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ON THE ∗-SEMISIMPLICITY OF THE ℓ^1-ALGEBRA ON AN ABELIAN ∗-SEMIGROUP

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Abstract

Towards an involutive analogue of a result on the semisimplicity of $\ell^1(S)$ by Hewitt and Zuckerman, we show that, given an abelian ∗-semigroup S, the commutative convolution Banach ∗-algebra $\ell^1(S)$ is ∗-semisimple if and only if Hermitian bounded semicharacters on S separate the points of S; and we search for an intrinsic separation property on S equivalent to ∗-semisimplicity. Very many natural involutive analogues of Hewitt and Zuckerman's separation property are shown not to work, thereby exhibiting intricacies involved in analysis on S.

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1. Introduction

Given an abelian semigroup S, Hewitt and Zuckerman showed in [5] that the commutative convolution Banach algebra $\ell^1(S)$ is semisimple if and only if bounded semicharacters on S separate the points of S, and that this is so if and only if S has the P_0-property: for $s, t \in S$, if $s^2 = t^2 = st$, then $s = t$; such semigroups were called *separating semigroups* in [5]. When S is a ∗-semigroup, the algebra $\ell^1(S)$ is a Banach ∗-algebra with the involution $f^*(s) = \overline{f(s^*)}$, $s \in S$. We search for an analogue of the Hewitt–Zuckerman result for ∗-semisimplicy of $\ell^1(S)$. Though ∗-semisimplicity is equivalent to bounded Hermitian semicharacters separating the points of S, a search for an analogue of the intrinsic P_0-property turns out to be involved. This leads to several closely related separation properties, some of which are necessary but not sufficient, and others sufficient but not necessary. For example, the most natural condition is the P_1-property: for $s, t \in S$, if $s^*s = t^*t = s^*t$, then $s = t$. It so happens that the P_1-property is necessary but not sufficient. The semigroup algebra $\ell^1(S)$ has remained an important object in Banach algebra theory. The interrelation between the

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semigroup structure of S and the Banach algebra structure of $\ell^1(S)$ is a fascinating aspect of harmonic analysis on semigroups (see [2–4, 6]). For Banach algebra terminologies, we refer to [1].

2. The $*$-semisimplicity of $\ell^1(S)$

Let S be an abelian $*$-semigroup. A *bounded semicharacter* on S is a nonzero map $\alpha : S \to \mathbb{C}$ such that $|\alpha(s)| \leq 1$ and $\alpha(st) = \alpha(s)\alpha(t)$ for all $s, t \in S$. Let

$$\begin{aligned}
\Phi_{bs}(S) &:= \text{the set of all bounded semicharacters on } S,\\
\Phi_s(S) &:= \{\alpha \in \Phi_{bs}(S) : |\alpha(s)| = 0 \text{ or } 1\ (s \in S)\},\\
\Psi_{bs}(S) &:= \{\alpha \in \Phi_{bs}(S) : \alpha(s^*) = \overline{\alpha(s)}\ (s \in S)\},\\
\Psi_s(S) &:= \Phi_s(S) \cap \Psi_{bs}(S).
\end{aligned}$$

Hewitt and Zuckerman [5, Theorems 3.5, 5.6, 5.8] proved the following theorem.

THEOREM 2.1. *The following statements are equivalent for an abelian semigroup S.*

(1) *$\ell^1(S)$ is semisimple.*
(2) *$\Phi_{bs}(S)$ separates the points of S.*
(3) *$\Phi_s(S)$ separates the points of S.*
(4) *S has the P_0-property.*

We search for an involutive analogue of Theorem 2.1. We prove the following, which exhibits the intricacies involved, showing that the complete analogue of Theorem 2.1 is not true.

THEOREM 2.2. *Consider the following statements for an abelian $*$-semigroup S.*

(1) *$\ell^1(S)$ is $*$-semisimple.*
(2) *$\Psi_{bs}(S)$ separates the points of S.*
(3) *$\Psi_s(S)$ separates the points of S.*
(4) *S has the P_1-property.*

Then $(1) \Leftrightarrow (2) \Leftarrow (3)$, $(2) \Rightarrow (4)$, $(2) \not\Rightarrow (3)$ *and* $(4) \not\Rightarrow (2)$.

