Generalized Descent Algebras

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Abstract. If A is a subset of the set of reflections of a finite Coxeter group W, we define a sub-$\mathbb{Z}$-module $\mathcal{D}_A(W)$ of the group algebra $\mathbb{Z}W$. We discuss cases where this submodule is a subalgebra. This family of subalgebras includes strictly the Solomon descent algebra, the group algebra and, if W is of type B, the Mantaci–Reutenauer algebra.

Introduction

Let (W, S) be a finite Coxeter system whose length function is denoted by $\ell \colon W \to \mathbb{N} = \{0, 1, 2, \dots\}$. In 1976, Solomon introduced a remarkable subalgebra ΣW of the group algebra $\mathbb{Z}W$, called the *Solomon descent algebra* [11]. Let us recall its definition. If $I \subset S$, let W_I denote the *standard parabolic subgroup* generated by I. Then

$$X_I = \{w \in W \mid \forall s \in I, \, \ell(ws) > \ell(w)\}$$

is a set of *minimal length coset representatives* of W/W_I. Let $x_I = \sum_{w \in X_I} w \in \mathbb{Z}W$. Then ΣW is defined as the sub-$\mathbb{Z}$-module of $\mathbb{Z}W$ spanned by $(x_I)_{I \subset S}$. The study of this algebra is strongly related to the study of many problems in symmetric groups and Coxeter groups, see for instance [2, 5, 10, 12].

In [3], the authors constructed a subalgebra $\Sigma'(W_n)$ of the group algebra $\mathbb{Z}W_n$ of the Coxeter group W_n of type B_n: it turns out that this subalgebra is defined from "generalized descent sets" relative to a larger set of reflections than S and that it contains ΣW_n. In fact $\Sigma'(W_n)$ is the Mantaci–Reutenauer algebra [8].

It is natural to ask whether this kind of construction can be generalized to other groups. Let us explain now what kind of subalgebras we are looking for.

Let $T = \{wsw^{-1} \mid w \in W \text{ and } s \in S\}$ be the set of reflections in W. Let A be a fixed subset of T. If $w \in W$, let $D_A(w) = \{s \in A \mid \ell(ws) < \ell(w)\} \subset A$ be the A-descent set of w. A subset I of A is said to be A-admissible if there exists $w \in W$ such that $D_A(w) = I$. Let $\mathcal{P}_{\mathrm{ad}}(A)$ denote the set of A-admissible subsets of A. If $I \in \mathcal{P}_{\mathrm{ad}}(A)$, we set $D_I^A = \{w \in W \mid D_A(w) = I\}$ and $d_I^A = \sum_{w \in D_I^A} w \in \mathbb{Z}W$. Now, let

$$\mathcal{D}_A(W) = \bigoplus_{I \in \mathcal{P}_{\mathrm{ad}}(A)} \mathbb{Z}d_I^A.$$

As an example, $\mathcal{D}_S(W) = \Sigma_S(W) = \Sigma W$. In fact, they are precisely the ΣW-modules defined in [9] (see Remark 1.12). The main theorem of this paper is the following (here, $C(w)$ denotes the conjugacy class of w in W).

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Theorem A *If there exist two subsets S_1 and S_2 of S such that $A = S_1 \cup (\bigcup_{s \in S_2} C(s))$, then $\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$.*

This theorem is a generalization of Atkinson's proof [1] of Solomon's result (take $A = S$). It provides a generalization of Atkinson's proof to the case of the Mantaci–Reutenauer algebra (take $A = \{s_1, \ldots, s_{n-1}\} \cup C(t)$, where $S = \{t, s_1, \ldots, s_{n-1}\}$ satisfies $C(t) \cap \{s_1, \ldots, s_{n-1}\} = \varnothing$). But for instance the theorem also gives another algebra in type B_n (take $A = \{t\} \cup C(s_1)$) and a new algebra in type F_4 of $\mathbb{Z}$-rank 300 (take $A = \{s_1, s_2\} \cup C(s_3)$, where $S = \{s_1, s_2, s_3, s_4\}$ satisfies $\{s_1, s_2\} \cap C(s_3) = \varnothing$). Moreover, if $A = T$, we get that $\mathcal{D}_A(W) = \mathbb{Z}W$ (see Example 1.7). In the case of dihedral groups, we get another family of algebras.

Theorem B *If W is a dihedral group of order $4m$ ($m \geq 1$), $S = \{s, t\}$ and $A = \{s, t, sts\}$ or $A = \{t, sts\}$, then $\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$.*

It must be noted that the algebras constructed in Theorems A and B are not necessarily unitary. More precisely, $1 \in \mathcal{D}_A(W)$ if and only if $S \subset A$. Moreover, if $S \subset A$, then $\Sigma W \subset \mathcal{D}_A(W)$. Some computations with GAP suggest that the following question has a positive answer. If $\mathcal{D}_A(W)$ is a unitary subalgebra of $\mathbb{Z}W$, is it true that A is one of the subsets mentioned in Theorems A and B?

This paper is organized as follows. Section 1 is essentially devoted to the proofs of Theorems A and B. In Section 2, we discuss more precisely the case of dihedral groups.

