{Re}\{b_o\}.f(0) + c.f'(0)$ ($L^*(f) = c.f(0)$ resp.) *for all* $f \in H(\Delta)$.

Proof. Since $L^*(q_t)$ ($L^*(q_t/z)$ resp.) as a function of t is analytic on $[-1, 1]$, we conclude that $L^*(f)$ ($L^*(f/z)$ resp.) is constant on $T_{\mathbb{R}}$ and we have therefore $L^*(z) = L^*(z+z^n/n) = c$ ($L^*(1) = L^*(1+z^{n-1}/n) = c$ resp.) for all $n = 2, 3, \ldots$. and the result follows. □

3. Extremal Problems for $S_{\mathbf{R}}^{-p}$. In this section we are interested in the class $S_{\mathbf{R}}^{-p}$ of univalent mappings defined on the unit disk Δ which have the following properties:

(1) $f(0) = f'(0) - 1 = 0$

(2) f is real on the interval $(-1, 1)$

(3) f omits a given point $-p$ on the negative real axis.

Since $S_{\mathbf{R}}^{-p}$ is empty for $0 < p < 1/4$ and contains only the Koebe mapping $q_1(z) = z/(1-z)^2$, if $p = 1/4$, we shall assume that $p > 1/4$. Observe also that for $1/4 < s < t$, $S_{\mathbf{R}}^{-p}$ is strictly included in $S_{\mathbf{R}}^{-t}$ and that $S_{\mathbf{R}}^{-\infty}$ is the usual class $S_{\mathbf{R}}$ of all normalized univalent typically real functions. Furthermore, for each $t \in [1/(2p) - 1, 1]$ the mapping $q_t(z) = z/(1 - 2tz + z^2)$ belongs to $S_{\mathbf{R}}^{-p}$. They are also extreme points and support points for this class. There are many other support or extreme points for $S_{\mathbf{R}}^{-p}$.

We start our investigation with an elementary automorphism on $S_{\mathbf{R}}^{-p}$. Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$. Then the correspondence $f \longrightarrow g_f$ defined by

$$\begin{aligned} (3.1) \qquad g_f(z) &= -p.f(-z)/[p + f(-z)] \\ &= z - (a_2 - 1/p)z^2 + (a_3 - 2a_2/p + 1/p^2)z^3 + \ldots \end{aligned}$$

is a homoeomorphism from $S_{\mathbf{R}}^{-p}$ onto itself. For instance, the function q_t is mapped onto $q_{-t+1/(2p)}$, $1/(2p) - 1 \leqq t \leqq 1$. As an immediate consequence we get the following elementary results.

Proposition 3.1. *For $f \in S_{\mathbf{R}}^{-p}$, $p \geqq 1/4$, we have*

(i) $2 \geqq a_s(f) \geqq -2 + 1/p$

(ii) $3 \geqq a_3(f) \geqq \begin{cases} (1 - 1/p)(3 - 1/p), & \text{if } 1/4 \leqq p \leqq 1/2 \\ -1, & \text{if } p \geqq 1/2 \end{cases}$

(iii) $q_{1/(2p)-1}(x) \leqq f(x) \leqq q_1(x)$, if $-1 < x < 1$.

For each case, equality holds only for $f = q_t$ with $t = 0$, $t = 1/(2p) - 1$ or $t = 1$.

Proof. Statement (*i*) follows from the fact that $a_2(f) \leqq 2$ for all $f \in T_{\mathbf{R}}$ and that $a_2(g_f) = (1/p) - a_2(f)$. The upper bound for $a_3(f)$ holds also for all $f \in T_{\mathbf{R}}$ and its lower bound follows from the inequality $a_3(f) \geqq a_2^2(f) - 1$ which is true for all $f \in T_{\mathbf{R}}$. Next, we have $q_{-1}(|x|) \leqq |f(x)| \leqq q_1(|x|)$ for all $x \in (-1, 1)$ and all $f \in T_{\mathbf{R}}$ which implies that $f(x) \leqq q_1(x)$ for all $x \in (-1, 1)$. Finally, $g_f(x) \leqq q_1(x)$ implies that $[1 - (2 - (1/p))x + x^2]f(-x) \geqq -x$. Replacing x by $-x$ statement (*iii*) follows. □

The following proposition creates a chain in $S_{\mathbf{R}}^{-p}$ from any mapping $f \in S_{\mathbf{R}}^{-p}$ to q_t, $1/(2p) - 1 \leqq t \leqq 1$.

Proposition 3.2. *Let $f \in S_{\mathbf{R}}^{-p}$, $p \geqq 1/4$, $0 < r \leqq 1$, and $1/(2p) - 1 \leqq t \leqq 1$. Put $\tau = p/[1 + (1 - r)(2p(1 + t) - 1)]$. Then $1/4 < \tau \leqq p$ and*

$$(3.2) \qquad F(\cdot, r, t) = f(q_t^{-1}(rq_t(\cdot))/r \in S_{\mathbf{R}}^{-\tau} \subset S_{\mathbf{R}}^{-p}$$

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Moreover, $F(\cdot, 0^+, t) = q_t$ *and* $F(\cdot, 1, t) = f$.

Proof. Fix $t \in [1/(2p) - 1, 1]$ and let $d = q_t^{-1}(rq_t(-1))$. Then $q_t(d) = -r/(2+2t) = 1/[(d+1/d) - 2t]$ and, by proposition 3.1 (*iii*), we have

$$F(-1, r, t) = f(d)/r \geqq q_{1/(2p)-1}(d)/r$$
$$= 1/\{r[(d+1/d) + 2 - 1/p]\} = -\tau \geqq -p. \qquad \square$$

As an immediate consequence we get

PROPOSITION 3.3.
(*i*) *If* $f \in S_{\mathbf{R}}^{-p}$ *and* $1/(2p) - 1 \leqq t \leqq 1$, *then*

$$F(\cdot, r, t) = f - [f'q_t/q_t' - f](1-r) + o(1-r) \in S_{\mathbf{R}}^{-p} \text{ as } r \text{ tends to } 1.$$

(*ii*) *The chain F satisfies the differential equation*

$$r\partial F/\partial r = (\partial F/\partial z)(q_t/q_t') - F,\ F(z, 0^+, t) \equiv q_t(z)$$
and $F(z, 1, t) \equiv f(z)$.

(*iii*) *Let* $L \in H'(\Delta)$ *and suppose that* $f \in S_{\mathbf{R}}^{-p}$ *is a solution of max* $L^*(S_{\mathbf{R}}^{-p})$. *Then we have* $L^*(f) \geqq L^*(f'q_t/q_t')$ *for all* $t \in [1/2p) - 1, 1]$.

The next result is a direct application of Proposition 3.2.

PROPOSITION 3.4. *Let* $L \in H'(\Delta)$ *and put* $\phi(p) = \max L^*(S_{\mathbf{R}}^{-p})$, $p > 1/4$. *Then* ϕ *satisfies locally the Lipschitz condition.*

Proof. For $p > 1/4$, choose $\epsilon > 0$ such that $p - \epsilon > 1/4$. Then $S_{\mathbf{R}}^{-(p-\epsilon)} \subset S_{\mathbf{R}}^{-p}$ and $\phi(p-\epsilon) \leqq \phi(p)$. Let $f \in S_{\mathbf{R}}^{-p}$ be an extremal function for L^* (i.e. $\phi(p) = L^*(f)$). Put $\tau = p - \epsilon$ and $t = 1$ in relation (3.2). Then $r = 1 - \epsilon/[(4p-1)(p-\epsilon)] \in (0, 1)$ and from Proposition 3.2 we conclude that $F(\cdot, r, 1) \in S_{\mathbf{R}}^{-(p-\epsilon)}$. On the other hand, by Proposition 3.3 (*i*), we have

$$0 \leqq \phi(p) - \phi(p-\epsilon) \leqq \phi(p) - L^*(F(\cdot, r, 1)$$
$$= L^*(f'q_1/q_1' - f)(1-r) + o(1-r),$$

where $r = 1 - \epsilon/[(4p-1)(p-\epsilon)]$ tends to 1 as ϵ tends to zero. Therefore $0 \leqq [\phi(p) - \phi(p-\epsilon)]/\epsilon \leqq L^*(f'q_1/q_1' - f)/[(4p-1)(p-\epsilon)] + o(1) = O(1)$ as ϵ tends to 0. $\square$

The class $S_{\mathbf{R}}^{-p}$ is closely related to the class $S_{\mathbf{R}}(M)$ which consists of all mappings in $S_{\mathbf{R}}$ which are bounded by M. Indeed, consider the transformation

(3.3) $\qquad g \rightarrow f_g = 4pq_1(g/(4p))$.

Then $f_g \in S_{\mathbf{R}}^{-p}$, if and only if $g \in S_{\mathbf{R}}(4p)$. Tammi [**12**] has extensively studied the class $S_{\mathbf{R}}(M)$. In particular he derived a Löwner-type differential equation for a dense subclass of $S_{\mathbf{R}}(M)$ consisting of mappings whose images are the disk $\{w: |w| < M\}$ minus two slits. Applying the transformation (3.3) one gets the differential equation

$$t.\,\partial F(z, t)/\partial t = -F^2(z, t)[1 + \cos(\theta(t))]/[2t + [1 - \cos(\theta(t))]F(z, t)],$$
$$F(\cdot, 1/4) = q_1 \quad \text{and } F(\cdot, p) = f,$$

where $\theta(t)$ is a real continuous function on [1/4, p].

We get the following nontrivial examples of support points.

