# A Stability Result for Families with Fixed Diameter

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Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets. The *diameter* of $\mathcal{F}$ is the maximum of the size of symmetric differences among pairs of members of $\mathcal{F}$. In 1966 Kleitman determined the maximum of $|\mathcal{F}|$ for fixed diameter. However, this important classical result lacked a characterization of the families meeting the bound. This is remedied in the present paper, where a best possible stability result is established as well.

In Section 4 we introduce a 'parity trick' that provides an easy way of deducing the odd case from the even case in both Kleitman's original theorem and its stability version.

## 1. Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard $n$-element set. The power set of $[n]$ is denoted by $2^{[n]}$. A subset $\mathcal{F}$ of $2^{[n]}$ is called a *family*.

For $i \in [n]$ we define $\mathcal{F}(i) = \{F - \{i\} : i \in F \in \mathcal{F}\}$ and $\mathcal{F}(\bar{i}) = \{F \in \mathcal{F} : i \notin F\}$. Note that both $\mathcal{F}(i)$ and $\mathcal{F}(\bar{i})$ are subsets of $2^{[n]-\{i\}}$, and $|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|$ holds.

Sometimes, in particular in coding theory, $2^{[n]}$ is considered a metric space with the distance of $A, B \subset [n]$ defined as the size of the symmetric difference. That is,

$$d(A, B) = |A \setminus B| + |B \setminus A|.$$

Here $2^{[n]}$ can be regarded as an elementary Abelian group of order $2^n$ as well. One defines the addition modulo 2, that is, $A + B = \{i : i \text{ is contained in exactly one of } A \text{ and } B\}$. Of course, $A + B = A \setminus B \cup B \setminus A$ is the symmetric difference of $A$ and $B$.

The *diameter* $\Delta(\mathcal{F})$ of a family $\mathcal{F} \subset 2^{[n]}$ is simply

$$\max\{d(A, B) : A, B \in \mathcal{F}\}$$

or equivalently $\quad \max\{|A + B| : A, B \in \mathcal{F}\}.$

Let us note the obvious inequality

$$|A + B| \leqq |A \cup B|. \tag{1.1}$$

One can argue that extremal set theory emerged as an independent field within combinatorics via the many problems and conjectures posed by Paul Erdős. Let us mention here two important classical results.

First define the function $m(n, s)$ for $n > s \geqq 0$ by

$$m(n, s) = \begin{cases} \displaystyle\sum_{0 \leqq i \leqq s/2} \binom{n}{i} & \text{if } s \text{ is even,} \\ 2 \cdot \displaystyle\sum_{0 \leqq i \leqq s/2} \binom{n-1}{i} & \text{if } s \text{ is odd.} \end{cases}$$

**Katona Theorem [4].**  Suppose that $\mathcal{F} \subset 2^{[n]}$ satisfies

$$|F \cup F'| \leqq s \quad \text{for all } F, F' \in \mathcal{F}. \tag{1.2}$$

Then, for all $n > s \geqq 0$, we obtain

$$|\mathcal{F}| \leqq m(n, s). \tag{1.3}$$

Moreover, if $n \geqq s + 2$ then equality holds in (1.3) if and only if $\mathcal{F}$ is of the following form:

(i) if $s$ is even,

$$\mathcal{F} = \left\{ F \subset [n] : |F| \leq \frac{s}{2} \right\} \overset{\text{def}}{=} \mathcal{K}(n, s),$$

(ii) if $s$ is odd,

$$\mathcal{F} = \left\{ F \subset [n] : |F \cap ([n] - \{y\})| \leqq \frac{s}{2} \right\} \overset{\text{def}}{=} \mathcal{K}_y(n, s),$$

for some fixed element $y \in [n]$.

**Kleitman Diameter Theorem [5].**  Suppose that $\mathcal{F} \subset 2^{[n]}$ satisfies

$$|F + F'| \leqq s \quad \text{for all } F, F' \in \mathcal{F}. \tag{1.4}$$

Then, for all $n > s \geqq 0$, we have

$$|\mathcal{F}| \leqq m(n, s). \tag{1.5}$$

In view of (1.1), the bound (1.5) is stronger than (1.3). On the other hand, no uniqueness is proved.

