THE BULLETIN OF SYMBOLIC LOGIC
Volume 21, Number 1, March 2015

EXISTENTIAL-IMPORT MATHEMATICS

JOHN CORCORAN AND HASSAN MASOUD

Hence, there can never be surprises in logic.
—Wittgenstein [22, 6.1251].

Abstract. First-order logic has *limited* existential import: the universalized conditional $\forall x\ [S(x) \rightarrow P(x)]$ implies its corresponding existentialized conjunction $\exists x\ [S(x)\ \&\ P(x)]$ in *some but not all* cases. We prove the *Existential-Import Equivalence*:

$\forall x\ [S(x) \rightarrow P(x)]$ implies $\exists x\ [S(x)\ \&\ P(x)]$ iff $\exists x\ S(x)$ is logically true.

The antecedent $S(x)$ of the universalized conditional alone determines whether the universalized conditional *has existential import*: implies its corresponding existentialized conjunction.

A *predicate* is a formula having only x free. An *existential-import* predicate $Q(x)$ is one whose existentialization, $\exists x\ Q(x)$, is logically true; otherwise, $Q(x)$ is *existential-import-free* or simply *import-free*. Existential-import predicates are also said to be *import-carrying*.

How widespread is existential import? How widespread are import-carrying predicates in themselves or in comparison to import-free predicates? To answer, let L be any first-order language with any interpretation INT in any [sc. nonempty] universe U. A subset S of U is *definable in* L *under* INT iff for some predicate $Q(x)$ in L, S is the truth-set of $Q(x)$ under INT. S is *import-carrying definable* iff S is the truth-set of an import-carrying predicate. S is *import-free definable* iff S is the truth-set of an import-free predicate.

Existential-Importance Theorem: Let L, INT, and U be arbitrary. Every nonempty definable subset of U is *both* import-carrying definable *and* import-free definable.

Import-carrying predicates are quite abundant, and no less so than import-free predicates. Existential-import implications hold as widely as they fail.

A particular conclusion cannot be validly drawn from a universal premise, or from any number of universal premises.—Lewis-Langford, 1932, p. 62.

§1. Introduction. The mathematical results in this paper could have been discovered and proved as early as the 1920s. Nevertheless, even though they clarify central aspects of standard first-order logic, they remained hidden for over 80 years.

Let us begin: The universalized conditional $\forall x\ (x = 0 \rightarrow x = (x + x))$ implies the corresponding existentialized conjunction $\exists x\ (x = 0\ \&\ x = (x + x))$. And $\exists x\ (x = 0)$ is logically true.

But $\forall x\ (x = (x + x) \rightarrow x = 0)$ does not imply $\exists x\ (x = (x + x)\ \&\ x = 0)$. And $\exists x\ (x = (x + x))$ is not logically true. The Existential-Import Equivalence says that these examples are typical.

Received December 3, 2013.
Key words and phrases. first-order logic, existential import, predicate, import-carrying, import-free.

1079-8986/15/2101-0001
DOI:10.1017/bsl.2015.2

In the first example, which apparently contradicts many logic books including Lewis and Langford's [13] *Symbolic Logic* quoted above, the "particular [sc. existential] conclusion" is not logically true. However, we should not overlook the exceptions to the quoted assertion that have logically true "particular conclusions": for example, the corresponding existentialized conjunction of $\forall x\ (x = x \rightarrow x = x)$ is logically true. Many logic texts—some otherwise highly competent such as Goldfarb [12, p. 108]—contain errors resulting from insufficient attention to the facts just stated. See Section 6 of Corcoran-Masoud [10] for details.

The "rule" that no existentialized conjunction is implied by the corresponding universalized conditional has exceptions, many of which are trivial but many of which are far from trivial—as we will see.

Implication and logical truth: As usual, a sentence or set of sentences *implies* a given sentence iff the given sentence is satisfied by every interpretation satisfying the sentence or set of sentences.[1]

One given sentence *is logically equivalent to* a second iff each implies the other. One given set of sentences *is logically equivalent to* or *is a logical equivalent of* a second set iff each set implies every member of the other. As usual, we occasionally use expressions such as 'the sentence $0 = 1$ implies ...' elliptically for corresponding expressions 'the unit set of the sentence $0 = 1$ implies ...'.

Sentences that are logically equivalent to the null set are said to be *logically true*, or to be *logical truths*.

It will be useful to notice that in order to show that a given sentence is logically true it is sufficient to show that it is an implication of the negation of one of its own implicants—in other words that it is implied by some sentence whose negation also implies it.

Scope and limits: This essay concerns standard one-sorted, first-order logics *in themselves*—without regard to their applications or to how their languages are used to translate normal or mathematical English.

One-sorted, first-order logics include the underlying logics—in the sense of Church [2]—of elementary number theory, set theory, geometry, and other sciences.

We do not consider *nonstandard* logics such as many-valued, paraconsistent, intuitionistic, or presupposition-free logics nor do we treat other *standard* logics such as many-sorted, equational, or second-order logics.

With these essential preliminaries settled, let us turn to the subject of the paper: "existential import".

