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FRACTAL AND RESISTANCE DIMENSIONS OF RANDOM TREES

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In this article we determine a formula for fractal and resistance dimensions of two models of uniformly bounded random trees. The type (transient or recurrent) of the random walk on such trees is ascribed, to some extent, to these dimensions. The results presented in this article generalize some of the results of [6] and [7].

1. INTRODUCTION

The structure of a tree plays a considerable role in the context of probability on trees. This role is essential, for instance, in determining the type of the random walk on a tree, whether it is transient or recurrent, which is equivalent to deciding whether the effective resistance is finite or infinite (if the tree is considered as an electric network by assigning conductivities to its edges). See [2], [5], [8], or [9]. The growth rate, $GR(\Gamma)$, of a tree Γ is defined roughly as the nth root of the size of the nth generation of the tree. As is seen from its definition, $GR(\Gamma)$ barely takes into account the process of branching of Γ. As such, we cannot rely much on $GR(\Gamma)$. The branching number, $BR(\Gamma)$, is more reliable in describing the branching process. $BR(\Gamma)$ is, roughly, the mean number of branches emerging out from a vertex of tree (outdegree). It is known that the random walk on a tree Γ is transient if $BR(\Gamma) > 1$ and could be of either type if $BR(\Gamma) = 1$ see [12]. The energy as well as the capacity of a flow are closely related to the type of the random walk. In this article we study two exponent measures of the structure of the tree: the fractal dimension and the resistance dimension. These dimensions are related to the random walk dimension via the *Einstein* equation $RWD = FD - RD + 2$, where RWD, FD, and RD stand respectively for the random walk dimension, the fractal dimension, and the resistance dimension. This relation holds true for sufficiently regular

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(dense and smooth) trees of polynomial growth. See [13] or [14]. It is shown for a spherically symmetric tree SSRT (where the degree of a vertex depends only on its distance from the root) of polynomial growth that FD = RD and, as such, RWD = 2. This entails that the mean time that the random walk on SSRT, starting at the root of Γ, takes to hit its nth level is n^{t_n} ; $t_n \to 2$.

2. BACKGROUND

To cast light on the connection between probability and electric networks, we consider a finite connected graph G with edge set E, endowed with nonnegative numbers $\{c_{xy} = c_{yx}, xy \in E\}$ called conductivities, and the reciprocals $\{r_{xy} = 1/c_{xy}, xy \in E\}$ are called resistances. The graph G associated with the assigned conductivities is called an electrical network and it is denoted by G^*. A real-valued function v defined on the vertices of G^* is called *harmonic* at a vertex x if

$$\sum_y \frac{c_{xy}}{c_x} v(y) = v(x),$$

where $c_x = \sum_y c_{xy}$. Two vertices a and z are respectively the *source* and the *sink* of the network G^*. According to the uniqueness theorem [2], the harmonic function v on the nodes of G^* is uniquely determined by its boundary values in the sense that if two harmonic functions on $G^*\backslash\{a, z\}$ have the same values on the boundary $\{a, z\}$, then they are equal. The *voltage* function v is harmonic on $G^*\backslash\{a, z\}$. See [2]. For this reason, we call any harmonic function on $G^*\backslash\{a, z\}$ a voltage if it has the same boundary values of the voltage function v.

A flow θ from a to z is a function on the oriented edges that is antisymmetric, $\theta_{xy} = -\theta_{yx}$ and that obeys *Kirchhof's node law* $\sum_y \theta_{xy} = 0$ for $x \notin \{a, z\}$. This is the requirement "flow in equals flow out" for any vertex $x \notin \{a, z\}$.

The voltage difference between a and z induces current that flows into the network. The current flow associated with the voltage v is defined for oriented edges by *Ohm's law*:

$$i_{xy} = \frac{v_x - v_y}{r_{xy}}.$$

The strength of the current flowing into the circuit at vertex a is defined by

$$i_a = \sum_x i_{ax}.$$

If you increase the voltage difference $v_a - v_z$ between a and z, the current i_a will increase such that the ratio $(v_a - v_z)/i_a$ remains constant. This constant is called the *effective resistance* of the network G^* and is denoted by $\mathfrak{R}(a \leftrightarrow z)$; that is,

$$\mathfrak{R}(a \leftrightarrow z) = \frac{v_a - v_z}{i_a}.$$

It is also shown that $\Re(a \leftrightarrow z)$ is the power (energy) dissipation of the unit current flow; see [2] or [9]:

$$\Re(a \leftrightarrow z) = \frac{1}{2}\sum_{x,y} i_{xy}^2 r_{xy}.$$

A weighted (nearest-neighbor) random walk on G is a Markov chain X_n on the vertex set of G with transition probabilities:

$$p_{xy} = p(X_{n+1} = y | X_n = x) = \frac{c_{xy}}{c_x}.$$

The probabilistic interpretation of the effective resistance is described as follows: Let $p(a \to z)$ denote the *escape probability* of the random walk that starts at a; that is

$$p(a \to z) = p_a(X_n \text{ hits } z \text{ before returning to } a).$$

It is shown in [2] that

$$p(a \to z) = \frac{1}{c_a \Re(a \leftrightarrow z)}. \tag{1}$$

We now consider an infinite leafless tree Γ for which the degree $d(y)$, the number of edges incident with y, of every vertex y satisfies $d(y) \geq 2$. If all of the vertices of level n is shorted (soldered) in one vertex z, then Eq. (1) gives the connection between the escape probability and the effective resistance of the portion of the tree between the root and the level n. Taking the limit in Eq. (1) as $n \to \infty$, we get

$$p(a \to \infty) = \frac{1}{c_a \Re(a \leftrightarrow \infty)}, \tag{2}$$

where $\Re(a \leftrightarrow \infty)$ stands for the effective resistance of the whole infinite tree and $p(a \to \infty)$ is the probability that the random walk will escape its starting point a. This entails that the random walk on Γ is *transient* if and only if $\Re(a \leftrightarrow \infty) < \infty$. This is the *Nash–Williams* result. See [1] or [11]. This criterion holds true for any infinite connected graph. Eventually, according to *Thomson's principle*, the random walk on a denumerable graph G is transient iff there is a unit flow on G having finite energy dissipation from some (every) vertex to ∞.