THEOREM 3.5. *For $f \in S_{\mathbf{R}}^{-p}$ we have the inequalities*

$$-1 - 1/(4p^2) \leqq a_3(f) - a_2(f)/p \leqq \begin{cases} 1 - 3/(8p^2), & \text{if } 1/4 \leqq p \leqq e/4 \\ 1 - 3/(8p^2) + (4p\sigma - 1)^2/(8p^2), & \text{if } p \geqq e/4, \end{cases}$$

where σ is the unique solution of the equation $4p\sigma \ln(\sigma) + 1 = 0$ in the interval $[1/e, 1)$. The lower bound is reached by the function $q_{1/(4p)}$ and the upper bound by the solutions $f = F(\cdot, p)$ of the above Löwner differential equation, where

(i) $\cos(\theta(t)) \equiv 0$ *and* $a_2(f) = 2/e$, *if* $1/4 \leqq p < e/4$,

(ii) $\cos(\theta(t)) = -2\lambda t$, $|\lambda| \leqq 2/e$, *and* $a_2(f) = \lambda + 2/e$, *if* $p = e/4$,

(iii) $\cos(\theta(t)) = \begin{cases} \pm 4\sigma t; \ 1/4 \leqq t \leqq 1/(4\sigma) \\ \pm 1; \ 1/(4\sigma) \leqq t \leqq p \end{cases}$.

and $a_2(f) = \mp 2\sigma + 2/e$, *if* $p > e/4$

All extremal functions map Δ either

(a) onto the complement of a three-fork slit consisting of the halfline $(-\infty, -p]$ and a bounded symmetric are cutting the interval $(-\infty, -2p]$ or

(b) onto the complement of three slits consisting of the halfline $(-\infty, -p]$ and two unbounded Jordan arcs which are symmetric with respect to the real axis and which contain no points of $\mathbb{R}$, or

(c) onto the complement of two slits consisting of the halfline $(-\infty, -p]$ and a two-fork slit which contains a real halfline $[a, \infty)$ for some $a > 0$.

Proof of Theorem 3.5. The lower estimate follows immediately from Proposition 3.1 (i) and the inequality $a_3(f) \geqq a_2^2(f) - 1, f \in T_{\mathbf{R}}$. Let $f \in S_{\mathbf{R}}^{-p}$ and $g = 4pq_1^{-1}(f/(4p))$. Then $g \in S_{\mathbf{R}}(4p)$ and $a_3(g) = a_3(f) - a_2(f)/p + 5/(16p^2)$. Apply now the results of Tammi to $a_3(g)$.

Remark. Tammi has also discussed the extremal functions in $S_{\mathbf{R}}(M)$ which correspond to the linear functional $a_4(g)$. The homeomorphism (3.3) transforms $a_4(g)$ to a concave functional on $S_{\mathbf{R}}^{-p}$.

In what follows, we give some geometric properties of extreme points and support points of $S_{\mathbf{R}}^{-p}$. The next Lemma will be useful later on.

LEMMA 3.6. *Let $f \in S_{\mathbf{R}}^{-p}$, $p > 1/4$, L, $\in H'(\Delta)$ and suppose that there is an infinite number of points $a \in \mathbb{C} \setminus f(\Delta)$ for which* *(i)* $L^*(f^2(f+p)/(f-a)^2) = 0$

or

(ii) $L^*(f(f+p)/(f-a)) + c/a = 0$, $c = const.$

holds. Then $L^*(f) = \text{Re}\{b_o\}f(0) + \text{Re}\{b_1\}f'(0)$.

Proof. Suppose that (i) holds. Let μ be a representing measure for L^* whose support lies in a compact set $K \subset \Delta$. Put $d\mu_1(w) = w^2(w+p)d\mu(f^{-1}(w))$. Then the function

$$(3.4) \qquad a \longrightarrow L^*(f^2(f+p)/(f-a)^2)$$

is analytic in a neighborhood of infinity and μ_1 is a complex Borel measure with support in $f(K)$. Observe that the function (3.4) vanishes at infinity and that all its Laurent coefficients are zero. In other words, we have

$$(3.5) \qquad (n-1)\int_K f^n(z)(f(z)+p)d\mu(z) = (n-1)\int_{f(K)} w^{n-2}d\mu_1(w) = 0$$

for all $n = 2, 3, \ldots$.

Define now L_o by $L_o(F) = \int_{f(K)} F(w)d\mu_1(w)$ and let $F \in H_{\mathbf{R}}(f(\Delta))$. By Runge's Theorem there is a sequence of polynomials p_n which converges uniformly to F on $f(K)$. Replacing p_n by p_n^* defined in (2.7) we may assume that the coefficients of p_n are real. From (3.5) we conclude that L_o vanishes on $H_{\mathbf{R}}(f(\Delta))$. Put $g_n(z) = z^n/[f^2(z)(f(z)+p)]$, $n = 2, 3, \ldots$. Then, $g_n(f^{-1}) \in H_{\mathbf{R}}(f(\Delta))$ and we have

$$L_o(g_n(f^{-1})) = \int_K g_n(z)d\mu_1(f(z)) = \int_K z^n d\mu(z) = L^*(z^n) = 0$$

for all $n = 2, 3, \ldots$, which shows the case (i). The proof for the case (ii) is similar. □

PROPOSITION 3.7. *If f is an extreme point or a support point of $S_{\mathbf{R}}^{-p}$, $p > 1.4$, then $f(\Delta)$ is dense in* $\mathbb{C}$.

Proof. Let $f \in S_{\mathbf{R}}^{-p}$ and suppose that $f(\Delta)$ omits an open set $\mathcal{D}$. Then there is a closed disk $\{w\colon |w-a| \leqq \epsilon\} \subset \mathcal{D} \cap \{w\colon \text{Im}\{w\} \neq 0\}$ and there is a $\delta > 0$, such that the functions

$$(3.6) \qquad \Phi_j(w) = w + (-1)^j\delta w^2(w+p)[(w-a)^{-2} + (w-\bar{a})^{-2}],\ j = 1, 2,$$

have the following properties:

(i) They are analytic and univalent in $\{w\colon |w-a| > \epsilon\} \cap \{w\colon |w-\bar{a}| > \epsilon\} = \Omega$. Indeed, $[\Phi_j(w) - \Phi_j(\omega)]/[w-\omega] = 1 + \delta.\,\Xi_j(w,\omega)$ where $\Xi_j(w,\omega)$ is bounded in $\Omega \times \Omega$.

(ii) They are strictly increasing on $\mathbb{R}$ and $\Phi_j'(0) = 1$.

(iii) $\Phi_j(w) > -p$ for all real $w > -p$. Indeed, $\Phi_j(-p) = -p$, $\Phi_j'(0) > 0$ and $\Phi_j(w)$ is univalent on the real axis. Therefore, $\Phi_j(f) \in S_{\mathbf{R}}^{-p}$ and $f = [\Phi_1(f) + \Phi_2(f)]/2$.

Suppose first that f is an extreme point of $S_{\mathbf{R}}^{-p}$. Then $f^2(f+p)[(f-a)^{-2} + (f-\bar{a})^{-2}] \equiv 0$ on Δ which leads to a contradiction.

Next, suppose that f is a support point of $S_{\mathbf{R}}^{-p}$. Then there is an $L \in H'(\Delta)$ for which L^* is not constant on $S_{\mathbf{R}}^{-p}$ and

$$\begin{aligned} L^*(f) &= \max L^*(S_{\mathbf{R}}^{-p}) \geqq L^*(\Phi_j(f)) \\ &= L^*(f) + 2(-1)^j \delta \operatorname{Re}\{L^*[f^2(f+p)/(f-a)^2]\}. \end{aligned}$$

This implies that $\operatorname{Re}\{L^*[f^2(f+p)/(f-a)^2]\} = 0$ for all a in the exterior of $f(\Delta)$. But this is impossible by Lemma 3.6. □

Suppose now that D is a simply connected domain of $\mathbb{C}$ and let a and b, $a \neq b$, be in $\mathbb{C}\backslash D$. Then both functions

$$(3.7) \qquad \Psi_j(w) = w + (-1)^j[(w-a)(w-b)]^{1/2}$$

are univalent and analytic on D and they have disjoint images. Historically, L. Brickman [**1**] has used this two functions to show that extreme points of S are monotonic slit mappings. Later, W. Koepf [**6**] adapted the method of Brickman to the class $(S_o)_{\mathbf{R}}$. Unfortunately, this method gives not so strong results for the class $S_{\mathbf{R}}^{-p}$. Indeed, if $f \in S_{\mathbf{R}}^{-p}$ and $f(-1) = -p$ in the sense of (1.3), then only one of the two mappings

$$[\Psi_j(f) - \Psi_j(0)]/\Psi_j'(0),\ b = \bar{a},$$

belongs to the class $S_{\mathbf{R}}^{-p}$. However, we have:

Proposition 3.8. *Let $f \in S_{\mathbf{R}}^{-p}$, $p > 1/4$, and $f(-1) > -p$ in the sense of (1.3). If there is a sequence of nonreal $a_n \in \mathbb{C}\backslash f(\Delta)$ which converges to infinity, then f is neither a support point nor an extreme point of $S_{\mathbf{R}}^{-p}$.*

Proof. Let $f(-1) = -p_1 > -p$, $a \in \mathbb{C}\backslash f(\Delta)$ with $\operatorname{Im}\{a\} \neq 0$, and let

$$(3.8) \qquad \Phi_j(f) = [\Psi_j j(f) - \Psi_j(0)]/\Psi_j'(0),\ j = 1, 2$$

where $\Psi_j(w)$ is defined in (3.7) with $b = \bar{a}$ and $(|a|^2)^{1/2} = |a|$. Then, for $j = 1, 2$, define $f_j = \Psi_j(f) \in S_{\mathbf{R}}$. Then $f = \lambda_1 f_1 + \lambda_2 f_2$, $\lambda_j = 1 - (-1)^j \operatorname{Re}\{a\}/|a| > 0$ and $\lambda_1 + \lambda_2 = 1$. Since $f_2(-1) = \Phi_2(-p_1) > -p_1 > -p$, we conclude that $f_2 \in S_{\mathbf{R}}^{-p}$. On the other hand, Φ_1 converges locally uniformly to the identity on a simply connected domain containing $f(\Delta)$ and $\{-p_1\}$ as a tends to infinity. Hence, we have $f_1(-1) = \Phi_1(-p_1) > -p$ for sufficiently large non real $a \in \mathbb{C}\backslash f(\Delta)$ and $f_1 \in S_{\mathbf{R}}^{-p}$. This shows that f is not an extreme point of $S_{\mathbf{R}}^{-p}$.

