

RAMSEY'S COHEIRS

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Abstract. We use the model theoretic notion of coheir to give short proofs of old and new theorems in Ramsey Theory. As an illustration we start from Ramsey's theorem itself. Then we prove Hindman's theorem and the Hales–Jewett theorem. Finally, we prove two Ramsey theoretic principles that have among their consequences partition theorems due to Carlson and to Gowers.

§1. Introduction. Ramsey theory has substantial and diverse applications to many parts of mathematics. In particular, Ramsey's theorem has foundational applications to model theory through the Ehrenfeucht–Mostowski construction of indiscernibles and generalizations thereof. In this paper we explore the converse direction, that is, we use model theory to obtain new proofs of classical results in Ramsey Theory.

The Stone–Čech compactification, obtained via ultrafilters, is a widely employed method for proving Ramsey theoretic results. One of its first major applications is the celebrated Galvin–Glazer proof of Hindman's theorem, see, e.g., [4]. Our methods are related, but alternative, to the ultrafilter approach. We replace βG (the Stone–Čech compactification of a semigroup G) with a large saturated elementary extension of G , i.e., a monster model of $\text{Th}(G/G)$. One immediate advantage is that we work with elements of a natural semigroup with a natural operation. In contrast, elements of βG are ultrafilters, that is, sets of sets, and the semigroup operation among ultrafilters is far from straightforward.

This idea is not completely new: in his seminal work on the applications of topological dynamics to model theory [14, 15], Newelski replaces the semigroup βG with the space of types over G with a suitably defined operation. Our approach is similar, except that, unlike Newelski, we do not pursue connections with topological dynamics, but rather offer an alternative realm of application. The investigation of alternative methods in the study of regularity phenomena has been called for by Di Nasso [5, Open problem #1]. This article contains a possible answer.

The model theoretic tools employed in this paper are relatively basic. Section 2 is meant to give an accessible overview of the necessary notions for readers whose expertise is not primarily in model theory. Our results do not require assumptions of model theoretic tameness such as stability, NIP, etc., much like those that use

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nonstandard analysis, for example in [6]. Investigating the effect of such assumptions remains as future work.

When this paper was essentially complete, we became aware of [2], which is worth mentioning since it employs similar methods in a related context.

The paper is divided into two parts. In the first part we prove that the notion of coheir leads to short and elegant proofs of well-known results. Most proofs in this part may be considered folklore, though they have not appeared in the literature so far. They are included here to provide a self-contained, gentle introduction to the techniques that are used in the second part.

As a preliminary illustrative step, we present a proof of Ramsey’s theorem (Theorem 3.1). Then we prove a generalization of Hindman’s theorem (Theorem 5.1), which is required in the second part of the paper. We also show how to combine Ramsey’s and Hindman’s theorems in a single proposition—the Milliken–Taylor theorem (Theorem 5.3). Finally, we prove an abstract algebraic version of the Hales–Jewett theorem (Theorem 6.4) due to Sabine Koppelberg [11].

In the second part of the paper we prove two Ramsey-theoretic properties of semigroups (Lemmas 7.1 and 8.1). As an application, we derive a generalization of Carlson’s theorem on colourings of variable words which we present in the style of Koppelberg (Theorem 7.2) and in its classical form (Corollary 7.3). Lemma 8.1 is a partition theorem that generalizes Gowers’ FIN_k Theorem [8] in a different direction than [12].

The extent of the generalizations mentioned above is limited, and they could be obtained in other ways, but our motivation here is to show the use and relevance of model theoretic methods. Numerous papers in the literature strengthen or generalize the partition theorems considered here. The comparison of the results that appear in these papers is not always straightforward—a few are compared in [1].

The proofs in this paper require a modicum of familiarity with model theory. However, the results can be stated in an elementary language, and in the rest of this introduction we introduce the necessary terminology.

Throughout the paper G is a semigroup and Σ a non-empty set of endomorphisms of G . For $\bar{a} \in G^{\leq \omega}$ we write

$$\text{fp}^\Sigma \bar{a} = \{ \sigma_0 a_{i_0} \cdots \sigma_k a_{i_k} : i_0 < \cdots < i_k < |\bar{a}|, \bar{\sigma} \in (\Sigma \cup \{\text{id}_G\})^{k+1}, k < |\bar{a}| \}.$$

Overlined symbols, such as \bar{a} or $\bar{\sigma}$, always denote a tuple, and a_i, σ_i denote the i th entry of that tuple.

When Σ is empty, we write $\text{fp} \bar{a}$.

EXAMPLE 1.1. For future reference, we instantiate the definition above in the context of free semigroups. Let G be the set of words on a finite alphabet $A \cup \{x\}$, where x is a symbol not in A which we call *variable*. Let C be the set of words on the alphabet A . Words in C are called *constant words*, while those in $G \setminus C$ are called *variable words*. When G is endowed with the operation of concatenation of words, C and $G \setminus C$ are subsemigroups of G . For $t \in G$ and $a \in A$, let $t(a)$ be the word obtained by replacing all the occurrences of x in t by a . Note that the map $\sigma_a : t \mapsto t(a)$ is an endomorphism of G . In the literature, when G is as above and $\Sigma = \{ \sigma_a : a \in A \}$, the elements of $\text{fp}^\Sigma \bar{s}$ are called *extracted words*. We say that a tuple $\bar{a} \in (\text{fp}^\Sigma \bar{s})^\omega$ is an *extracted sequence* if $a_i \in \text{fp}^\Sigma s_{[n_i, n_{i+1})}$ for some increasing

sequence of positive integers $\langle n_i : i < \omega \rangle$. If, moreover, $a_i \notin C$ for all i , we say that \bar{a} is an *extracted variable sequence* of \bar{s} .

The following definition will be used to express our results in the general context of semigroups.

DEFINITION 1.2. Let $<$ be a binary relation on G . We say that G is *$<$ -covered* if for every finite $A \subseteq G$ there is a c such that $A < c$. If c can be found in some fixed $B \subseteq G$, we say *$<$ -covered by B* . We say that G is *$<$ -closed* if $a < b < c$ implies $a < b \cdot c$ for all $a, b, c \in G$. A *$<$ -chain* in G is a tuple $\bar{a} \in G^{\leq \omega}$ such that $a_i < a_{i+1}$.

The preorder relation given by the length of the words on a free semigroup G is a natural example that is both $<$ -closed and $<$ -covered. A less straightforward relation is used in the proof of Theorem 7.2.

Finally, we recall two standard notions. Let $C \subseteq G$ be a subsemigroup. We say that C is *nice* if $a \cdot b \in C$ implies $a, b \in C$. A homomorphism $\sigma : G \rightarrow C$ such that $\sigma \upharpoonright C = \text{id}_C$ is called *retraction* of G onto C . Note that the set of constant words in Example 1.1 is a nice subsemigroup and that the maps σ_a are retractions.