1   Descent Sets

Let (W, S) be a finitely generated Coxeter system (not necessary finite). If $s, s' \in S$, we denote by $m(s, s')$ the order of $ss' \in W$. If W is finite, we denote by w_0 its longest element.

1.1   Root System

Let V be an $\mathbb{R}$-vector space endowed with a basis indexed by S denoted by $\Delta = \{\alpha_s \mid s \in S\}$. Let $B \colon V \times V \to \mathbb{R}$ be the symmetric bilinear form such that

$$B(\alpha_s, \alpha_{s'}) = -\cos\left(\frac{\pi}{m(s, s')}\right)$$

for all $s, s' \in S$. If $s \in S$ and $v \in V$, we set $s(v) = v - 2B(\alpha_s, v)\alpha_s$. Thus s acts as the reflection in the hyperplane orthogonal to α_s (for the bilinear form B). This extends to an action of W on V as a group generated by reflections. It stabilizes B.

We recall some basic terminology on root systems (see for instance [4, 6]). The *root system* of (W, S) is the set $\Phi = \{w(\alpha_s) \mid w \in W, s \in S\}$ and the elements of Δ are the *simple roots*. The roots contained in

$$\Phi^+ = \left(\sum_{\alpha \in \Delta} \mathbb{R}^+ \alpha\right) \cap \Phi$$

are said to be *positive*, while those contained in $\Phi^- = -\Phi^+$ are said to be *negative*. Moreover, Φ is the disjoint union of Φ^+ and Φ^-. If $w \in W$, $\ell(w) = |N(w)|$, where

$$N(w) = \left\{ \alpha \in \Phi^+ \mid w(\alpha) \in \Phi^- \right\}.$$

Let $\alpha = w(\alpha_s) \in \Phi$; then $s_\alpha = wsw^{-1}$ acts as the reflection in the hyperplane orthogonal to α and $s_\alpha = s_{-\alpha}$. Therefore, the *set of reflections of W*

$$T = \bigcup_{w \in W} wSw^{-1}$$

is in bijection with Φ^+ (and thus Φ^-).

Let us recall the following well-known result.

Lemma 1.1 *Let $w \in W$.*

(i)   *If $\alpha \in \Phi^+$, then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) \in \Phi^+$.*

(ii) *If $s \in S$, then*

$$N(sw) = \begin{cases} N(w) \coprod \{w^{-1}(\alpha_s)\} & \text{if } \ell(sw) > \ell(w), \\ N(w) \setminus \{-w^{-1}(\alpha_s)\} & \text{otherwise.} \end{cases}$$

Henceforth, we fix a subset A of T. We start with easy observations.

As a consequence of Lemma 1.1(i), we get that

$$D_A(w) = \left\{ s_\alpha \in A \mid \alpha \in \Phi^+ \text{ and } w(\alpha) \in \Phi^- \right\}.$$

We also set

$$N_A(w) = \{ \alpha \in \Phi^+ \mid s_\alpha \in A \text{ and } w(\alpha) \in \Phi^- \}.$$

The map $N_A(w) \to D_A(w)$, $\alpha \mapsto s_\alpha$ is then a bijection.

1.2   Properties of the Map D_A

First, using Lemma 1.1(ii), we get the following.

Corollary 1.2   *If $s \in S$ and if $w \in W$ is such that $w^{-1}sw = s_{w^{-1}(\alpha_s)} \notin A$, then $N_A(w) = N_A(sw)$ (and $D_A(w) = D_A(sw)$).*

Remark 1.3   *If $A_1 \subset A_2 \subset T$, then $D_{A_1}(w) = D_{A_2}(w) \cap A_1$ for all $w \in W$. Therefore if W is finite, $\mathcal{D}_{A_1}(W) \subset \mathcal{D}_{A_2}(W)$.*

Proposition 1.4   *We have*

(i)   *$\varnothing$ is A-admissible;*

(ii) *$D_\varnothing^A = \{1\}$ if and only if $S \subset A$.*

Proof   We have $D_A(1) = \varnothing$ so (i) follows. If $s \in S \setminus A$, then $D_A(s) = \varnothing$. This shows (ii). ∎

The notion of *A*-descent set is obviously compatible with direct products:

Proposition 1.5   *Assume that $W = W_1 \times W_2$ where W_1 and W_2 are standard parabolic subgroups of W. Then for all $I \in \mathcal{P}_{\mathrm{ad}}(A)$, we have*

$$D_I^A = D_{I \cap W_1}^{A \cap W_1} \times D_{I \cap W_2}^{A \cap W_2}.$$

Corollary 1.6   *Assume that W is finite and that $W = W_1 \times W_2$ where W_1 and W_2 are standard parabolic subgroups of W. Then $\mathcal{D}_A(W) = \mathcal{D}_{A \cap W_1}(W_1) \otimes_{\mathbb{Z}} \mathcal{D}_{A \cap W_2}(W_2)$.*

Example 1.7   Consider the case where $A = T$ (then $N_A(w) = N(w)$). It is well known [4, Chapter VI, Exercise 16] that the map $w \mapsto N(w)$ from W onto the set of subsets of Φ^+ is injective (observe that if $\alpha \in N(w_1 w_2^{-1})$, then $\pm w_2^{-1}(\alpha)$ lives in the union, but not in the intersection, of $N(w_1)$ and $N(w_2)$). Therefore, the map $W \to \mathcal{P}_{\mathrm{ad}}(T)$, $w \mapsto D_T(w)$ is injective. In particular, if W is finite, then $\mathcal{D}_T(W) = \mathbb{Z}W$.