For any $L \in H'(\Delta)$ for which L^* is not constant on $S_{\mathbf{R}}^{-p}$ we conclude from Proposition 3.7 that $\max L^*(S_{\mathbf{R}}^{-p}) > L^*(f_j)$ and therefore f is also not a support point of $S_{\mathbf{R}}^{-p}$. □

Remark. Proposition 3.8 is not in contradiction with the examples of support points we have given in Theorem 3.5, since we require here that $f(-1) > -p$.

We present now an analogue of the Goluzin variation (see [**5**, p. 99] for the general form and p. 106 for the specific choice of $Q(w)$). Let A_k, $1 \leqq k \leqq n$, be n arbitrary complex numbers and let a_k, $1 \leqq k \leqq n$, be n arbitrary nonreal numbers. For $p > 1/4$, consider the function

$$w^*(w, \lambda) = w + \lambda Q(w), \quad \text{where} \tag{3.9}$$

$$Q(w) = \sum_{k=1}^{n} A_k \frac{w(w+p)}{w-a_k} + \overline{A_k}\frac{w(w+p)}{w-\overline{a_k}}.$$

Then w^* is analytic and univalent in w on any domain

$$\{w \in \mathbb{C} : |w - a_k| > \delta \quad \text{and } |w - \overline{a_k}| > \delta,\ 1 \leqq k \leqq n\} \tag{3.10}$$

whenever

$$|\lambda| < \left[2\sum_{k=1}^{n} |A_k|(1 + |a_k|\,|a_k + p|/\delta^2)\right]^{-1}. \tag{3.11}$$

Indeed, this follows from

$$\frac{w^*(w,\lambda) - w^*(u,\lambda)}{w-u} = 1 + \lambda \sum_{k=1}^{n} A_k\left(1 - \frac{a_k(a_k+p)}{(w-a_k)(u-a_k)}\right) + \overline{A_k}\left(1 - \frac{\overline{a_k}(\overline{a_k}+p)}{(w-\overline{a_k})(u-\overline{a_k})}\right).$$

Let $f \in S_{\mathbf{R}}^{-p}$ and suppose that for some r, $0 < r < 1$, all the points a_k are in $f(\{z\colon |z| < r\})$. Then, for sufficiently small $|\lambda|$, the function $w^*(w, \lambda)$ is analytic and univalent on the annulus $\{z : r < |z| < 1\}$.

Choose n nonreal numbers z_k, $1 \leqq k \leqq n$, in Δ such that $a_k = f(z_k)$. Then the function $Q(f)/[z.f']$ has only simple poles in Δ which lie on the set $\{z_k\colon 1 \leqq k \leqq n\}$. The Goluzin interior variation $f^{\#}$ of f as given in equation 2 in [5, p. 100] takes the form

$$f^{\#} = f + \lambda \sum_{k=1}^{n} [A_k H_{a_k} + \overline{A_k} H_{\overline{a_k}}] + O(\lambda^2) \in S_{\mathbf{R}}^{-p+O(\lambda^2)},\ \lambda \in \mathbb{R},\ \lambda \to 0, \tag{3.12}$$

under the condition that

$$\operatorname{Re}\left\{\sum_{k=1}^{n} A_k H'_{a_k}(0)\right\} = 0, \tag{3.13}$$

where

$$(3.14)\quad G_\zeta(z) = zf'(z)(1-z^2)/[(1-\zeta z)(1-z/\zeta)]$$

and

$$(3.15)\quad H_\zeta(z) = f(\zeta)[f(\zeta)+p]G_\zeta(z)/[\zeta f'(\zeta)]^2 + f(z)[f(z)+p]/[f(z)-f(\zeta)].$$

Observe that $H_\zeta(z) \in H(\Delta \times \Delta)$ as a function of the variables z and ζ and that $f^\#$ is typically real.

From the variation formula (3.12), (3.13) we deduce the following Schiffer-type differential equation.

THEOREM 3.9. *Let f be a support point of $S_{\mathbf{R}}^{-p}$, $p > 1/4$, and let $L \in H'(\Delta)$ for which L^* is not constant on $S_{\mathbf{R}}^{-p}$ and for which $L^*(f) = \max L^*(S_{\mathbf{R}}^{-p})$. Using the notations* (3.14) *and* (3.15) *the following conclusions hold:*

(*i*) *$g(\zeta) = L^*(H_\zeta)/H_\zeta'(0)$ is constant on Δ.*

(*ii*) *For all $\zeta \in \Delta$ we have:*

$$(3.16)\quad \left\{L^*\left(\frac{f(f+p)}{f-f(\zeta)}\right) + \frac{pC}{f(\zeta)}\right\} \frac{f(\zeta)}{f(\zeta)+p}\left[\frac{\zeta f'(\zeta)}{f(\zeta)}\right]^2 + L^*(G_\zeta) - C = 0$$

(*iii*) *$C = g(0) = 2L^*(f+p[1-(1-z^2)f'])/(1-2pa_2(f)$ whenever $a_2(f) \neq 1/(2p)$. If $a_2(f) = 1/(2p)$, then $L^*(f+p[1-(1-z^2)f']) = 0$.*

Proof. Take $n = 2$ in (3.12) and (3.13). By Proposition 3.4, the function $\Phi(p) = \max L^*(S_{\mathbf{R}}^{-p})$, $p > 1/4$, satisfies the local Lipschitz condition. Since $L^*(H_{\bar z}) \equiv \overline{L^*(H_z)}$, we get

$$\begin{aligned} L^*(f^\#) &= \Phi(p) + 2\lambda \operatorname{Re}\left\{L^*\left(\sum_{k=1}^{2} A_k H_{a_k}\right)\right\} + O(\lambda^2) \\ &\leqq \Phi(p + 0(\lambda^2)) = \Phi(p) + 0(\lambda^2) \end{aligned}$$

as λ tends to zero. Therefore we have

$$(3.17)\quad \operatorname{Re}\left\{\sum_{k=1}^{2} A_k L^*(H_{a_k})\right\} = 0 \quad \text{and} \operatorname{Re}\left\{\sum_{k=1}^{2} A_k H_{a_k}'(0)\right\} = 0.$$

Next, put in (3.17) $a_1 = \zeta$, $a_2 = \eta$, $A_1 = e^{i\alpha}$ and $A_2 = \gamma e^{i\beta}$, where α, β and γ are real numbers. Then we get

$$(3.18)\quad e^{i\alpha}L^*(H_\zeta) + e^{-i\alpha}L^*(H_{\bar\zeta}) + \gamma e^{i\beta}L^*(H_\eta) + \gamma e^{-i\beta}L^*(H_{\bar\eta}) = 0$$

which holds under the restriction

$$(3.19)\quad \operatorname{Re}\{e^{i\alpha}H_\zeta'(0) + \gamma e^{i\beta}H_\eta'(0)\} = 0.$$

Now, we multiply both sides of (3.18) by

$$2\ \mathrm{Re}\{e^{i\beta}H'_\eta(0)\} = e^{i\beta}H'_\eta(0) + e^{-i\beta}H'_{\bar\eta}(0)$$

and we use (3.19) in order to eliminate γ. Since α and β are arbitrary real number, we conclude that

$$H'_\eta(0)L^*(H_\zeta) = H'_\zeta(0)L^*(H_\eta), H'_{\bar\eta}(0)L^*(H_\zeta) = H'_\zeta(0)L^*(H_{\bar\eta})$$

and therefore

$$H'_\zeta(0)\{H'_\eta(0)L^*(H_{\bar\eta}) - H'_{\bar\eta}(0)L^*H_\eta)\} = 0 \quad \text{for all } \zeta \text{ and } \eta \text{ in } \Delta.$$

The function $H'_\zeta(0) = f(\zeta)(f(\zeta)+p)/(\zeta f'(\zeta))^2 - p/f(\zeta)$ is analytic on Δ and is identically zero if and only if

$$(3.20) \quad f(z) = f_0(z) = 4pq_1(z/(4p)) = z + z^2/(2p) + \dots .$$

Observe that f_0 is bounded and belongs to the class $S_{\mathbf{R}}^{-p}$ but is not a support point of $S_{\mathbf{R}}^{-p}$ (see Proposition 3.7). Hence

$$H'_{\bar\eta}(0)L^*(H_\eta) \in \mathbb{R} \quad \text{for all } \eta \in \Delta.$$

On the other hand, $g(\eta) = L^*H_\eta)/H'_\eta(0)$ is meromorphic and real on Δ and therefore g is constant on Δ.