We are now ready to state Lemma 7.1.

LEMMA. *Let Σ be a finite set of retractions of G onto a nice subsemigroup C . Let $<$ be a relation on G that makes it $<$ -closed and $<$ -covered by $G \setminus C$. Then, for every finite coloring of G , there is a $<$ -chain $\bar{a} \in (G \setminus C)^\omega$ such that $\text{fp}^\Sigma \bar{a} \setminus C$ is monochromatic.*

When C and Σ are empty and $<$ holds for all pairs, the lemma reduces to Hindman's theorem (Theorem 5.1).

The appropriate choice of G , C , Σ , and $<$ yields Carlson's partition theorem (in particular no model theoretic argument is necessary, see Theorem 7.2 and its Corollary 7.3).

In the last section we prove Lemma 8.1 which is similar to the lemma above but deals with composition of homomorphisms. This is also stated in an elementary language and a general version of a partition theorem by Gowers is derived from it.

§2. Coheirs, and coheir sequences. We assume that the reader is familiar with undergraduate model theory and in this section we only review the few prerequisites that go beyond that. Proofs are omitted. The reader may consult any standard model theory textbook, e.g., [18] (the intrepid reader may consult [19], some lecture notes which use the same notation and quirks as this paper). The notation and terminology are standard with the possible exception of Definitions 2.3 and 2.5.

A *sequence* is a function whose domain is a linear order. A *tuple* is a sequence whose domain is an ordinal. The domain of the tuple c is denoted by $|c|$ and is called the *length* of c .

NOTATION 2.1. *Sometimes (i.e., not always) we may overline tuples as mnemonic. When a tuple \bar{c} is introduced, c_i denotes the i th element of \bar{c} . We write $c \upharpoonright I$, where $I \subseteq |c|$, for the tuple which is naturally associated with the restriction of \bar{c} to I . The bar is dropped for ease of notation.*

We denote the monster model by \mathcal{U} or, when dealing with semigroups, by \mathcal{G} . We always work over a fixed set of parameters $A \subseteq \mathcal{U}$. When this set is a model, as it will often be, we denote it by M , or G in the case of semigroups.

We say that a type $p(x)$ is *finitely satisfied* in A if every conjunction of formulas in $p(x)$ has a solution in $A^{|x|}$. A global type that is finitely satisfiable in A is invariant over A .

If M is a model every consistent type $p(x) \subseteq L(M)$ is finitely satisfied in M . For this reason in a few points in this paper it is necessary to work over a model. For simplicity, we always assume this.

The following is an easy, well-known fact.

PROPOSITION 2.2. *Every type $q(x) \subseteq L(\mathcal{U})$ that is finitely satisfiable in M has an extension to a global type finitely satisfiable in M .*

If $p(x)$ is finitely satisfied in M , the extensions of $p(x)$ that are also finitely satisfied in M are called *coheirs* of $p(x)$.

In many cases it is useful to focus on elements instead of their types. We introduce the following notation to express that $\text{tp}(a/M, b)$ is finitely satisfied in M . (The notion is standard in model-theory, and it has no standard notation though.)

DEFINITION 2.3. For every $a \in \mathcal{U}^{|x|}$ and $b \in \mathcal{U}^{|z|}$ we define

$$a \Downarrow_M b \Leftrightarrow \varphi(a; b) \text{ for all } \varphi(x; z) \in L(M) \text{ such that } M^{|x|} \subseteq \varphi(\mathcal{U}^{|x|}; b).$$

We call this the *coheir–heir* relation. We define the type

$$x \Downarrow_M b = \{ \varphi(x; b) : \varphi(x; b) \in L(M) \text{ and } M^{|x|} \subseteq \varphi(\mathcal{U}^{|x|}; b) \}.$$

The tuples a realizing this type are those such that $a \Downarrow_M b$. We will use the symbol $a \equiv_A x \Downarrow_M b$ for the union of the types $x \Downarrow_M b$ and $\text{tp}(a/M)$.

We imagine $a \Downarrow_M b$ as saying that a is *independent* from b over M . This is a very strong form of independence. In general it is not symmetric, that is, $a \Downarrow_M b$ is not the same as $b \Downarrow_M a$ (symmetry is equivalent to stability).

We shall use, sometimes without reference, the following easy lemma.

LEMMA 2.4. *The following properties hold for all small M, a, b , and c :*

1. $a \Downarrow_M b \Rightarrow fa \Downarrow_M fb$ for every $f \in \text{Aut}(\mathcal{U}/M)$ *invariance;*
2. $a \Downarrow_M b \Leftrightarrow a_0 \Downarrow_M b_0$ for all finite $a_0 \subseteq a$ and $b_0 \subseteq b$ *finite character;*
3. $a \Downarrow_M b, c$ and $b \Downarrow_M c \Rightarrow a, b \Downarrow_M c$ *transitivity;*
4. $a \Downarrow_M b \Rightarrow$ there exists $a' \equiv_{M, b} a$ such that $a' \Downarrow_M b, c$ *coheir extension.*

Note that $a \equiv_M x \Downarrow_M b$ is the intersection of all types in $S(M, b)$ that are coheirs of $\text{tp}(a/M)$. As there may be more than one of such coheirs, $a \equiv_M x \Downarrow_M b$ need not be a complete over M, b . In fact, completeness is a rather strong property.

DEFINITION 2.5. If $a \equiv_M x \Downarrow_M b$ is a complete type (over M, b) for every $a \in \mathcal{U}^{<\omega}$, every $b \in \mathcal{U}^{|x|}$, and every tuple of variables x , then we say that \Downarrow_M is *stationary*. We say *n-stationary* if the requirement above is restricted to $|x| = n$.

Stationarity is often ensured by the following property.

PROPOSITION 2.6. *Fix a tuple of variable x of length n . If for every $\varphi(x) \in L(\mathcal{U})$ there is a formula $\psi(x) \in L(M)$ such that $\varphi(M^{|x|}) = \psi(M^{|x|})$ then \Downarrow_M is n -stationary.*

REMARK 2.7. Stationarity of \Downarrow_M over every model M is equivalent to the stability of T . However, in unstable theories the assumption may hold for some particular model. For example, if every subset of M^n is the trace of a definable set, then \Downarrow_M is n -stationary by the proposition above. This simple observation will be of help in the proof of Theorem 5.1. For natural example let $T = T_{\text{dlo}}$ and let $M \subseteq \mathcal{U}$ have the order-type of \mathbb{R} . By quantifier elimination every definable of \mathcal{U} is union of finitely many intervals. By Dedekind completeness, the trace on A of any interval of \mathcal{U} coincides with that of an M -definable interval.