PROOF. First we note that the Gel'fand space $\Delta(\ell^1(S))$ of $\ell^1(S)$ can be identified with $\Phi_{bs}(S)$ [5, Theorem 2.7] via the mapping $\alpha \mapsto \varphi_\alpha$, where

$$\varphi_\alpha(f) = \sum_{s\in S} f(s)\alpha(s) \quad \Big(f = \sum_{s\in S} f(s)\delta_s \in \ell^1(S)\Big).$$

Let $\widetilde{\Delta}(\ell^1(S)) := \{\varphi \in \Delta(\ell^1(S)) : \varphi(f^*) = \overline{\varphi(f)},\ \ f \in \ell^1(S)\}$ be the Hermitian Gel'fand space of $\ell^1(S)$. Then we can identify $\widetilde{\Delta}(\ell^1(S))$ with $\Psi_{bs}(S)$ by the restriction of the above mapping. Thus $\varphi_\alpha \in \widetilde{\Delta}(\ell^1(S))$ if and only if $\alpha \in \Psi_{bs}(S)$.

$(1) \Rightarrow (2)$ Let $s, t \in S$ be such that $\alpha(s) = \alpha(t)$ $(\alpha \in \Psi_{bs}(S))$. Then

$$\alpha(s^*s) = \alpha(t^*t) = \alpha(s^*t) = \alpha(t^*s) \quad (\alpha \in \Psi_{bs}(S)).$$

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Set $f = \delta_s - \delta_t$. Then $f^*f = (\delta_s - \delta_t)^*(\delta_s - \delta_t) = \delta_{s^*s} + \delta_{t^*t} - \delta_{s^*t} - \delta_{t^*s}$. So $\varphi_\alpha(f)^2 = \varphi_\alpha(f^*f) = 0\ (\alpha \in \Psi_{bs}(S))$. Thus $\varphi_\alpha(f) = 0\ (\alpha \in \Psi_{bs}(S))$. Hence $f \in \mathrm{srad}\ell^1(S)$, the $*$-radical of $\ell^1(S)$. Since $\ell^1(S)$ is $*$-semisimple, $\mathrm{srad}\ell^1(S) = \{0\}$. So $f = 0$, that is, $s = t$.

(2) $\Rightarrow$ (1) The proof is the same as that of [5, Theorem 3.4].

(3) $\Rightarrow$ (2) This is clear since $\Psi_s(S) \subset \Psi_{bs}(S)$.

(2) $\Rightarrow$ (4) Let $s, t \in S$ be such that $s^*s = t^*t = s^*t$. Then we have $|\alpha(s)|^2 = |\alpha(t)|^2 = \overline{\alpha(s)}\alpha(t)\ (\alpha \in \Psi_{bs}(S))$, that is, $\alpha(s) = \alpha(t)\ (\alpha \in \Psi_{bs}(S))$. Since $\Psi_{bs}(S)$ separates the points of S, we have $s = t$.

(2) $\not\Rightarrow$ (3) Consider the semigroup $\mathbb{N}$ with usual addition and the involution being $n^* = n\ (n \in \mathbb{N})$. Then $\Phi_{bs}(\mathbb{N}) \cong \mathbb{D}^\bullet := \{z \in \mathbb{C} : |z| \leq 1,\ z \neq 0\}$ via the mapping $z \mapsto \alpha_z$, where $\alpha_z(n) = z^n\ (n \in \mathbb{N})$, $\Psi_{bs}(\mathbb{N}) \cong [-1, 1] \setminus \{0\}$, and $\Psi_s(\mathbb{N}) \cong \{-1, 1\}$. Then $\Psi_{bs}(\mathbb{N})$ separates the points of $\mathbb{N}$ but $\Psi_s(\mathbb{N})$ does not.

