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ON BITOPOLOGICAL QUASI-PSEUDOMETRIZATION

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Abstract

In this paper we give a sufficient condition of quasi-pseudometrization for bitopological spaces. From this condition we obtain, as immediate corollaries, some known results.

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1. Introduction

A quasi-pseudometric space is a pair (X, d) where X is a set and d is a mapping from $X \times X$ into the real numbers $\mathbf{R}$ satisfying for all $x, y, z \in X$: (i) $d(x, y) \geqslant 0$, (ii) $d(x, x) = 0$, (iii) $d(x, y) \leqslant d(x, z) + d(z, y)$. From a quasi-pseudometric, we can determine two topologies on X in a natural way: $\tau_d = \{A \subset X: \forall x \in A$ there is a $r > 0$ such that $B_d(x, r) \subset A\}$ with $B_d(x, r) = \{y \in X: d(x, y) < r\}$ and $\tau^d = \{A \subset X: \forall x \in A$ there is a $r > 0$ such that $B^d(x, r) \subset A\}$ with $B^d(x, r) = \{y \in X: d(y, x) < r\}$.

Kelly [2] began the study of bitopological spaces and its quasi-pseudometrizability. Also Patty [4], Lane [3] and Salbany [5] have contributions. Salbany gives an interesting sufficient condition for quasi-pseudometrizability from which we deduce a generalization of the Nagata-Smirnov theorem, solving with it a conjecture of Patty. Here we obtain (Theorem 1) another sufficient condition of quasi-pseudometrization from which we deduce (Corollary 1.1) that is $(X, \mathcal{P}, \mathcal{Q})$ is a pairwise perfectly normal space, $\mathcal{P}$ has a $\mathcal{Q}$-σ-locally finite base and $\mathcal{Q}$ has a

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$\mathcal{P}$-σ-locally finite base, then $(X, \mathcal{P}, \mathcal{Q})$ is quasi-pseudometrizable. This result modifies an incorrect assertion of Salbany (Example 1). Several standard theorems of the theory of quasi-pseudometrizability are easily deduced from Corollary 1.1.

2. Results

THEOREM 1. *Let $(X, \mathcal{P}, \mathcal{Q})$ be a bitopological space such that $\mathcal{P}$ has a $\mathcal{Q}$-σ-locally finite base $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ and $\mathcal{Q}$ has a $\mathcal{P}$-σ-locally finite base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. Then, $(X, \mathcal{P}, \mathcal{Q})$ is quasi-pseudometrizable if, and only if, we verify*:

(a) *For every $A \in \mathcal{A}$ there is a function f_A: $X \to [0, 1]$ $\mathcal{Q}$-upper semi-continuous such that $f_A^{-1}(0) = X - A$.*

(b) *For every j and every sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}_j$, if $\mathcal{P} - \lim_n x_n = x$ then $\lim_n \max[f_{A_n}(x) - f_{A_n}(x_n), 0] = 0$.*

(a′) *For every $B \in \mathcal{B}$ there is a function g_B: $X \to [0, 1]$ $\mathcal{P}$-upper semi-continuous such that $g_B^{-1}(0) = X - B$.*

(b′) *For every j and every sequence $\{B_n\}_{n=1}^{\infty} \subset \mathcal{B}_j$, if $\mathcal{Q} - \lim_n x_n = x$ then $\lim_n \max[g_{B_n}(x) - g_{B_n}(x_n), 0] = 0$.*

PROOF. For each couple of points $x, y \in X$ and each $n \in \mathbf{N}$ we can define $d_n(x, y) = \sup_{A \in \mathcal{A}_n}\{\max[f_A(x) - f_A(y), 0]\}$ and $d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x, y)$.

The mapping d is a quasi-pseudometric for X. We have to prove that $\tau_d = \mathcal{P}$ and $\tau^d \subset \mathcal{Q}$.