Existential import: Aristotle's logic has *unlimited* existential import: the universal affirmative 'P *belongs-to-every* S' implies the corresponding existential affirmative 'P *belongs-to-some* S' in *every* case (Corcoran [3]). In other words, also in keeping with Aristotle's own terminological alternatives, the universal affirmative '*every* S *is a* P' implies the corresponding existential

[1] See Tarski [18] and Corcoran-Sagüillo [11]. For other uses of 'implies' see Corcoran [4].

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

affirmative '*some* S *is a* P' in *every* case. In this respect, it is similar to modern many-sorted logic (Corcoran [8]):

$$\forall s \exists p\ (s = p) \text{ implies } \exists s \exists p\ (s = p).$$

Timothy Smiley [17], William Parry [14], and others showed how Aristotle's categorical syllogistic can be faithfully represented in modern symbolic logic—if many-sorted first-order logic is used instead of the usual one-sorted first-order logic.[2] In many-sorted logic, each sort of variable suggests a common noun.

In contrast to Aristotle's logic, as noted in the abstract above, one-sorted first-order logic has *limited* existential-import: the universalized conditional sentence $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ implies the corresponding existentialized conjunction $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$ in *some but not all* cases. The *Existential-Import Equivalence* in Corcoran [7, p. 144] determines which implications hold:

$$\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x)) \text{ implies } \exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x)) \text{ iff } \exists x\ \mathrm{S}(x) \text{ is logically true.}$$

To be clear, the Existential-Import Equivalence is the proposition that in order for a universalized conditional sentence $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ to imply the corresponding existentialized conjunction $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$ it is necessary and sufficient for $\exists x\ \mathrm{S}(x)$, the existentialization of the antecedent predicate, to be logically true. Sufficiency, "if", is obvious enough. The easily proved necessity, "only if", can be established using reasoning similar to that used by Corcoran and Masoud [9]. A new proof is given in Section 2 below.

Notice that a consequence of the Existential-Import Equivalence is that whether an existential-import implication holds is independent of the form and content of the consequent $\mathrm{P}(x)$—in the sense that if $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ implies $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$, then $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ implies $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{Q}(x))$ no matter which predicate $\mathrm{Q}(x)$ is used. Of course, it also follows that if $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ does not imply $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$ then $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ does not imply $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{Q}(x))$ no matter which predicate $\mathrm{Q}(x)$ is used.

To discuss further consequences of the Existential-Import Equivalence it is useful to officially recognize and expand terminology already used here and elsewhere.

Existential-Import Terminology: Let us say that a given universalized conditional *has existential import* if it implies the corresponding existentialized conjunction. It may seem awkward at first but we will also say that a given existentialized conjunction *has existential import* if it is implied by the corresponding universalized conditional.

By an *implication,* we mean a metalogical proposition to the effect that a certain sentence implies another. By an *existential-import implication,* we mean an implication to the effect that a certain universalized conditional

[2]Recently Neil Tennant, one of Smiley's PhD students from the 1970s, made some interesting remarks related to these points in [21]. This is a convenient place to point out that Quine [15, p. 25] is simply mistaken if he thinks that many-sorted logic is reducible to one-sorted. Otherwise, Aristotle's logic would be reducible to monadic first-order.

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

implies the corresponding existentialized conjunction. In other words, an *existential-import implication* is a metalogical proposition of the form

$$\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x)) \text{ implies } \exists x\ (\mathrm{S}(x) \,\&\, \mathrm{P}(x)),$$

where $\mathrm{S}(x)$ and $\mathrm{P}(x)$ are specific predicates—possibly long and complex.

The *premise* is $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$; the *conclusion* is $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$; the *determinant* is $\exists x\ \mathrm{S}(x)$; the *determinant predicate* is $\mathrm{S}(x)$; and the *second predicate* is $\mathrm{P}(x)$.

In every existential-import implication, the premise $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$, the conclusion $\exists x\ (\mathrm{S}(x) \,\&\, \mathrm{P}(x))$, and the determinant $\exists x\ \mathrm{S}(x)$ are all sentences, "closed" sentences having no free occurrences of variables.

The Existential-Import Equivalence says that in order for an existential-import implication to hold it is necessary and sufficient for the determinant to be logically true. The fact that the truth or falsity of an existential-import implication is independent of the second predicate means that in determining the truth-value of an existential-import implication it is never necessary to consider the nonlogical constants not in the determinant. To show that an existential-import implication is true it is sufficient to deduce the determinant either from a known logical truth or from its own negation or from the negation of one of its known implicants. To show that an existential-import implication is false it is sufficient to produce an interpretation that falsifies the determinant and this need not consider any nonlogical constants not in the determinant. These observations greatly simplify study of existential-import implications.

For example, to show that $\forall x\ [x = 1 \rightarrow {\sim}\exists y\ x = (y + 0)]$ *has* existential import it is sufficient to deduce the determinant $\exists x\ (x = 1)$ from $(1 = 1)$.

Also, to show that $\forall x\ [{\sim}(x = 1) \rightarrow \exists y\ x = (y + 0)]$ *does not have* existential import it is sufficient to notice that the determinant $\exists x\ {\sim}(x = 1)$ is false under any singleton interpretation.