COROLLARY 3.10. *If f is a support point of $S_{\mathbb{R}}^{-p}$, $p > 1/4$, then $\mathbb{C}\backslash f(\Delta)$ is a finite union of analytic arcs.*

Proof. Let $f \in \sigma(S_{\mathbf{R}}^{-p})$ and let $L \in H'(\Delta)$ for which L^* is not constant on $S_{\mathbf{R}}^{-p}$ and for which $L^*(f) = \max L^*(S_{\mathbf{R}}^{-p})$. By Lemma 3.6 (ii), the function $b(w) = L^*(f(f+p)/(f-w)) + pC/w$, C defined in Theorem 3.9 (iii), is not constant on $\mathbb{C}\backslash f(\Delta)$. Furthermore, $L^*(G_\zeta)$ is real on the unit circle $\partial\Delta$. Hence, except for a finite number of points of $\partial\Delta$, the differential equation (3.16) determines a finite system of analytic arcs (see e.g. [3, 5, 7]). □

4. A completion of a result of W. Koepf. In this section we weaken the conditions which we have imposed on the mappings in $T_{\mathbf{R}}^{-p}$ and $S_{\mathbf{R}}^{-p}$. We shall no more require that $f'(0) = 1$, but rather let it be free. There is no essential importance which negative prescribed value $-p$ has to be omitted. Historically, one finds rather the normalization $f(0) = 1$ and the omitted point is $w = 0$. In concordance with this fact, we shall use the notations:

$$
\begin{aligned}
(4.1) \quad & (T_0)_{\mathbf{R}} = \{f \in H(\Delta) : f(0) = 1,\ w = 0 \in \mathbb{C}\backslash f(\Delta) \\
& \text{and } \mathrm{Im}\{f(z)\}.\mathrm{Im}\{z\} > 0 \quad \text{on } \Delta\}, \\
& (S_0)_{\mathbf{R}} = \{f \in H_{\mathbb{R}}(\Delta) : f \quad \text{univalent on } \Delta, f(0) = 1 \\
& \text{and } w = 0 \in \mathbb{C}\backslash f(\Delta)\}, \\
& (S_0)^+_{\mathbf{R}} = \{f \in (S_0)_{\mathbf{R}} : f'(0) > 0\}.
\end{aligned}
$$

Observe, that $f'(x) > 0$ for all $x \in (-1, 1)$ and all $f \in (T_0)_{\mathbf{R}}$ and that $(S_0)_{\mathbf{R}}$ is not contained in $(T_0)_{\mathbf{R}}$. The following result was shown in [11] but we give a new proof for it.

THEOREM 4.1. *The following relations hold.*
(*i*) $\mathcal{E}((T_0)_{\mathbf{R}}) = \{q_t/q_{-1} : -1 \leqq t \leqq 1\}$,
(*ii*) $(T_0)_{\mathbf{R}} = \overline{co}\{(S_0)_{\mathbf{R}}^{+}\} = \{f/q_{-1} : f \in T_{\mathbf{R}}\} = \{f_\mu = \int_{[-1,1]}[q_t/q_{-1}]d\mu(t) : \mu \in \mathbb{P}_{[-1,1]}\}$.
(*iii*) *Furthermore, we have* $f_\mu(-1) = \mu(\{-1\})$.

Proof. Let $f \in (T_0)_{\mathbf{R}}$. Then, except for $f \equiv 1$, the function $(f-1)/f'(0) \in T_{\mathbf{R}}^{-1/f'(0)}$ and therefore there is a positive measure λ on the Borel σ-algebra over $[-1, 1]$ such that for all r, $0 < r < 1$,

$$-1 \leqq f(-r) - 1 = \int_{[-1,1]} q_t(-r)d\lambda(t) = -\int_{[-1,1]} q_{-t}(r)d\lambda(t).$$

Define λ_r by $d\lambda_r(t) = q_{-t}(r)d\lambda(t)$. Then λ_r is again a positive measure on the Borel σ-algebra over $[-1, 1]$ whose total mass is $\lambda_r([-1, 1]) \leqq 1$. By the Banach-Alaoglu Theorem and by the Riesz representation Theorem for $C'([-1, 1])$ there exists a Borel measure μ_1 such that λ_r converges to μ_1 in the weak*-topology as r tends to 1. (Strictly speaking one should take a weak* convergent sequence of λ_r as r tends to 1. However, by Lemma 2.1, μ_1 is uniquely determined.) For each fixed $z \in \Delta$ we have

$$f(z) = 1 + 2\int_{[-1,1]} (1+t)q_t(z)d\lambda_r(t) + (\sqrt{r} - 1/\sqrt{r})^2 \int_{[-1,1]} q_t(z)d\lambda_r(t).$$

Letting r tend to 1, we get

$$f(z) = 1 + 2\int_{[-1,1]} (1+t)q_t(z)d\mu_1(t) = \int_{[-1,1]} [q_t/q_{-1}](z)d\mu(t),$$

where $\mu = \mu_1 + [1 - \mu_1([-1, 1])]\delta_{-1} \in \mathbb{P}_{[-1,1]}$. Thus,

$$(T_0)_{\mathbf{R}} \subset \overline{co}\{q_t/q_{-1} : -1 \leqq t \leqq 1\} \subset (T_0)_{\mathbf{R}}.$$

Hence, we have

$$(T_0)_{\mathbf{R}} = \{f/q_{-1} : f \in T_{\mathbf{R}}\} \quad \text{and } \mathcal{E}((T_0)_{\mathbf{R}}) = \{f/q_{-1} : f \in \mathcal{E}(T_{\mathbf{R}})\}$$

and (i) follows.

Statement (ii) follows from (i) and the facts that $(S_0)_{\mathbf{R}}^{+}$ is contained in $(T_0)_{\mathbf{R}}$ which is compact and convex and that $q_t/q_{-1} = 1 + 2(1+t)q_t$ is univalent on Δ for each t, $-1 < t \leqq 1$. Finally, the inequality

$$\left| f_\mu(x) - \mu(\{-1\}) - \int_{[y,1]} [q_t/q_{-1}](x)d\mu(t) \right|$$
$$= \left| \int_{(-1,y)} [q_t/q_{-1}](x)d\mu(t) \right| \leqq \mu((-1, y))$$

holds for all $x \in (-1, 0)$ and all $y \in (-1, 1)$. Letting first x tend to -1 and then y tend to -1 statement (iii) follows. □

For the class $(S_0)_{\mathbf{R}}$ W. Koepf [**6**] has obtained the following result:

THEOREM 4.2. *The following relations hold:*
(*i*) $\mathcal{E}((S_0)_{\mathbf{R}}) = \{q_t/q_{-1} : -1 < t \leqq 1\} \cup \{q_t/q_1 : -1 \leqq t < 1\}$.
(*ii*) $\sigma((S_0)_{\mathbf{R}}) \subset \{q_t/q_s : -1 \leqq s,\, t \leqq 1,\, s \neq t\}$.

We shall now complete statement (ii) of the above Theorem.

THEOREM 4.3. *We have* $\sigma((S_0)_{\mathbf{R}}) = \{q_t/q_{-1} : -1 < t \leqq 1\} \cup \{q_t/q_1 : -1 \leqq t < 1\} = \mathcal{E}((S_0)_{\mathbf{R}})$.

Proof. Let $f \in \sigma((S_0)_{\mathbf{R}})$. If $f'(0) > 0$, then, by Lemma 2.2 and Theorem 4.1 (i), we conclude that $f \in \sigma((T_0)_{\mathbf{R}}) \subset co\{q_t/q_{-1} : -1 \leqq t \leqq 1\}$. Moreover, since f is univalent, there is a $\lambda \in [0, 1)$ and an $s \in (-1, 1]$ such that $f = (1-\lambda)q_s/q_{-1} + \lambda$. Moreover, there is an $L \in H'(\Delta)$ such that $L^*(f) = \max L^*((S_0)_{\mathbf{R}})$ and L^* is not constant on $(S_0)_{\mathbf{R}}$. In particular, we have $L^*(f - q_t/q_{\pm 1}) \geqq 0$ for all $t \in [-1, 1]$ which implies that

$$(4.2) \qquad (1-\lambda)(1+s)L^*(q_s) - (\pm 1 + t)L^*(q_t) \geqq 0 \quad \text{for all } t \in [-1, 1].$$

Put $t = 1$. Then (4.2) becomes $(1-\lambda)(1+s)L^*(q_s) \geqq 0$ which is satisfied if either $\lambda = 1$ or $L^*(q_s) \geqq 0$. The first case is excluded since $f \equiv 1$ is not univalent. Next, put $t = s$. Then (4.2) reduces to $-\lambda(1+s)L^*(q_s) \geqq 0$ which holds if either $\lambda = 0$ or $L^*(q_s) \leqq 0$. Suppose that $\lambda \neq 0$. Then $L^*(q_s) = 0$ and, by (4.2), we have $(\pm 1 - t)L^*(q_t) \geqq 0$ for all $t \in [-1, 1]$. Therefore, $L^*(q_t) = 0$ for all $t \in (-1, 1)$. By Lemma 2.4, we conclude that L^* is constant on $(S_0)_{\mathbf{R}}$. Therefore, the only possible case is $\lambda = 0$, i.e. $f = q_s/q_{-1}$ for some $s \in (-1, 1]$.

Let now $f'(0) < 0$. Put $f_1(z) \equiv f(-z)$ and $L_1(f) = L(f_1)$ and apply the above proof. Therefore we have

$$\sigma((S_0)_{\mathbf{R}}) \subset \{q_t/q_{-1} : -1 < t \leqq 1\} \cup \{q_t/q_1 : -1 \leqq t < 1\}.$$

It remains to show that the converse inclusion holds. Fix $s \in (-1, 1]$ and consider the continuous linear functional

$$(4.3) \qquad L(f) = \sum_{k=1}^{n+1} b_k(n)a_k(f),$$

where $-(n-1)/(n+1) < s$ and $L(q_t) = [(n+1)s + n - t]^n$. The coefficients $b_k(n)$ exist since the polynomials $a_1(q_t), a_2(q_t), \ldots, a_{n+1}(q_t)$ form an algebraic basis for the linear space of all real polynomials of degree at most n. First, observe that $(n+1)s + n - t > 1 - t \geqq 0$ for all $t \in [-1, 1]$ and that $(1+t)[(n+1)s+n-t]^n$ has the unique global maximum at the point $t = s$ on the interval $[-1, 1]$. Since $L(q_t/q_{-1}) - L(q_t/q_q) = 4L(q_t) > 0$, we conclude that

$$L(q_t/q_1) \leqq L(q_t/q_{-1}) = 2(1+t)L(q_t) \leqq L(q_s/q_{-1}) \quad \text{for all } t \in [-1, 1].$$

Therefore, $\{q_s/q_{-1} : -1 < s \leqq 1\} \subset \sigma((S_0)_{\mathbf{R}})$.