Let $p(x) \in S(\mathcal{U})$ be a global type that is finitely satisfiable in M . We say that the tuple \bar{c} is a *coheir sequence* of $p(x)$ over M if $c_i \models p_{\upharpoonright M, c_{\upharpoonright i}}(x)$ for every $i < |\bar{c}|$. The following is a convenient characterization of coheir sequences.

LEMMA 2.8. *For \bar{c} a tuple of length ω , the following are equivalent:*

1. \bar{c} is a coheir sequence over M ;
2. $c_n \Downarrow_M c_{\upharpoonright n}$ and $c_{n+1} \equiv_{M, c_{\upharpoonright n}} c_n$ for every $n < \omega$.

Let $I, <_I$ be a linear order. We call a function $\bar{a} : I \rightarrow \mathcal{U}^{|\mathcal{X}|}$ an I -sequence, or simply a *sequence* when I is clear.

If $I_0 \subseteq I$ we call $a_{\upharpoonright I_0}$, the restriction of \bar{a} to I_0 , a *subsequence* of \bar{a} . When I_0 is finite we identify $a_{\upharpoonright I_0}$ with a tuple of length $|I_0|$.

DEFINITION 2.9. Let $I, <_I$ be an infinite linear order and let \bar{a} be an I -sequence. We say that a is a *sequence of indiscernibles* over A or, a *sequence of A -indiscernibles*, if $a_{\upharpoonright I_0} \equiv_A a_{\upharpoonright I_1}$ for every $I_0, I_1 \subseteq I$ of equal finite cardinality.

The following can be easily derived from the lemma above by induction.

PROPOSITION 2.10. *Every sequence of coheirs over M is M -indiscernible.*

§3. Ramsey’s theorem from coheir sequences. We illustrate the relation between coheirs and Ramsey phenomena in the simplest possible case: Ramsey’s theorem. The subsequent sections build on this proof for more sophisticated results.

In this chapter we deal with finite partitions. The partition of a set X into k subsets is often represented by a map $f : X \rightarrow [k]$. The elements of $[k] = \{1, \dots, k\}$ are also called *colors*, and the partition a *coloring*, or k -coloring, of X . We say that $Y \subseteq X$ is *monochromatic* if f is constant on Y .

Let M be an arbitrary infinite set. Fix $n, k < \omega$ and fix a coloring f of the set of all n -subsets of M , i.e., the *complete n -uniform hypergraph* with vertex set M ,

$$f : \binom{M}{n} \rightarrow [k].$$

We say that $H \subseteq M$ is a *monochromatic subgraph* if the subgraph induced by H is monochromatic. In the literature monochromatic subgraphs are also called *homogeneous sets*.

The following is a very famous theorem which we prove here in an unusual way. The proof will serve as a blueprint for other constructions in this paper.

RAMSEY’S THEOREM 3.1. *Let M be an infinite set. Then for every positive integer n and every finite coloring of the complete n -uniform hypergraph with vertex set M there is an infinite monochromatic subgraph.*

PROOF. Let L be a language that contains k relation symbols r_1, \dots, r_k of arity n . Given a k -coloring f we define a structure with domain M . The interpretation of the relation symbols is

$$r_i^M = \{a_1, \dots, a_n \in M : f(\{a_1, \dots, a_n\}) = i\}.$$

We may assume that M is an elementary substructure of some large saturated model \mathcal{U} . Pick any type $p(x) \in S(\mathcal{U})$ finitely satisfied in M but not realized in M and let $\bar{c} = \langle c_i : i < \omega \rangle$ be a coheir sequence of $p(x)$.

There is a first-order sentence saying that the formulas $r_i(x_1, \dots, x_n)$ are a coloring of $\binom{M}{n}$. Then by elementarity the same holds in \mathcal{U} . By indiscernibility, all tuples of n distinct elements of \bar{c} have the same color, say 1. We now prove that there is a sequence $\bar{a} = \langle a_i : i < \omega \rangle$ in M with the same property.

We construct $a_{\uparrow i}$ by induction on i as follows.

Assume as induction hypothesis that the subsequences of length n of $a_{\uparrow i}, c_{\uparrow n}$ all have color 1. Our goal is to find $a_i \in M$ such that the same property holds for $a_{\uparrow i}, a_i, c_{\uparrow n}$. By the indiscernibility of \bar{c} , the property holds for $a_{\uparrow i}, c_{\uparrow n}, c_n$. And this can be written by a formula $\varphi(a_{\uparrow i}, c_{\uparrow n}, c_n)$. As \bar{c} is a coheir sequence, by Lemma 2.8 we can find $a_i \in M$ such that $\varphi(a_{\uparrow i}, c_{\uparrow n}, a_i)$. So, as the order is irrelevant, $a_{\uparrow i}, a_i, c_{\uparrow n}$ satisfies the induction hypothesis. \dashv

§4. Idempotent orbits in semigroups. In this and the following sections we fix a semigroup G which we identify with a first-order structure. The language contains, among others, the symbol \cdot which is interpreted as a binary associative operation on G . We write \mathcal{G} for a large saturated elementary extension of G .

For any two sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$ we define

$$\mathcal{A} \cdot_G \mathcal{B} = \{a \cdot b : a \in \mathcal{A}, b \in \mathcal{B} \text{ and } a \downarrow_G b\}.$$

In this and the next section we abbreviate $\mathcal{O}(a/G)$, the orbit of a under $\text{Aut}(\mathcal{G}/G)$, with a_G . We write $a \cdot_G \mathcal{B}$ for $\mathcal{O}(a/G) \cdot_G \mathcal{B}$. Similarly for $\mathcal{A} \cdot_G b$ and $a \cdot_G b$.

LEMMA 4.1. *If \mathcal{A} is type definable over G then so is $\mathcal{A} \cdot_G b$ for any b .*

PROOF. The set $\mathcal{A} \cdot_G b$ is the union of $\mathcal{A} \cdot_G \{c\}$ as c ranges in b_G . The set $\mathcal{A} \cdot_G \{c\}$ is type definable, say by the type $\exists y p(x, y, c)$ where

$$p(x, y, c) = y \downarrow_G c \wedge y \cdot c = x \wedge y \in \mathcal{A}.$$

Note that, by the invariance of \downarrow_G , if $f \in \text{Aut}(\mathcal{G}/G)$, then $\exists y p(x, y, fc)$ defines $\mathcal{A} \cdot_G \{fc\}$. Therefore if $q(z) = \text{tp}(b/G)$ then $\exists y, z [q(z) \cup p(x, y, z)]$ defines $\mathcal{A} \cdot_G b$. \dashv

By the invariance of \downarrow_G , for every $f \in \text{Aut}(\mathcal{G}/G)$ we have $f[\mathcal{A} \cdot_G \mathcal{B}] = f[\mathcal{A}] \cdot_G f[\mathcal{B}]$. Therefore when \mathcal{A} and \mathcal{B} are invariant over G , also $\mathcal{A} \cdot_G \mathcal{B}$ is invariant over G . Below we mainly deal with invariant sets.

