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Optimal growth of harmonic functions frequently hypercyclic for the partial differentiation operator

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We solve a problem posed by Blasco, Bonilla and Grosse-Erdmann in 2010 by constructing a harmonic function on $\mathbb{R}^N$, that is frequently hypercyclic with respect to the partial differentiation operator $\partial/\partial x_k$ and which has a minimal growth rate in terms of the average L^2-norm on spheres of radius $r > 0$ as $r \to \infty$.

Keywords: Frequent hypercyclicity; partial differentiation operator; harmonic functions; growth rate

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1. Introduction

For a separable Fréchet space X, the continuous linear operator $T\colon X \to X$ is *hypercyclic* if there exists $x \in X$ (called a *hypercyclic vector*) such that its orbit under T is dense in X, that is,

$$\overline{\{T^n x : n \geqslant 0\}} = X.$$

A stronger property was introduced by Bayart and Grivaux in [6], where they defined $T\colon X \to X$ to be *frequently hypercyclic* if there exists $x \in X$ such that for any nonempty open set $U \subset X$ one has

$$\liminf_{m\to\infty} \frac{\#\{n : T^n x \in U,\ 0 \leqslant n \leqslant m\}}{m+1} > 0.$$

Here # denotes the cardinality of the set. The definition states that the set of iterations, for which the orbit of x visits any given neighbourhood of X, has positive lower density and such an $x \in X$ is called a *frequently hypercyclic vector* for T. Comprehensive introductions to the area of linear dynamics can be found in [7, 12].

It was shown in [6, example 2.4] that the differentiation operator $f \mapsto f'$ is frequently hypercyclic on the space of entire holomorphic functions on $\mathbb{C}$ and estimates for the growth of its frequently hypercyclic vectors, in terms of average L^p-norms on spheres of radius $r > 0$ as $r \to \infty$, were found by Blasco, Bonilla and

Grosse-Erdmann [8, theorems 2.3, 2.4] and Bonet and Bonilla [9, corollary 2.4]. The minimal growth rates were subsequently established by Drasin and Saksman [11].

Aldred and Armitage [2] previously identified sharp growth rates, in terms of the average L^2-norm on spheres of radius $r > 0$ as $r \to \infty$, of hypercyclic vectors for the partial differentiation operators on the space $\mathcal{H}(\mathbb{R}^N)$ of harmonic functions on $\mathbb{R}^N$, which we denote by

$$\frac{\partial}{\partial x_k} : \mathcal{H}(\mathbb{R}^N) \to \mathcal{H}(\mathbb{R}^N)$$

for $N \geqslant 2$ and $1 \leqslant k \leqslant N$.

Subsequently, Blasco *et al.* [8, theorem 4.2] computed growth rates, again in terms of the L^2-norm on spheres of radius $r > 0$, in the frequently hypercyclic case and they asked about the minimal growth rates of frequently hypercyclic vectors for $\partial/\partial x_k$ on $\mathcal{H}(\mathbb{R}^N)$. In this paper, we answer their question by explicitly constructing a frequently hypercyclic harmonic function with the prescribed growth rate.

2. Frequent hypercyclicity of the partial differentiation operator

In this section, we recall the exact question posed by Blasco *et al.* [8] and we state our main result. We first introduce the notions and background required to discuss the problem in precise terms.

Denote by $S(r)$ the sphere of radius r in the euclidean metric $|\cdot|$ centred at the origin of $\mathbb{R}^N$ and let σ_r be the normalized $(N-1)$-dimensional measure on $S(r)$ so that $\sigma_r(S(r)) = 1$. For $h \in \mathcal{H}(\mathbb{R}^N)$ and $r > 0$, we let

$$M_2(h, r) = \left(\int_{S(r)} |h|^2 \, \mathrm{d}\sigma_r \right)^{1/2} \tag{2.1}$$

denote the 2-integral mean of h on $S(r)$ and for g, $h \in \mathcal{H}(\mathbb{R}^N)$ the corresponding inner product is written as

$$\langle g, h \rangle_r = \int_{S(r)} g h \, \mathrm{d}\sigma_r .$$

The space $\mathcal{H}(\mathbb{R}^N)$ of harmonic functions is a Fréchet space when equipped with the complete metric

$$d(g, h) = \sum_{n=1}^{\infty} 2^{-n} \frac{|g - h|_{S(n)}}{1 + |g - h|_{S(n)}}$$

for g, $h \in \mathcal{H}(\mathbb{R}^N)$ and it corresponds to the topology of local uniform convergence. Above we set $|f|_{S(n)} = \sup_{|x|=n} |f(x)|$ for $f \in \mathcal{H}(\mathbb{R}^N)$.

Aldred and Armitage [2, theorem 1] proved that given any function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$, there exists a harmonic function $h \in \mathcal{H}(\mathbb{R}^N)$ which is a

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$\partial/\partial x_k$-hypercyclic vector, for $1 \leqslant k \leqslant N$, such that

$$M_2(h,r) \leqslant \varphi(r)\frac{e^r}{r^{(N-1)/2}}$$

for $r > 0$ sufficiently large. Furthermore, they showed there does not exist a $\partial/\partial x_k$-hypercyclic vector $h \in \mathcal{H}(\mathbb{R}^N)$ that satisfies

$$M_2(h,r) \leqslant C\frac{e^r}{r^{(N-1)/2}} \tag{2.2}$$

for $r > 0$ and any constant $C > 0$. Strictly speaking, the results in [2] are stated for the more general concept of universality of the family $\{D^\alpha : \alpha \in \mathbb{N}^N\}$ of all partial derivatives and the preceding hypercyclicity growth results for $\partial/\partial x_k$ are implicit in the proofs.

Subsequently, Blasco *et al.* [8, § 4] considered the frequently hypercyclic case, where they obtained the following L^2-growth rates for $1 \leqslant k \leqslant N$.

1. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be any function with $\varphi(r) \to \infty$ as $r \to \infty$. Then there exists a $\partial/\partial x_k$-frequently hypercyclic function $h \in \mathcal{H}(\mathbb{R}^N)$ with

$$M_2(h,r) \leqslant \varphi(r)\frac{e^r}{r^{N/2-3/4}}$$

for $r > 0$ sufficiently large.

2. Let $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be any function with $\psi(r) \to 0$ as $r \to \infty$. Then there is no $\partial/\partial x_k$-frequently hypercyclic vector $h \in \mathcal{H}(\mathbb{R}^N)$ with

$$M_2(h,r) \leqslant \psi(r)\frac{e^r}{r^{N/2-3/4}}$$

for $r > 0$ sufficiently large.