To verify the first equality we use a technique of Borges [1]: If $\lim_n d(x, x_n) = 0$ but $\mathcal{P} - \lim_n x_n \neq x$ then there exists $A \in$ some $\mathcal{A}_m$ which is a neighbourhood of x and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $A \cap \{x_{n_k}\}_{k=1}^{\infty} = \varnothing$. Then $f_A(x_{n_k}) = 0$ for every k, and $f_A(x) \neq 0$, therefore $\lim_k d(x, x_{n_k}) \neq 0$. Now, if $\mathcal{P} - \lim_n x_n = x$, for every $\varepsilon > 0$ there is an $m \in \mathbf{N}$ such that $2^{-m+2} < \varepsilon$; then $\varepsilon/2 > \sum_{r=m}^{\infty} 2^{-r} d_r(x, x_n)$ for every x_n. If $r < m$, $\lim_n d_r(x, x_n) = 0$: otherwise, there exists $p < m$ such that $\lim_n d_p(x, x_n) \neq 0$; so we can find $\delta > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $d_p(x, x_{n_k}) \geqslant \delta$ for every k. Consequently, there is an $A_k \in \mathcal{A}_p$ for every k, with $\max[f_{A_k}(x) - f_{A_k}(x_k), 0] > \delta/2$; then we have $\lim_k \max[f_{A_k}(x) - f_{A_k}(x_k), 0] \geqslant \delta$ which is a contradiction with (b). Then $\mathcal{P} = \tau_d$.

To prove $\tau^d \subset \mathcal{Q}$ it suffices to show that, for every $x \in X$ and $\varepsilon > 0$, $\{y \in X: d(y, x) < \varepsilon\}$ is a $\mathcal{Q}$-open set. For a fixed $x_0 \in X$, let $x \in X$ and $\varepsilon > 0$, then there is a $\mathcal{Q}$-neighbourhood W of x with $W \cap A_i \neq \varnothing$, $i = 1, 2, \ldots, p$, being $A_i \in \mathcal{A}_n$, and $W \cap A = \varnothing$ for every $A \in \mathcal{A}_n - \{A_1, A_2, \ldots, A_p\}$. Since f_{A_i} is $\mathcal{Q}$-upper semi-continuous, $i = 1, 2, \ldots, p$, there is a $\mathcal{Q}$-neighbourhood V_i of x with $f_{A_i}(x) - f_{A_i}(y) < \varepsilon$ for every $y \in V_i$. Let $V = W \cap \left(\bigcap_{i=1}^{p} V_i\right)$; V is a $\mathcal{Q}$-neighbourhood of x and for every $y \in V$ we get that:

$$d_n(y, x_0) - d_n(x, x_0) \leq d_n(y, x) = \sup_{A \in \mathcal{Q}_n} \{\max[f_A(y) - f_A(x), 0]\}$$
$$= \max\left\{\max\left[f_{A_i}(y) - f_{A_i}(x), 0\right], i = 1, 2, \ldots, p\right\} < \varepsilon.$$

Then $\tau^d \subset \mathcal{Q}$.

Similarly, we can define for every $x, y \in X$ and every $n \in \mathbf{N}$ $d_n'(x, y) = \sup_{B \in \mathcal{B}_n}\{\max[g_B(x) - g_B(y), 0]\}$ and $d'(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n'(x, y)$, d' is a quasi-pseudometric for X with $\tau_{d'} = \mathcal{Q}$ and $\tau^{d'} \subset \mathcal{P}$. Then, the quasi-pseudometric $d''(x, y) = d'(y, x)$ verifies $\tau^{d''} = \mathcal{Q}$ and $\tau_{d''} \subset \mathcal{P}$. Now, let $\bar{d}(x, y) = d(x, y) + d''(x, y)$. This quasi-pseudometric for X verifies $\mathcal{P} = \tau_{\bar{d}}$ and $\mathcal{Q} = \tau^{\bar{d}}$.

Conversely, let $(X, \mathcal{P}, \mathcal{Q})$ be a quasi-pseudometrizable bitopological space. There is a quasi-pseudometric d such that $\tau_d = \mathcal{P}$ and $\tau^d = \mathcal{Q}$, we can suppose that d has diameter lower than 1. Let A be a $\mathcal{P}$-open set in X and $f_A(x) = d(x, X - A)$. Obviously, f_A verifies (a) and (b). In the same way we verify (a′) and (b′). The proof is complete.