PROPOSITION 4.2. *For all G -invariant sets \mathcal{A}, \mathcal{B} , and \mathcal{C}*

$$\mathcal{A} \cdot_G (\mathcal{B} \cdot_G \mathcal{C}) \subseteq (\mathcal{A} \cdot_G \mathcal{B}) \cdot_G \mathcal{C}.$$

PROOF. Let $a \cdot b \cdot c$ be an arbitrary element of the l.h.s. where $a \downarrow_G b \cdot c$ and $b \downarrow_G c$. By extension (Lemma 2.4), there exists a' such that $a \equiv_{G, b \cdot c} a' \downarrow_G b \cdot c, b, c$. By

transitivity (again Lemma 2.4), $a' \cdot b \Downarrow_G c$. Therefore $a' \cdot b \cdot c$ belongs to the r.h.s. Finally, as $a' \equiv_{G, b \cdot c} a$, also $a \cdot b \cdot c$ belongs to the r.h.s. by invariance. \dashv

Let A be a non-empty set. When $A \cdot_G A \subseteq A$, we say that it is *idempotent* (over G).

COROLLARY 4.3. *Assume $B \subseteq A$ are both G -invariant. Then if A is idempotent, also $A \cdot_G B$ is idempotent.*

PROOF. We check that if A is idempotent so is $A \cdot_G B$

$$\begin{aligned} (A \cdot_G B) \cdot_G (A \cdot_G B) &\subseteq A \cdot_G (A \cdot_G B) && \text{because } A \cdot_G B \subseteq A \\ &\subseteq (A \cdot_G A) \cdot_G B && \text{by the lemma above} \\ &\subseteq A \cdot_G B. && \dashv \end{aligned}$$

We show that, under the assumption of stationarity, the operation \cdot_G is associative. The quotient map $\mathcal{G} \rightarrow \mathcal{G}/\equiv_G$ is almost a homomorphism.

PROPOSITION 4.4. *Assume \Downarrow_G is 1-stationary, see Definition 2.5. Fix $a \Downarrow_G b$ arbitrarily. Then $a' \cdot b' \equiv_G a \cdot b$ for every $a' \equiv_G a$ and $b' \equiv_G b$ such that $a' \Downarrow_G b'$. Or, in other words, $(a \cdot b)_G = a \cdot_G b$.*

PROOF. We prove two inclusions, only the second one requires stationarity.

\subseteq As $a \Downarrow_G b$ holds by hypothesis, $a \cdot b \in a \cdot_G b$. The inclusion follows by invariance.

\supseteq By invariance it suffices to show that the l.h.s. contains $a \cdot_G \{b\}$. Let $a' \in a_G$ such that $a' \Downarrow_G b$. We claim that $a' \cdot b \in (a \cdot b)_G$. Both a and a' satisfy $a \equiv_G x \Downarrow_G b$. By 1-stationarity, $a \equiv_{G, b} a'$. Hence $a \cdot b \equiv_G a' \cdot b$. \dashv

COROLLARY 4.5 (Associativity). *Assume \Downarrow_G is 1-stationary. Then for all G -invariant sets A, B , and C*

$$A \cdot_G (B \cdot_G C) = (A \cdot_G B) \cdot_G C.$$

PROOF. We can assume that A, B , and C are G -orbits. Say of a, b , and c respectively. We can assume that $a \Downarrow_G b \cdot c$ and $b \Downarrow_G c$. By Proposition 4.4 the set on the l.h.s. equals $(a \cdot b \cdot c)_G$. By a similar argument the set on the r.h.s. equals $(a' \cdot b' \cdot c')_G$ for some elements a', b' , and c' . Proposition 4.2 proves that inclusion \subseteq holds in general. But inclusion between orbits amounts to equality. \dashv

The following lemma proves the existence of idempotent orbits. The proof is self-contained, i.e., it does not use Ellis' theorem on the existence of idempotents in compact left topological semigroups (however, the argument is very similar). As a comparison, finding a proof in the setting of nonstandard analysis is listed as an open problem in [5].

LEMMA 4.6. *Assume \Downarrow_G is 1-stationary. If A is minimal among the idempotent sets that are type-definable over G , then $A = b_G$ for some (any) $b \in A$.*

PROOF. Fix arbitrarily some $b \in A$. By Corollary 4.3, the set $A \cdot_G b$ is contained in A , idempotent and type-definable over G by Lemma 4.1. Therefore by minimality $A \cdot_G b = A$. Let $A' \subseteq A$ be the set of those a such that $a \cdot_G b = b_G$. This set is non-empty because $b \in A \cdot_G b$. It is easy to verify that A' is type-definable over G, b . As it is clearly invariant over G , it is type-definable over G . By associativity it is idempotent. Hence, by minimality, $A' = A$. Then $b \in A'$, which implies $b \cdot_G b = b_G$. That is, b has idempotent orbit. Finally, by minimality, $A = b_G$. \dashv

COROLLARY 4.7. *Under the same assumptions of the lemma above, every idempotent set that is type-definable over G contains an element with an idempotent orbit.*

§5. Hindman’s theorem. In this section we merge the theory of idempotents presented in Section 4 with the proof of Ramsey’s theorem to obtain Hindman’s theorem [10].

Let \bar{a} be a tuple of elements of G of length $\leq \omega$. In Section 1 we defined $\text{fp } \bar{a}$ and the notions of $-$ -closed and $<$ -covered. The relation $<$ is introduced mainly for future reference. The classical Hindman’s theorem is obtained with the positive integers (as an additive semigroup) for G and $<$ for $<$.

HINDMAN THEOREM 5.1. *Let $<$ be a relation on G that makes it $-$ -closed and $<$ -covered. Then for every finite coloring of G there is a $<$ -chain \bar{a} such that $\text{fp } \bar{a}$ is monochromatic. If there is no $g \in G$ such that $G < g$, we may further assume that the elements of the $<$ -chain are all distinct.*

PROOF. We interpret G as a structure in a language that extends the language of semigroups with a symbol for $<$ and one for each subset of G . Let \mathcal{G} be a saturated elementary superstructure of G . As observed in Remark 2.7, the language makes \perp_G trivially 1-stationary.

We write \mathcal{G}' for the type-definable set $\{g : G < g\}$, which is non-empty because G is $<$ -covered. We claim that \mathcal{G}' is idempotent. In fact, if $a, b \in \mathcal{G}'$ then, as $G < a, b$ and $a \perp_G b$, we must have that $a < b$. Therefore, from the $-$ -closedness of G we infer $a \cdot b \in \mathcal{G}'$.