It is useful to notice that in many cases, when a given universalized conditional $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ has existential import, its logical equivalents that are universalized conditionals do not all have existential import. In particular, there are cases where $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ has existential import but its contrapositive $\forall x\ ({\sim}\mathrm{Q}(x) \rightarrow {\sim}\mathrm{S}(x))$ does not. Examples are given below. Perhaps the most extreme examples involve logically true antecedents and consequents. An example given above is $\forall x\ (x = x \rightarrow x = x)$, whose corresponding existentialized conjunction is logically true. The contrapositive is $\forall x\ ({\sim}(x = x) \rightarrow {\sim}(x = x))$, also logically true of course. But $\exists x\ ({\sim}(x = x)\ \ \&\ \ {\sim}(x = x))$, the existentialized conjunction corresponding to the contrapositive, is not only nonlogically true; it is self-contradictory.

By the contrapositive of a universalized conditional $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ is meant $\forall x\ ({\sim}\mathrm{Q}(x) \rightarrow {\sim}\mathrm{S}(x))$, the universalization of the conditional's contrapositive. And by the contrapositive of a conditional predicate $(\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ is meant $({\sim}\mathrm{Q}(x) \rightarrow {\sim}\mathrm{S}(x))$, the result of negating the components of the conditional's converse.

To be sure, there are cases where $\forall x\ (S(x) \rightarrow Q(x))$ has existential import and its contrapositive $\forall x\ ({\sim}Q(x) \rightarrow {\sim}S(x))$ does too. Examples are given below in the next paragraph. Perhaps the most extreme examples involve logically true antecedents and self-contradictory consequents as $\forall x\ (x = x \rightarrow {\sim}(x = x))$, whose corresponding existentialized conjunction is self-contradictory. The contrapositive is $\forall x\ ({\sim}{\sim}(x = x) \rightarrow {\sim}(x = x))$, also self-contradictory, of course. But $\exists x\ ({\sim}{\sim}(x = x)\ \&\ {\sim}(x = x))$, the existentialized conjunction corresponding to the contrapositive, is also self-contradictory.

There are also examples that are familiar sentences of arithmetic. Consider the sentence $\forall x\ (x = 0 \rightarrow {\sim}(\mathrm{x} = 1))$, which has existential import and whose contrapositive $\forall x\ ({\sim}{\sim}(x = 1) \rightarrow {\sim}(x = 0))$ has it too. Consider the original sentence and an equivalent of its contrapositive $\forall x\ (x = 1 \rightarrow {\sim}(x = 0))$ obtained by deleting the double negation. Notice that these two sentences are in the same logical form. It follows from the principle of form for implication that any two sentences in the same logical form both have existential import or both lack it (Corcoran [5]).

Notice that the Existential-Import Equivalence somewhat justifies the terminology introduced in the abstract: a one-place predicate ("open formula" having only x free) $Q(x)$ is *import-carrying* iff $\exists x\ Q(x)$ is logically true. An existential-import predicate is one that is import carrying.

Import-carrying predicates: Consider the existentialized conjunctions $\exists x\ (x = t\ \&\ P(x))$ used in Gödel's Diagonal Lemma (Boolos et al. [1, pp. 221f])—where t is a numeral. These all involve import-carrying predicates. Since t is a numeral, $\exists x\ (x = t)$ is logically true. Thus, $x = t$ is an existential-import predicate and $\exists x\ (x = t\ \&\ P(x))$ has existential import. If t is the standard numeral denoting the Gödel number of $P(x)$, then $\exists x\ (x = t\ \&\ P(x))$ is the "diagonalization" of $P(x)$.

As we will see below, the observation that for every $P(x)$, the diagonalization of $P(x)$, $\exists x\ (x = t\ \&\ P(x))$, has existential import leads to the at-first-surprising result that every sentence containing an individual constant is logically equivalent to a universalized conditional having existential import.

The most obvious examples of import-carrying predicates are logically true predicates, for example, (1) truth-functionally logically true predicates as $(x = t \rightarrow x = t)$ and (2) identity-logically true predicates as $x = x$, $(x = t \rightarrow t = x)$, etc.— where of course t is any constant term such as a numeral, a "sum" of two numerals $(n + m)$, and so on. Examples of import-carrying predicates other than logically true trivialities are readily produced. 1) $x = t$, where t is any constant term; 2) $\exists y\ (x = t(y))$, where $t(y)$ is any term in one free variable y, such as a successor term sy, ss$y$, etc., a "sum" of two such terms, and so on; 3) $(\exists x\ Q(x) \rightarrow Q(x))$, where $Q(x)$ is any predicate whatever, e.g. $\forall y\ [x \neq (y + (y + 1))]$.

It is obvious that in any standard first-order language with identity there are infinitely-many import-carrying predicates. But still import-carrying predicates might be isolated special cases that would not be regarded as "extensive" or "widespread" in an intuitive sense. Perhaps the

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

traditional doctrine, or "rule", that no universalized conditional implies its corresponding existentialized conjunction is one that needs tweaking rather than full-fledged refuting.

Much of the rest of this essay is devoted to precise deliberations that will tend to settle such vague questions. One relevant precise point already mentioned is that, although all logically true existentialized conjunctions are obviously implied by their corresponding universalized conditionals, many existentialized conjunctions implied by their corresponding universalized conditionals are not logically true. One example is $\exists x\,(x = 0 \,\&\, x = (x + x))$, which was mentioned in the beginning of this essay.