In the case of finite Coxeter groups, the multiplication on the left by the longest element has the following easy property.

Proposition 1.8   *If W is finite and if $w \in W$, then $D_A(w_0 w) = A \setminus D_A(w)$.*

Corollary 1.9   *If W is finite, then*

(i)   *A is A-admissible;*
(ii)   *$I \in \mathcal{P}_{\mathrm{ad}}(A)$ if and only if $A \setminus I \in \mathcal{P}_{\mathrm{ad}}(A)$;*
(iii) *$D_A^A = \{w_0\}$ if and only if $S \subset A$.*

Proof $D_A(w_0) = A$, so (i) follows. (ii) follows from Proposition 1.8. (iii) follows from Proposition 1.8 and Proposition 1.4(ii). ∎

1.3   Left-Connectedness

Atkinson gave a new proof of Solomon's result by using an equivalence relation to describe descent sets [1]. We extend his result to A-descent sets. It shows in particular that the subsets D_I^A are left-connected (recall that a subset E of W is said to be *left-connected* if, for all $w, w' \in E$, there exists a sequence $w = w_1, w_2, \ldots, w_r = w'$ of elements of E such that $w_{i+1} w_i^{-1} \in S$ for every $i \in \{1, 2, \ldots, r-1\}$).

Let w and w' be two elements of W. We say that w is an *A-descent neighborhood* of w', and write $w \smile_A w'$, if $w'w^{-1} \in S$ and $w^{-1}w' \notin A$. It is easily seen that $\smile_A$ is a symmetric relation. The reflexive and transitive closure of the A-descent neighborhood relation is called *the A-descent equivalence*, and is denoted by $\sim_A$. The next proposition characterizes this equivalence relation in terms of A-descent sets.

Proposition 1.10   *Let $w, w' \in W$. Then*

$$w \sim_A w' \Leftrightarrow D_A(w) = D_A(w') \Leftrightarrow N_A(w) = N_A(w').$$

Proof The second equivalence is clear. If $w \smallsmile_A w'$, then it follows from Corollary 1.2 that $N_A(w) = N_A(w')$. It remains to show that if $N_A(w) = N_A(w')$, then $w \sim_A w'$.

So, assume that $N_A(w) = N_A(w')$. Write $x = w'w^{-1}$ and let $m = \ell(x)$. If $\ell(x) = 0$, then $w = w'$ and we are done. Assume that $m \geq 1$, and write $x = s_1 s_2 \cdots s_m$ with $s_i \in S$. We now want to prove by induction on m that

$$(*) \qquad w \smallsmile_A s_m w \smallsmile_A s_{m-1} s_m w \smallsmile_A \cdots \smallsmile_A s_2 \cdots s_m w \smallsmile_A s_1 s_2 \cdots s_m w = w'.$$

First, assume that $w \not\smallsmile_A s_m w$. In other words, $w^{-1} s_m w \in A$. For simplification, let $\alpha_i = \alpha_{s_i}$. By Lemma 1.1(ii), we have

$$N_A(s_m w) = N_A(w) \coprod \{w^{-1}(\alpha_m)\} \quad \text{or} \quad N_A(s_m w) = N_A(w) \setminus \{-w^{-1}(\alpha_m)\}.$$

In the first case, as $N_A(w) = N_A(w')$, and by applying Lemma 1.1(ii) again, there exists a step $i \in \{1, 2, \ldots, m-1\}$ between $N_A(w)$ to $N_A(w')$ where $w^{-1}(\alpha_m)$ is removed from $N_A(s_i \cdots s_m w)$, that is, $(s_{i+1} \cdots s_m w)^{-1}(\alpha_i) = -w^{-1}(\alpha_m)$. In the same way, we get the same result in the second case. In other words, we have proved that there exists $i \in \{1, 2, \ldots, m-1\}$ such that $s_m \cdots s_{i+1}(\alpha_i) = -\alpha_m$, so, by Lemma 1.1(i), we have $\ell(s_m \cdots s_{i+1} s_i) < \ell(s_m \cdots s_{i+1})$. This contradicts the fact that $m = \ell(x)$. So $w \smallsmile_A s_m w$, and then $N_A(w) = N_A(s_m w)$. Hence $N_A(s_m w) = N_A(w')$ and $w'(s_m w)^{-1} = s_1 s_2 \cdots s_{m-1}$. We then get by induction that

$$s_m w \smallsmile_A s_{m-1} s_m w \smallsmile_A \cdots \smallsmile_A s_1 s_2 \cdots s_m w = w',$$

which shows $(*)$. ∎

Corollary 1.11 *If $I \in \mathcal{P}_{\mathrm{ad}}(A)$, then D_I^A is left-connected.*

Remark 1.12 The above corollary was first stated by Tits [13, Theorem 2.19] as follows: Let $s \in T$; we denote by X_s the set of $w \in W$ such that $\ell(ws) > \ell(w)$. Then for all $J \subset A \subset T$, the set $Y_J^A = \bigcap_{s \in J} X_s \cap \bigcap_{s \in A \setminus J}(W \setminus X_s)$ is left-connected. An easy computation shows that $Y_J^A = D_{A \setminus J}^A$ (possibly empty). Proposition 1.10 provides a new (shorter) proof of Tits' theorem.