One of the aims of the present paper is to remedy this problem and for $n \geqq s + 2$ determine all families attaining equality in (1.5).

Let us mention that Kleitman proves (1.5) by reducing the problem on $|F + F'|$ to that on $|F \cup F'|$. For this reason he introduces the very useful operation of *down-shift*, $S_j$.

**Definition.** Let $\mathcal{F} \subset 2^{[n]}$, $j \in [n]$. Define $S_j(\mathcal{F}) = \{S_j(F) : F \in \mathcal{F}\}$, where

$$S_j(F) = \begin{cases} F - \{j\} & \text{if } j \in F \in \mathcal{F} \text{ and } (F - \{j\}) \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

**Claim 1.1 (Kleitman [5]).** $|S_j(\mathcal{F})| = |\mathcal{F}|$ *and* $\Delta(S_j(\mathcal{F})) \leqq \Delta(\mathcal{F})$ *hold.*

The following are easy to verify:

$$\widetilde{\mathcal{F}}(j) = \mathcal{F}(j) \cap \mathcal{F}(\bar{j}), \tag{1.6}$$

$$\widetilde{\mathcal{F}}(\bar{j}) = \mathcal{F}(j) \cup \mathcal{F}(\bar{j}) \quad \text{and} \tag{1.7}$$

$$|\widetilde{\mathcal{F}}(i)| = |\mathcal{F}(i)|, |\widetilde{\mathcal{F}}(\bar{i})| = |\mathcal{F}(\bar{i})| \quad \text{for all } i \neq j. \tag{1.8}$$

**Definition.** For a family $\mathcal{G} \subset 2^{[n]}$ and a set $S \subset [n]$, we define $\mathcal{G} + S$, the *translation* of $\mathcal{G}$ by $S$, in the following way:

$$\mathcal{G} + S = \{G + S : G \in \mathcal{G}\}.$$

Note that $\Delta(\mathcal{G} + S) = \Delta(\mathcal{G})$ holds.

Our main result is the following.

**Theorem 1.2.** *Let $n \geq s + 2$, $s \geq 0$ and let $\mathcal{F} \subset 2^{[n]}$ satisfy $\Delta(\mathcal{F}) \leqq s$. Then*

$$\begin{aligned}|\mathcal{F}| = m(n, s) \text{ implies that } \mathcal{F} \text{ is a translate of} \\ \mathcal{K}(n, s) \text{ (for } s \text{ even) or of } \mathcal{K}_y(n, s) \text{ (for } s \text{ odd).}\end{aligned} \tag{1.9}$$

*Moreover, the following stability results hold.*
*If $s = 2d$ and $\mathcal{F}$ is not contained in any translate of $\mathcal{K}(n, s)$, then*

$$|\mathcal{F}| \leqq \sum_{0 \leqq i \leqq d} \binom{n}{i} - \binom{n - d - 1}{d} + 1. \tag{1.10}$$

*If $s = 2d + 1$ and $\mathcal{F}$ is not contained in any translate of $\mathcal{K}_y(n, s)$ (for any $y \in [n]$), then*

$$|\mathcal{F}| \leqq 2 \sum_{0 \leqq i \leqq d} \binom{n - 1}{i} - \binom{n - d - 2}{d} + 1. \tag{1.11}$$

Let us mention that both (1.10) and (1.11) are best possible. We shall discuss this in the next section.

## 2. Tools of proofs

Let us recall that a family $\mathcal{C} \subset 2^{[n]}$ is called a *complex* (or down-set) if for all $D \subset C \in \mathcal{C}$ we have $D \in \mathcal{C}$.