We ask exactly how extensive are import-carrying predicates? To answer, let L be any first-order language with any interpretation INT in any universe U. A subset S of U is *definable* [in L under INT] iff S is the truth-set of some predicate Q(x). S is *import-carrying definable* (respectively, *import-free definable*) iff S is the truth-set of an import-carrying (respectively, import-free) predicate.[3] A predicate Q(x) *defines* a subset S of U [in L under INT] iff S is the truth-set of Q(x) [in L under INT], i.e., iff S is the set of individuals in U that satisfy Q(x) [in L under INT].

Given suitable L and INT, the even-number set is the truth-set of the import-carrying $\exists y\ x = (y + y)$ *and* of the import-free $\forall y\ x \neq (y + (y + 1))$. This set is typical. Whether the existential-import implication holds is independent of the content (truth-set) of the antecedent S(x) if it is nonempty—just as the existential-import implication's holding is independent of the form and content of the consequent P(x), as indicated above.

As stated in the abstract, the *Existential-Importance Theorem* is as follows: Let L, INT, and U be arbitrary. Every nonempty definable subset of U is *both* import-carrying definable *and* import-free definable.

Thus, import-carrying predicates are quite widespread, and no less so than import-free predicates.

The existential-importance theorem reduces to two lemmas that are largely unrelated and are best dealt with separately.

The import-carrying-predicate lemma: Let L, INT, and U be arbitrary. Every nonempty definable subset of U is import-carrying definable.

The import-free-predicate lemma: Let L, INT, and U be arbitrary. Every nonempty definable subset of U is import-free definable.

The next section §2 of the paper will prove the Existential-Import Equivalence. The two succeeding sections, §3 and §4, will prove the import-carrying-predicate lemma and the import-free-predicate lemma. The following section §5 discusses some equivalence relations useful for thinking about existential import. The final section §6 presents some concluding remarks.

[3]The word 'definable' is used in Tarski's semantic sense of 'definable in an interpretation (model)' as opposed to the syntactic concept 'definable in a theory'. See Tarski [19, pp. xxiii, 118, 194, etc.]. The concept of arithmetical definability is a special case of this semantic concept. See Boolos et al. [1, pp. 199f, 286f]. The concept of definability-in-T found in Boolos et al. [1, pp. 207] is a third notion.

§2. The Existential-Import Equivalence. *The Existential-Import Equivalence*: In any first-order logic, for a universalized conditional to imply the corresponding existentialized conjunction it is necessary and sufficient for the existentialization of the antecedent predicate to be logically true. More succinctly: Let $\mathrm{S}(x)$ and $\mathrm{P}(x)$ be any predicates in a first-order language.

$$\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x)) \text{ implies } \exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x)) \text{ iff } \exists x\ \mathrm{S}(x) \text{ is logically true.}$$

There are several proofs independently found by John Corcoran, Sriram Nambiar (personal communication), and Joel Friedman (personal communication). Perhaps the easiest way to see it is as follows.

"If" is almost immediate: evidently the two sentences $\exists x\ \mathrm{S}(x)$ and $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ together imply $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$. Moreover, if one of two sentences implying a third is logically true, then the other by itself implies the third. Thus if $\exists x\ \mathrm{S}(x)$ is logically true, then $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ implies $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$. ⊣

"Only if" uses the fact that in order for a given sentence to be logically true it is sufficient for it to be implied by the negation of one of its implicants.

Assume that $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ implies $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$. $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ implies $\exists x\ \mathrm{S}(x)$.

Thus to see that $\exists x\ \mathrm{S}(x)$ is logically true, it is sufficient to see that the negation $\sim\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ implies $\exists x\ \mathrm{S}(x)$. Thus $\exists x\ \mathrm{S}(x)$ is logically true. ⊣

This equivalence is remarkable in several respects. One consequence already mentioned is that whether an existential-import implication holds or fails is entirely independent of the consequent.

Another conclusion that can be drawn, given the obvious fact that there are infinitely-many logically true existentials $\exists x\ \mathrm{S}(x)$, is that there are infinitely-many universalized conditionals that imply their respective corresponding existentialized conjunctions.

The independence of the consequents, i.e., that substituting arbitrary predicates for the consequent leaves existential import unchanged, suggests that the contrapositive of a universalized conditional implying its corresponding existentialized conjunction need not imply its own corresponding existentialized conjunction—in other words, that the contrapositive of a universalized conditional having existential import need not itself have existential import. This would suggest another source of logically equivalent universalized conditionals some but not all of which imply their respective corresponding existentialized conjunctions. We will see that these suggestions are fruitful.

Corollaries: There are several corollaries of this equivalence that are very close in content. Some people might call them variants of the equivalence. We present a few.

The First Existential-Import Sentence Equivalence:

$\{\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x)) \rightarrow \exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))\}$ is logically equivalent to $\exists x\ \mathrm{S}(x)$.

This is essentially the same as the main result in Corcoran-Masoud [9].

The Second Existential-Import Sentence Equivalence:

$\{\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x)) \rightarrow \exists x\ \mathrm{S}(x)\}$ is logically equivalent to $\exists x\ \mathrm{S}(x)$.