We mention that, using this result and the terminology of Tits, Moszkowski [9] has shown that for all $A \subset T$, $\mathcal{D}_A(W)$ is a ΣW-module for the left multiplication.

1.4   Nice Subsets of T

We say that A is *nice* if for every $s \in A$ and $w \in W$ such that $w^{-1} s w \notin A$, we have $D_A(sw) = D_A(w)$. Notice that every subset of S is nice, by Corollary 1.2.

If $w \in W$ and $I, J \in \mathcal{P}_{\mathrm{ad}}(A)$, we set

$$D_A(I, J, w) = \{(u, v) \in D_I^A \times D_J^A \mid uv = w\}.$$

The next lemma gives a characterization of the fact that $\mathcal{D}_A(W)$ is an algebra in terms of these sets.

Lemma 1.13   *Assume that W is finite. Then the following are equivalent.*

(i) $\mathcal{D}_A(W)$ *is a subalgebra of* $\mathbb{Z}W$.
(ii)   *For all I, J* $\in \mathcal{P}_{\mathrm{ad}}(A)$ *and for all w, w'* $\in W$ *such that* $D_A(w) = D_A(w')$, *we have* $|D_A(I, J, w)| = |D_A(I, J, w')|$.

If these conditions are fulfilled, we choose for any I $\in \mathcal{P}_{\mathrm{ad}}(A)$ *an element* z_I *in* D_I^A. *Then*

$$d_I^A d_J^A = \sum_{K \in \mathcal{P}_{\mathrm{ad}}(A)} |D_A(I, J, z_K)| d_K^A.$$

Proof   That (ii) implies (i) is obvious.
    If (i) is true, then, on the one hand,

$$d_I^A d_J^A = \sum_{K \in \mathcal{P}_{\mathrm{ad}}(A)} c_K d_K^A$$

for some $c_K \in \mathbb{Z}$. On the other hand,

$$d_I^A d_J^A = \sum_{D_A(u)=I} \sum_{D_A(v)=J} uv = \sum_{K \in \mathcal{P}_{\mathrm{ad}}(A)} \sum_{D_A(w)=K} |D_A(I, J, w)| w.$$

(ii) follows by identification.   ∎

    Let us now fix $s \in S$ and let $(u, v) \in W \times W$. If $u \smile_A su$, we set $\psi_s^A(u, v) = (su, v)$. If $u \not\smile_A su$, then we set $\psi_s^A(u, v) = (u, u^{-1}suv)$. Note that in the last case, $u^{-1}su \in A$. We have $(\psi_s^A)^2 = \mathrm{Id}_{W \times W}$. In particular, ψ_s^A is a bijection. Using ψ_s^A, one can relate the notion of nice subsets to the property (ii) stated in Lemma 1.13.

Proposition 1.14   *Assume that A is nice. Let I, J* $\in \mathcal{P}_{\mathrm{ad}}(W)$, *let w* $\in W$, *and let s* $\in S$ *be such that* $w \smile_A sw$. *Then* $\psi_s^A(D_A(I, J, w)) = D_A(I, J, sw)$.

Proof   Let (u, v) be an element of $D_A(I, J, w)$. By symmetry, we only need to prove that $\psi_s^A(u, v) \in D_A(I, J, sw)$. If $u \smile_A su$, then $D_A(su) = D_A(u) = I$ by Proposition 1.10, so $\psi_s^A(u, v) = (su, v) \in D_A(I, J, sw)$. So, we may assume that $u \not\smile_A su$. Let $s' = u^{-1}su \in A$. Note that $w^{-1}sw = v^{-1}s'v \notin A$. Then $\psi_s^A(u, v) = (u, s'v)$ and $us'v = sw$. So we only need to prove that $D_A(s'v) = D_A(v)$. But this just follows from the definition of nice subset of T.   ∎

Corollary 1.15   *If W is finite and if A is nice, then* $\mathcal{D}_A(W)$ *is a subalgebra of* $\mathbb{Z}W$. *It is unitary if and only if* $S \subset A$.

Proof   This follows from Lemma 1.13 and from Propositions 1.10 and 1.14.   ∎

1.5 Proof of Theorems A and B

Using Corollary 1.15, we see that Theorems A and B are direct consequences of the following theorem (which holds also for infinite Coxeter groups).

Theorem 1.16 *Assume that one of the following holds.*

(i) *There exist two subsets S_1 and S_2 of S such that $A = S_1 \cup \left(\bigcup_{s \in S_2} C(s) \right)$.*

(ii) *$S = \{s, t\}$, $m(s, t)$ is even or ∞, and $A = \{s, t, sts\}$ or $A = \{t, sts\}$.*

Then A is nice.