Let g_0 be an element of \mathcal{G}' with idempotent orbit as given by Corollary 4.7. We can assume that $g_0 \notin G$ otherwise the sequence that is identically g_0 trivially proves the theorem. We can assume that $g_0 \notin G$ otherwise the sequence that is identically g_0 trivially proves the theorem. If we want the elements of the chain \bar{a} to be distinct it suffices require that $g_0 \notin G$. By definition of g_0 , this can be directly assumed when there is no $g \in G$ such that $G < g$.

Let $p(x) \in \mathcal{S}(\mathcal{G})$ be a global coheir of $\text{tp}(g_0/G)$. Let \bar{g} be a coheir sequence of $p(x)$, that is $g_i \models p_{\perp G, g_i}(x)$.

We write $\bar{g}_{\perp i}$ for the tuple g_{i-1}, \dots, g_0 . By the idempotency of $(g_0)_G$ and Proposition 4.4, $h \equiv_G g_0$ for all $h \in \text{fp } \bar{g}_{\perp i}$ and all i . It follows in particular that $\text{fp } \bar{g}_{\perp i}$ is monochromatic, say all its elements have color 1. Now, we use the sequence \bar{g} to define $\bar{a} \in G^\omega$ such that all elements of $\text{fp } \bar{a}$ have color 1.

Assume as induction hypothesis that $\text{fp}(a_{\perp i}, g_0)$ is monochromatic of color 1. Our goal is to find a_i such that the same property holds for $\text{fp}(a_{\perp i+1}, g_0)$.

First we claim that from the induction hypothesis it follows that, for all j , all elements of $\text{fp}(a_{\perp i}, \bar{g}_{\perp j})$ have color 1. In fact, the elements of $\text{fp}(a_{\perp i}, \bar{g}_{\perp j})$ have the form $b \cdot h$ for some $b \in \text{fp}(a_{\perp i})$ and $h \in \text{fp}(\bar{g}_{\perp j})$. As $h \equiv_G g_0$, we conclude that $b \cdot h \equiv_G b \cdot g_0$, which proves the claim.

Let $\varphi(a_{\perp i}, g_{i+1}, g_{\perp i+1})$ say that all elements of $\text{fp}(a_{\perp i}, \bar{g}_{\perp i+2})$ have color 1. As \bar{g} is a coheir sequence we can find a_i such that $\varphi(a_{\perp i}, a_i, g_{\perp i+1})$. Hence all elements of $\text{fp}(a_{\perp i+1}, \bar{g}_{\perp i+1})$ have color 1. Therefore a_i is as required. \dashv

Hindman's theorem generalizes to a proposition that subsumes Ramsey's theorem. It is usually referred to as the Milliken–Taylor theorem [13] and [17]. By the following observation, we may use virtually the same proof.

PROPOSITION 5.2. *Assume \downarrow_G is 1-stationary. Let $\bar{g} \in \mathcal{G}^\omega$ be a coheir sequence of some global coheir of $tp(g/G)$ where g has idempotent orbit. Let $\bar{h} \in \mathcal{G}^\omega$ be such that $h_i \in fp(\bar{g} \upharpoonright_{I_i})$ for some finite non-empty $I_i \subseteq \omega$ such that $I_i < I_{i+1}$. Then $\bar{h} \equiv_G \bar{g}$.*

PROOF. Write n_i for the minimum of I_i . It suffices to prove that $h_i \equiv_{G, g \upharpoonright_{n_i}} g_{n_i}$. Note that the type $g \equiv_G x \downarrow_{G, g \upharpoonright_{n_i}}$ is satisfied both by h_i and g_{n_i} , hence the claim follows by stationarity. \dashv

Write $fp(\bar{a})_n$ for the n -uniform hypergraph with vertex set $fp(\bar{a})$ and as edges those sets $\{h_1, \dots, h_n\}$ such that $h_i \in fp(a \upharpoonright_{I_i})$ for some finite sets $I_1 < \dots < I_n$.

MILLIKEN–TAYLOR THEOREM 5.3. *Let $<$ be a relation on G that makes it \bar{c} -closed and \bar{c} -covered. Then for every positive integer n and every finite coloring of the complete n -uniform hypergraph with vertex set G there is a \bar{c} -chain \bar{a} such that $fp(\bar{a})_n$ is monochromatic.*

PROOF. Given a coheir sequence \bar{g} as in the proof of Theorem 5.1 we want to define $\bar{a} \in G^\omega$ such that $fp(\bar{a})_n$ is monochromatic. By the proposition above, $fp(\bar{g} \upharpoonright_{I_i})_n$ is monochromatic for every $i \geq n$. As in the proof of Theorem 5.1, we define by induction $\bar{a} \in G^\omega$ in such a way that $fp(a \upharpoonright_{I_i}, \bar{g} \upharpoonright_{I_i})_n$ is a finite monochromatic subgraph of G . \dashv

§6. The Hales–Jewett theorem. The Hales–Jewett theorem is a purely combinatorial statement that implies the van der Waerden theorem. The original proof by Alfred Hales and Robert Jewett is combinatorial [9]. An alternative proof, also combinatorial, is due by Saharon Shelah [16]. Our proof is similar to the proof by Andreas Blass in [4] (based on ideas from [3]), but we use saturated models where he uses Stone–Čech compactification. We present three versions of the main theorem.

First we prove an abstract algebraic version due to Sabine Koppelberg [11] which is easier to state and to prove (this version comes in two variants). The classical version follows easily from the algebraic one.

We work with the same notation as in Section 4. We say that an element c is *left-minimal* (w.r.t. A) if $c \in A \cdot_G g$ for every $g \in A \cdot_G c$.

PROPOSITION 6.1. *Assume \downarrow_G is 1-stationary. Let A be idempotent and type-definable over G . Then A contains a left-minimal element c with idempotent orbit.*

PROOF. Construct by induction a chain of type-definable idempotent sets $\mathcal{B}_\alpha \subseteq A$ and elements $b_\alpha \in \mathcal{B}_\alpha$ such that $\mathcal{B}_0 = A$ and $\mathcal{B}_{\alpha+1} = A \cdot_G b_\alpha$. For α limit take the intersection. By idempotency of A , it is straightforward to check that $\mathcal{B}_{\alpha+1} \subseteq \mathcal{B}_\alpha$. The sets \mathcal{B}_α are type-definable and idempotent by Lemma 4.1 and Corollary 4.3. For α limit \mathcal{B}_α is non-empty by compactness, as it is intersection of a chain of closed sets.