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

§3. The import-carrying-predicate lemma. *The import-carrying-predicate lemma*: Let L, INT, and U be arbitrary. Every nonempty definable subset of U is import-carrying definable.

Excluding the empty set, in any interpretation, the import-carrying definable sets coincide with the definable sets and, thus, import-carrying definable sets are as widespread as they can be.

A Transformation: To see this we will first prove that for every predicate $\mathrm{Q}(x)$, the conditional predicate $(\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$ is import-carrying, i.e., that its existentialization $\exists x\ (\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$ is logically true. Because this transformation is so important in this work, it deserves a name: $(\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$ is *the existential qualification of* $\mathrm{Q}(x)$.

Notice that all occurrences of the variable x in the antecedent of an existential qualification are bound but there are free occurrences of x in the consequent and thus in the existential qualification itself. The fact that it is the same variable occurring bound in the antecedent and free in the consequent is irrelevant and thus potentially confusing. In certain contexts therefore it would be convenient to replace the antecedent $\exists x\ \mathrm{Q}(x)$ by one of its logically equivalent alphabetic variants say $\exists y\ \mathrm{Q}(y)$ as $(\exists y\ \mathrm{Q}(y) \rightarrow \mathrm{Q}(x))$ to emphasize logical form. However, this proves to complicate the exposition. See Section 5 below for more on formal equivalence and logical equivalence in relation to existential import.

Using the new terminology, our first result then will be that every existential qualification is import-carrying.

To see this we first make two useful logical observations.

The first is that $\exists x\ \mathrm{Q}(x)$ implies $\exists x\ (\mathrm{P} \rightarrow \mathrm{Q}(x))$, no matter whether P is a one-place predicate $\mathrm{R}(x)$ or a sentence. The second observation is that $\sim$P implies $\exists x\ (\mathrm{P} \rightarrow \mathrm{R}(x))$, no matter which predicate $\mathrm{R}(x)$ is.

By the first observation, $\exists x\ \mathrm{Q}(x)$ implies $\exists x\ (\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$.

By the second observation, $\sim\exists x\ \mathrm{Q}(x)$ implies $\exists x\ (\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$.

Thus, $\exists x\ (\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$ is logically true since it is implied by the negation of one of its implicants. This proves that for every predicate $\mathrm{Q}(x)$, the existential qualification $(\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$ is import-carrying.

To prove the lemma: Let L, INT, and U be arbitrary. Let Q be any nonempty definable subset of U. Without loss of generality let Q be defined by $\mathrm{Q}(x)$. Since Q is nonempty, $\exists x\ \mathrm{Q}(x)$ is true. Thus $(\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$ is coextensive with $\mathrm{Q}(x)$: a given member of U satisfies the first iff it satisfies the second. Thus $(\exists x\ \mathrm{Q}(x) \rightarrow \mathrm{Q}(x))$ defines Q.

Thus every nonempty definable subset of U is import-carrying definable. ⊣

To see that nonemptiness is required notice that the empty set is not import-carrying definable. The empty set is defined in any interpretation by $x \neq x$ but the existential qualification $(\exists x\ x \neq x \rightarrow x \neq x)$ defines U. In fact, in order for a predicate to be import-free it is necessary and sufficient for it to define the null set under some interpretation.

§4. The import-free-predicate lemma. *The import-free-predicate lemma*: Let L, INT, and U be arbitrary. Every nonempty definable subset of U is import-free definable.

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

In this case, the restriction to nonempty subsets is unnecessary. Our task here then is to find, for any given predicate, a coextensive import-free predicate, a coextensive predicate whose existentialization is not logically true. We give two proofs that analyze the facts in two ways. The first uses a transformation that sheds more light on how extensive or widespread import-free predicates are—without considering what they define.

Another Transformation: We will first prove that for every predicate $Q(x)$, the conjunctional predicate $(\exists x \sim Q(x)\ \&\ Q(x))$ is import-free, i.e., that its existentialization $\exists x\ (\exists x \sim Q(x)\ \&\ Q(x))$ is not logically true. Because this transformation is so important in this work, it deserves a name: $(\exists x \sim Q(x)\ \&\ Q(x))$ is *the existential-negative conjunctification of* $Q(x)$. Of course, for every predicate $Q(x)$, the conjunctional predicate $(\exists x \sim Q(x)\ \&\ Q(x))$ is called *an existential-negative conjunctification.*

Using the new terminology, our first result then will be that every existential-negative conjunctification is import-free. We have to show that for every predicate $Q(x)$, there is an interpretation under which $\exists x\ (\exists x \sim Q(x)\ \&\ Q(x))$ is false. Notice that $\exists x\ (\exists x \sim Q(x)\ \&\ Q(x))$ is logically equivalent to the conjunction $(\exists x \sim Q(x)\ \&\ \exists x\ Q(x))$. In any interpretation INT having a singleton universe, either $Q(x)$ defines the universe—and thus INT falsifies $\exists x \sim Q(x)$—or $\sim Q(x)$ defines the universe, and thus INT falsifies $\exists x\ Q(x)$.

Thus, we have a one-one transformation that carries each predicate to a closely related predicate that is import-free. This suggests that import-free predicates are widespread. However, this result can be used to get stronger suggestions.