**Claim 2.1 (Kleitman [5]).** *If $\mathcal{C}$ is a complex, then*

$$|C \cup C'| \leq \Delta(\mathcal{C}) \text{ holds for all } C, C' \in \mathcal{C}.$$

**Proof.** Set $C'' = C' \setminus C$. Then $C'' \in \mathcal{C}$ and $C \cup C' = C \cup C'' = C + C''$, implying the statement. $\square$

In fact Kleitman deduced (1.5) from (1.3) by repeatedly applying the down-shift $S_j$, $1 \leq j \leq n$. If we start with a family $\mathcal{F} \subset 2^{[n]}$ satisfying $\Delta(\mathcal{F}) \leq s$, then we end up with a complex $\mathcal{C}$. In view of Claims 1.1 and 2.1, the complex $\mathcal{C}$ satisfies $|\mathcal{C}| = |\mathcal{F}|$, $\Delta(\mathcal{C}) \leq s$ and even $|C \cup C'| \leq s$ for all $C, C' \in \mathcal{C}$. Thus, applying (1.3) to $\mathcal{C}$ yields (1.5).

We are going to imitate Kleitman's approach. However, since $S_j(\mathcal{F})$ changes the structure of $\mathcal{F}$, we have to be careful. On the other hand, if the 'end-product' $\mathcal{C}$ is a complex which is not contained in $\mathcal{K}(n,s)$ or $\mathcal{K}_y(n,s)$, then we can apply the following recent stability theorem.

**Theorem 2.2 ([1]).** *Suppose that $\mathcal{F} \subset 2^{[n]}$ ($n \geq s+2 \geq 2$) satisfies $|F \cup F'| \leq s$ for all $F, F' \in \mathcal{F}$.*
(i) *If $s = 2d$ and $\mathcal{F} \not\subset \mathcal{K}(n,s)$, then*

$$|\mathcal{F}| \leq \sum_{0 \leq i \leq d} \binom{n}{i} - \binom{n-d-1}{d} + 1. \tag{2.1}$$

(ii) *If $s = 2d+1$ and there is no $y \in [n]$ such that $\mathcal{F} \subset \mathcal{K}_y(n,s)$, then*

$$|\mathcal{F}| \leq \sum_{0 \leq i \leq d} \binom{n}{i} + \binom{n-1}{d} - \binom{n-d-2}{d} + 1. \tag{2.2}$$

For the inexperienced reader it might not be clear that the right-hand side of (2.2) is less than $m(n, 2d+1)$. However, it follows from

$$2 \sum_{0 \leq i \leq d} \binom{n-1}{i} = \sum_{0 \leq i \leq d} \binom{n}{i} + \binom{n-1}{d}.$$

Let us show the constructions giving equality in (2.1) and (2.2). First we suppose that $s = 2d$ and let $D \in \binom{[n]}{d+1}$ be a fixed $d+1$-element set. Define

$$\mathcal{H}(n,s) = \{H \subset [n] : |H| \leq d\} \cup \{D\} \setminus \left\{ H \in \binom{[n]}{d} : H \cap D = \emptyset \right\}.$$

Next consider the case $s = 2d+1$ and fix a $D \in \binom{[n-1]}{d+1}$. First define the intersecting family $\mathcal{H}_0(n,s)$:

$$\mathcal{H}_0(n,s) = \left\{ H \in \binom{[n]}{d+1} : n \in H, \ H \cap D \neq \emptyset \right\} \cup \{D\}.$$

Then define

$$\mathcal{H}(n,s) = \{H \subset [n] : |H| \leq d\} \cup \mathcal{H}_0(n,s).$$

It is easy to check that in both cases $|H \cup H'| \leq s$ holds for all $H, H' \in \mathcal{H}(n,s)$. Consequently, $\Delta(\mathcal{H}(n,s)) \leq s$. Therefore $\Delta(\mathcal{H}(n,s) + S) \leq s$ for all $S \subset [n]$ as well.

In [1] it is also proved that unless $s = 5$, $\mathcal{H}(n,s)$ are the only families for which equality holds in (2.1) and (2.2).

A family $\mathcal{H} \subset \binom{[n]}{k}$ is called *intersecting* if $H \cap H' \neq \emptyset$ for all $H, H' \in \mathcal{H}$. It is called *non-trivial* if $\bigcap_{H \in \mathcal{H}} H = \emptyset$. We shall use the following classical result.