Moreover, they asked [8, § 6] whether there exists a $\partial/\partial x_k$-frequently hypercyclic vector $h \in \mathcal{H}(\mathbb{R}^N)$ such that the above function φ can be replaced with a constant in the growth rate.

We answer this question in the positive in the following theorem, using a modification of the approach of Drasin and Saksman [11] in the case of the entire functions.

THEOREM 2.1. *Let $N \geqslant 2$ and $1 \leqslant k \leqslant N$. Then for any constant $C > 0$ there exists a $\partial/\partial x_k$-frequently hypercyclic harmonic function $h \in \mathcal{H}(\mathbb{R}^N)$ such that*

$$M_2(h,r) \leqslant C\frac{e^r}{r^{N/2-3/4}} \tag{2.3}$$

for all $r > 0$.

Similar to [11], the argument involves the explicit construction of a function in $\mathcal{H}(\mathbb{R}^N)$ that is a frequently hypercyclic vector for $\partial/\partial x_k$. By contrast, [8] applies a generalization of the Frequent Hypercyclicity Criterion in an associated separable weighted Banach space of harmonic functions which is densely embedded in

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$(\mathcal{H}(\mathbb{R}^N),\, d)$, but this general technique does not appear to be available in the case of the minimal growth rate (2.3). It is also worthwhile to note the qualitative difference between (2.3) and the corresponding behaviour (2.2) of the $\partial/\partial x_k$-hypercyclic harmonic functions from [**2**].

Furthermore, for $N = 2$ the claim can be deduced from the entire function case, as given in [**11**], by considering the real part of a corresponding frequently hypercyclic entire function possessing minimal L^2-growth (cf. the comments which appear at the beginning of the proof of proposition 5.2). Thus we are essentially concerned with the case $N \geqslant 3$, which turns out to involve different tools compared to the case $N = 2$. The required harmonic function is constructed in § 4 and we prove it is frequently hypercyclic for $\partial/\partial x_k$ in § 5. The argument is completed in § 6 by showing it has the desired minimal growth rate (2.3).

3. Harmonic polynomials

We recall in this section the crucial background and auxiliary results from [**2**, **8**, **13**] regarding harmonic polynomials on $\mathbb{R}^N$ needed to prove theorem 2.1. The space of homogeneous harmonic polynomials on $\mathbb{R}^N$ of homogeneity degree $m \geqslant 0$ is denoted by $\mathcal{H}_m(\mathbb{R}^N)$. The harmonic analogue of the standard power series representation of holomorphic functions states that any $h \in \mathcal{H}(\mathbb{R}^N)$ has a unique expansion of the form

$$h = \sum_{m=0}^{\infty} H_m \tag{3.1}$$

where $H_m \in \mathcal{H}_m(\mathbb{R}^N)$ for each $m \geqslant 0$ and the expansion converges in the metric d, see [**5**, corollary 5.34]. Moreover, $\langle H_j,\, H_k\rangle_r = 0$ when $j \neq k$, so by orthogonality one has for any $r > 0$ that

$$M_2^2(h, r) = \sum_{m=0}^{\infty} M_2^2(H_m, r).$$

The references [**4**, **5**] contain further useful background information on harmonic functions and the spaces $\mathcal{H}_m(\mathbb{R}^N)$ are discussed in detail in [**5**, chapter 5] and [**4**, chapter 2].

It will be enough to prove theorem 2.1 in the case of $\partial/\partial x_1$ and this will be our standing assumption in the sequel. The cases $\partial/\partial x_k$ for $k = 2, \ldots, N$ can be dealt with analogously.

For any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we recall a function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be x_1-*axial* if $f(x)$ depends only on x_1 and $(x_2^2 + \cdots + x_N^2)^{1/2}$. This means f is invariant under rotation around the x_1-axis, that is

$$f(x_1, x_2, \ldots, x_N) = f(y_1, y_2, \ldots, y_N)$$

whenever $x_1 = y_1$ and $x_2^2 + \cdots + x_N^2 = y_2^2 + \cdots + y_N^2$.

Kuran [**13**] used x_1-axial polynomials to construct a specific orthogonal representation of $\mathcal{H}_m(\mathbb{R}^N)$, see (3.2) below, which will be crucial for our construction. The starting point is the following fact due to Brelot and Choquet [**10**, proposition 4] (see also [4, 2.3.8]), where $d_{m,N}$ denotes the dimension of $\mathcal{H}_m(\mathbb{R}^N)$: there exists an x_1-axial polynomial $I_{m,N}$ in $\mathcal{H}_m(\mathbb{R}^N)$ satisfying $I_{m,N}(1, 0, \ldots, 0) = 1$, for which every x_1-axial element of $\mathcal{H}_m(\mathbb{R}^N)$ is proportional to $I_{m,N}$ and

$$M_2(I_{m,N}, r) = r^m \, (d_{m,N})^{-1/2}$$

for $r > 0$. It can be shown [5, proposition 5.8] that $d_{0,2} = 1$ and

$$d_{m,N} = \frac{N + 2m - 2}{N + m - 2} \binom{N + m - 2}{m}$$

for $N + m \geqslant 3$.

Kuran [13] defined homogeneous (but not necessarily harmonic) polynomials of degree $m > 0$ on $\mathbb{R}^N$ by

$$I^*_{m,N+2p}(x_1, \ldots, x_N) = I_{m,N+2p}(x_1, \ldots, x_N, \overbrace{0, \ldots, 0}^{2p})$$

for $p \in \mathbb{N}$. We denote by $\mathcal{H}^0_m(\mathbb{R}^N)$ the subspace

$$\mathcal{H}^0_m(\mathbb{R}^N) = \left\{H \in \mathcal{H}_m(\mathbb{R}^N) : \partial H/\partial x_1 \equiv 0\right\}.$$

We will need the following reformulation of [13, theorems 2,3] which is recalled from [2, lemma 3].

Proposition 3.1. *Let $m \in \mathbb{N}$.*

1. *For $p \in \mathbb{N}$, if $u \in \mathcal{H}^0_p(\mathbb{R}^N)$ then $uI^*_{m,N+2p} \in \mathcal{H}_{m+p}(\mathbb{R}^N)$ and*

$$d_{m,N+2p} M_2^2(uI^*_{m,N+2p}, 1) = M_2^2(u, 1).$$

2. *If $H \in \mathcal{H}_m(\mathbb{R}^N)$ then H has a unique representation*

$$H = \sum_{p=0}^{m} u_p I^*_{m-p,N+2p} \tag{3.2}$$

where $u_p \in \mathcal{H}^0_p(\mathbb{R}^N)$ for $p = 0, \ldots, m$ and the terms in (3.2) are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle_r$.