The above result is a weak bitopological version of Borges' topological theorem [1], page 801, since we only obtain a sufficient condition for quasi-pseudometrization.

Corollary 1.1. *If $(X, \mathcal{P}, \mathcal{Q})$ is a pairwise perfectly normal space, $\mathcal{P}$ has a $\mathcal{Q}$-σ-locally finite base $\mathcal{Q} = \bigcup_{n=1}^{\infty} \mathcal{Q}_n$ and $\mathcal{Q}$ has a $\mathcal{P}$-σ-locally finite base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, then $(X, \mathcal{P}, \mathcal{Q})$ is quasi-pseudometrizable.*

Proof. For each $A \in \mathcal{Q}$, there is by [3] a $\mathcal{P}$-lower semi-continuous and $\mathcal{Q}$-upper semi-continuous function f_A: $X \to [0, 1]$ such that $f_A^{-1}(0) = X - A$. By the $\mathcal{Q}$-local finiteness of every $\mathcal{Q}_j$ condition (b) of Theorem 1 follows. Likewise are verified conditions (a′) and (b′). By Theorem 1, $(X, \mathcal{P}, \mathcal{Q})$ is quasi-pseudometrizable.

Corollary 1.2 (Salbany). *If $(X, \mathcal{P}, \mathcal{Q})$ is a pairwise regular space, $\mathcal{P}$ has a $\mathcal{Q}$-σ-locally finite base and $\mathcal{Q}$ has a $\mathcal{P}$-σ-locally finite base, then $(X, \mathcal{P}, \mathcal{Q})$ is quasi-pseudometrizable.*

Proof. By [4] $(X, \mathcal{P}, \mathcal{Q})$ is pairwise perfectly normal and by Corollary 1.1 we have the result.

Note. This corollary is also obtained by Salbany as corollary of another more general result.

COROLLARY 1.3 (Kelly). *If* $(X, \mathcal{P}, \mathcal{Q})$ *is a pairwise regular space and* $\mathcal{P}$ *and* $\mathcal{Q}$ *have countable basis, then* $(X, \mathcal{P}, \mathcal{Q})$ *is quasi-pseudometrizable.*

Salbany asserts, [5], page 302, that if $(X, \mathcal{P}, \mathcal{Q})$ is pairwise normal and $\mathcal{P}$ has a $\mathcal{Q}$-σ-locally finite base, then $\mathcal{P} = \tau_d$ and $\mathcal{Q} \supset \tau^d$ for some quasi-pseudometric d. The following example proves that this result is not correct.

EXAMPLE 1. Let $(X, \mathcal{P}, \mathcal{Q})$ be such that $X = \mathbf{R}$ (the set of reals), $\mathcal{P}$ is the euclidean topology and $\mathcal{Q}$ is the trivial topology. $(X, \mathcal{P}, \mathcal{Q})$ is pairwise normal and if d is a quasi-pseudometric such that $\mathcal{P} = \tau_d$ and $\mathcal{Q} \supset \tau^d$, then $\mathcal{Q} = \tau^d$ which is a contradiction since $(X, \mathcal{P}, \mathcal{Q})$ is not pairwise regular.

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References

[1] C. J. Borges, 'On metrizability of topological spaces', *Canad. J. Math.* **20** (1968), 795–804.
[2] J. C. Kelly, 'Bitopological spaces', *Proc. London Math. Soc.* **13** (1963), 71–89.
[3] E. P. Lane, 'Bitopological spaces and quasi-uniform spaces', *Proc. London Math. Soc.* **17** (1967), 241–256.
[4] C. W. Patty, "Bitopological spaces', *Duke Math. J.* **34** (1967), 387–392.
[5] S. Salbany, 'Quasi-metrization of bitopological spaces', *Arch. Math.* (*Basel*) **23** (1972), 299–306.

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