**Hilton–Milner Theorem [3].** If $\mathcal{H} \subset \binom{[n]}{k}$ is a non-trivial intersecting family, $n > 2k$, then

$$|\mathcal{H}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

During the proof we shall also need the following inequality. Let us recall that two families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are called *cross-intersecting* if $A \cap B \neq \emptyset$ holds for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

**Proposition 2.3 (Frankl and Tokushige [2]; see also Wang and Zhang [6]).** *Suppose that* $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$, $n > 2k$, *are cross-intersecting, non-empty and* $\mathcal{A} \cap \mathcal{B} = \emptyset$. *Then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \binom{n-k}{k}. \tag{2.3}$$

Note that (2.3) is a slight improvement over an inequality used by Hilton and Milner [3].

## 3. Proof of Theorem 2.2

We start with a family $\mathcal{F} \subset 2^{[n]}$ satisfying $\Delta(\mathcal{F}) \leq s$. Define

$$S = \{i : |\mathcal{F}(i)| > |\mathcal{F}(\bar{i})|\} \subset [n].$$

Then the translated family $\widetilde{\mathcal{F}} = \mathcal{F} + S$ satisfies $\Delta(\widetilde{\mathcal{F}}) = \Delta(\mathcal{F})$ and $|\widetilde{\mathcal{F}}(i)| \leq |\widetilde{\mathcal{F}}(\bar{i})|$ for all $i \in [n]$. Without loss of generality we can consider $\mathcal{F} + S$ instead of $\mathcal{F}$. Thus we assume that $|\mathcal{F}(i)| \leq |\mathcal{F}(\bar{i})|$ holds for all $i \in [n]$ at the start.

If $\mathcal{F}$ is a complex, then by Claim 2.1 we have $|F \cup F'| \leq s$ for all $F, F' \in \mathcal{F}$ and the statements follow directly from Theorem 1.2.

Suppose that $\mathcal{F}$ is not a complex. Since repeated applications of the down-shift $S_j$, $1 \leq j \leq n$, eventually turn $\mathcal{F}$ into a complex, there must be an intermediary family $\mathcal{G}$ satisfying the following:

$$|\mathcal{G}| = |\mathcal{F}|, \quad \Delta(\mathcal{G}) \leq \Delta(\mathcal{F}) \leq s,$$

and $\mathcal{G}$ is not a complex but $S_j(\mathcal{G})$ is a complex. To have part of the argument unified for $s$ even and $s$ odd, we use the symbol $\mathcal{K}(n,s)$ for $s$ odd as well, having $\mathcal{K}_y(n,s)$ in mind for some unspecified $y \in [n]$. If $S_j(\mathcal{G}) \not\subset \mathcal{K}(n,s)$, applying Theorem 1.2 concludes the proof. Thus we assume that $S_j(\mathcal{G}) \subset \mathcal{K}(n,s)$ holds. It might happen that $\mathcal{G} \subset \mathcal{K}(n,s)$. In that case we backtrack and consider the last $\mathcal{G}$ with $\mathcal{G} \not\subset \mathcal{K}(n,s)$ but $S_j(\mathcal{G}) \subset \mathcal{K}(n,s)$ (by abuse of notation we use the same letter $j$).

Let us first consider the case $s = 2d$. There must exist some $D \in \mathcal{G} \cap \binom{n}{d+1}$ such that $j \in D$ and $D - \{j\} \notin \mathcal{G}$. Define the two families $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\mathcal{A} = \left\{ A \in \binom{[n]}{d} : j \notin A, \ (A \cup \{j\}) \in \mathcal{G} \right\},$$

$$\mathcal{B} = \left\{ B \in \binom{[n]}{d} : B \in \mathcal{G} \right\}.$$

**Claim 3.1.**

  (i) $\mathcal{A} \neq \emptyset$.
 (ii) *For $A \in \mathcal{A}$ we have $A \notin \mathcal{G}$.*
(iii) $\mathcal{A} \cap \mathcal{B} = \emptyset$.
(iv) $\mathcal{A}$ *and $\mathcal{B}(\bar{j})$ are cross-intersecting.*

**Proof.** **(i)** This follows from $D \in \mathcal{A}$.

**(ii)** Since $S_j(\mathcal{G})$ contains no members of size exceeding $d$, $S_j(A \cup \{j\}) = A$ must hold for $A \in \mathcal{A}$. This implies $A \notin \mathcal{G}$.