For some α we cannot properly extend this construction. For this α , for every $c, g \in \mathcal{B}_\alpha$ we have $A \cdot_G c = \mathcal{B}_\alpha = A \cdot_G g$. Hence every $c \in \mathcal{B}_\alpha$ is left-minimal. As \mathcal{B}_α is idempotent, by Corollary 4.7 there is some $c \in \mathcal{B}_\alpha$ with idempotent orbit. \dashv

PROPOSITION 6.2. Assume \Downarrow_G is 1-stationary. Let \mathcal{A} be idempotent and type-definable over G . Let c_G be idempotent and such that $c \cdot_G \mathcal{A}, \mathcal{A} \cdot_G c \subseteq \mathcal{A}$. Then

1. $c \cdot_G \mathcal{A} \cdot_G c$ contains some g with idempotent orbit;
2. if moreover c is left-minimal, then $c \equiv_G g$ for every g as in 1.

Note, parenthetically, that the set in 1 may not be type-definable, therefore Corollary 4.7 does not apply directly and we need an indirect argument.

PROOF.

1. From $c \cdot_G \mathcal{A} \subseteq \mathcal{A}$ we obtain that $\mathcal{A} \cdot_G c$ is idempotent. As it is also type-definable, $\mathcal{A} \cdot_G c$ contains a b with idempotent orbit by Corollary 4.7. There is an $a \in \mathcal{A}$ such that $b_G = a \cdot_G c$, then $b \cdot_G c = b_G$. From this we obtain that $c \cdot_G b$ is idempotent and contained in $c \cdot_G \mathcal{A} \cdot_G c$.
2. From $g \in c \cdot_G \mathcal{A} \cdot_G c$ and the idempotency of c_G we obtain $g_G = c \cdot_G g$. As $g \in \mathcal{A} \cdot_G c$, from the left-minimality of c_G we obtain $c \in \mathcal{A} \cdot_G g$. Hence $c_G = c \cdot_G g$, by the idempotency of g_G . Therefore $c_G = g_G$, which proves 2. ⊖

The following is a technical lemma that is required in many proofs below.

PROPOSITION 6.3. Assume \Downarrow_G is 1-stationary. Let $\sigma : \mathcal{G} \rightarrow \mathcal{G}$ be a semigroup homomorphism definable over G . Then for every $a, b \in \mathcal{G}$

1. $\sigma[a_G] = (\sigma a)_G$,
2. $\sigma[a \cdot_G b] = \sigma a \cdot_G \sigma b$.

PROOF.

1. As $a \equiv_G a'$ implies $\sigma a \equiv_G \sigma a'$, inclusion \subseteq is clear. For the converse, note that the type $\exists y [\sigma y = x \wedge y \equiv_G a]$ is trivially realized by σa . Therefore it is realized by all elements of $(\sigma a)_G$. Hence all elements of $(\sigma a)_G$ are the image of some element in a_G .
2. Let $a \equiv_G a' \Downarrow_G b' \equiv_G b$. By Proposition 4.4 we have $\sigma[a \cdot_G b] = \sigma[(a' \cdot b')_G]$. Then it suffices to prove that $\sigma[(a' \cdot b')_G] \subseteq \sigma a \cdot_G \sigma b$, because by 1 and Proposition 4.4 both sides of the equality are orbits. As σ preserves \Downarrow_G and orbits, we obtain that $\sigma(a' \cdot b')$ is in $\sigma a \cdot_G \sigma b$, as well as all other elements of $\sigma[(a' \cdot b')_G]$. ⊖

HALES–JEWETT THEOREM 6.4 (Koppelberg’s version). Let G be an infinite semigroup and let $C \subset G$ be a nice subsemigroup. Let Σ be a finite set of retractions of G onto C . Then, for every finite coloring of C , there is an $a \in G \setminus C$ such that $\{\sigma a : \sigma \in \Sigma\}$ is monochromatic.

PROOF. Let $G \preceq \mathcal{G}$. Here \mathcal{G} is a monster model in a language that expands the natural one with a symbol for all subsets of G and for every retraction in Σ . As observed in Remark 2.7, this makes \Downarrow_G trivially 1-stationary. Let \mathcal{C} be the definable set such that $C = G \cap \mathcal{C}$. By elementarity, \mathcal{C} is a nice subsemigroup of \mathcal{G} . The language contains also symbols for the retractions $\sigma : \mathcal{G} \rightarrow \mathcal{C}$.

By Proposition 6.1, there is a left-minimal $c \in \mathcal{C}$ with idempotent orbit.

By niceness, $\mathcal{G} \setminus \mathcal{C}$ and c satisfy the assumptions of Proposition 6.2. Hence, by the first claim of that proposition, there is an idempotent $g \in c \cdot_G (\mathcal{G} \setminus \mathcal{C}) \cdot_G c$. In particular, $g \in \mathcal{G} \setminus \mathcal{C}$. Now apply the second claim of Proposition 6.3, with \mathcal{C} for

A to obtain $\sigma g \in c \cdot_G \mathcal{C} \cdot_G c$ for all $\sigma \in \Sigma$. As σg is also idempotent, we apply Proposition 6.2 to conclude that $\sigma g \equiv_G c$. In particular the set $\{\sigma g : \sigma \in \Sigma\}$ is monochromatic.

Though the element g above does not need to belong to $G \setminus C$, by elementarity $G \setminus C$ contains some a with the same property and this proves the theorem. \dashv

Finally we show how the classical Hales–Jewett theorem follows from its abstract version.

If C and X are two semigroups we denote by $C * X$ their free product. That is, $C * X$ contains finite sequences of elements of $C \cup X$, below called *words*, that alternate elements in C with elements in X . The product of two words is obtained concatenating them and, when it applies, replacing two contiguous elements of the same semigroup by their product. Note that C and X are nice subsemigroups of $C * X$. When X is the free semigroup generated by a variable x , we denote $C * X$ by $C[x]$. If $w(x)$ is an element of $C[x]$ and $a \in C$ we denote by $w(a)$ the result of replacing x by a in $w(x)$.

HALES–JEWETT THEOREM 6.5 (Classical version). *Let C be a semigroup generated by some finite set A . Let x be a variable. Then for every finite coloring of $C[x]$ there is a $w(x) \in C[x] \setminus C$ such that $\{w(a) : a \in A\}$ is monochromatic.*

PROOF. Let $G = C[x]$. For every $a \in A$ the homomorphism $\sigma_a : w(x) \mapsto w(a)$ is a retraction of G onto C . Hence we can apply the theorem above. \dashv

We conclude with a variant of Theorem 6.4 that applies to a broader class of semigroup homomorphisms. This result is not required for the following.

For Σ a set of maps $\sigma : G \rightarrow C$ and $c \in C$ we define

$$\Sigma^{-1}[c] = \bigcap_{\sigma \in \Sigma} \sigma^{-1}[c].$$

Clearly, when the maps in Σ are retractions, $\Sigma^{-1}[c]$ is non-empty for all $c \in C$ because it contains at least c .