The preceding result allowed Aldred and Armitage [2] to define linear maps $P_k : \mathcal{H}_m(\mathbb{R}^N) \to \mathcal{H}_{m+k}(\mathbb{R}^N)$, for $k \geqslant 0$, by

$$P_k(H) = \sum_{p=0}^{m} \frac{(m-p)!}{(m-p+k)!} u_p I^*_{m-p+k,N+2p} \tag{3.3}$$

where $H \in \mathcal{H}_m(\mathbb{R}^N)$ has the representation (3.2). In view of the following fundamental lemma, taken from [2, lemma 4], we will refer to $P_k(H)$ as the k^{th} *primitive* of H.

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Lemma 3.2. *Let m, $k \geqslant 0$ and $N \geqslant 2$. If $H \in \mathcal{H}_m(\mathbb{R}^N)$ then $P_k(H) \in \mathcal{H}_{m+k}(\mathbb{R}^N)$,*

$$\frac{\partial^k}{\partial x_1^k} P_k(H) = H$$

and

$$M_2^2(P_k(H), 1) \leqslant c_{k,m,N} M_2^2(H, 1) \tag{3.4}$$

where

$$c_{k,m,N} = \frac{(N+2m-2)!}{k!(N+2m+k-3)!(N+2m+2k-2)}.$$

For fixed m, we will use the simpler estimate

$$c_{k,m,N} \leqslant \frac{c_m}{(k+m)!^2(k+m+1)^{N-2}} \tag{3.5}$$

for $k \in \mathbb{N}$, (cf. line (4.2) in [8, p. 52]). Here

$$c_m = c(m, N) \tag{3.6}$$

and the exact bound in (3.6) is not important for our purposes, but we may assume that $m \mapsto c_m$ is increasing.

Finally, the following compatibility property of the different maps P_k defined by (3.3) will be technically convenient.

Lemma 3.3. *Let $H \in \mathcal{H}_m(\mathbb{R}^N)$ and k, $\ell \geqslant 0$. Then*

$$P_{k+\ell}(H) = P_k(P_\ell(H)).$$

Proof. $P_\ell(H) \in \mathcal{H}_{m+\ell}(\mathbb{R}^N)$ and hence

$$\begin{aligned} P_k(P_\ell(H)) &= P_k\left(\sum_{p=0}^{m} \frac{(m-p)!}{(m-p+\ell)!} u_p I^*_{m-p+\ell,N+2p}\right) \\ &= \sum_{p=0}^{m} \frac{(m-p)!}{(m-p+k+\ell)!} u_p I^*_{m-p+k+\ell,N+2p} = P_{k+\ell}(H). \end{aligned}$$

□

4. Construction of the harmonic function h

Let $N \geqslant 2$ be fixed. The set of harmonic polynomials on $\mathbb{R}^N$ is dense in the separable Fréchet space $(\mathcal{H}(\mathbb{R}^N), d)$, so we can fix a d-dense sequence of harmonic polynomials $(F_k) \subset \mathcal{H}(\mathbb{R}^N)$. For technical simplicity, we will also assume that each polynomial is repeated infinitely often in the sequence. For each $k \geqslant 0$, we let m_k be the degree of F_k and by (3.1) there is a unique representation

$$F_k = \sum_{j=0}^{m_k} H_{k,j} \tag{4.1}$$

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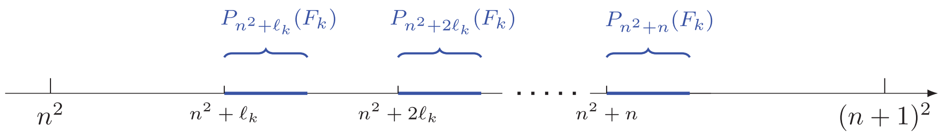

Figure 1. Scope of the degrees of the polynomials contained in Q_n.

where $H_{k,j} \in \mathcal{H}_j(\mathbb{R}^N)$ for $j = 0, \ldots, m_k$. We associate with each F_k an odd integer $\ell_k \in \mathbb{N}$, so that the sequence (ℓ_k) is strictly increasing and

$$\ell_k \geqslant 2m_k + c_{m_k} 2^k \left(M_2^2\left(F_k, 1\right) + 1\right) \tag{4.2}$$

where c_{m_k} is as defined in (3.6). The final choice for (ℓ_k) will be made later in (6.8), depending on a given constant $C > 0$.

We can unambiguously define the n^{th} primitive of F_k for all $n \in \mathbb{N}$ as

$$P_n(F_k) = \sum_{j=0}^{m_k} P_n(H_{k,j})$$

where each $P_n(H_{k,j}) \in \mathcal{H}_{n+j}(\mathbb{R}^N)$ is as defined in (3.3). It follows from lemma 3.2 that

$$\frac{\partial^n}{\partial x_1^n} P_n(F_k) = F_k.$$

We next introduce the sets

$$\mathcal{A}_k = \left\{(2m-1)\ell_k 2^k : m \geqslant 1\right\} \tag{4.3}$$

for each $k \geqslant 1$. Since each integer ℓ_k is odd, we note that the $\mathcal{A}_k$ are pairwise disjoint infinite arithmetic sequences, so that $\bigcup_{k\geqslant 1} \mathcal{A}_k$ is a partition of some subset of the even natural numbers.

To construct the required harmonic function, we define for each $n \in \mathcal{A}_k$ a harmonic polynomial Q_n which is a finite sum of suitable primitives of the harmonic polynomial F_k.

We first let $Q_n = 0$ whenever $n \notin \bigcup_{k\geqslant 1} \mathcal{A}_k$ (this includes all odd integers n) or $n = 0$. Suppose next that the even integer $n \in \mathcal{A}_k$, for a fixed unique k. If $n < 10\ell_k$, we set $Q_n = 0$ and for $n = (2m-1)\ell_k 2^k \geqslant 10\ell_k$, we define

$$Q_n = \sum_{j=1}^{(2m-1)2^k} P_{n^2+j\ell_k}(F_k).$$

We note that the degrees of the primitives of the associated harmonic polynomial F_k contained in Q_n are supported on the interval $(n^2, n^2+n+m_k]$, as illustrated in figure 1.