**(iii)** If $A \in \mathcal{A} \cap \mathcal{B}$ then $j \notin A$ and both $A$ and $A \cup \{j\}$ are in $\mathcal{G}$. Thus $S_j(A \cup \{j\}) = A \cup \{j\}$, contradicting $S_j(\mathcal{G}) \subset \mathcal{K}(n,s)$.

**(iv)** This follows from $|(A \cup \{j\}) + B| \leq 2d$. $\qquad\square$

Let us note that $\mathcal{B}(\bar{j}) = \emptyset$ would imply

$$|\mathcal{G}| \leq 2|\mathcal{G}(\bar{j})| \leq 2 \sum_{0 \leq i < d} \binom{n-1}{i} = \sum_{0 \leq i < d} \binom{n}{i} + \binom{n-1}{d-1},$$

which is smaller than the right-hand side of (1.10). Thus we may assume that $\mathcal{B}(\bar{j}) \neq \emptyset$ holds.

Applying Proposition 2.3 to $\mathcal{A}$ and $\mathcal{B}(\bar{j}) \subset \binom{[n]-\{j\}}{d}$ gives

$$|\mathcal{A}| + |\mathcal{B}(\bar{j})| \leq \binom{n-1}{d} - \binom{n-1-d}{d}.$$

Since $|\mathcal{B}(j)| \leq \binom{n-1}{d-1}$, we find that

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{d} - \binom{n-d-1}{d}.$$

Note that there are at most $|\mathcal{A}| + |\mathcal{B}|$ sets of size $d$ in $S_j(\mathcal{G})$. Using $S_j(\mathcal{G}) \subset \mathcal{K}(n,s)$, we find that

$$|\mathcal{F}| = |S_j(\mathcal{G})| \leq \sum_{0 \leq i \leq d} \binom{n}{i} - \binom{n-d-1}{d}.$$

Now we consider the case $s = 2d + 1$. We set $\mathcal{K}(n,s) = \mathcal{K}_y(n,s)$ for an unspecified $y \in [n]$, that is,

$$\mathcal{K}(n,s) = \{K \subset [n] : |K \cap ([n] - \{y\})| \leq d\}, \quad \mathcal{G} \not\subset \mathcal{K}(n,s)$$

together with $S_j(\mathcal{G}) \subset \mathcal{K}(n,s)$ imply the existence of some $D \subset [n]$ satisfying $|D \cap ([n] - \{y\})| \geqq d+1$, $D \in \mathcal{G}$, $D \notin S_j(\mathcal{G})$ and, more importantly, $|S_j(D) \cap ([n] - \{y\})| \leqq d$. This is possible only for $j \neq y$ and $|D \cap ([n] - \{y\})| = d+1$ and only in the case $(D - \{j\}) \notin \mathcal{G}$.

First we deal with the case

$$|G| \leqq d+1 \text{ holds for all } G \in \mathcal{G}. \tag{3.1}$$

Let us consider the subfamily $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{G} \cap \binom{[n]}{d+1}$ of all $(d+1)$-sets in $\mathcal{G}$.

Since $\Delta(\mathcal{H}) \leqq \Delta(\mathcal{G}) \leqq 2d+1$, $\mathcal{H}$ is an intersecting family. If $\mathcal{H}$ is a *star*, that is, if there exists an element $z \in [n]$ with $z \in H$ for all $H \in \mathcal{H}$, then using (3.1), $\mathcal{G} \subset \mathcal{K}_z(n,s)$ follows. This is a contradiction.

Consequently we may apply the Hilton–Milner Theorem to $\mathcal{H}$ and obtain

$$|\mathcal{H}| \leqq \binom{n-1}{d} - \binom{n-d-2}{d} + 1.$$

This yields

$$|\mathcal{G}| \leqq \sum_{0 \leqq i \leqq d} \binom{n}{i} + \binom{n-1}{d} - \binom{n-d-2}{d} + 1,$$

as desired. Now suppose that there is some $G \in \mathcal{G}$ with $|G| \geqq d+2$.