Proof Assume that (i) or (ii) holds. Let $r \in A$ and let $w \in W$ be such that $w^{-1}rw \notin A$. We want to prove that $D_A(rw) = D_A(w)$. By symmetry, we only need to show that $D_A(rw) \subset D_A(w)$. If $r \in S$, then this follows from Corollary 1.2. So we may assume that $r \notin S$.

Assume that (i) holds. Write $A' = \bigcup_{s \in S_2} C(s)$ and $S' = A \setminus A'$. Then $S' \subset S$, $A = A' \coprod S'$ and A' is stable under conjugacy. Then $r \in A'$ and $w^{-1}rw \in A' \subset A$, which contradicts our hypothesis.

Assume that (ii) holds. If $m(s, t) = 2$, then A is contained in S and therefore is nice by Corollary 1.2. So we may assume that $m(s, t) \geq 4$. Since $r \notin S$, we have $r = sts$. Assume that $D_A(rw) \not\subset D_A(w)$. Let

$$\Phi_A = \{\alpha \in \Phi^+ \mid s_\alpha \in A\} \subset \{\alpha_s, \alpha_t, s(\alpha_t)\}.$$

There exists $\alpha \in \Phi_A$ such that $rw(\alpha) \in \Phi^-$ and $w(\alpha) \in \Phi^+$. So $w(\alpha) \in N(r) = \{\alpha_s, s(\alpha_t), st(\alpha_s)\}$. Since $w^{-1}rw \notin A$, we have that $ws_\alpha w^{-1} \neq r$, so $w(\alpha) \neq s(\alpha_t)$. So $w(\alpha) = \alpha_s$ or $st(\alpha_s)$. But the roots α_s and α_t lie in different W-orbits, so $\alpha = \alpha_s$ and $w(\alpha_s) \in \{\alpha_s, st(\alpha_s)\}$. If W is infinite, this gives that $w \in \{1, st\}$, which contradicts the fact that $w^{-1}rw \notin A$. If W is finite, this gives that $w \in \{1, st, w_0s, stsw_0\}$. But again, this contradicts the fact that $w^{-1}rw \notin A$. ∎

1.6 A Remark Concerning the Solomon Homomorphism

Let $\mathbb{Z}\,\mathrm{Irr}\,W$ denote the characters algebra of W, ΣW is endowed with a $\mathbb{Z}$-linear map $\theta \colon \Sigma W \to \mathbb{Z}\,\mathrm{Irr}\,W$ satisfying $\theta(x_I) = \mathrm{Ind}_{W_I}^W 1_{W_I}$. This is an algebra homomorphism [11].

If W is the symmetric group $\mathfrak{S}_n$, then θ becomes an epimorphism and the pair $(\Sigma\mathfrak{S}_n, \theta)$ provides a nice construction [7] of $\mathrm{Irr}(\mathfrak{S}_n)$, which is the first ingredient of several recent works, see for instance [12]. However, the morphism θ is surjective if and only if W is a product of symmetric groups.

In [3], the authors have shown that the Mantaci–Reutenauer algebra is also endowed with an algebra homomorphism $\theta' \colon \Sigma'(W_n) \to \mathbb{Z}\,\mathrm{Irr}\,W_n$ extending θ. Moreover, θ' is surjective and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ker}\,\theta'$ is the radical of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_n)$. This leads to a construction of the irreducible characters of W_n following Jőllenbeck's strategy.

However, the situation seems to be much more complicated in the other types. Let us define another sub-$\mathbb{Z}$-module of $\mathbb{Z}W$. If $I \subset A$, we still denote by W_I the subgroup

of W generated by I and we still set

$$X_I = \{w \in W \mid \forall\, x \in W_I,\ \ell(wx) \geq \ell(w)\}.$$

Then X_I is again a set of representatives for W/W_I. Now, let $x_I = \sum_{w \in X_I} w \in \mathbb{Z}W$. Then $\Sigma_A(W) = \sum_{I \subset A} \mathbb{Z}x_I$ is a sub-$\mathbb{Z}$-module of $\mathbb{Z}W$. However, it is not in general a subalgebra of $\mathbb{Z}W$.

The bad point in the above construction is that in many cases one has $\Sigma_A(W) \neq \mathcal{D}_A(W)$. Also, we are not able to construct in general a morphism of algebras $\theta_A \colon \mathcal{D}_A(W) \to \mathbb{Z}\operatorname{Irr}W$ extending θ if $S \subset A$ (see Remark 2.2 and Proposition 2.3 below).

Example 1.17   Assume here that W is of type F_4, that $S = \{s_1, s_2, s_3, s_4\}$ and that $A = C(s_1) \cup S$. Then, using GAP, one can see that $\operatorname{rank}_{\mathbb{Z}} \mathcal{D}_A(W) = 300$ and $\operatorname{rank}_{\mathbb{Z}} \Sigma_A(W) = 149$. Moreover, $\Sigma_A(W)$ is not a subalgebra of $\mathcal{D}_A(W)$.

2   Example: The Dihedral Groups

The aim of this section is to study the unitary subalgebras $\mathcal{D}_A(W)$ constructed in Theorems A and B whenever W is finite and dihedral. Henceforth, we assume that $S = \{s, t\}$ with $s \neq t$ and that $m(s, t) = 2m$, with $2 \leq m < \infty$.