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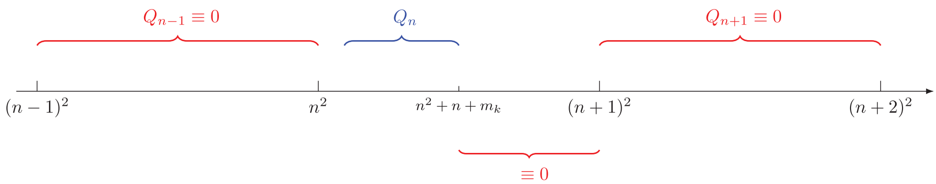

Figure 2. Scope of the degrees of the polynomials Q_n.

Finally, we define

$$h = \sum_{n=1}^{\infty} Q_n = \sum_{k=1}^{\infty} \sum_{n \in \mathcal{A}_k} Q_n \tag{4.4}$$

and we proceed to show in §§ 5 and 6 that h satisfies the claims of theorem 2.1. Note we must also verify that h is defined on the whole of $\mathbb{R}^N$ and that $h \in \mathcal{H}(\mathbb{R}^N)$. Since the details are closely related to the estimates in § 5 we defer this discussion until remark 5.3.

The estimates in §§ 5 and 6 will frequently use the fact that the respective sets of homogeneity degrees of the harmonic polynomials appearing in Q_n and $Q_{n'}$ are disjoint whenever $n \neq n'$. That is $\langle Q_n, Q_{n'} \rangle_r = 0$ for any $r > 0$, so that Q_n and $Q_{n'}$ are orthogonal for $M_2(\,\cdot\,, r)$. In fact, if $F_k = \sum_{q=0}^{m_k} H_{k,q}$ and $n = (2m-1)\ell_k 2^k \in \mathcal{A}_k$, then the homogeneity degrees related to $Q_n = \sum_{j=1}^{(2m-1)2^k} P_{n^2+j\ell_k}(F_k)$ are contained in the interval $(n^2, n^2+n+m_k]$. These intervals are pairwise disjoint for different n, as illustrated in figure 2, since in view of (4.2)

$$n^2 + n + m_k < (n+1)^2.$$

5. Frequent hypercyclicity of *h*

The aim of this section is to prove in proposition 5.2 that the function h defined in (4.4) is a frequently hypercyclic vector in $\mathcal{H}(\mathbb{R}^N)$ for the partial differentiation operator $\partial/\partial x_1$. Towards this end, we first point out that convergence in the average L^2-norm on the sphere $S(2r)$ of radius $2r$ gives convergence in the sup-norm on $S(r)$ for any $r > 0$. This depends on basic facts about the Poisson kernel, which we first recall (the details can be found in [**5**, chapter 1] or [**4**, § 1.3]).

We denote the open ball of radius $r > 0$ centred at the origin of $\mathbb{R}^N$ by $B(r)$ and put $B = B(1)$. Moreover, let $\overline{B}(r)$ be the closed ball and $S = S(1)$ the unit sphere. The *Poisson kernel* of B is the function $P\colon B \times S \to \mathbb{R}$ defined as

$$P(x,y) = \frac{1-|x|^2}{|x-y|^N}$$

for $x \in B$ and $y \in S$. It is well known for any harmonic function $h \in \mathcal{H}(\overline{B})$ that we have

$$h(x) = \int_S P(x,y)h(y)\,\mathrm{d}\sigma(y) \tag{5.1}$$

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for all $x \in B$ and where $\sigma = \sigma_1$ is the normalized $(N-1)$-dimensional measure on S, as introduced in § 2.

LEMMA 5.1. *Let h be a harmonic function on $\overline{B}(2r) \subset \mathbb{R}^N$, for $r > 0$ and $N \geqslant 2$. Then*

$$\sup_{|x|=r} |h(x)| \leqslant c_N M_2(h, 2r)$$

where c_N is a constant depending only on N.

Proof. For any $x \in B$ with $|x| \leqslant 1/2$, it follows from (5.1) and the Cauchy-Schwarz inequality that

$$\begin{aligned} |h(x)| &\leqslant \int_S P(x,y)|h(y)| \, \mathrm{d}\sigma(y) \\ &\leqslant \left(\int_S (P(x,y))^2 \, \mathrm{d}\sigma(y) \right)^{1/2} M_2(h,1) = c_N M_2(h,1) \end{aligned} \tag{5.2}$$

where c_N depends only on N.

Since dilations preserve harmonicity, we can extend (5.2) to any ball of radius $2r > 0$. To see this let $h_{2r}(x) = h(2rx)$ and notice for any $x \in B$ with $|x| \leqslant r$ that according to the appropriate normalizations

$$\begin{aligned} \sup_{|x| \leqslant 1/2} |h_{2r}(x)| &\leqslant c_N M_2(h_{2r}, 1) \\ &= c_N \left(\int_S |h(2ry)|^2 \, \mathrm{d}\sigma(y) \right)^{1/2} \\ &= c_N \left(\int_{S(2r)} |h(y)|^2 \, \mathrm{d}\sigma_{2r}(y) \right)^{1/2} = c_N M_2(h, 2r). \end{aligned}$$

This yields the claim. □

PROPOSITION 5.2. *Let $h \in \mathcal{H}(\mathbb{R}^N)$ be as defined in (4.4). Then h is a frequently hypercyclic vector in $\mathcal{H}(\mathbb{R}^N)$ for the partial differentiation operator $\partial/\partial x_1$ for any strictly increasing sequence (ℓ_k) satisfying (4.2).*

Proof. We first note that for $N = 2$ the complete result in theorem 2.1 can be deduced from the corresponding case for the entire functions in [**11**]. In fact, let $f_0 = u_0 + \mathrm{i}v_0$ be a frequently hypercyclic entire function for the differentiation operator $g \mapsto g'$ having minimal L^2-growth on $S(r)$. Then $u_0 = \mathrm{Re}(f_0)$ is a harmonic function on $\mathbb{C} = \mathbb{R}^2$ and $M_2(u_0, r) \leqslant M_2(f_0, r)$ for all $r > 0$. It is not difficult to check that u_0 is a frequently hypercyclic vector for $\partial/\partial x_1$ in $\mathcal{H}(\mathbb{R}^2)$. Consequently, we may (and will) assume for the rest of the argument that $N \geqslant 3$.