Proof. Let L, INT, and U be arbitrary. Let Q be any definable subset of U—empty or not. Without loss of generality, let Q be defined by $Q(x)$. Either $Q(x)$ defines the universe or not.

If $Q(x)$ defines the universe U, it is coextensive with $\forall y\ x = y$ or $\exists y\ x \neq y$ according as U is singleton or not. Moreover, both are import-free: their existentializations are informative.

If $Q(x)$ does not define the universe U, $\exists x \sim Q(x)$ is true. Thus $(\exists x \sim Q(x)\ \&\ Q(x))$ is coextensive with $Q(x)$: a given member of U satisfies the first iff it satisfies the second. Thus $(\exists x \sim Q(x)\ \&\ Q(x))$ defines Q and is import-free, as we saw above.

Thus, whether $Q(x)$ defines the universe or not, it is coextensive with an import-free predicate. Thus, every nonempty definable subset of U is import-free definable.[4] ⊣

§5. Equivalence relations. 1. *Formal equivalence*: Two sentences are defined to be *formally equivalent* or *equivalent in form* iff they have the same logical form (Corcoran [5, p. 445]).[5]

[4] We proved the stronger result that every definable subset of U is import-free definable, but the weaker result stated is sufficient for purposes of the above-stated Existential-Importance Theorem. The same applies to the second proof.

[5] In other words, using the poorly chosen expression "almost identical" in the sense of Tarski and Givant [20, p. 43], two sentences are *formally equivalent* or *equivalent in form*

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

Two sentences are formally equivalent if some 1-1 category-preserving function from the set of nonlogical constants of the language onto itself carries one sentence exactly into the other. A property of sentences is said to be *formal* if it belongs to every sentence formally equivalent to one it belongs to, i.e., if formal equivalence preserves it. As mentioned above in different words, being a universalized conditional with existential import is a formal property. Formal equivalence preserves existential import in the sense that every formal equivalent of a universalized conditional with existential import also has this property, i.e., is a universalized conditional with existential import.[6]

2. *Logical equivalence*: Logical equivalence does not preserve existential import: not every logical equivalent of a universalized conditional with existential import also has this property. This was exemplified above with contrapositives. This remark answers no to the question: does every universalized conditional logically equivalent to some universalized conditional with existential import have existential import?

However, what was not discussed was how extreme the lack of preservation is. For example, is every universalized conditional whose antecedent predicate is import-carrying logically equivalent to some universalized conditional whose antecedent predicate is import-free? In other words, is every universalized conditional with existential import logically equivalent to some universalized conditional without existential import? As far as we know, this is an open question.

If this is answered no, we can ask instead which universalized conditionals with existential import are logically equivalent to universalized conditionals without existential import.

Of course, once the above questions are raised, one immediately asks the converse of the first: is every universalized conditional whose antecedent predicate is import-free logically equivalent to some universalized conditional whose antecedent predicate is import-carrying? In other words, is every universalized conditional without existential import logically equivalent to a universalized conditional with existential import? Perhaps surprisingly, this is not an open question. The answer is yes. However, some proofs also show that the question is less interesting than might have been thought. There is less to this than meets the eye.

Import-Creation Corollary: every universalized conditional without existential import is logically equivalent to a universalized conditional with existential import.

Proof. Assume that $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{P}(x))$ does not imply $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$. Now consider $\forall x\ \{x = x \rightarrow (\mathrm{S}(x) \rightarrow \mathrm{P}(x))\}$. The required logical

iff they are either almost identical or one is an alphabetic variant of a sentence almost identical to the other. Being instances of the same schema does not imply formal equivalence (Corcoran [6]).

[6]Several senses of 'form' play important roles in Russell [16, p. 199], Church [2, pp. 2, 10ff], and Goldfarb [12, pp. 5, 48, 150, passim].

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

equivalence is obvious; it is a special case of the fact that for every $P(x)$, $\forall x\ P(x)$ is logically equivalent to $\forall x\ \{x = x \rightarrow P(x)\}$. ⊣

The implied existential is $\exists x\ \{x = x\ \&\ (S(x) \rightarrow P(x))\}$.

Notice that the Import-Creation Corollary implies that absolutely every universalized conditional—regardless of antecedent predicate—is logically equivalent to a universalized conditional whose antecedent predicate is import-carrying. In other words, every universalized conditional is logically equivalent to a universalized conditional with existential import.

This brings us to the result announced above about sentences containing an individual constant.

The Individual-Constant Import-Creation Lemma: every sentence containing an individual constant is logically equivalent to a universalized conditional having existential import.

PROOF. Let P be any sentence containing an individual constant, say t. Without loss of generality, assume that x does not occur in P and that $P(x)$ is the predicate obtained by replacing every occurrence of t by x. As noted by Boolos et al. [1, p. 221], it is easy to see that P is logically equivalent to $\exists x\ (x = t\ \&\ P(x))$. But it is also easy to see that the latter existentialized conjunction, $\exists x\ (x = t\ \&\ P(x))$, is in turn logically equivalent to its own corresponding universalized conditional $\forall x\ (x = t \rightarrow P(x))$, which has existential import. ⊣

This result admits of generalization in view of the fact that every sentence is logically equivalent to one containing an individual constant: P is logically equivalent to $(P\ \&\ t = t)$. Thus, we have proved:

The Strong Import-Creation Corollary: every sentence is logically equivalent to a universalized conditional whose antecedent predicate is import-carrying. In other words, every sentence—whether a universalized conditional or not—is logically equivalent to a universalized conditional with existential import.