HALES–JEWETT THEOREM 6.6 (Yet another variant). *Let C be a semigroup and let Σ be a finite set of homomorphisms $\sigma : G \rightarrow C$ such that $\Sigma^{-1}[c]$ is non-empty for all $c \in C$. Then, for every finite coloring of C , there is a $g \in G$ such that the set $\{\sigma g : \sigma \in \Sigma\}$ is monochromatic.*

PROOF. Let $G * C$ be the free product of the two semigroups. Any homomorphism $\sigma : G \rightarrow C$ extends canonically to a retraction of $G * C$ onto C . The elements of G that occur in a word are replaced by their image under σ , finally the elements in the resulting sequence are multiplied. This extension is denoted by the same symbol σ .

Apply Theorem 6.4 to obtain some $w \in G * C$ such that $\{\sigma w : \sigma \in \Sigma\}$ is monochromatic. Suppose $w = c_0 \cdot g_0 \cdots c_n \cdot g_n$ for some $g_i \in G$ and $c_i \in C$, where one or both of c_0 or g_n could be absent. Pick some $h_i \in \Sigma^{-1}[c_i]$ and let $g = h_0 \cdot g_0 \cdots h_n \cdot g_n$. Then $\{\sigma g : \sigma \in \Sigma\}$ is monochromatic as required to complete the proof. \dashv

§7. Carlson’s theorem. This section is devoted to the following lemma and some of its consequences.

LEMMA 7.1. *Let Σ be a finite set of retractions of G onto a nice subsemigroup C . Let $<$ be a relation on G that makes it $<$ -closed and $<$ -covered by $G \setminus C$. Then, for every finite coloring of G , there is a $<$ -chain $\bar{a} \in (G \setminus C)^\omega$ such that $\text{fp}^\Sigma \bar{a} \setminus C$ is monochromatic.*

PROOF. The models \mathcal{G} and \mathcal{C} are as in the proof of Theorem 6.4. The language is the same with $<$ included. Let $\mathcal{B} = \{g \in \mathcal{G} \setminus \mathcal{C} : G < g\}$. By Proposition 6.1 there is some left-minimal $c \in \mathcal{C}$ with idempotent orbit. As G is $<$ -covered by $G \setminus C$, the set \mathcal{B} is non-empty. As G is $<$ -closed and C is nice, \mathcal{B} and c satisfy the assumptions of Proposition 6.2. Then, $c \cdot_G \mathcal{B} \cdot_G c$ contains some g_0 with idempotent orbit. By Proposition 6.3, we obtain that $\sigma g_0 \in c \cdot_G \mathcal{C} \cdot_G c$ for all $\sigma \in \Sigma$. As $(\sigma g_0)_G$ is also idempotent, we apply the second claim of Proposition 6.3, with \mathcal{C} for \mathcal{A} to conclude that $\sigma g_0 \equiv_G c$ for all $\sigma \in \Sigma$. Now, let \bar{g} be a coheir sequence as in Theorem 5.1, and assume the notation thereof. As $g_0 \in c \cdot_G \mathcal{B} \cdot_G c$ then $c \cdot_G g_0 = g_0 \cdot_G c = (g_0)_G$. Hence $h \equiv_G g_0$ for all i and all $h \in \text{fp} \bar{g}_{\uparrow i} \setminus \mathcal{C}$. In particular all these h have the same color, say color 1. Now, we can use the sequence \bar{g} to define $\bar{a} \in (G \setminus C)^\omega$ such that all elements of $\text{fp}^\Sigma \bar{a} \setminus C$ have color 1 by reasoning as in the proof of Theorem 5.1. ⊣

Carlson’s theorem is a result that combines the theorems of Hindman and Hales–Jewett and has a number of important consequences. We refer the reader to [7] for a discussion of some of these consequences. The definitions in Example 1.1 will help matching the notation.

We first present a Koppelberg-style version of the theorem. It is obtained from the lemma above applying a suitable coding.

CARLSON THEOREM 7.2 (À la Koppelberg). *Let Σ be a finite set of retractions of G onto a nice subsemigroup C . Let $\bar{s} \in (G \setminus C)^\omega$. Then for every finite coloring of G , there is an increasing sequence of positive integers $\langle n_i : i < \omega \rangle$ and some $a_i \in \text{fp}^\Sigma s_{\uparrow [n_i, n_{i+1})} \setminus C$ such that $\text{fp}^\Sigma \bar{a} \setminus C$ is monochromatic.*

PROOF. Let G_* be the free semigroup generated by the alphabet

$$\{\langle \sigma, g \rangle : \sigma \in \Sigma \cup \{\text{id}_G\}, g \in G \setminus C\}.$$

The semigroup C_* is defined as G_* , only σ is restricted to range over Σ . Clearly C_* is a nice subsemigroup of G_* . We associate to each $\sigma \in \Sigma$ the endomorphism of G_* that substitutes σ for every occurrence of id_G in a word. These maps, which we denote by σ_* , are retractions of G_* onto C_* .

If $g_* \in G_*$ has the form $\langle \sigma_1, g_1 \rangle \cdots \langle \sigma_n, g_n \rangle$ we call $\sigma_1 g_1 \cdots \sigma_n g_n \in G$ the *evaluation* of g_* . We denote the evaluation by $\text{eval}(g_*)$. As $\tau \sigma = \sigma$ for every $\tau, \sigma \in \Sigma$, we have that $\text{eval}(\sigma_* g_*) = \sigma \text{eval}(g_*)$. The evaluation of $g_* \in C_*$ belongs to C and, as C is nice, and the evaluation of $g_* \in G_* \setminus C_*$ belongs to $G \setminus C$.

We color each element of G_* with the color of its evaluation.

We define the relation $<$ on G_* . First, we need to define the *well-formed* elements of G_* . These are elements of the form $\langle \sigma_1, s_{i_1} \rangle \cdots \langle \sigma_n, s_{i_n} \rangle$ for some $i_1 < \cdots < i_n$. Now,

for $h_*, g_* \in G_*$ we define $h_* < g_*$ if one of the following holds:

1. h_* is not well-formed while g_* is;
2. the product (i.e., concatenation) h_*g_* is well-formed.

It is immediate to verify that $<$ is G_* is ω -closed and $<$ -covered by $G_* \setminus C_*$. Therefore by Lemma 7.1 there is a $<$ -chain $\bar{a}_* \in (G_* \setminus C_*)^\omega$ such that $\text{fp}^\Sigma \bar{a}_* \setminus C_*$ is monochromatic. We can assume that all elements of \bar{a}_* are well-formed (only the first element might be ill-formed, but we can drop it). Then the sequence $\langle \text{eval}(a_{i*}) : i \in \omega \rangle$ is as required by the lemma. \dashv

From the algebraic version of Carlson’s theorem we obtain the classical one in the same way as for the Hales–Jewett theorem (Theorem 6.5), which we refer to for the notation.