To begin the actual argument, for any $k \geqslant 1$ and $n = (2m-1)\ell_k 2^k \in \mathcal{A}_k$ with $n \geqslant 10\ell_k$ let

$$\mathcal{B}_{n,k} = \left\{ n^2 + j\ell_k \, : \, 1 \leqslant j \leqslant (2m-1)2^k \right\}.$$

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We claim that the union

$$\mathcal{B}_k = \bigcup_{n\in\mathcal{A}_k} \mathcal{B}_{n,k}$$

has positive lower density for any $k \geqslant 1$. In fact, suppose $n = (2m-1)\ell_k 2^k \in \mathcal{A}_k$ for some $m \geqslant 1$ and consider a given integer $t \in [n^2, n^2+2n]$. By inspection $\mathcal{B}_k$ contains $(2u-1)2^k$ integers from the interval

$$\left[(2u-1)^2\ell_k^2 2^{2k},\, (2u+1)^2\ell_k^2 2^{2k}\right)$$

for each $u = 1, \ldots, m-1$, so that

$$\frac{\#\left(\mathcal{B}_k \cap \{s\in\mathbb{N} : 0 \leqslant s \leqslant t\}\right)}{t} \geqslant \frac{2^k\left(\sum_{u=1}^{m-1}(2u-1)\right)}{(2m+1)^2\,\ell_k^2\,2^{2k}} = \frac{(m-1)^2}{(2m+1)^2\,\ell_k^2\,2^k}$$

which tends to $1/(4\ell_k^2, 2^k)$ as $m \to \infty$. Clearly this estimate yields that $\mathcal{B}_k$ has positive lower density.

Fix $r > 0$ and let $\widetilde{F} \in (F_u)_{u\geqslant 1}$ as well as $\varepsilon > 0$ be given. By construction, we can find an F_k from our dense sequence (F_k) such that $\widetilde{F} = F_k$ and the corresponding integer $\ell_k \geqslant (\mathrm{e}r)^2$. We claim for k large enough it holds that

$$M_2\left(\widetilde{F} - \frac{\partial^s}{\partial x_1^s}h, r\right) \leqslant \varepsilon \tag{5.3}$$

for all $s \in \mathcal{B}_{n,k}$, where $n = (2m-1)\ell_k 2^k \in \mathcal{A}_k$ and $n \geqslant 10\ell_k$.

Suppose $s = n^2 + t\ell_k \in \mathcal{B}_{n,k}$ for some integer $1 \leqslant t \leqslant (2m-1)2^k$. To compute $(\partial^s/\partial x_1^s)h$ note first that $(\partial^s/\partial x_1^s)Q_{\{n'\}} = 0$ for $n' < n$. Moreover, by construction and lemma 3.3, we get after relabelling that

$$\begin{aligned}\frac{\partial^s}{\partial x_1^s}Q_n &= \sum_{j=1}^{(2m-1)2^k} \frac{\partial^s}{\partial x_1^s}P_{n^2+j\ell_k}(F_k) = F_k + \sum_{j=t+1}^{(2m-1)2^k} \frac{\partial^s}{\partial x_1^s}P_{n^2+j\ell_k}(F_k) \\ &= F_k + \sum_{j=1}^{(2m-1)2^k-t} P_{j\ell_k}(F_k).\end{aligned}$$

Here we used that $(\partial^s/\partial x_1^s)P_{\{n^2+j\ell_{\{k\}}\}}(F_k) = 0$ for $j < t$, since $n^2 + j\ell_k + m_k < s$. The next term in h is $Q_{n+1} \equiv 0$, since $n+1$ is odd and hence

$$\frac{\partial^s}{\partial x_1^s}h - F_k = \sum_{j=1}^{(2m-1)2^k-t} P_{j\ell_k}(F_k) + \sum_{j=n+2}^{\infty} \frac{\partial^s}{\partial x_1^s}Q_j =: F + G. \tag{5.4}$$

Before we proceed to estimate $M_2^2\left((\partial^s/\partial x_1^s)h - F_k, r\right)$, it is convenient to calculate first an upper bound that will be needed later in this proof. For any $n,\, k \in \mathbb{N}$

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we consider the primitive $P_n(F_k)$, where F_k is a harmonic polynomial from our dense sequence (F_k). By orthogonality and (4.1), we have for any $r > 0$ that

$$M_2^2\left(P_n(F_k), r\right) \leqslant \sum_{i=0}^{m_k} M_2^2\left(P_n(H_{k,i}), r\right)$$

and furthermore, by (3.4), (3.5) and homogeneity it follows for each $i = 0, \ldots, m_k$ that

$$M_2^2\left(P_n(H_{k,i}), r\right) \leqslant \frac{c_i r^{2(n+i)} M_2^2\left(H_{k,i}, 1\right)}{(n+i)!^2(n+i+1)^{N-2}}.$$

Applying Stirling's formula and the fact that $c_i \leqslant c_{m_k}$ for $i = 0, \ldots, m_k$, it follows that

$$\frac{c_i r^{2(n+i)}}{(n+i)!^2(n+i+1)^{N-2}} \leqslant \frac{c_{m_k}(\mathrm{e}r)^{2(n+i)}}{2\pi(n+i)^{2(n+i)}(n+i)^{N-1}}.$$

Combining the above and using orthogonality, we get that

$$M_2^2\left(P_n(F_k), r\right) \leqslant \frac{c_{m_k}}{2\pi n^{N-1}} \sum_{i=0}^{m_k} \frac{(\mathrm{e}r)^{2(n+i)} M_2^2\left(H_{k,i}, 1\right)}{(n+i)^{2(n+i)}}. \tag{5.5}$$

Following these preparations we next estimate $M_2^2(F, r)$, where F is as defined on the right-hand side of (5.4). For each harmonic polynomial $P_{j\ell_k}(F_k)$ we get from (5.5) that

$$M_2^2\left(P_{j\ell_k}\left(F_k\right), r\right) \leqslant \frac{c_{m_k}}{2\pi \ell_k^{N-1}} \sum_{i=0}^{m_k} \frac{(\mathrm{e}r)^{2(j\ell_k+i)} M_2^2\left(H_{k,i}, 1\right)}{(j\ell_k+i)^{2(j\ell_k+i)}}.$$