Note that $w_0 = (st)^m$ is central. In what follows, we will need some facts on the character table of W. Let us recall here the construction of $\operatorname{Irr}W$. First, let H be the subgroup of W generated by st. It is normal in W, of order $2m$ (in other words, of index 2). We choose the primitive $(2m)$-th root of unity $\zeta \in \mathbb{C}$ of argument π/m. If $i \in \mathbb{Z}$, we denote by $\xi_i \colon H \to \mathbb{C}^\times$ the unique linear character such that $\xi_i(st) = \zeta^i$. Then $\operatorname{Irr}H = \{\xi_i \mid 0 \leq i \leq 2m - 1\}$. Now let $\chi_i = \operatorname{Ind}_H^W \xi_i$. Then $\chi_i = \chi_{2m-i}$ and, if $1 \leq i \leq m - 1$, $\chi_i \in \operatorname{Irr}W$. Also, χ_i has values in $\mathbb{R}$. More precisely, for $1 \leq i \leq m - 1$ and $j \in \mathbb{Z}$

$$\chi_i((ts)^j) = \zeta^j + \zeta^{-j} = 2\cos\left(\frac{ij\pi}{m}\right) \quad \text{and} \quad \chi_i(s(ts)^j) = 0.$$

Let 1 denote the trivial character of W, let ε denote the sign character and let $\gamma \colon W \to \{1, -1\}$ be the unique linear character such that $\gamma(s) = -\gamma(t) = 1$. Then

(1) $$\operatorname{Irr}W = \{1, \varepsilon, \gamma, \varepsilon\gamma\} \cup \{\chi_i \mid 1 \leq i \leq m - 1\}.$$

In particular, $|\operatorname{Irr}W| = m + 3$.

2.1   The Subset $A = \{s, t, sts\}$

From now on, we assume that $A = \{s, t, sts\}$. We set $\bar{s} = A \setminus \{s\} = \{t, sts\}$ and $\bar{t} = A \setminus \{t\} = \{s, sts\}$. It is easy to see that $\mathcal{P}_{\mathrm{ad}}(A) = \{\varnothing, \{s\}, \{t\}, \bar{s}, \bar{t}, A\}$. For simplification, we will denote by d_I the element d_I^A of $\mathbb{Z}W$ (for $I \in \mathcal{P}_{\mathrm{ad}}(W)$) and we

set $d_s = d_{\{s\}}$ and $d_t = d_{\{t\}}$. We have

$$d_\varnothing = 1, \quad d_{\bar{s}} = w_0 s, \quad d_s = s, \quad d_A = w_0,$$

$$d_t = \sum_{i=1}^{m-1} \left((st)^i + (ts)^{i-1} t \right), \quad d_{\bar{t}} = \sum_{i=1}^{m-1} \left((st)^i s + (ts)^i \right).$$

The multiplication table of $\mathcal{D}_A(W)$ is given by

	1	d_s	$d_{\bar{s}}$	d_A	d_t	$d_{\bar{t}}$
1	1	d_s	$d_{\bar{s}}$	d_A	d_t	$d_{\bar{t}}$
d_s	d_s	1	d_A	$d_{\bar{s}}$	d_t	$d_{\bar{t}}$
$d_{\bar{s}}$	$d_{\bar{s}}$	d_A	1	d_s	$d_{\bar{t}}$	d_t
d_A	d_A	$d_{\bar{s}}$	d_s	1	$d_{\bar{t}}$	d_t
d_t	d_t	$d_{\bar{t}}$	d_t	$d_{\bar{t}}$	z_A	z_A
$d_{\bar{t}}$	$d_{\bar{t}}$	d_t	$d_{\bar{t}}$	d_t	z_A	z_A

where $z_A = (m-1)\left(1 + d_A + d_s + d_{\bar{s}} \right) + (m-2)\left(d_t + d_{\bar{t}} \right)$. We now study the sub-$\mathbb{Z}$-module $\Sigma_A(W)$; we will show that it coincides with $\mathcal{D}_A(W)$. First, it is easily seen that $\mathcal{P}_0(A) = \{\varnothing, \{s\}, \{t\}, \{sts\}, \bar{s}, A\}$ is the set of subsets I of A such that $W_I \cap A = I$ and that

$$
\begin{aligned}
x_A &= 1 \\
x_{\bar{s}} &= 1 + d_s \\
x_{sts} &= 1 + d_s + d_t \\
x_t &= 1 + d_s \phantom{{} + d_t} + d_{\bar{t}} \\
x_s &= 1 \phantom{{} + d_s} + d_t \phantom{{} + d_{\bar{t}}} + d_{\bar{s}} \\
x_\varnothing &= 1 + d_s + d_t + d_{\bar{t}} + d_{\bar{s}} + d_A
\end{aligned}
$$

Therefore, $\Sigma_A(W) = \mathcal{D}_A(W) = \bigoplus_{I \in \mathcal{P}_0(A)} \mathbb{Z} x_I$. So we can define a map

$$\theta_A \colon \Sigma_A(W) \to \mathbb{Z}\operatorname{Irr} W$$

by $\theta_A(x_I) = \operatorname{Ind}_{W_I}^W 1_{W_I}$.