**Claim 3.2.** $\{j,y\} \subset G$ *and* $|G| = d+2$.

**Proof.** $S_j(\mathcal{G}) \subset \mathcal{K}_y(n,s)$ implies $|S_j(G) \cap ([n] - \{y\})| \leqq d$. Thus $S_j(G) \neq G$, that is, $j \in G$ and $S_j(G) = G - \{j\}$. Since $|S_j(G)| = |G| - 1$, $|G| = d+2$ and $y \in G$ follow as well. $\qquad \square$

Let us again define two families of sets:

$$\mathcal{A} = \{A \subset ([n] - \{j\}) : |A \cap ([n] - \{y\})| = d,\ A \cup \{j\} \in \mathcal{G}\},$$
$$\mathcal{B} = \{B \subset ([n] - \{j\}) : |B \cap ([n] - \{y\})| = d,\ B \in \mathcal{G}\},$$

In view of $S_j(\mathcal{G}) \subset \mathcal{K}_y(n,s)$, we obtain $\mathcal{A} \cap \mathcal{B} = \emptyset$.

We are going to consider the four families

$$\mathcal{A}(y), \quad \mathcal{A}(\bar{y}), \quad \mathcal{B}(y), \quad \mathcal{B}(\bar{y}) \subset \binom{[n] - \{y,j\}}{d}.$$

Here $\Delta(\mathcal{G}) \leqq 2d+1$ implies that $\mathcal{A}(y)$ and $\mathcal{B}(\bar{y})$ and also $\mathcal{A}(\bar{y})$ and $\mathcal{B}(y)$ are cross-intersecting.

By definition, $G$ from Claim 3.2 provides us with a set, namely $G - \{j,y\}$, belonging to $\mathcal{A}(y)$. If $\mathcal{B}(\bar{y}) \neq \emptyset$, then (2.3) yields

$$|\mathcal{A}(y)| + |\mathcal{B}(\bar{y})| \leqq \binom{n-2}{d} - \binom{n-2-d}{d}.$$

Since $\mathcal{A} \cap \mathcal{B} = \emptyset$, we find that also

$$|\mathcal{A}(\bar{y})| + |\mathcal{B}(y)| \leqq \binom{n-2}{d}.$$

Clearly,

$$|\{G \in \mathcal{G} : j \in G, \ |G \cap ([n] - \{y\})| = d\}| \leqq 2 \binom{n-2}{d-1}.$$

We infer that

$$|\mathcal{G}| \leqq 2 \sum_{0 \leqq i < d} \binom{n-1}{i} + 2 \binom{n-2}{d-1} + 2 \binom{n-2}{d} - \binom{n-2-d}{d}$$

$$= 2 \sum_{0 \leqq i \leqq d} \binom{n-1}{i} - \binom{n-2-d}{d},$$

as desired. Thus we may assume that $\mathcal{B}(\bar{y}) = \emptyset$. Exactly the same argument works if both $\mathcal{B}(y)$ and $\mathcal{A}(\bar{y})$ are non-empty. To conclude the proof, we distinguish the two cases $\mathcal{A}(\bar{y}) = \emptyset$ or $\mathcal{B}(y) = \emptyset$.

**(a)** $\mathcal{A}(\bar{y}) = \emptyset$

Recall that $|\mathcal{F}(\bar{y})| \geq |\mathcal{F}(y)|$ held at the start, and this is not altered by the down-shift (see (1.6)–(1.8)). Therefore,

$$|\mathcal{G}| = |\mathcal{G}(y)| + |\mathcal{G}(\bar{y})| \leqq 2|\mathcal{G}(\bar{y})|.$$

Note that the only sets of size at least $d$ in $\mathcal{G}(\bar{y}) = \mathcal{G}(\bar{y}) - (\mathcal{A}(\bar{y}) \cup \mathcal{B}(\bar{y}))$ are sets $H \subset \binom{[n]-\{y\}}{d}$ with $j \in H$. These are at most $\binom{n-2}{d-1}$ sets, implying

$$|\mathcal{G}(\bar{y})| \leqq \sum_{0 \leqq i < d} \binom{n-1}{i} + \binom{n-2}{d-1}.$$

Therefore,

$$|\mathcal{G}| \leqq 2 \sum_{0 \leqq i \leqq d-1} \binom{n-1}{i} + 2 \binom{n-2}{d-1}.$$

To conclude the proof in this case we need

$$2 \binom{n-2}{d-1} \leqq 2 \binom{n-1}{d} - \binom{n-2-d}{d}.$$

Rearranging, we get

$$\binom{n-d-2}{d} \leqq 2 \binom{n-2}{d},$$

which holds trivially.