By the fact that $\ell_k \geqslant (\mathrm{e}r)^2$ it follows that

$$\frac{(\mathrm{e}r)^{2(j\ell_k+i)}}{(j\ell_k+i)^{(j\ell_k+i)}} \leqslant 1$$

and hence by orthogonality

$$M_2^2\left(P_{j\ell_k}\left(F_k\right), r\right) \leqslant \frac{c_{m_k}}{2\pi \ell_k^{N-1} \ell_k^j} M_2^2(F_k, 1).$$

By summing up, we have that

$$\begin{aligned} \sum_{j=1}^{(2m-1)2^k - t} M_2^2\left(P_{j\ell_k}\left(F_k\right), r\right) &\leqslant \frac{c_{m_k}}{2\pi \ell_k^{N-1}} M_2^2(F_k, 1) \sum_{j=1}^{\infty} \frac{1}{\ell_k^j} \\ &\leqslant \frac{1}{2\pi \ell_k^{N-1}} \cdot \frac{c_{m_k} M_2^2(F_k, 1)}{\ell_k} \left(\frac{1}{1-(1/\ell_k)}\right) \\ &\leqslant \frac{1}{2\ell_k^{N-1}} \end{aligned}$$

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where we applied (4.2) and a geometric series estimate. We conclude that

$$M_2^2(F, r) \leqslant \frac{1}{2\ell_k^{N-1}}. \tag{5.6}$$

Next, we estimate $M_2^2(G, r)$, where G is defined on the right-hand side of (5.4). Suppose $j \geqslant n+2$ with $j = (2m'-1)\ell_{k'}2^{k'} \in \mathcal{A}_{k'}$ for some k', $m' \geqslant 1$. By orthogonality, we have

$$M_2^2\left(\frac{\partial^s}{\partial x_1^s}Q_j, r\right) = \sum_{q=1}^{(2m'-1)2^{k'}} M_2^2\left(P_{j^2-s+q\ell_{k'}}(F_{k'}), r\right)$$

and by (5.5) it follows for each harmonic polynomial $P_{j^2-s+q\ell_{k'}}(F_{k'})$ that

$$\begin{aligned} &M_2^2\left(P_{j^2-s+q\ell_{k'}}(F_{k'}), r\right) \\ &\qquad \leqslant \frac{c_{m_{k'}}}{2\pi(j^2-s+q\ell_{k'})^{N-1}} \sum_{i=0}^{m_{k'}} \frac{(\mathrm{e}r)^{2(j^2-s+q\ell_{k'}+i)} M_2^2(H_{k',i}, 1)}{(j^2-s+q\ell_{k'}+i)^{2(j^2-s+q\ell_{k'}+i)}}. \end{aligned}$$

Recall next that $1 \leqslant t \leqslant (2m-1)2^k$ and $n \geqslant t\ell_k$ so that

$$j^2 - s \geqslant (n+2)^2 - n^2 - t\ell_k > 2n. \tag{5.7}$$

Applying the fact that $n \geqslant 10\ell_k \geqslant (\mathrm{e}r)^2$ one also has

$$\frac{(\mathrm{e}r)^{2(j^2-s+q\ell_{k'}+i)}}{(2n+q\ell_{k'})^{j^2-s+q\ell_{k'}+i}} \leqslant 1$$

for $i = 0, \ldots, m_{k'}$. By adding these estimates, applying (5.7) and using the fact that $n \geqslant 10\ell_k$, we get again by orthogonality that

$$M_2^2\left(P_{j^2-s+q\ell_{k'}}(F_{k'}), r\right) \leqslant \frac{c_{m_{k'}} M_2^2(F_{k'}, 1)}{2\pi\ell_k^{N-1}(2n+q\ell_{k'})^{j^2-s+q\ell_{k'}}}.$$

From this estimate, we get that

$$\begin{aligned} M_2^2\left(\frac{\partial^s}{\partial x_1^s}Q_j, r\right) &= \sum_{q=1}^{(2m'-1)2^{k'}} M_2^2\left(P_{j^2-s+q\ell_{k'}}(F_{k'}), r\right) \\ &\leqslant \sum_{q=1}^{\infty} \frac{c_{m_{k'}} M_2^2(F_{k'}, 1)}{2\pi\ell_k^{N-1}(2n+q\ell_{k'})^{j^2-s+q\ell_{k'}}} \\ &\leqslant \frac{c_{m_{k'}} M_2^2(F_{k'}, 1)}{2\pi\ell_k^{N-1}(2n)^{j^2-s}} \sum_{q=1}^{\infty} \frac{1}{\ell_{k'}^q} \\ &\leqslant \frac{1}{2\pi\ell_k^{N-1}(2n)^{j^2-s}} \cdot \frac{c_{m_{k'}} M_2^2(F_{k'}, 1)}{\ell_{k'}} \left(\frac{1}{1-(1/\ell_{k'})}\right) \\ &\leqslant \frac{1}{2\ell_k^{N-1}(2n)^{j^2-s}} \end{aligned}$$

where we again applied (4.2) and a crude geometric series estimation.

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By summing over $j \geqslant n+2$, we arrive at

$$\sum_{j=n+2}^{\infty} M_2^2\left(\frac{\partial^s}{\partial x_1^s} Q_j, r\right) \leqslant \frac{1}{2\ell_k^{N-1}} \sum_{j=n+2}^{\infty} \frac{1}{(2n)^{j^2-s}} \leqslant \frac{1}{2\ell_k^{N-1}} \tag{5.8}$$

where we again used a geometric series estimation and that $n \geqslant 10\ell_k$.

By combining (5.6) and (5.8), and taking into account orthogonality and the fact that $N \geqslant 3$, we finally get that

$$M_2^2\left(\widetilde{F} - \frac{\partial^s}{\partial x_1^s} h, r\right) \leqslant \frac{1}{\ell_k^2}.$$

Since $\widetilde{F} = F_k$ for infinitely many k and the sequence (ℓ_k) is strictly increasing, we can certainly find k such that $\ell_k^{-2} < \varepsilon$.

In conclusion, we have shown that we can estimate $\widetilde{F}$ up to any given $\varepsilon > 0$ by partial derivatives of h associated with $\mathcal{A}_k$ in the L^2-norm on the sphere $S(r)$ for any fixed $r > 0$. By applying lemma 5.1 we obtain a similar estimate in the sup-norm on the closed ball $\overline{B}(r/2)$. This completes the proof of proposition 5.2, since for each k the set $\mathcal{B}_k$ corresponding to $\mathcal{A}_k$ has positive lower density. □

This is a suitable point to verify that h defined in (4.4) does indeed define a harmonic function on $\mathbb{R}^N$.