Similarly, for fixed $s \in [-1, 1)$ and $(n-1)/(n+1) > s$, we have $-(n+1)s+n+t > 1+t \geqq 0$, $t \in [-1, 1]$. Put $L(q_t) = [-(n+1)s+n+t]^n$. Then the functional (4.3) has the property

$$\begin{aligned} L(q_t/q_{-1}) &\geqq L(q_t/q_1) = -2(1-t)\,L(q_t) \\ &\geqq L(q_s/q_1) \quad \text{for all } t \in [-1, 1]. \end{aligned}$$

Therefore, $\{q_s/q_1 : -1 \leqq s < 1\} \subset \sigma((S_0)_{\mathbf{R}})$ and Theorem 4.3 is established.

5. Extremal problems for the class $T_{\mathbf{R}}^{-p}$. In this section we solve some extremal problems for the class $T_{\mathbf{R}}^{-p}$ of all normalized ($f(0) = f'(0) - 1 = 0$) typically real functions which omit a given point $-p$ on the negative real axis. Again, $T_{\mathbf{R}}^{-p}$ is empty for $0 < p < 1/4$ and contains only the Koebe mapping $q_1(z) = z/(1-z)^2$, if $p = 1/4$. Furthermore, if $1/4 < s < t$, then $T_{\mathbf{R}}^{-p}$ is strictly included in $T_{\mathbf{R}}^{-t}$ and $T_{\mathbf{R}}^{-\infty}$ is the usual class $T_{\mathbf{R}}$ of all normalized typically real functions. The mappings

$$q_t(z) = z/(1-2tz+z^2);\ t \in [1/(2p)-1,\ 1]$$

belong to $T_{\mathbf{R}}^{-p}$ and are extreme points and support points for this class. However, there are many other support or extreme points for $T_{\mathbf{R}}^{-p}$ which are different from those for $S_{\mathbf{R}}^{-p}$. The first proposition shows that there is no similar relation to (1.4) for this class.

PROPOSITION 5.1. *For each $p > 1/4$ we have the strict inclusions*

$$\overline{co}\{q_t : 1/(2p) - 1 \leqq t \leqq 1\} \subset \overline{co}\{S_{\mathbf{R}}^{-p}\} \subset T_{\mathbf{R}}^{-p}.$$

Proof. Both inclusions are obvious. Let us show that they are strict. For any $f \in T_{\mathbf{R}}$ we have the unique Robertson representation

$$f(z) = \int_{[0,\pi]} q_{\cos(t)}(z)\,d\mu(t), \tag{5.1}$$

where $\mu = \mu_f \in \mathbb{P}_{[0,\pi]}$. Each μ_f is the weak* limit of the sequence $\mu_n \in \mathbb{P}_{[0,\pi]}$ defined by

$$d\mu_n = (2/\pi)\ \mathrm{Im}\{f((1-1/n)e^{it})\}\ \sin(t)dt,\ 0 \leqq t \leqq \pi.$$

(a) The mappings $f_r(z) = q_1(rz)/r$ belong to $S_{\mathbf{R}}^{-p}$ for all r close to 1, $0 < r < 1$. The unique measure μ of the representation (5.1) for f_r is

$$d\mu_r = (2/\pi).\ \mathrm{Im}\{q_1(re^{it})/r\}\ \sin(t)dt.$$

Since $\mu_r(\cos^{-1}[-1, 1/(2p)-1)) > 0$, we conclude that f_r does not belong to $\overline{co}\{q_t : 1/(2p)-1 \leqq t \leqq 1\}$. Also the examples given in Theorem 3.5 show that the first inclusion is strict.

(b) Let $f = (1-\lambda)q_s + \lambda q_t$, $0 < \lambda < 1$, $-1 < s < 1/(2p) - 1 < t < 1$, such that $f(-1) = -p$. Then $f \in T_{\mathbf{R}}^{-p} \backslash S$. In particular, f is not an extreme point of the closed convex hull of $S_{\mathbf{R}}^{-p}$ (Krein Milman). Suppose now that f is a convex combination of two other functions f_1 and f_2 in $T_{\mathbf{R}}^{-p}$. Then the support of the representing measures (5.1) of both functions consists of at most two points. Since $f_1(-1) = f_2(-1) = -p$ in the sense of (1.3), we conclude that $f_1 = f_s = f$ and therefore $f \in \mathcal{E}(T_{\mathbf{R}}^{-p})$. □

In the next theorem we determine the set of all extreme points for the class $T_{\mathbf{R}}^{-p}$.

THEOREM 5.2. *Let $p > 1/4$. Then the following relations hold:*

(*i*) $\mathcal{E}(T_{\mathbf{R}}^{-p}) = \{q_s q_t / q_{2p(1+s)(1+t)-1} : -1 \leqq s \leqq 1/(2p) - 1 \leqq t \leqq 1\}$.
and hence

(*ii*) $T_{\mathbf{R}}^{-p} = \{f_\mu = \int_E [q_s q_t / q_{2p(1+s)(1+t)-1}] \cdot d\mu(s, t) : \mu \in \mathbb{P}_E\}$,
where $E = \{(s, t) \in \mathbb{R}^2 : -1 \leqq s \leqq 1/(2p) - 1 \leqq t \leqq 1\}$.

Remark. Observe that for every $(s, t) \in E$ we have

$$q_s q_t / q_{2p(1+s)(1+t)-1} = (1-\lambda)q_s + \lambda q_t,$$

where $\lambda = [1-2p(1+s)](1+t)/(t-s) \in [0, 1]$, if $s < t$, and $\lambda = 0$ or 1 if $s = t = 1/(2p) - 1$. Furthermore, if $-1 < s \leqq t \leqq 1$, then $q_s q_t / q_{2p(1+s)(1+t)-1}(-1) = -p$.

Proof. We have $T_{\mathbf{R}}^{-p} = \{f : f/p + 1 \in (T_0)_{\mathbf{R}}$ and $f'(0) = 1\}$. Using Theorem 4.1 (ii) and the relation $q_t/q_{-1} = 1 + 2(1+t)q_t$ we get

$$T_{\mathbf{R}}^{-p} = \{f_\mu = 2p \int_{[-1,1]} (1+t) q_t d\mu(t) : \mu \in \mathbb{P}_{[-1,1]} \text{ and } 2p \int_{[-1,1]} (1+t) d\mu(t) = 1\}.$$

Next, we want to apply Lemma 2.3. Since the correspondence $\mu \longrightarrow f_\mu$ is an affine homeomorphism, we get

$$\mathcal{E}(T_{\mathbf{R}}^{-p}) = \left\{ f_\mu : \mu \in \mathcal{E} \left\{ \nu \in \mathbb{P}_{[-1,1]} \text{ such that } 2p \int_{[-1,1]} (1+t) d\nu(t) = 1 \right\} \right\}.$$

Putting $\Phi(\nu) = 2p \int_{[-1,1]} (1+t) d\nu(t)$ we get from (2.5) that f is an extreme point of $T_{\mathbf{R}}^{-p}$ if and only if $f = 2p(1-\lambda)(1+s)q_s + 2p\lambda(1+t)q_t$ under the condition that $f'(0) = 1 = 2p(1-\lambda)(1+s) + 2p\lambda(1+t)$. Without loss of generality we may assume that $s \leqq t$. The last condition shows that $-1 \leqq s \leqq 1/(2p) - 1 \leqq t \leqq 1$ and the result follows. □

Let $L \in H'(\Delta)$ and let L^* be as defined in (2.6). Then any linear optimisation problem over the class $T_{\mathbf{R}}^{-p}$, $M = \max L^*(T_{\mathbf{R}}^{-p})$ ($\min L^*(T_{\mathbf{R}}^{-p})$ resp.), can be reduced by means of Theorem 5.2 to a classical optimization problem involving a real-valued differentiable function of two real variables, i.e.

$$M = \max(\min)\{F(s, t) : -1 \leqq s \leqq 1/(2p) - 1 \leqq t \leqq 1\},$$

where

(5.2) $$F(s, t) = L^*(q_s q_t / q_{2p(1+s)(1+t)-1}).$$

In all the calculations it is convenient to use the following relations:

(5.3) $$\begin{aligned} &(q_t - q_s)/(t - s) = 2q_s q_t, \quad q_t/q_s = 1 + 2(t - s)q_t, \\ &\partial^n(q_t)/\partial t^n = 2^m n! q_t^{n+1}, \quad q_t' = (z^{-2} - 1)q_t^2, \\ &((1+t)q_t - (1+s)q_S)/(t - s) = q_s q_t / q_{-1}, \\ &(t - s)F(s, t) = [1 - 2p(1+s)](1+t)L^* q_t) \\ &\qquad\qquad - [1 - 2p(1+t)](1+s)L^*(q_s). \end{aligned}$$

The classical necessary and sufficient conditions for a local maximum (local.minimum resp.) of F are summarized in the Lemma below.