(4) $\not\Rightarrow$ (2) Let S be the abelian group $\mathbb{Z} \times \mathbb{Z}$. For $(m, n) \in S$, define the involution as $(m, n)^* = (-m, n)$. Then S has the P_1-property. It is easy to see that $\Psi_{bs}(S) \cong \Gamma \times \{-1, 1\}$ via the map $(z, w) \mapsto \alpha_{z,w}$, where $\alpha_{z,w}(m, n) = z^m w^n\ ((m, n) \in S)$ and Γ is the unit circle. Then $\alpha_{z,w}(1, 2) = z^1 w^2 = z = z^1 w^{-2} = \alpha_{z,w}(1, -2)$ for all $(z, w) \in \Gamma \times \{-1, 1\}$. Hence $\Psi_{bs}(S)$ does not separate the points of S. □

A $*$-semigroup S is *$*$-idempotent* if s^*s is idempotent for all $s \in S$ and if, for every $s \in S$, there exists $e_s \in S$ such that $s = e_s s^* = s^* e_s$. For example, every abelian group G with $g^* = g^{-1}\ (g \in G)$ is a $*$-idempotent semigroup. For this class of $*$-semigroups, the following theorem implies that the semisimplicity and the $*$-semisimplicity are equivalent.

Theorem 2.3. *Let S be a $*$-idempotent, abelian $*$-semigroup. Then $\ell^1(S)$ is semisimple if and only if it is $*$-semisimple.*

Proof. Let $\ell^1(S)$ be semisimple. By Theorems 2.1 and 2.2, it is enough to show that $\Phi_{bs}(S) = \Psi_{bs}(S)$. Let $\alpha \in \Phi_{bs}(S)$ and let $s \in S$. First assume that $\alpha(s) = 0$. Since S is $*$-idempotent, there exists $e_s \in S$ such that $s^* = e_s s$, and hence $\alpha(s^*) = 0$. Thus $\alpha(s^*) = \overline{\alpha(s)}$. Secondly, assume that $\alpha(s) \neq 0$. Then, by the above argument, we have $\alpha(s^*) \neq 0$. Since s^*s is an idempotent, we have $\alpha(s^*)\alpha(s) = \alpha(s^*s) = 1$. Since α is a bounded semicharacter, $\alpha(s^*) = \overline{\alpha(s)}$. This proves that $\alpha \in \Psi_{bs}(S)$. The converse holds for any Banach $*$-algebra. □

3. Some relevant conditions

Theorem 2.2 shows that the natural condition P_1 is not the correct involutive analogue of the condition P_0 so as to be equivalent to $*$-semisimplicity. Our search for a correct intrinsic condition leads to the following conditions. Though experimental, they seem to be of some relevance.

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Definition 3.1. Label properties on a $*$-semigroup S as follows.

$$\begin{aligned}
&P_1 : s = t \quad \text{whenever } s^*s = t^*t = s^*t,\\
&P_2 : s = t \quad \text{whenever } ss^*s = ss^*t = tt^*t = st^*t,\\
&P_3 : s = t \quad \text{whenever } ss^*s = tt^*t = s^3 = t^3,\\
&P_4 : s = t \quad \text{whenever } ss^*s = tt^*t = s^2t = t^2s,\\
&P_5 : s = t \quad \text{whenever } ss^*s = tt^*t,\\
&Q_1 : ss^*s = s \quad (s \in S),\\
&Q_2 : s = t \quad \text{whenever } s^*t = t^*s,\\
&Q_3 : s = t \quad \text{whenever } s^*ts = st^*s.
\end{aligned}$$

Proposition 3.2. *Let S be an abelian $*$-semigroup. Then:*

(1) $P_2 \Leftrightarrow P_1 \Rightarrow P_0$;
(2) $P_5 \Rightarrow P_3$ *and* $P_5 \Rightarrow P_4$; *and*
(3) $Q_3 \Leftrightarrow Q_2 \Rightarrow Q_1$.