**(b)** $\mathcal{B}(y) = \emptyset$

Since $\mathcal{B}(\bar{y}) = \emptyset$ as well, $\mathcal{B} = \emptyset$ follows. This means that $\mathcal{G} + \{j\} \subset \mathcal{K}_y(n, s)$. What are the members of $\mathcal{G}(\bar{j})$? If $H \in \mathcal{G}(\bar{j})$, then $j \notin H$ implies $H \in S_j(\mathcal{G})$. Thus $|H \cap ([n] - \{y\})| \leqq d$. Using $\mathcal{B} = \emptyset$, we also obtain that $|H \cap ([n] - \{y\})| \leqq d - 1$. Since $j \notin H$ implies $H \subset ([n] - \{j\})$, we infer that

$$|\mathcal{G}(\bar{j})| \leqq 2 \sum_{0 \leqq i \leqq d-1} \binom{n-2}{i} = \sum_{0 \leqq i \leqq d} \binom{n-1}{i} - \binom{n-2}{d}.$$

Consequently,

$$|\mathcal{G}| \leqq 2|\mathcal{G}(\bar{j})| \leqq 2 \sum_{0 \leqq i \leqq d} \binom{n-1}{i} - 2\binom{n-2}{d}.$$

Since

$$2\binom{n-2}{d} > \binom{n-2}{d} > \binom{n-2-d}{d},$$

the proof is complete.

## 4. The parity trick

Let $p(n,s)$ denote the maximum-value of $\mathcal{F} \subset 2^{[n]}$ satisfying $\Delta(\mathcal{F}) \leqq s$. By Kleitman's theorem we know that $p(n,s)$ equals $m(n,s)$ from Katona's theorem.

In this section we are going to give a simple proof of the following result.

**Proposition 4.1.** *We have*

$$p(n, 2d+1) = 2p(n-1, 2d) \quad \text{for } n \geqq 2d+2. \tag{4.1}$$

Looking at the formulae for $m(n,s)$, we can easily verify that (4.1) holds. However, we are going to prove it without assuming *any* knowledge of the actual formula for $m(n,s)$ or $p(n,s)$.

**Proof.** Let $\mathcal{A} \subset 2^{[n-1]}$ satisfy $|\mathcal{A}| = p(n-1, 2d)$ and $\Delta(\mathcal{A}) \leqq 2d$. Define $\mathcal{B} = \mathcal{A} \cup \{A \cup \{n\} : A \in \mathcal{A}\}$. Then $|\mathcal{B}| = 2|\mathcal{A}|$ and $\Delta(\mathcal{B}) = \Delta(\mathcal{A}) + 1 \leqq 2s + 1$ hold. This proves $p(n, 2d+1) \geqq 2p(n-1, 2d)$.

Let us prove the opposite inequality. For $\mathcal{F} \subset 2^{[n]}$ satisfying $\Delta(\mathcal{F}) \leqq 2d+1$ and $|\mathcal{F}| = p(n, 2d+1)$, define the partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ by

$$\mathcal{F}_i = \{F \in \mathcal{F} : |F| \equiv i \ (\text{mod } 2)\}.$$

Note that for $F, F' \in \mathcal{F}_i$ we have $|F + F'| \equiv 0 \ (\text{mod } 2)$, implying

$$\Delta(\mathcal{F}_i) \leqq 2d \quad \text{for } i = 0, 1.$$

At first sight this gives only the bound $|\mathcal{F}_i| \leqq p(n, 2d)$, which is insufficient to prove (4.1).