3. *Extensional Equivalence*: We move on to another equivalence relation useful in surveying the distribution of existential import and in seeing how widespread it is. Two universal sentences are defined to be *extensionally equivalent [with each other] under a given interpretation* INT iff their predicates define the same set under INT. If two universal sentences are both true, then their predicates define the universe and thus they are extensionally equivalent. In addition, if they have different truth-values, they are not extensionally equivalent. However, if they are both false, it is necessary to dig a little deeper.

Two false universal sentences are extensionally equivalent iff they have the same counterexamples: For any two predicates $S(x)$ and $P(x)$, $S(x)$ and $P(x)$ define the same subset of the universe iff their negations $\sim S(x)$ and $\sim P(x)$ define the same subset.

Our examples use standard first-order logic with the class of numbers [non-negative integers] as universe of discourse. A number n is a *counterexample* for a universal sentence $\forall x\ P(x)$ iff n satisfies $\sim P(x)$. In some familiar cases,

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

logically equivalent false sentences have the same counterexamples. "Every number that is not even is prime" and "Every number that is not prime is even" are both *counterexemplified* by the nonprime odd numbers.

Moving along a spectrum, we find cases that share some but not all counterexamples. "Every number divides every other number" is counterexemplified by every number except one, whereas its equivalent "Every number is divided by every other number" is counterexemplified by every number except zero. On the other end of the spectrum there are cases having no counterexamples in common: "Every even number precedes every odd number" is counterexemplified only by even numbers, whereas its equivalent "Every odd number is preceded by every even number" is counterexemplified only by odd numbers. One easy result is that, given any nonempty finite set of numbers, every false universal sentence is logically equivalent to another counterexemplified exclusively by numbers in the given set.

Let us return to the main issue: how extensional equivalence can shed light on how widespread existential import is. Due to space limitations, we will give only two results both easily proved as corollaries of above considerations. Actually, both are essentially restatements of previous results.

The Extensional Import-Creation Corollary: Under any given interpretation, every universal sentence is extensionally equivalent under that interpretation to some universalized conditional having existential import.

The Extensional Import-Destruction Corollary: Under any given interpretation, every universal sentence whose predicate does not define the null set is extensionally equivalent under that interpretation to some universalized conditional not having existential import.

4. *Biextensional Equivalence:* We move on to yet another equivalence relation useful in surveying the extensiveness and distribution of existential import. Two universalized conditional sentences are defined to be *biextensionally equivalent [with each other] under a given interpretation* INT iff their antecedent predicates define the same set under INT and their consequent predicates define the same set under INT.

The import-carrying-predicate lemma—that under any INT, every nonempty definable subset of U is import-carrying definable—implies that under any INT, every universalized conditional is biextensionally equivalent to some universalized conditional having existential import.

In other words, this lemma implies the following:

The Biextensional Import-Creation Corollary: In any interpretation, every universalized conditional without existential import is biextensionally equivalent to some universalized conditional with existential import.

§**6. Concluding remarks.** Exemplifying the widely heralded failure of existential-import in modern logic, we can point out that

"Every number that is even precedes some odd number"
does not imply
"Some number that is even precedes some odd number".

https://doi.org/10.1017/bsl.2015.2 Published online by Cambridge University Press

In addition, analogously we note that

$$\forall x\,(\mathrm{E}x \rightarrow \exists y\,(x < y \,\&\, \mathrm{O}y)) \text{ does not imply } \exists x\,(\mathrm{E}x \,\&\, \exists y\,(x < y \,\&\, \mathrm{O}y)).$$

However, replacing Ex, "x is even", by the coextensive predicate $(\exists y\ x = (y + y))$, "x is the sum of a number with itself", an implication results.

"Every number that is the sum of a number with itself precedes some odd number"

does imply

"Some number that is the sum of a number with itself precedes some odd number".

$$\forall x\,(\exists y\ x = (y + y) \rightarrow \exists y\,(x < y \,\&\, \mathrm{O}y)) \text{ implies}$$
$$\exists x\,(\exists y\ x = (y + y) \,\&\, \exists y\,(x < y \,\&\, \mathrm{O}y)).$$

This is typical. As shown in this paper, whenever a universalized conditional proposition fails to imply the corresponding existentialized conjunction, there is another universalized conditional—with the same consequent and a coextensive antecedent—that does imply its corresponding existentialized conjunction. The exception, of course, is the case where the antecedent of the universalized conditional defines the null set.

