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A CONSTRUCTION IN GENERAL RADICAL THEORY

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1. Introduction. Given an arbitrary associative ring R we consider the ring $R[x]$ of polynomials over R in the commutative indeterminate x. For each radical property S we define the function S^* which assigns to each ring R the ideal

$$S^*(R) = S(R[x]) \cap R$$

of R. It is shown that the property S_A (that a ring R be equal to $S^*(R)$) is a radical property. If S is semiprime, then S_A is semiprime also. If S is a special radical, then S_A is a special radical. S_A is always contained in S. A necessary and sufficient condition that S and S_A coincide is given.

The results are generalized in the last section to include extensions of R other than $R[x]$. One such extension is the semigroup ring $R[A]$, where A is a semigroup with an identity adjoined. Hence one may consider polynomial rings in several indeterminates which need not commute with each other.

This work was motivated by the papers of Amitsur [**2**] and McCoy [**4**; **5**]. For the terminology used the reader may refer to [**3**].

2. Preliminaries. A radical property S will be said to be *inherited by ideals* (*subrings*) if every ideal (subring) of an S-ring is itself an S-ideal (S-ring). By a *subring of invariants* of R we shall mean a set $\{a \in R | \, ah = a\}$, where h is some endomorphism of R. Correspondingly, there are properties which are *inherited by subrings of invariants*; e.g., quasi-regularity.

We will say that a radical property S is *semiprime* [**6**] (or a *Z-property* [**1**]) if for all rings R, $S(R)$ is a semiprime ideal of R.

We shall make use of some results of McCoy [**4**; **5**]. If P is an ideal of R, then there exists an ideal P' of $R[x]$ such that $R[x]/P' \cong R/P$, $P' \cap R = P$, and $P[x] \subset P'$. Clearly if P is prime, then P' is prime. If P is primitive, then P' is primitive. If Q is any prime ideal of $R[x]$, then $Q \cap R$ is a prime ideal of R. We cannot have a similar result for primitive ideals, for then we would have $J(R) \subset J(R[x]) \cap R$, where J is the Jacobson radical property. But $J(R[x]) \cap R$ is contained in the nil radical $N(R)$ of R. (See [**2**, p. 357, Lemma 3J].) Thus we would have $J = N$, a contradiction.

3. Main results.

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THEOREM 1. *If S is a radical property, then the property S_A (that a ring R be equal to $S^*(R)$) is a radical property. For all rings R, $S_A(R) \subset S^*(R)$. If $S^*(S^*(R)) = S^*(R)$, then $S_A(R) = S^*(R)$.*

Proof. If h is a homomorphism of R such that $\ker h \subset S^*(R)$, then h may be extended to a homomorphism h' of $R[x]$ such that

$$\ker h' = (\ker h)[x] \subset S^*(R)[x] = S^*(R) \cdot R^1[x] \subset S(R[x]) \cdot R^1[x] = S(R[x]),$$

where R^1 is the usual ring with identity in which R is embedded as an ideal. Since S is a radical property, $S(R[x]h') = (S(R[x]))h'$. Intersecting with Rh we obtain $S^*(Rh) = S^*(R)h$. In particular, every homomorphic image of an S_A-ring is an S_A-ring.

If I is an ideal of any ring R, then $S(I[x]) \subset S(R[x])$ [**3**, p. 125, Corollary 1] and we have $S^*(I) \subset S^*(R)$. Hence, if R is not an S_A-ring, then $R/S^*(R)$ is a non-zero homomorphic image of R without non-zero S_A-ideals. For if $I/S^*(R)$ is an S_A-ideal of $R/S^*(R)$, then $I/S^*(R) = S^*(I/S^*(R)) \subset S^*(R/S^*(R)) = 0$.

Therefore S_A is a radical property [**3**]. Since $S^*(R)$ contains all S_A-ideals of R, $S_A(R) \subset S^*(R)$. The rest is clear.

THEOREM 2. *$S_A \leqq S$ for all radical properties S.*

Proof. By the results of McCoy stated above, there exists an ideal P' of $R[x]$ such that $R[x]/P' \cong R/S(R)$ and $P' \cap R = S(R)$. Hence $R[x]/P'$ is S-semisimple and $P' \supset S(R[x])$. Intersecting with R, $S(R) \supset S^*(R)$. Therefore $S_A \leqq S$.

THEOREM 3. *If S is a semiprime radical property, then S_A is again a semiprime radical property.*

Proof. If R is a zero ring, then $R[x]$ is a zero ring. Hence $S(R[x]) = R[x]$ and $S^*(R) = R$. Therefore R is an S_A-ring.