Fortunately we can do one more trick. Define $\mathcal{G}_i = \{F \cap [n-1] : F \in \mathcal{F}_i\} \subset 2^{[n-1]}$. The point is that if $F, F' \in \mathcal{F}_i$ are distinct then $|F| \equiv |F'| \,(\text{mod } 2)$ implies $|F + F'| \geqq 2$. Consequently $F \cap [n-1] \neq F' \cap [n-1]$ holds, and therefore we obtain $|\mathcal{G}_i| = |\mathcal{F}_i|$ and $\Delta(\mathcal{G}_i) \leqq 2d$. By definition,

$$p(n, 2d+1) = |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{G}_0| + |\mathcal{G}_1| \leqq 2p(n-1, 2d)$$

follows, concluding the proof of (4.1). $\square$

Proposition 4.1 shows that the odd case is a consequence of the even case in Kleitman's Diameter Theorem.

Let us elaborate this approach and sketch how the above parity trick can be used to derive the odd case of Theorem 1.2 from the even case.

Let $\mathcal{F} \subset 2^{[n]}$ satisfy $\Delta(\mathcal{F}) \leqq 2d + 1$. Define $\mathcal{F}_i$ and $\mathcal{G}_i$, $i = 0, 1$, as above.

If $|\mathcal{F}| = p(n, 2d + 1) = 2p(n - 1, 2d)$, then $|\mathcal{F}_i| = |\mathcal{G}_i| = p(n - 1, 2d)$ holds for $i = 0, 1$. Applying the even case of Theorem 1.2, we infer that there exist subsets $S_0$ and $S_1$ of $[n - 1]$ such that

$$\mathcal{G}_i = \{G \subset [n - 1] : |G + S_i| \leqq d\}, \quad \text{for } i = 0, 1.$$

There are three cases: $S_0 = S_1$, $|S_0 + S_1| = 1$ and $|S_0 + S_1| \geqq 2$.

In the first case $\mathcal{G}_0 = \mathcal{G}_1$, and thus

$$\mathcal{F}_0 \cup \mathcal{F}_1 = \{F \subset [n] : |F \cap [n - 1]| \leqq d\}.$$

In the second case, defining $j$ by $S_0 + S_1 = \{j\}$, we find that

$$|\mathcal{F}_0 \cup \mathcal{F}_1| + S' = \{F \subset [n] : |F \cap ([n] - \{j\})| \leqq d\}$$

for some appropriate $S'$. In the third case one can easily find $E_i \subset [n - 1]$, $|E_i| \leqq d$, $i = 0, 1$ such that

$$|(S_0 + E_0) + (S_1 + E_1)| \geqq \min\{n - 1, 2d + |S_0 + S_1|\} \geqq 2d + 2.$$

Since $S_0 + E_0 \in \mathcal{G}_0$, $S_1 + E_1 \in \mathcal{G}_1$, we infer the existence of $F_i \in \mathcal{F}_i$, $i = 0, 1$ with $|F_0 + F_1| \geqq 2d + 2$, a contradiction.

If either $\mathcal{G}_0$ or $\mathcal{G}_1$ satisfy $\mathcal{G}_i \not\subset \mathcal{K}(n - 1, 2d) + S$ for all choices of $S \subset [n]$, then applying (1.10) with $n - 1$ gives

$$|\mathcal{G}_i| \leqq \sum_{0 \leqq i \leqq d} \binom{n - 1}{i} - \binom{n - d - 2}{d} + 1 \tag{4.2}$$

for the corresponding $i$. For $i' = 1 - i$ we still have

$$|\mathcal{G}_{i'}| \leqq \sum_{0 \leqq i \leqq d} \binom{n - 1}{i}.$$

Adding these two inequalities gives (1.11). The hardest case is when $\mathcal{G}_i \subset \mathcal{K}(n, s) + S_i$ holds for an appropriate choice of $S_i$, $i = 0, 1$. In the case $|S_0 + S_1| \leqq 1$, we find t hat $\mathcal{F}_0 \cup \mathcal{F}_1 = \mathcal{F} \subset \mathcal{K}(n, s) + S$ for a suitable choice of $S$.

Finally, if $|S_0 + S_1| \geqq 2$ and neither $\mathcal{G}_0$ nor $\mathcal{G}_1$ verifies (4.2), then $\Delta(\mathcal{G}_0 \cup \mathcal{G}_1) \geqq 2d + 2$ can be shown easily, the final contradiction.

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