Proposition 2.1 *Assume that $S = \{s, t\}$ with $s \neq t$, that $m(s, t) = 2m$ with $m \geq 2$ and that $A = \{s, t, sts\}$. Then*

(i) $\Sigma_A(W) = \mathcal{D}_A(W)$ *is a subalgebra of $\mathbb{Z}W$ of $\mathbb{Z}$-rank 6.*
(ii) θ_A *is a morphism of algebras.*
(iii) $\operatorname{Ker}\theta_A = \mathbb{Z}(x_t - x_{sts})$.
(iv) $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A$ *is the radical of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma_A(W)$.*
(v) θ_A *is surjective if and only if $m = 2$ that is, if and only if W is of type B_2.*

Proof (i) has already been proven. For proving the other assertions, we need to

compute explicitly the map θ_A. It is given by the following table:

d_I	1	d_s	$d_{\bar{s}}$	d_A	d_t	$d_{\bar{t}}$
$\theta_A(d_I)$	1	$\varepsilon\gamma$	γ	ε	$\displaystyle\sum_{i=1}^{m-1}\chi_i$	$\displaystyle\sum_{i=1}^{m-1}\chi_i$

(iii) This shows that $\operatorname{Ker}\theta_A = \mathbb{Z}(d_{\bar{t}} - d_t) = \mathbb{Z}(x_t - x_{sts})$, so (iii) holds.

(ii) To prove that θ_A is a morphism of algebras, the only difficult point is to prove that $\theta_A(d_t^2) = \theta_A(d_t)^2$. Let ρ denote the regular character of W. Then

$$\theta_A(d_t) = \frac{1}{2}(\rho - 1 - \gamma - \varepsilon\gamma - \varepsilon).$$

Therefore, $\theta_A(d_t)^2 = (m-2)\rho + 1 + \gamma + \varepsilon\gamma + \varepsilon$. But, $d_t^2 = z_A = (m-2)x_\varnothing + 1 + d_s + d_{\bar{s}} + d_A$. This shows that $\theta_A(d_t^2) = \theta_A(d_t)^2$.

(iv) Let R denote the radical of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma_A(W)$. We only need to prove that $\mathbb{C} \otimes_{\mathbb{Z}} R = \mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A$. Since $\mathbb{C}\operatorname{Irr} W$ is a split semisimple commutative algebra, every subalgebra of $\mathbb{C}\operatorname{Irr} W$ is semisimple. So $(\mathbb{C} \otimes_{\mathbb{Z}} \Sigma_A(W))/(\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A)$ is a semisimple algebra. This shows that R is contained in $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A$. Moreover, since $(x_t - s_{sts})^2 = (d_t - d_{\bar{t}})^2 = 0$, $\operatorname{Ker}\theta_A$ is a nilpotent two-sided ideal of $\Sigma_A(W)$. So $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A$ is contained in $\mathbb{C} \otimes_{\mathbb{Z}} R$. This shows (iv).

(v) If $m = 2$, then $\operatorname{Irr} W = \{1, \gamma, \varepsilon\gamma, \varepsilon, \chi_1\} = \theta_A(\{1, d_s, d_{\bar{s}}, d_A, d_t\})$ so θ_A is surjective. Conversely, if θ_A is surjective, then $|\operatorname{Irr} W| = 5$ (by (i) and (iii)). Since $|\operatorname{Irr} W| = m + 3$, this gives $m = 2$. ∎

We close this subsection by giving a complete set of orthogonal primitive idempotents for $\Sigma_A(W)$, extending to our case those given in [2]:

$$E_\varnothing = \frac{1}{4m}x_\varnothing, \quad E_s = \frac{1}{2}\left(x_s - \frac{1}{2}x_\varnothing\right), \quad E_t = \frac{1}{2}\left(x_t - \frac{1}{2}x_\varnothing\right),$$

$$E_{\bar{s}} = \frac{1}{2}\left(x_{\bar{s}} - \frac{1}{2}x_t - \frac{1}{2}x_{sts} + \frac{m-1}{2m}x_\varnothing\right),$$

$$E_A = 1 - \frac{1}{2}x_s - \frac{1}{4}x_t + \frac{1}{4}x_{sts} - \frac{1}{2}x_{\bar{s}} + \frac{1}{4}x_\varnothing.$$

2.2   The Subset $B = \{s\} \cup C(t)$

Let $B = \{s\} \cup C(t)$ (so that $|B| = m + 1$). It is easy to see that $\mathcal{P}_{\mathrm{ad}}(B)$ consists of the sets

$$\varnothing = D_B(1), \quad B = D_B(w_0), \quad \{s\} = D_B(s), \quad C(t) = D_B(w_0 s),$$

$$D_B((ts)^i) = D_B(s(ts)^i), \quad 1 \le i \le m - 1,$$

$$D_B((st)^j) = D_B((ts)^{j-1}t), \quad 1 \le j \le m - 1.$$

Therefore $\mathcal{D}_B(W)$ is a subalgebra of $\mathbb{Z}W$ of $\mathbb{Z}$-rank $(2m + 2)$.