THEOREM 4. *If S is a radical property which is inherited by ideals (subrings, subrings of invariants), then S_A is also inherited by ideals (subrings, subrings of invariants).*

Proof. If S is as described and T is an ideal (subring, subring of invariants), then $S^*(R) \cap T \subset S^*(T)$.

LEMMA 5. *If P' is a semiprime ideal of $R[x]$ such that $R[x]/P'$ is S-semisimple, then $S^*(R/(P' \cap R)) = 0$.*

Proof. Let $P = P' \cap R$. Then $P[x] \cdot R[x] \subset P'$ and since P' is semiprime, $P[x] \subset P'$. If h is the natural homomorphism of $(R/P)[x]$ onto $R[x]/P'$ and if $a + P \in S^*(R/P) \subset S((R/P)[x])$, then

$$a + P' = (a + P)h \in S(R[x]/P') = 0.$$

Hence $a \in P$ and $S^*(R/P) = 0$.

THEOREM 6. *If S is special, then S_A is special.*

Proof. If T is the intersection of all ideals P of R for which R/P is prime and S_A-semisimple, then $S_A(R) \subset T$. On the other hand, if P' is an ideal of $R[x]$ such that $R[x]/P'$ is prime and S-semisimple, then $S^*(R/(P' \cap R)) = 0$. Hence $R/(P' \cap R)$ is prime and S_A-semisimple. Therefore

$$S^*(R) = \cap\{P' \cap R \mid R[x]/P' \text{ is prime and } S\text{-semisimple}\} \supset T.$$

Since T is an ideal, $S^*(R) \cap T \subset S^*(T)$. Hence $T \subset S^*(T)$ and T is an S_A-ideal.

Amitsur has shown [**2**, Lemma 2J] that if S is a semiprime radical property which is inherited by subrings of invariants, then $S(R[x]) = S^*(R)[x]$. In this case, $S^*(R) \subset S(R[x]) = S(S(R[x])) = S(S^*(R)[x])$. Hence $S^*(R)$ is an S_A-ideal and $S^*(R) = S_A(R)$. He has also shown that if S is such that $R[x]$ is an S-ring whenever R is an S-ring, then $S = S_A$. Conversely, suppose that $S = S_A$. If R is an S-ring, then $R[x] = S(R)[x] = S^*(R)[x] \subset S(R[x])$. Hence $R[x]$ is an S-ring.

4. Generalizations. Let $'$ denote a function from the class of all rings into itself such that for each ring R, R is a subring of R' and suppose that $'$ satisfies the following condition for all rings R:

(P.1) Every homomorphism h of R may be extended to a homomorphism h' of R' such that $R'h' = (Rh)'$ and

$$\ker h' = (\ker h)' \subset (\ker h) \cdot (R^1)'.$$

If S is a radical property and if one defines

$$S^*(R) = S(R') \cap R,$$

then Theorem 1 is valid. If P^* is a semiprime ideal of R' and if P is an ideal of R such that $P \subset P^*$, then $P' \subset P^*$. If S is hereditary, then

$$S^*(R) \cap I \subset S^*(I)$$

for any ideal I of R, and hence S_A is hereditary. If $'$ satisfies the further property:

(P.2) If P^* is a prime ideal of R', then $P^* \cap R$ is a prime ideal of R, then Lemma 5 (modified) holds, and S_A is special whenever S is special. S_A is semiprime when S is semiprime if the following property is satisfied (independently of (P.2)):

(P.3) If R is a zero ring, then R' is a zero ring.

In particular, one may take R' to be the semigroup ring $R[A]$, where A is a semigroup with an identity adjoined; i.e., $ab = 1$ if and only if $a = b = 1$. It is easy to see that $R[A]$ satisfies conditions (P.1), (P.2), and (P.3).

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References

1. S. A. Amitsur, *A general theory of radicals.* II, Amer. J. Math. *76* (1954), 100–125.
2. ——— *Radicals of polynomial rings*, Can. J. Math. *8* (1956), 355–361.
3. N. J. Divinsky, *Rings and radicals* (Univ. Toronto Press, Toronto, Ontario, 1965).
4. N. H. McCoy, *The prime radical of a polynomial ring*, Publ. Math. Debrecen *4* (1956), 161–162.
5. ——— *Certain classes of ideals in polynomial rings*, Can. J. Math. *9* (1957), 352–362.
6. M. Nagata, *On the theory of radicals in a ring*, J. Math. Soc. Japan *3* (1951), 330–344.

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