LEMMA 5.3. *Let $J(t) = L^*(q_t)$. Then the following statements hold.*

(i) If (s, t) is a critical point of F in $\{(s, t) : -1 < s < 1/(2p) - 1 < t < 1\}$, then it is a solution of the two equations

(5.4) $$(J(t) - J(s))/(t - s) = J'(t)(1+t)/(1+s) = J'(s)(1+s)/(1+t).$$

(ii) Let (s, t), $-1 < s < 1/(2p) - 1 < t < 1$, be a critical point of F. Then (s, t) is a local maximum (local minimum resp.) of F if

(5.5) $$(1+s)J''(s) + 2J'(s) < 0 \quad \text{and } (1+t)J''(t) + 2J'(t) < 0$$

(5.6) $$((1+s)J''(s) + 2J'(s) > 0 \quad \text{and } (1+t)J''(t) + 2J'(t) > 0 \text{ resp.}).$$

(iii) If s is a critical point of $G(s) = F(s, 1), -1 < s < 1/(2p) - 1$, then it is a solution of the equation:

(5.7) $$(J(1) - J(s))/(1 - s) = J'(s)(1+s)/2.$$

If, in addition, $(1+s)J''(s) + 2J'(s) < 0$ (> 0 resp.), then s is a local maximum (local minimum resp.*) of G.*

In Proposition 5.1 we have seen that the convex hull of $S_{\mathbf{R}}^{-p}$ is contained in $T_{\mathbf{R}}^{-p}$ but there are functions in $T_{\mathbf{R}}^{-p}$ which do not belong to the convex closure of $S_{\mathbf{R}}^{-p}$. The next Proposition is an easy application of Lemma 5.3 and could also be shown by the same way as Proposition 3.1 (iii).

PROPOSITION 5.4. *If $f \in T_{\mathbf{R}}^{-p}$, $p \geqq 1/4$, then we have for all x, $-1 < x < 1$, the sharp inequalities:*

$$q_{1/(2p)-1}(x) \leqq f(x) \leqq q_1(x).$$

Proof. The upper bound holds for all functions in $T_{\mathbf{R}}$. Fix $x \in (-1,0) \cup (0,1)$ and put $L^*(f) = f(x)$. Condition (5.4) becomes

$$\begin{aligned}(q_t(x) - q_s(x))/(t-s) &= 2q_s(x)q_t(x) = 2q_t^2(x)(1+t)/(1+s)\\ &= 2q_s^2(x)(1+s)/(1+t) > 0\end{aligned}$$

which implies that $q_t(x)(1+t) = q_s(x)(1+s)$. But there is no point $(s,\, t)$, $-1 < s < 1/(2.p) - 1 < t < 1$ such that

$$((1+t)q_t(x) - (1+s)q_s(x))/(t-s) = q_s(x)q_t(x)/q_{-1}(x) = 0.$$

In other words, there is no critical point in $-1 < s < 1/(2p) - 1 < t < 1$. On the boundary part $\{(s,\, 1) : -1 \leqq s \leqq 1/(2p) - 1\}$ the function $G(s) = F(s,\, 1)$ is a homography of the variable s. Therefore, G has no critical point on $\{(s,\, 1) : -1 < s < 1/(2p) - 1\}$. The same fact holds for the function $H(t) = F(-1,\, t) = J(t)$ on the boundary part $\{(-1,\, t) : 1/(2p) - 1 \leqq t \leqq 1\}$. Observe furthermore that $F(s,\, 1/(2p) - 1) \equiv F(1/(2p) - 1,\, t) \equiv q_{1/(2p)-1}(x)$. Therefore, the extremal functions are q_1 and $q_{1/(2p)-1}$. The first gives the maximum value and the second the minimum value of $f(x)$. □

THEOREM 5.5. *Fix $p > 1/4$ and $z \in \Delta$, $\mathrm{Im}\{z\} \neq 0$. Then the set*

$$E = \{w = f(z) : f \in T_{\mathbf{R}}^{-p}\} \tag{5.8}$$

is the closed circular lens which is bounded by the two arcs

$$\begin{aligned}&\gamma_1 = \{q_t(z) : 1/(2p) - 1 \leqq t \leqq 1\} \quad \text{and}\\ &\gamma_2 = \{q_1(z)q_s(z)/q_{4p(1+s)-1}(z) : -1 \leqq s \leqq 1/(2p) - 1\}.\end{aligned}$$

Furthermore, γ_2 is tangent to the straightline segment from $q_{-1}(z)$ to $q_1(z)$ at the point $q_1(z)$.

Proof. Consider the functional $L(f) = e^{i\alpha}f(z)$ where α is a real number. The equations (5.4) lead to the equalities

$$\mathrm{Re}\{e^{i\alpha}q_s(z)q_t^2(z)/q_{-1}(z)\} = \mathrm{Re}\{e^{i\alpha}q_t(z)q_s^2(z)\, q_{-1}(z)\} = 0.$$

Hence, $q_s(z)/q_t(z) \in \mathbb{R}$. Since $q_s(z)$ and $q_t(z)$ lie on a circle passing through the origin, this situation is impossible for $t \neq s$. Therefore, there is no critical point of F in $-1 < s < 1/(2p) - 1 < t < 1$. On the other hand, through each boundary point of the compact convex set E defined by (5.8) passes a straight line which supports E. Thus, by theorem 5.2 (i),

$$\begin{aligned} E &= co\{[q_s q_t / q_{2p(1+s)(1+t)-1}](z) : (s, t) \in \partial([-1, 1/(2p) - 1] \\ &\quad \times [1/(2p) - 1, 1])\} \\ &= co\{\gamma_1 \cup \gamma_2\}. \quad \text{Finally, } \gamma_1 \cup \gamma_2 = \partial E. \end{aligned}$$ □

The next Theorem gives estimates for the derivative of f at a given point in $(-1, 1)$.

THEOREM 5.6. *Let $p > 1/4$ and let x_o be the unique solution in $(-1, 0)$ of $x_o + 1/x_o = -1 - (1 + 2/p)^{1/2}$. Put $s = [(x+1/x)^2 + 2(x+1/x) - 4]/4$. If $f \in T_{\mathbf{R}}^{-p}$, then*

(*i*) $q'_{1/(2p)-1}(x) \leqq f'(x) \leqq q'_1(x)$, *if* $0 \leqq x < 1$.
(*ii*) $q'_1(x) \leqq f'(x) \leqq q'_{1/(2p)-1}(x)$, *if* $x_o \leqq x \leqq 0$.
(*iii*) $q'_1(x) \leqq f'(x) \leqq [q_1 q_s / q_{4p(1+s)-1}]'(x)$, *if* $-1 < x \leqq x_o$.

Proof. First observe that $xf'(x) \leqq xq'_1(x)$ for all $x \in (-1, 1)$. Let $L(f) = f'(x)$, $-1 < x < 1$, $x \neq 0$, and put $J(t) = q'_t(x) = (x^{-2} - 1)q_t^2(x)$. We now show that $F(s, t)$ has no critical points in $-1 < s < 1/(2p) - 1 < t < 1$. Indeed, Condition (5.4) becomes

$$\begin{aligned} (q_t^2(x) - q_s^2(x))/(t - s) &= 2q_s(x)q_t(x)[q_s(x) + q_t(x)] \\ &= 4q_t^3(x)(1+t)/(1+s) = 4q_s^3(x)(1+s)/(1+t) \end{aligned}$$

which implies that $[q_s(x)q_t(x)(q_s(x)+q_t(x))]^2 = 4q_s^3(x)q_t^3(x)$ or $[q_s(x)q_t(x)(q_s(x) - q_t(x))]^2 = 0$ which leads to a contradiction for $s \neq t$. Therefore there are no critical points in $-1 < s < 1/(2p) - 1 < t < 1$. Consider now the function $H(t) = F(-1, t) = J(t)$ on the boundary part $\{(-1, t) : 1/(2p) - 1 \leqq t \leqq 1\}$. Since $xJ'(t) > 0$ for all $x \in (-1, 1)\backslash\{0\}$, we have

$$q'_{1/(2p)-1}(x) = J(1/(2p) - 1) \leqq J(t) \leqq J(1) = q'_1(x), \quad \text{if } 0 < x < 1$$

and

$$q'_{1/(2p)-1}(x) = J(1/(2p) - 1) \geqq J(t) \geqq J(1) = q'_1(x), \quad \text{if } -1 < x < 0.$$

Consider now the function $G(s) = F(s, 1)$ on the boundary part $\{(s, 1) : -1 \leqq s \leqq 1/(2p) - 1\}$. Then the condition (5.4) becomes

$$(5.9) \qquad q_1(x)[q_1(x) + q_s(x)] = (1+s)q_s^2(x), \ -1 < s < 1/(2p) - 1.$$

Substituting $Q = q_1(x)/q_s(x)$ in (5.9) we get $Q(Q+1) = 1 + s$. Since $Q > 0$, we obtain $Q = [-1 + (1 + 4(1+s))^{1/2}]/2$. Put $w = (x + 1/x)/2$. Then $|w| \geqq 1$ and

$$5 + 4s = (2Q+1)^2 = [2(w-s)/(w-1) + 1]^2,$$

and therefore

$$(s-1)(s - w^2 - w + 1) = 0.$$

Since $-1 < s < 1/(2p) - 1$, the only possible solution is $s^* = w^2 + w - 1$ which implies that $-[1 + (1 + 2/p)^{1/2}]/2 < w < -1$ or $-1 < x < x_o$. Therefore, if $x_o \leqq x < 1$, the only extremal functions are $q_{1/(2p)-1}$ and q_1 and the statements (i) and (ii) are proved.