Using GAP, we can see that, in general, $\Sigma_B(W) \neq \mathcal{D}_B(W)$. First examples are given in the following table

m	2	3	4	5	6	7	8	9	10	11
$\mathbb{Z}$-rank of $\mathcal{D}_B(W)$	6	8	10	12	14	16	18	20	22	24
$\mathbb{Z}$-rank of $\Sigma_B(W)$	6	8	10	10	14	12	18	18	22	16

Remark 2.2 The linear map $\theta_B \colon \Sigma_B(W) \to \mathbb{Z}\operatorname{Irr} W$, $x_I \mapsto \operatorname{Ind}_{W_I}^W 1_{W_I}$, is well-defined and surjective if and only if $m \in \{2, 3\}$ (recall that $m \geq 2$). Indeed, the image of θ_B cannot contain a non-rational character. But all characters of W are rational if and only if W is a Weyl group, that is, if and only if $2m \in \{2, 3, 4, 6\}$.

Moreover, if $m = 2$, then $A = B$ and $\theta_B = \theta_A$ is surjective by Proposition 2.1. If $m = 3$, then this follows from Proposition 2.3 below.

2.3 The Algebra $\mathcal{D}_B(G_2)$

From now on $m = 3$, that is, W is of type $I_2(6) = G_2$. For convenience, we keep the same notation as in §2.1. We have

$$d_\varnothing = 1, \qquad\qquad d_1 = d_{\{t\}}^B = t + st,$$

$$d_s = s, \qquad\qquad d_2 = d_{\{s,sts\}}^B = ts + sts,$$

$$d_{\bar{s}} = d_{C(t)}^B = w_0 s, \qquad\qquad d_3 = d_{\{s,sts,tstst\}}^B = tsts + ststs,$$

$$d_A = d_B^B = w_0, \qquad\qquad d_4 = d_{\{t,tstst\}}^B = tst + stst.$$

Let us now show that $\Sigma_B(W) = \mathcal{D}_B(W)$. First, it is easily seen that

$$\mathcal{P}_0(B) = \{\varnothing, \{s\}, \{t\}, \{sts\}, \bar{s}, \{tstst\}, \{s, tstst\}, B\}$$

is the set of subsets I of B such that $W_I \cap B = I$ and that

$$
\begin{aligned}
x_A = x_B \quad &= 1 \\
x_{\bar{s}} \quad &= 1 + d_s \\
x_{\{s,tstst\}} &= 1 \qquad\quad + d_1 \\
x_{tstst} \quad &= 1 + d_s + d_1 + d_2 \\
x_t \quad &= 1 + d_s \qquad\quad + d_2 + d_3 \\
x_{sts} \quad &= 1 + d_s + d_1 \qquad\qquad\quad + d_4 \\
x_s \quad &= 1 \qquad\quad + d_1 \qquad\qquad\quad + d_4 + d_{\bar{s}} \\
x_\varnothing \quad &= 1 + d_s + d_1 + d_2 + d_3 + d_4 + d_{\bar{s}} + d_A
\end{aligned}
$$

Therefore, $\Sigma_B(W) = \mathcal{D}_B(W) = \bigoplus_{I \in \mathcal{P}_0(B)} \mathbb{Z}x_I$. So we can define a map

$$\theta_B \colon \Sigma_B(W) \to \mathbb{Z}\operatorname{Irr} W$$

by $\theta_B(x_I) = \operatorname{Ind}_{W_I}^W 1_{W_I}$.

Proposition 2.3   *Assume that $S = \{s, t\}$ with $s \neq t$, that $m(s, t) = 6$ and that $B = \{s, t, sts, tstst\}$. Then*

(i)   *$\Sigma_B(W) = \mathcal{D}_B(W)$ is a subalgebra of $\mathbb{Z}W$ of $\mathbb{Z}$-rank 8.*

(ii)   *θ_B is a surjective linear map (and not a morphism of algebras).*

(iii) *$\operatorname{Ker} \theta_B = \mathbb{Z}(x_{tstst} - x_t) \oplus \mathbb{Z}(x_{tstst} - x_{sts})$.*

(iv)   *$\operatorname{Irr} W = \theta_B\left(\{1, d_s, d_{\bar{s}}, d_A, d_1, d_2\}\right)$.*

Proof   With computations similar to §2.1, we obtain the following table for the map θ_B.

d_I^B	1	d_s	$d_{\bar{s}}$	d_A	d_1	d_2	d_3	d_4
$\theta_B(d_I^B)$	1	$\varepsilon\gamma$	γ	ε	χ_2	χ_1	χ_2	χ_1

This shows (ii) and (iv). As $d_1 - d_3 = x_{\{tstst\}} - x_t$ and $d_2 - d_4 = x_{tstst} - x_{sts}$, (iii) is proved. Finally, the fact that θ_B is not a morphism of algebras follows from $\theta_B(d_1^2)(w_0) = (1 + \varepsilon\gamma + \chi_1)(w_0) = -2 \neq \theta_B(d_1)^2(w_0) = \chi_2(w_0)^2 = 4$.   ∎

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