Although this essay concerns exclusively one-place predicates that define sets of individuals under interpretations, similar results hold for two-place predicates.

$$\forall x \forall y\,[(x = 0 \,\&\, y = 1) \rightarrow x < y] \text{ implies}$$
$$\exists x \exists y\,[(x = 0 \,\&\, y = 1) \,\&\, x < y].$$
$$\forall x \forall y\,[x < y \rightarrow (x = 0 \,\&\, y = 1)] \text{ does not imply}$$
$$\exists x \exists y\,[x < y \,\&\, (x = 0 \,\&\, y = 1)].$$

As might have been anticipated, we can prove the following *Two-place Existential-Import Equivalence.*

$$\forall xy\,[\mathrm{S}(x,y) \rightarrow \mathrm{P}(x,y)] \text{ implies } \exists xy\,[\mathrm{S}(x,y) \,\&\, \mathrm{P}(x,y)] \text{ iff}$$
$$\exists xy\ \mathrm{S}(x,y) \text{ is logically true.}$$

From here, it is easy to formulate and prove the general case for n-place predicates.

Acknowledgments. The meticulous anonymous BSL referees deserve credit for many improvements and corrections. We are also grateful to Gabriela Fulugonia of the University of Buenos Aires, Leonard Jacuzzo of Fredonia University, Calvin Jongsma of Dordt College, Justin Legault of Ottawa, Canada, Edwin Mares of Victoria University at Wellington, Sriram Nambiar of the Indian Institute of Technology at Madras, José Miguel Sagüillo of the University of Santiago de Compostela, and Michael Scanlan of Oregon State University. Valuable help and advice were also received from Edgar Andrade, Mark Brown, Newton da Costa, William Demopoulos, William Frank, Joel Friedman, James Gasser, Marc Gasser, Warren Goldfarb, Luis Estrada González, Anil Gupta, David Hitchcock, Wilfrid Hodges, Joaquin Miller, Bo Mou, Frango Nabrasa, Jacek Paśniczek, Wagner Sanz, Wilfried Sieg, Alasdair Urquhart, Jan von Plato, George Weaver, and others. This paper overlaps Corcoran-Masoud [10]: a much longer, more widely accessible, and more expository treatment that contains further complementary philosophical, historical, and pedagogical deliberations.

REFERENCES

[1] Boolos, G., J. Burgess, and R. Jeffrey, ***Computability and logic***, Cambridge University Press, Cambridge, 2007.

[2] Church, A., ***Introduction to Mathematical Logic***, Princeton University Press, Princeton, 1956.

[3] Corcoran, J., *Completeness of an ancient logic*. ***Journal of Symbolic Logic***, vol. 37 (1972), pp. 696–702.

[4] ———, *Meanings of implication*. ***Diálogos***, vol. 9 (1973), pp. 59–76.

[5] ———, *First-order logical form*, this Bulletin, vol. 10 (2004), p. 445.

[6] ———, *Schemata: the Concept of Schema in the History of Logic*, this Bulletin, vol. 12 (2006), pp. 219–40.

[7] ———, *Existential import*, this Bulletin, vol. 13 (2007), pp. 143–144.

[8] ———, *Aristotle's many-sorted logic*, this Bulletin, vol. 14 (2008), pp. 155–156.

[9] Corcoran, J., and H. Masoud, *Existential-import sentence schemas: classical and relativized*, this Bulletin, vol. 20 (2014), pp. 402–403.

[10] ———, *Existential import today: New metatheorems; historical, philosophical, and pedagogical misconceptions*. ***History and Philosophy of Logic***, vol. 36 (2015), pp. 39–61.

[11] Corcoran, J., and J. M. Sagüillo, *Absence of multiple universes of discourse in the 1936 Tarski consequence-definition paper*. ***History and Philosophy of Logic***, vol. 32 (2011), pp. 359–374.

[12] Goldfarb, W., ***Deductive Logic***, Hackett, Indianapolis, IN, 2003.

[13] Lewis, C. I. and C. H. Langford, ***Symbolic Logic***, Century, New York, NY, 1932.

[14] Parry, W., *Quantification of the Predicate and Many-sorted Logic*. ***Philosophy and Phenomenological Research***, vol. 22 (1966), pp. 342–360.

[15] Quine, W., ***Philosophy of logic***, Harvard University Press, Cambridge, 1970/1986.

[16] Russell, B., ***Introduction to Mathematical Philosophy***, Dover, New York, 1919.

[17] Smiley, T., *Syllogism and quantification*. ***The Journal of Symbolic Logic***, vol. 27 (1962), pp. 58–72.

[18] Tarski, A., *On the concept of logical consequence*, ***Logic, Semantics, Metamathematics, Papers from 1923 to 1938*** (John Corcoran, editor), Hackett, Indianapolis, IN, 1983, pp. 409–420.

[19] ———, ***Logic, Semantics, Metamathematics, Papers from 1923 to 1938*** (John Corcoran, editor), Hackett, Indianapolis, IN, 1983.

[20] Tarski, A., and S. Givant, ***A Formalization of Set Theory without Variables***, American Mathematical Society, Providence, RI, 1987.

[21] Tennant, N., Aristotle's syllogistic and core logic. ***History and Philosophy of Logic***, vol. 35 (2014), pp. 120–147.

[22] Wittgenstein, L., ***Tractatus Logico-Philosophicus***, Kegan Paul, London, 1922.

DEPARTMENT OF PHILOSOPHY
UNIVERSITY AT BUFFALO, BUFFALO
NY 14260-4150, USA
E-mail: corcoran@buffalo.edu

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF ALBERTA, EDMONTON
AB T6G2E7, CANADA
E-mail: hassan.masoud@ualberta.ca