It remains the case (iii). For all $x \in (-1, 0)$, we have

$$(1+s^*)J''(s^*) + 2J'(s^*) = 8(x^{-2} - 1)q_{s^*}^3(x)[3(1+s^*)q_{s^*}(x) + 1] < 0.$$

Therefore, the function $G(s)$ has a local maximum at s^* and we have the inequalities

$$q_1'(x) \leqq f'(x) \leqq \max\{q'_{1/(2p)-1}(x),\ [q_1 q_{s^*}/q_{4p(1+s^*)-1}]'(x)\},$$
$$-1 < x \leqq x_o.$$

It remains to show that for all x, $-1 < x \leqq x_o$,

$$(5.10)\qquad [q_1 q_{s^*}/q_{4p(1+s^*)-1}]'(x) \geqq q'_{1/(2p)-1}(x).$$

For convenience put $u = -w$ and $m = 1/(2p) - 1$. Then we have $s^* = u^2 - u - 1$ and $1 < u < [1 + (1 + 2/p)^{1/2}]/2$. First, observe that

$$(5.11)\qquad q_1 q_{s^*}/q_{4p(1+s^*)-1} = (1-\lambda)q_{s^*} + \lambda q_1$$

where $\lambda = [2 - 4p(1+s^*)]/(1-s^*) = [2 - 4pu(u-1)]/[(1+u)(2-u)] \in (0, 1)$. Using the fact that $u + s^* = u^2 - 1$, we are lead to show that

$$(5.10')\qquad [(1-\lambda) + \lambda(u-1)^2]/(u^2-1)^2 - 1/(u+m)^2 \geqq 0.$$

But

$$[(1-\lambda) + \lambda(u-1)^2]/(u^2-1)^2 = [1 - \lambda u(2-u)]/(u^2-1)^2$$
$$= (4pu^2 - 1)/[(u^2-1)(u+1)^2].$$

From the identity

$$(4pu^2 - 1)(u+m)^2 - (u^2-1)(u+1)^2 = (4p-1)(u^2 - u - 1 - m)^2$$

we conclude that (5.10) holds. Equality holds if and only if $x = x_o$. □

The following Lemma will be useful for our next result.

LEMMA 5.7. *For all positive integers and all* $x \in (o, \pi)$ *we have*

$$\sin(nx)/[\text{nsin}(x)] < [2+\cos(nx)]/[2+\cos(x)]. \tag{5.12}$$

Proof. Put $u(x) = \sin(x)/[x(2+\cos(x))]$ and $v(x) = xu(x)$. Since

$$[u'(x)x^2(2+\cos(x))^2]' = 2.\sin(x)[\sin(x)-x] < 0$$

we conclude that $u'(x) < 0$ for all $x \in (0, \pi)$ and therefore $u(x)$ is strictly decreasing on $[0, \pi]$. Moreover, $v(x)$ is strictly increasing on $[0, 2\pi/3]$ and decreasing on $[2\pi/3, \pi]$. Therefore, for $0 < x \leqq \pi/n$, we have:

$$\begin{aligned} \sin(nx)/[2+\cos(nx)] &= v(nx) = nxu(nx) < nxu(x) = nv(x) \\ &= n\sin(x)/[2+\cos(x)]. \end{aligned} \tag{5.13}$$

Next we show that (5.12) holds for all $x \in [\pi(n-1)/n, \pi)$. If n is even and $\pi(n-1)/n \leqq x < \pi$, then

$$\sin(nx)/[\text{nsin}(x)] = -\sin(n(\pi-x))/[\text{nsin}(\pi-x)] \leqq 0.$$

For odd n and $\pi(n-1)/n \leqq x < \pi$ we have $\cos(x) < \cos(nx)$ and, according to (5.13), we get

$$\begin{aligned} \sin(nx)/[\text{nsin}(x)] &= \sin(n(\pi-x))/[\text{nsin}(\pi-x)] \\ &< [2+\cos(n(\pi-x))]/[2+\cos(\pi-x)] \\ &= [2-\cos(nx)]/[2-\cos(x)] \\ &< [2+\cos(nx)]/[2+\cos(x))]. \end{aligned}$$

Observe also that it is sufficient to show (5.12) for the subset of $x \in (0, \pi)$ for which $\sin(nx) \geqq 0$, i.e. if the integer part of nx/π is even. Let now $\pi/n < x < (n-1)\pi/n$ be fixed and let the integer part of nx/π be equal to 2k where k is an integer in $(0, (n-1)/2)$. Then we have $0 \leqq x - 2k\pi/n < \pi/n$ and, by (5.13), we conclude that

$$v(nx) = v(n(x-2\pi k/n)) \leqq nv(x-2\pi k/n) < nv(\pi/n).$$

If $x \in (\pi/n, 2\pi/3]$, then, by the monotonicity of v we have $v(nx) < nv(\pi/n) < nv(x)$. Similarly, if $x \in (2\pi/3, \pi - \pi/n)$, then

$$nv(x) > nv(\pi-\pi/n) \geqq nv(\pi/n) > v(nx).$$

This completes the proof. □

In what follows, we are interested in sharp estimates of some coefficients of functions in the class $T_{\mathbf{R}}^{-p}$. Using the same proof as for Proposition 3.1 (i) and (ii) we have

PROPOSITION 5.8. *If $f \in T_{\mathbf{R}}^{-p}$, $p \geqq 1/4$, then the following sharp estimates hold:*

(i) $2 \geqq a_2(f) \geqq -2 + 1/p$

(ii) $3 \geqq a_e(f) \geqq \begin{cases} (1 - 1/p)(3 - 1/p), & \text{if } 1/4 \leqq p \leqq 1/2 \\ -1, & \text{if } p \geqq 1/2 \end{cases}$. *The extremal functions are $q_{1/(2p)+1}$ or q_0 for the minimum and q_1 for the maximum.*

Evidently, $a_n(f) \leqq n$ for all $n, \in \mathbb{N}$, since $q_1 \in T_{\mathbf{R}}^{-p}$ for all $p \geqq 1/4$. The situation is quite different for the minimum of $a_4(f)$. We shall use the same method as we have applied for the previous Theorems. Put $J(t) = a_n(q_t)$ and

$$(5.14) \qquad F(s, t) = a_n(q_s q_t / q_{2p(1+s)(1+t)-1}) = (1 - \lambda)a_n(q_s) + \lambda a_n(q_t),$$

where $\lambda = [1-2p(1+s)](1+t)/(t-s) \in [0, 1]$ and $-1 \leqq s \leqq 1/(2p)-1 \leqq t \leqq 1$.

LEMMA 5.9. *Let $p > 1/4$ and put $A_n(p) = \min\{a_n(q_t) : (2p) - 1 \leqq t \leqq 1\}$. Denote by B the set of all critical points of* (5. 14) *in the open rectangle $\{(s, t) : -1 < s < 1/(2p) - 1 < t < 1\}$. Then we have:*

$$\min\{a_n(T_{\mathbf{R}}^{-p})\} = \min\{A_n(p), \min\{F(s, t) : (s, t) \in B\}\}.$$

Proof. First, observe that $F(-1, t) = a_n(q_t)$, $F(s, 1/(2p) - 1) = F(1/(2p) - 1, t) = a_n(q_{1/(2p)-1})$. Put $s = \cos(x)$. Then, by Lemma 5.7, we conclude that

$$[\partial F/\partial s](s, 1) = (4p - 1)n(2 + \cos(x)) \times \left\{ \frac{\sin(nx)}{n\sin(x)} - \frac{2 + \cos(nx)}{2 + \cos(x)} \right\} < 0$$

for all $x \in (0, \pi)$. □

In contrast to the cases of $\min\{a_2(T_{\mathbf{R}}^{-p})\}$ and $\min\{a_3(T_{\mathbf{R}}^{-p})\}$ we get for the problem $\min\{a_4(T_{\mathbf{R}}^{-p})\}$ extremal functions which are not univalent for some values of p.

THEOREM 5.10. *If $f \in T_{\mathbf{R}}^{-p}$, $p > 1/4$, then we have the sharp estimate*

$$a_4(f) \geqq \begin{cases} 4m(2m^2 - 1), & \text{if } 1/4 < p \leqq 3 - \sqrt{7} \text{ or } p \geqq 3 + \sqrt{7} \\ -1 - p/4, & \text{if } 3 - \sqrt{7} \leqq p \leqq 3 + \sqrt{7} \end{cases}$$

where $m = 1/(2p) - 1$. The extremal function is q_m for the upper case and $q_{s^*} q_{t^*} / q_{2p(1+s^*)(1+t^*)-1}$ for the lower case where $s^* = -(1 + \sqrt{7})/4$ and $t^* = (\sqrt{7} - 1)/4$.

Proof. Put $J(t) = a_4(q_t) = 4t(2t^2 - 1)$ and $Y(s, t) = (1 + s)(1 + t)(J(t) - J(s))/(t - s)$. Then $Y(t, t) = (1 + t)^2 J'(t)$ and (5.4) can be written in the form

$$(5.4') \qquad Y(s, t) = Y(t, t) = Y(s, s), \; -1 < s < 1/(2p) - 1 < t < 1.$$

or

$$(5.4'') \qquad [Y(s, t) - Y(t, t)]/[(t - s)(1 + t)] = [Y(s, t) - Y(s, s)]/[(t - s)(1 + s)] = 0, \; -1 < s < 1/(2p) - 1 < t < 1.$$

But $Y(s, t) = 4(1 + s)(1 + t)[2(t^2 + ts + s^2) - 1]$ and, by (5.4''), the critical points in $\{(s, t) : -1 < s < 1/(2p) - 1 < t < 1\}$ have to satisfy the equations

$$2(1 + s)(t - s) - 6t(1 + s + t) + 1 = 0,$$
$$2(1 + t)(t - s) + 6s(1 + s + t) - 1 = 0.$$

The only critical point in $\{(s, t) : -1 < s < 1/(2p) - 1 < t < 1\}$ is

$$s^* = -(1 + \sqrt{7})/4 \quad \text{and } t^* = (\sqrt{7} - 1)/4$$

which is, by (5.6), a local minimum provided that $3 - \sqrt{7} < p < 3 + \sqrt{7}$. The correspondent value for a_4 is $F(s^*, t^*) = -(p + 4)/4 = -1 - 1/[8(m + 1)]$. Let $A_4(p)$ be as in Lemma 5.9 and put $m = 1/(2p) - 1$. Then we get

$$A_4(p) = \begin{cases} 4m(2m^2 - 1), & \text{if } 1/4 < p \leqq (6 - \sqrt{6})/10 \\ & \text{or } p \geqq (3 + \sqrt{6})/2 \\ -4\sqrt{6}/9, & \text{if } (6 - \sqrt{6})/10 \leqq p \leqq (3 + \sqrt{6})/2 \end{cases}$$

Next, observe that

$$4m(2m^2 - 1) + 1 + 1/[8(m + 1)] = (8m^2 + 4m - 3)^2/[8(m + 1)] \geqq 0.$$

Furthermore, we have $-1 - p/4 < -4\sqrt{6}/9$, whenever $p \geqq 3 - \sqrt{7}$, and the interval $[(6 - \sqrt{6})/10, (3 + \sqrt{6})/2]$ is contained in the interval $[3 - \sqrt{7}, 3 + \sqrt{7}]$. By Lemma 5.9, we conclude that for the case $3 - \sqrt{7} < p < 3 + \sqrt{7}$ the function F attains its global minimum at the point (s^*, t^*). For the remaining values of p the extremal function is q_m. □

It is a natural question to ask under what conditions the extremal functions are univalent. The following Lemma gives a partial answer to it.