REMARK 5.3. We claim that $h \in \mathcal{H}(\mathbb{R}^N)$. Let $r > 0$ be fixed. Observe that if $n \in \mathcal{A}_k$ and we take $s = 0$ in the upper bound in (5.8) for the remainder $G_0 = \sum_{j=n+2}^{\infty} Q_j$ defined in (5.4), then by following the argument from proposition 5.2 we obtain that

$$M_2^2\left(\sum_{j=n+2}^{\infty} Q_j, r\right) \leqslant \frac{1}{\ell_k^2}$$

for all large enough k (depending on r). This implies by lemma 5.1 that the remainder term of the series defining h in (4.4) converges uniformly to 0 on the closed ball $\overline{B}(r/2)$ for any fixed $r > 0$. By completeness, the partial sums of h then converge to a harmonic function defined on the whole of $\mathbb{R}^N$.

6. Growth rate of h

In this section, we complete the proof of theorem 2.1 by showing that the frequently hypercyclic harmonic function h from (4.4) has the desired minimal L^2-growth rate as soon as the sequence (ℓ_k) from (4.2) grows fast enough. For this purpose, we need the following useful lemma which is a variant of [**8**, lemma 2.2].

LEMMA 6.1. *Let $N \geqslant 2$ be given. Then there exists a constant $C > 0$ such that for all given integers $\ell \geqslant 1$, $u \in \{0, \ldots, \ell-1\}$ and radii $r > 0$ it holds that*

$$\sum_{k=\ell}^{\infty} \frac{r^{2(\ell k+u)}}{(\ell k+u)!^2(\ell k+u+1)^{N-2}} \leqslant \frac{C}{\ell} \cdot \frac{\mathrm{e}^{2r}}{r^{N-3/2}}.$$

For the proof of the lemma we record the following useful observation.

Lemma 6.2. *Let $(a_n)_{n\geqslant 0}$ be a summable sequence of non-negative real numbers, for which there exists $n_0 \geqslant 2$ such that the elements $n \mapsto a_n$ are increasing for $n \leqslant n_0$ and decreasing for $n \geqslant n_0$. Then for any $\ell \geqslant 2$ and any $u \in \{0, \ldots, \ell-1\}$ we have*

$$\sum_{k=1}^{\infty} a_{k\ell+u} \;\leqslant\; \ell^{-1}\left(\sum_{n=0}^{\infty} a_n\right) + 2\sup_{n\geqslant 0} a_n.$$

In the case when the sequence $(a_n)_{n\geqslant 0}$ is decreasing we have the stronger estimate

$$\sum_{k=1}^{\infty} a_{k\ell+u} \;\leqslant\; \ell^{-1}\left(\sum_{n=0}^{\infty} a_n\right).$$

Proof. Let $k \geqslant 0$ be given. If $k\ell + u \leqslant n_0 - \ell + 1$ we obtain that

$$a_{k\ell+u} \leqslant \ell^{-1}(a_{k\ell+u} + a_{k\ell+u+1} + \cdots + a_{k\ell+u+\ell-1})$$

since the sequence (a_n) is increasing for these indices. Correspondingly, if $k\ell + u \geqslant n_0 + \ell$ we get that

$$a_{k\ell+u} \leqslant \ell^{-1}(a_{k\ell+u-\ell+1} + a_{k\ell+u-\ell+2} + \cdots + a_{k\ell+u})$$

since (a_n) is decreasing here. We obtain the claim by summing these estimates over all possible values of k and noting that all indices k such that $k\ell + u \notin [n_0 - \ell + 1, n_0 + \ell]$ are covered, apart from at most two. □

Proof of lemma 6.1. We may assume without loss of generality that $r \geqslant 2$. We start by recalling the following estimate, which can be found in [8, lemma 2.2]

$$\sum_{k=0}^{\infty} \frac{r^{2n}}{n!^2} \lesssim \frac{\mathrm{e}^{2r}}{r^{1/2}}, \quad \text{for } r > 0. \tag{6.1}$$

Here (and below) $\lesssim$ denotes an inequality up to a numerical constant. By noting that

$$\frac{r^{2(n+1)}/(n+1)!^2}{r^{2n}/n!^2} = \left(\frac{r}{n+1}\right)^2 \tag{6.2}$$

and using Stirling's formula to estimate both the $\lfloor r \rfloor^{\text{th}}$ and the $(\lfloor r \rfloor + 1)^{\text{th}}$ terms, we easily verify that the maximal term of the above series satisfies the estimate

$$\frac{r^{2n}}{n!^2} \lesssim \frac{\mathrm{e}^{2r}}{r}. \tag{6.3}$$

From (6.1), we deduce immediately that

$$\sum_{n\geqslant \lfloor r/2 \rfloor} \frac{r^{2n}}{n!^2(n+1)^{N-2}} \lesssim \frac{\mathrm{e}^{2r}}{r^{N-3/2}}, \quad \text{for } r > 0. \tag{6.4}$$

On the other hand, by the monotonicity of the terms and Stirling's formula, we have the crude bound

$$\sum_{n<\lfloor r/2\rfloor} \frac{r^{2n}}{n!^2(n+1)^{N-2}} \leqslant \sum_{n<\lfloor r/2\rfloor} \frac{r^{2n}}{n!^2} \leqslant \lfloor r/2\rfloor \left(\frac{\lfloor r/2\rfloor^{\lfloor r/2\rfloor}}{\lfloor r/2\rfloor!}\right)^2$$
$$\lesssim r\left(\frac{\mathrm{e}^{r/2}}{r^{1/2}}\right)^2 = \mathrm{e}^r \lesssim \mathrm{e}^{2r} r^{3/2-N}. \tag{6.5}$$

In combination with (6.4) this yields that

$$\sum_{n=0}^{\infty} \frac{r^{2n}}{n!^2(n+1)^{N-2}} \lesssim \frac{\mathrm{e}^{2r}}{r^{N-3/2}}, \quad \text{for } r>0. \tag{6.6}$$

We next observe that (6.5) implies that the maximal term among the first $\lfloor r/2\rfloor$ terms of the series

$$\sum_{n=0}^{\infty} \frac{r^{2n}}{n!^2(n+1)^{N-2}}$$

is dominated by e^r. On the other hand, in view of (6.3) the remaining terms of this series have the upper bound $\lesssim \mathrm{e}^{2r} r^{1-N}$. A fortiori, the latter bound is an upper bound for all the terms of the sum (6.6).