Proof. (1) Assume that S has the P_2-property. Let $s, t \in S$ be such that $s^*s = t^*t = s^*t$. Then $ss^*s = st^*t = ss^*t$ and $ts^*s = tt^*t = ts^*t$. Therefore, $ss^*s = ss^*t = tt^*t = st^*t$. Hence, by the assumption, $s = t$. Thus S has the P_1-property. Conversely, assume that S has the P_1-property. Let $s, t \in S$ be such that

$$ss^*s = ss^*t = tt^*t = st^*t. \tag{3.1}$$

Set $u = s^*s$, $v = t^*t$ and $w = s^*t$. Then using (3.1) we can show that $u^*u = v^*v = w^*w = u^*v = u^*w$. Since S has the P_1-property, $u = v = w$, that is, $s^*s = t^*t = s^*t$. Again using the P_1-property, we get $s = t$.

Next, assume that S has the P_1-property. Let $s, t \in S$ be such that

$$s^2 = t^2 = st. \tag{3.2}$$

Set $u = s^*s$, $v = t^*t$ and $w = s^*t$. Then, using (3.2) we have the following relations:

$$\begin{aligned}
u^*u &= (s^*s)^*(s^*s) = s^*ss^*s = (s^*)^2s^2 = (s^2)^*s^2\\
&= (t^2)^*t^2 = t^*t^*tt = t^*tt^*t = vv = v^*v,\\
u^*v &= (s^*s)^*(t^*t) = s^*st^*t = s^*t^*st = (st)^*st\\
&= (s^2)^*s^2 = (s^*)^2s^2 = (s^*s)(s^*s) = u^*u,\\
u^*u &= (s^*s)(s^*s) = (s^*)^2s^2 = (s^2)^*s^2\\
&= (st)^*(st) = (s^*t)^*(s^*t) = w^*w,\\
u^*w &= (s^*s)(s^*t) = (s^*)^2(st) = (s^*)^2s^2.
\end{aligned}$$

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Thus $u^*v = v^*v = u^*u = w^*w = u^*w$. Since S has the P_1-property, $u = v = w$, that is, $s^*s = t^*t = s^*t$. Again by the P_1-property of S, we have $s = t$.

(2) This follows directly from the definitions.

(3) $Q_3 \Rightarrow Q_2$ is clear. Assume that S has the Q_2-property. Let $s, t \in S$ be such that $s^*ts = st^*s$. Set $u = s^*t$ and $v = t^*s$. Then $u^*v = v^*u$. Since S has the Q_2-property, $u = v$, that is, $s^*t = t^*s$. Again, since S has the Q_2-property, $s = t$. So S has the Q_3-property. Finally, let $s \in S$. Take $t = ss^*s$. Then $s^*t = s^*(ss^*s) = (ss^*s)^*s = t^*s$. Since S has the Q_2-property, $ss^*s = t = s$. Hence S has the Q_1-property. □

The following result gives some necessary, but not sufficient, conditions for the $*$-semisimplicity of $\ell^1(S)$.

Theorem 3.3. *Let S be an abelian $*$-semigroup. If $\ell^1(S)$ is $*$-semisimple, then S has the P_i-property ($i = 0, 1, \ldots, 5$).*

Proof. First we show that S satisfies the P_2-property (and hence P_0 and P_1 due to Proposition 3.2(1)). Let $s, t \in S$ be such that $ss^*s = ss^*t = tt^*t = st^*t$. Then $\alpha(ss^*s) = \alpha(ss^*t) = \alpha(tt^*t) = \alpha(st^*t)$ $(\alpha \in \Psi_{bs}(S))$. Therefore, $|\alpha(s)|^2\alpha(s) = |\alpha(s)|^2\alpha(t) = |\alpha(t)|^2\alpha(t) = \alpha(s)|\alpha(t)|^2$. This implies that $\alpha(s) = \alpha(t)$ $(\alpha \in \Psi_{bs}(S))$. Since $\Psi_{bs}(S)$ separates the points of S, $s = t$. Thus S has the P_2-property.