LEMMA 5.11. *Let $L \in H'(\Delta)$ and $J(t) = L^*(q_t)$, $-1 \leqq t \leqq 1$. Suppose that there is a $t^* \in [-1, 1)$ such that J is convex and increasing on $[t^*, q]$ and J attains the global minimum at t^*. Then*

$$\min L^*(T_{\mathbf{R}}^{-p}) = \min L^*(S_{\mathbf{R}}^{-p}) = \min\{L^*(q_t) : 1/(2p) - 1 \leqq t \leqq q\}$$

for all $p > 1/4$.

Proof. Let $m = 1/(2p) - 1$ be fixed. If $-1 < m \leqq t^*$, then $q_{t^*} \in S_{\mathbf{R}}^{-p}$ and $J(t^*) = \min L^*(T_{\mathbf{R}}) \leqq \min L^*(T_{\mathbf{R}}^{-p}) \leqq \min L^*(S_{\mathbf{R}}^{-p}) \leqq J(t^*)$. Hence, the result follows for this case. It remains to verify the case $-1 \leqq t^* \leqq m < 1$. Consider the linear functional $K(f) = a_2(f)/2 + i.L^*(f), f \in H(\Delta)$. Since $K(q_t) = t + iJ(t)$, we conclude from Proposition 5.8 that $K(T_{\mathbf{R}}^{-p})$ lies in the strip $\{w : m \leqq \mathrm{Re}\{w\} \leqq 1\}$. Furthermore, Theorem 5.2 and the above assumptions on t^* imply that $K(T_{\mathbf{R}}^{-p})$ is contained in the set $\{w : m \leqq \mathrm{Re}\{w\} \leqq 1$ and $\mathrm{Im}\{w\} \geqq J(\mathrm{Re}\{w\})\}$. Therefore, we get

$$\begin{aligned} J(m) &\leqq \min\{\mathrm{Im}\{w\} : w \in K(T_{\mathbf{R}}^{-p})\} = \min L^*(T_{\mathbf{R}}^{-p}) \\ &\leqq \min L^*(S_{\mathbf{R}}^{-p}) \leqq L^*(q_m) = J(m). \end{aligned}$$ □

The next result is an application of the above Lemma.

THEOREM 5.12. *For all odd integers $n \geqq 3$ and all $p > 1/4$, we have* $\min a_n(T_{\mathbf{R}}^{-p}) = \min a_n(S_{\mathbf{R}}^{-p}) = \min\{\sin(nx(/\sin(x) : 1/(2p)- \leqq \cos(x) \leqq 1\}$.

Proof. Put $t = \cos(x)$, $t_k = \cos(x_k)$ and $x_k = k\pi/n$, $k = 1, 2, \ldots, n-1$. It is sufficient to check that the polynomial

$$J(t) = a_n(q_t) = 2^{n-1} \prod_{k=1}^{n-1} (t - t_k) = \omega(x) = \sin(nx)/\sin(x)$$

satisfies Lemma 5.11 for a suitable t^*. For $n = 3$, $J(t)$ is a convex parabole. If $n = 5$, then $J(t) = 16t^4 - 12t^2 + 1$ satisfies Lemma 5.11 with $t^* = \sqrt{6}/4$. Let now $n \geqq 7$. Then $J(-t) = J(t)$ and $J'(t)$ has exactly $n - 2$ distinct zeros $s_k \in (t_{k+1}, t_k)$, $k = 1, 2, \ldots, n-2$, on the interval $(-1, 1)$. Moreover, $J''(t)$ has exactly $n-3$ distinct zeros $r_k \in (s_{k+1}, s_k)$, $k = 1, 2, \ldots, n-3$, on $(-1, 1)$. Thus we conclude $J > 0$ on $(t_1, 1]$, $J' > 0$ on $(s_1, 1]$ and $J'' > 0$ on $(r_1, 1]$. Put $t^* = s_1$. Then, J is convex and increasing on $(t^*, 1)$. It remains to show that the global minimum of J is attained at t^*. Observe that the local minima of J are at the points $s_{2k-1} = \cos(x_{2k-1}^*)$, $x_{2k-1} < x_{2k-1}^* < x_{2k}$. By the symmetry it is sufficient to check the interval $0 < x < \pi/2$. Put $\xi_{2k-1} = x_{2k-1}^* - 2(k-1)\pi/n \in (x_1, x_2)$. Then we get

$$\omega(x_{2k-1}^*) = \sin(n\xi_{2k-1})/\sin(x_{2k-1}^*) \geqq \sin(n\xi_{2k-1})/\sin(\xi_{2k-1}) \geqq \omega(x_1^*)$$

and Theorem 5.12 is shown. □

The problem of sharp lower bounds for even coefficients of functions in $S_{\mathbf{R}}^{-p}$ is still open. However, for p large enough (depending on n), there is a q_{t^*} which minimizes $a_n(f)$.

THEOREM 5.13. *For every $L \in H'(\Delta)$ there is a constant p_L such that for all $p > p_L$ we have* $\min L^*(T_{\mathbf{R}}^{-p}) = \min L^*(S_{\mathbf{R}}^{-p}) = \min\{L^*(q_t) : 1/(2p)-1 \leqq t \leqq 1\}$.

Proof. Let $J(t) = L^*(q_t), -1 \leqq t \leqq 1$, and suppose that J attains its global minimum at a point $t^* \in (-1, 1]$. Then Theorem 5.13 holds for $p_L = 1/(2+2t^*)$. Assume therefore that $t^* = -1$ is the only global minimum of J. We shall proceed in two steps.

Step 1. Denote by B the set of all critical points of the function $F(s, t) = L^*(q_s q_t / q_{2p(1+s)(1+t)-1})$ on the domain $\{(s, t) : -1 < s < t < 1\}$. Suppose first that B is nonempty. Then J'' is not identical zero on $[-1, 1]$. We want to show that $s_o = \inf\{s : (s, t) \in B\} > -1$. Assume that the contrary holds. Then there is sequence $(s_n, t_n) \in B$ such that $\lim_{n\to\infty} s_n = -1$ and $\lim_{n\to\infty} t_n = \tau \in [-1, 1]$. The case $\tau \neq -1$ is excluded. Indeed, if $\tau \neq -1$, then (5.4) implies that

$$[J(\tau) - J(-1)]/(\tau + 1) = \lim_{n\to\infty} J'(s_n)(1 + s_n)/(1 + t_n) = 0$$

which contradicts the assumption $t^* = -1$ is the unique global minimum of J. Since J is analytic on $[-1, 1]$ and J'' does not vanish identically there, there is a $\delta > 0$ such that $J'(t)J''(t) \neq 0$ for all $t \in (-1, -1 + \delta)$. From the fact that $0 < [J(t) - J(-1)]/(t + 1) = J'(\theta)$ for all $t \in (-1, -1 + \delta)$ and some $\theta \in (-1, t)$ we conclude that $J' > 0$ on $(-1, -1 + \delta)$. Moreover, if n is sufficiently large, then, by (5.4), we get $-1 < s_n < t_n < -1 + \delta$ and $0 < J'(t_n) < J'(t_n)(1+t_n)/(1+s_n) = J'(s_n)(1+s_n))/(1+t_n) < J's_n)$. In other words we have $J''(t) < 0$ for all $t \in (-1, -1 + \delta)$ and we conclude that $J'(-1) > 0$.

Next we use again (5.4) and (5.3) and we get for points $(s, t) \in B$

$$2L^*(q_s q_t) = L^*(q_t - q_s)/(t - s) = 2L^*(q_t^2)(1 + t)/(1 + s)$$

and hence

$$0 = (1 + s)L^*(q_s . q_t) - (1 + t)L^*(q_t^2) = (s - t)L^*(q_s q_t^2 / q_{-1}).$$

In particular, $J'(-1) = 2L^*(q_{-1}^2) = 2 \lim_{n\to\infty} L^*(q_{s_n} q_{t_n}^2 / q_{-1}) = 0$ which leads to a contradiction. Therefore, if B is nonempty, $s_o > -1$. Put $p_1 = 1/4$, if B is empty and $p_1 = 1/(2 + 2s_o)$, if B is otherwise.

Step 2. Let $G(s) = F(s, 1), -1 < s < 1$ and consider the condition (5.7). First, we claim that there are only finitely many solutions of (5.7). Indeed, if not, then (5.7) holds for all $s \in [-1, 1]$, since J is analytic on $[-1, 1]$. But the only analytic solution for (5.7) on $[-1, 1]$ is the constant function. Therefore, there is an interval $(-1, -1 + \rho)$, $\rho > 0$, which contains no critical points of G. Put $p_2 = 1/(2\rho)$.

Finally, put $p_L = \max\{p_1, p_2\}$. By Lemma 5.3, Theorem 5.13 follows.

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