Next observe that for each $r>0$ the function

$$n \mapsto \log\left(\frac{r^{2n}}{n!^2(n+1)^{N-2}}\right)$$

is concave for $n \geqslant n_0(N)$, where the bound only depends on N. Assuming this for a moment, we complete the proof of lemma 6.1 as follows. The concavity allows us to invoke lemma 6.2 for large enough $\ell \geqslant \ell_0(N)$ and we deduce that

$$\sum_{k=2\ell}^{\infty} \frac{r^{2(\ell k+u)}}{(\ell k+u)!^2(\ell k+u+1)^{N-2}} \lesssim \frac{C}{\ell}\cdot\frac{\mathrm{e}^{2r}}{r^{N-3/2}} + \mathrm{e}^{2r} r^{1-N}.$$

For $r \geqslant \ell^2$ this inequality immediately yields the desired estimate. On the other hand, if $r<\ell^2$ one notes that the left-hand series is decreasing starting from the index $k=\ell$, whence the claim follows directly from the second statement in lemma 6.2.

To verify the claim about concavity we consider the related function

$$\psi(x) := 2x\log r - (\log(\Gamma(x+1)) + (N-2)\log(x+1)).$$

It follows from (31) of [**1**, p. 200] that the second derivative of the logarithmic gamma function is

$$\psi''(x) = -\sum_{n=0}^{\infty} \frac{1}{(x+n+1)^2} + \frac{N-2}{(x+1)^2}.$$

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This implies that $\psi''(x) < 0$ for all $x > x_N$, since

$$\sum_{n=0}^{\lfloor x\rfloor} \frac{1}{(x+1+n)^2} \geqslant \frac{x}{(2x+1)^2}.$$

□

The proof of theorem 2.1 is completed by the following proposition.

Proposition 6.3. *Let $h \in \mathcal{H}(\mathbb{R}^N)$ be as defined in (4.4). Then for any given constant $C > 0$ there exists a choice of (ℓ_k) so that the $\partial/\partial x_1$-frequently hypercyclic harmonic function $h \in \mathcal{H}(\mathbb{R}^N)$ satisfies*

$$M_2(h,r) \leqslant C\frac{\mathrm{e}^r}{r^{N/2-3/4}}$$

for all $r > 0$.

Proof. Recall that h has the representation $h = \sum_{k=1}^{\infty}\sum_{n\in\mathcal{A}_k} Q_n$ whence

$$M_2^2(h,r) = \sum_{k=1}^{\infty}\sum_{n\in\mathcal{A}_k} M_2^2(Q_n,r)$$

by orthogonality for any $r > 0$. Moreover, for any fixed k and any $n = (2m-1)\ell_k 2^k \in \mathcal{A}_k$ we further obtain, by orthogonality, (4.1), (3.4), (3.5), as well as changing the order of summation, that

$$\begin{aligned}
M_2^2(Q_n,r) &= \sum_{j=1}^{(2m-1)2^k} M_2^2\left(P_{n^2+j\ell_k}(F_k),r\right)\\
&= \sum_{j=1}^{(2m-1)2^k}\sum_{q=0}^{m_k} M_2^2\left(P_{n^2+j\ell_k}(H_{k,q}),r\right)\\
&\leqslant c_{m_k}\sum_{q=0}^{m_k} M_2^2(H_{k,q},1)\sum_{j=1}^{(2m-1)2^k}\frac{r^{2(n^2+j\ell_k+q)}}{(n^2+j\ell_k+q)!^2(n^2+j\ell_k+q+1)^{N-2}}.
\end{aligned}$$

We also used above that $c_q \leqslant c_{m_k}$ for $0 \leqslant q \leqslant m_k$.

Recall next that the sets $\{n^2 + j\ell_k : 1 \leqslant j \leqslant (2m-1)2^k\}$ are pairwise disjoint as the $n = (2m-1)\ell_k 2^k \in \mathcal{A}_k$ vary. Hence we may add the above estimates over the disjoint blocks of indices corresponding to $n \in \mathcal{A}_k$. Using the facts that $n \geqslant 10\ell_k$, ℓ_k divides n, by orthogonality and applying lemma 6.1, we obtain that

$$\begin{aligned}
\sum_{n\in\mathcal{A}_k} M_2^2(Q_n,r) &\leqslant c_{m_k}\sum_{q=0}^{m_k} M_2^2(H_{k,q},1)\sum_{j=2\ell_k}^{\infty}\frac{r^{2(j\ell_k+q)}}{(j\ell_k+q)!^2(j\ell_k+q+1)^{N-2}}\\
&\leqslant c_{m_k}\sum_{q=0}^{m_k}\frac{C' M_2^2(H_{k,q},1)}{\ell_k}\cdot\frac{\mathrm{e}^{2r}}{r^{N-3/2}}\\
&= C'\frac{c_{m_k} M_2^2(F_k,1)}{\ell_k}\cdot\frac{\mathrm{e}^{2r}}{r^{N-3/2}} \qquad (6.7)
\end{aligned}$$

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where the numerical constant $C' > 0$ from lemma 6.1 is independent of ℓ_k and $q = 0, \ldots, m_k$.

As the final step, we successively choose the integers of the increasing sequence (ℓ_k) large enough so that (4.2) is satisfied and in addition,

$$C' \sum_{k=1}^{\infty} \frac{c_{m_k} M_2^2(F_k, 1)}{\ell_k} \leqslant C^2 \tag{6.8}$$

where $C > 0$ is the given constant in the proposition. By summing (6.7) over k and taking (6.8) into account, we arrive at the desired growth estimate

$$M_2^2(h, r) = \sum_{k=1}^{\infty} \sum_{n \in \mathcal{A}_k} M_2^2(Q_n, r) \leqslant C^2 \frac{\mathrm{e}^{2r}}{r^{N-3/2}}.$$

□

Combining propositions 5.2 and 6.3, we have established theorem 2.1.

7. Concluding remarks

A natural further question is, what are the precise growth rates of the L^p-norm on $S(r)$ for $\partial/\partial x_k$-frequently hypercyclic harmonic functions $h \in \mathcal{H}(\mathbb{R}^N)$ for $p \neq 2$? A comparison with the entire functions case [**11**] indicates that additional tools will be required. Note that [**3**] and [**8**] also contain results related to general partial differentiation operators D^α on $\mathcal{H}(\mathbb{R}^N)$ for $\alpha = (\alpha_1, \ldots, \alpha_N)$.

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