Now we show that S has the P_5-property (and so P_3 and P_4 by Proposition 3.2(2)). Let $s, t \in S$ be such that $ss^*s = tt^*t$. Then $\alpha(ss^*s) = \alpha(tt^*t)$, that is, $\alpha(s)|\alpha(s)|^2 = \alpha(t)|\alpha(t)|^2$. If $\alpha(s) = 0$, then $\alpha(t) = 0$. If $\alpha(t) \neq 0$, then $|\alpha(s)|^3 = |\alpha(t)|^3$, that is, $|\alpha(s)| = |\alpha(t)|$ and so $\alpha(s) = \alpha(t)$. Therefore, $\alpha(s) = \alpha(t)$ $(\alpha \in \Psi_{bs}(S))$. Since $\Psi_{bs}(S)$ separates the points of S, $s = t$. Thus S has the P_5-property. □

The following gives a sufficient, but not necessary, condition for the $*$-semisimplicity of $\ell^1(S)$.

Theorem 3.4. *Let S be an abelian $*$-semigroup. If S has the Q_1-property, then $\ell^1(S)$ is $*$-semisimple.*

Proof. First we show that $\Phi_{bs}(S)$ separates the points of S. By Theorem 2.1, it is enough to show that S has the P_0-property. Let $s, t \in S$ be such that $s^2 = t^2 = st$. Then

$$\begin{aligned} s = ss^*s = s^2s^* = sts^* = stt^*ts^* &= (st)(s^*t^*)t \\ &= (st)(s^*t^*)(tt^*t) = (st)(s^*t^*)t^2t^* = (st)(s^*t^*)(st)t^* \\ &= (ss^*s)(tt^*t)t^* = stt^* = t^2t^* = tt^*t = t. \end{aligned}$$

Next we show that $\Psi_{bs}(S) = \Phi_{bs}(S)$. Let $\alpha \in \Phi_{bs}(S)$ and $s \in S$. Since $s = ss^*s$ and $s^* = s^*ss^*$, $\alpha(s) = 0$ if and only if $\alpha(s^*) = 0$. Therefore, $\alpha(s^*) = \overline{\alpha(s)}$. Let $\alpha(s) \neq 0$. Since s^*s is an idempotent, $\alpha(s^*s) = 1$. Since α is a bounded semicharacter, $\alpha(s^*) = \overline{\alpha(s)}$. Hence $\alpha \in \Psi_{bs}(S)$. Since $\Phi_{bs}(S)$ separates points of S, $\Psi_{bs}(S)$ separates the points of S. Therefore, $\ell^1(S)$ is $*$-semisimple. □

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TABLE 1. Table of examples.

	P_0	P_1	P_3	P_4	P_5	Q_1	Q_2	$\ell^1(S)$ is $*$-s.s.
S_1	✓	✓	✓	✓	✓	✓	✓	✓
S_2	✓	✓	✓	✓	✓	✓	×	✓
S_3	✓	✓	✓	✓	✓	×	×	✓
S_4	✓	✓	✓	✓	✓	×	×	×
S_5	✓	✓	×	✓	×	×	×	×
S_6	✓	×	✓	×	×	×	×	×
S_7	×	×	×	×	×	×	×	×

The following examples and Table 1 show that the reverses of the implications discussed above do not hold.

EXAMPLES 3.5. Consider the following abelian $*$-semigroups.

(1) $S_1 = \mathbb{Z} \times \mathbb{Z}$ with the usual addition and $(m, n)^* = (-m, -n)$.
(2) $S_2 = \mathbb{N}$ with multiplication $st = \max\{s, t\}$ and $s^* = s$.
(3) $S_3 = \mathbb{N}$ with the usual addition and $n^* = n$.
(4) $S_4 = \mathbb{Z} \times \mathbb{Z}$ with the usual addition and $(m, n)^* = (-m, n)$.
(5) $S_5 = \mathbb{C}^\bullet$ with the usual multiplication and $s^* = s$.
(6) $S_6 = \mathbb{Z}$ with $st = 0$ if $s \neq t$ and $st = s$ if $s = t$ and $s^* = -s$.
(7) $S_7 = \mathbb{Z}_4$ with usual multiplication modulo 4 and $s^* = s$.

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