COROLLARY 7.3 (Carlson’s theorem, classical version). *Let C be a semigroup generated by some finite set A . Let x be a variable. Let $\bar{s} \in (C[x] \setminus C)^\omega$. Let Σ contain, for every $a \in A$, the function $w(x) \mapsto w(a)$. Then, for every finite coloring of $C[x]$, there is an increasing sequence of positive integers $\langle n_i : i < \omega \rangle$ and some $a_i \in \text{fp}^\Sigma s_{\upharpoonright [n_i, n_{i+1})} \setminus C$ such that $\text{fp}^\Sigma \bar{a} \setminus C$ is monochromatic (with the terminology of Example 1.1, \bar{a} is an extracted variable sequence of \bar{s}).*

§8. Gowers’ partition theorem. The following is similar to Lemma 7.1 but here Σ contains compositions of homomorphisms.

LEMMA 8.1. *For $0 < i < n$, let G_i be a nice subsemigroup of G_{i+1} and let $\sigma_i : G_{i+1} \rightarrow G_i$ be homomorphisms. Let $<$ be a relation on G_n that makes it ω -closed and $<$ -covered by $G_n \setminus G_{n-1}$. Finally, let $\Sigma = \{\sigma_i \circ \dots \circ \sigma_{n-1} : 0 < i < n\}$. Then, for every finite coloring of G_n , there is a $<$ -chain $\bar{a} \in (G_n \setminus G_{n-1})^\omega$ such that $\text{fp}^\Sigma \bar{a} \setminus G_{n-1}$ is monochromatic.*

PROOF. For convenience, we let i run from 0, hence we agree that $\sigma_0 : G_1 \rightarrow G_0 = G_1$ is the identity. Let $\mathcal{B}_n = \{b \in \mathcal{G}_n \setminus \mathcal{G}_{n-1} : G_n < b\}$ and $\mathcal{B}_i = \sigma_i[\mathcal{B}_{i+1}]$. Note that the \mathcal{B}_i are non-empty because G_n is $<$ -covered by $G_n \setminus G_{n-1}$. Also, as \mathcal{G}_i is a nice subsemigroup of \mathcal{G}_{i+1} , we have that $\mathcal{B}_i \cdot_G \mathcal{B}_{i+1}, \mathcal{B}_{i+1} \cdot_G \mathcal{B}_i \subseteq \mathcal{B}_{i+1}$.

We claim there is some $b_n \in \mathcal{B}_n$ with idempotent orbit such that, if we define $b_i = \sigma_i b_{i+1}$ for $0 \leq i < n$, the following holds:

$$b_n \cdot_G b_i = b_i \cdot_G b_n = (b_n)_G.$$

Note that these equalities may be replaced by

$$b_i \cdot_G b_{i+1} = b_{i+1} \cdot_G b_i = (b_{i+1})_G. \tag{\#i}$$

Let $b_0 = b_1$ be any element of \mathcal{B}_0 with idempotent orbit. We assume as induction hypothesis that we have $b_i \in \mathcal{B}_i$ for $i \leq k$, with idempotent orbits, such that $b_i = \sigma_i b_{i+1}$ and $\#i$ hold for all $i < k$. We show how to find b_{k+1} .

We prove that b_k and the set $\mathcal{B}_{k+1} \cap \sigma_k^{-1}[b_k]$, which below we denote by \mathcal{A} for short, satisfy the assumptions of Proposition 6.2. The proof of the idempotency of \mathcal{A} is left to the reader. We prove that $b_k \cdot_G \mathcal{A} \subseteq \mathcal{A}$, the proof of $\mathcal{A} \cdot_G b_k \subseteq \mathcal{A}$ is similar. As $b_k \cdot_G \mathcal{B}_{k+1} \subseteq \mathcal{B}_{k+1}$ by nicety, it suffices to prove that $b_k \cdot_G \sigma_k^{-1}[b_k]$ is contained

in $\sigma_k^{-1}[b_k]$. This latter inclusion holds because, by the induction hypothesis,

$$\sigma_k \left[b_k \cdot_G \sigma_k^{-1}[b_k] \right] = \sigma_k[b_k] \cdot_G b_k = b_{k-1} \cdot_G b_k = (b_k)_G.$$

Now we apply Proposition 6.2 to find an idempotent $b_{k+1} \in b_k \cdot_G \mathcal{A} \cdot_G b_k$. Therefore $\#_k$ is satisfied. Moreover $\sigma_k b_{k+1} \in (b_k)_G$ by Proposition 6.3, hence we can assume $b_k = \sigma_k b_{k+1}$ as claimed above.

Finally, as in the proof of Theorem 5.1, the required chain \bar{a} is obtained from a coher sequence of a global coher of $\text{tp}(b_n/G)$. -1

REMARK 8.2. The lemma above continues to hold, with essentially the same proof, if for Σ we take a set of the form

$$\Sigma = \bigcup_{i=1}^{n-1} \Sigma_i \circ \dots \circ \Sigma_{n-1},$$

where

$$\Sigma_i \circ \dots \circ \Sigma_{n-1} = \{ \sigma_i \circ \dots \circ \sigma_{n-1} : \sigma_i \in \Sigma_i, \dots, \sigma_{n-1} \in \Sigma_{n-1} \},$$

and where Σ_i are some finite sets of homomorphisms $G_{i+1} \rightarrow G_i$ such that for every $g \in G_i$ the set $\Sigma_i^{-1}[g]$ is non-empty.

Let G_i be the set of functions $a : \omega \rightarrow \{0, \dots, i\}$ with finite support that is, the set $\text{supp}(a) = \{x \in \omega : a x \neq 0\}$ is finite. We introduce a semigroup operation on G_i by defining $(a \cdot b)x = \max\{ax, bx\}$. This makes G_i a nice subsemigroup of G_{i+1} .

COROLLARY 8.3 (Gowers' partition theorem). *With G_i as above, let $\sigma_i : G_{i+1} \rightarrow G_i$ be homomorphisms and let Σ be as in Lemma 8.1. Then for every finite coloring of G_n there is an $\bar{a} \in (G_n \setminus G_{n-1})^\omega$ such that $\text{fp}^\Sigma \bar{a} \setminus G_{n-1}$ is monochromatic and $\text{supp}(a_i) < \text{supp}(a_{i+1})$.*

The homomorphisms σ_i usually considered in the literature are so-called *tetris* operations, i.e., $(\sigma_i a)x = \max\{a x - 1, 0\}$, or generalizations thereof. However the theorem is more general.

PROOF. Let $<$ be the relation $\text{supp}(a) < \text{supp}(b)$ and apply Theorem 8.1. -1

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