Let Γ_n, $n \geq 0$, denote the set of vertices of level n (having distance n from the root) of Γ and let $|\Gamma_n|$ denote its cardinality. Then $|\Gamma_0| = 1$. Let $B_n = \sum_{k=0}^{n} |\Gamma_k|$ be the number of vertices in the first n levels. The geometric, resistance, and mean hitting time exponents are involved respectively in the following definitions of fractal, resistance, and random walk dimensions.

Consider a general infinite connected graph G with a vertex r to be identified as a *reference vertex*. The fractal dimension of G is defined to be

$$\mathrm{FD} = \limsup_{n\to\infty} \frac{\log B_n}{\log n}, \tag{3}$$

where B_n is the number of vertices of a ball of radius n and centered at r.

If the set of all vertices of G having distance n from r is shorted in one vertex b and R_n denotes the effective resistance between r and b, then the resistance dimension of G, having effective resistance R_{eff}, is defined as

$$\mathrm{RD} = \begin{cases} 2 - \limsup_{n\to\infty} \dfrac{\log R_n}{\log n} & \text{if } R_{eff} = \infty \\ 2 - \limsup_{n\to\infty} \dfrac{\log (R - R_n)}{\log n} & \text{if } R_{eff} = R < \infty. \end{cases} \tag{4}$$

The random walk dimension of G is defined to be

$$\mathrm{RWD} = \limsup_{n\to\infty} \frac{\log E(T_n)}{\log n}, \tag{5}$$

where $E(T_n)$ is the mean time that the random walk takes to exit a ball of radius n and centered at r.

Note 1: We notice that if $\mathrm{RD} < 2$, $R_{eff} = \infty$ and the random walk on G is recurrent, whereas if $\mathrm{RD} > 2$, $R_{eff} < \infty$ and the random walk on G is transient.

It is assumed in [13, 14] that the G is of polynomial growth and it is shown for dense graphs that $\mathrm{FD} \geq \mathrm{RD}$ and hence $\mathrm{RWD} \geq 2$. For smooth graphs, the three exponents dimensions are related by the *Einstein* equation: $\mathrm{RWD} = \mathrm{FD} + 2 - \mathrm{RD}$. It can easily be shown for the binary tree (every vertex has degree 3 except the root has degree 2) that $\mathrm{RWD} = 1$. This is due to the exponential growth of that tree. One important remark is that the random walk on all integer cubical lattices Z^d has $\mathrm{RWD} = 2$, whereas the random walk on Z^d is transient if and only if $d > 2$.

3. AUXILIARY LEMMAS

The following three lemmas are presented in [6].

LEMMA 1: *Suppose that X_n, $n \geq 1$, are independent random variables such that $0 \leq X_n \leq K$ for some constant K. Set $S_n = \sum_{j=1}^{n} X_j$. If $E(S_n) \to \infty$ as $n \to \infty$, then*

$$\frac{S_n}{E(S_n)} \longrightarrow 1 \quad a.s. \text{ as } n \longrightarrow \infty.$$

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LEMMA 2: *Let $\{b_n\}$ be a sequence of real numbers such that $\lim_n nb_n$ exists.*

(i) If $b_n > 0$, then

$$\lim_n nb_n = \lim_n \frac{\sum_{i=1}^{n} b_j}{\log n}$$

(ii) If $b_n > -1$ and $b_n \to 0$, then

$$\lim_n nb_n = \lim_n \frac{\sum_{j=1}^{n} \log(1+b_j)}{\log n}.$$

LEMMA 3: *Let $\{b_n\}$ be a positive sequence such that*

$$\lim_n \frac{\log b_n}{\log n} = L < \infty.$$

(i) If $L \geq -1$, then

$$\lim_n \frac{\log \sum_{j=1}^{n} b_j}{\log n} = L + 1.$$

(ii) If $L < -1$, then

$$\lim_n \frac{\log \sum_{j=n}^{\infty} b_j}{\log n} = L + 1.$$

The following lemma is presented in [8]. We now need to introduce two models of random trees. As mentioned in Section 1, a tree is called *spherically symmetric* if the degree of any vertex depends only on its distance from the root and this type of trees will be denoted by Γ. Let d_n stand for the outdegree of any vertex of level n. This sequence is called the degree sequence of Γ. It will be assumed that d_n, $n \geq 0$, are independent random variables having a probability distribution that depends on n. To introduce a tree that corresponds to a branching process in random environments, let us consider a doubly-indexed family of independent random variables $\{d_{nk}, n \geq 0, k \geq 1\}$ such that for fixed n, they are identically distributed following the same distribution of d_n. If we let d_{nk} denote the out degree of the kth vertex of level n, the resulting tree is called *branching process in random environments tree*, which will be abbreviated as BPRET and will be denoted by Γ^*.

Obviously for Γ, $|\Gamma_{n+1}| = \prod_{j=0}^{n} d_j$, and for Γ^*, $|\Gamma^*_{n+1}| = \sum_{j=0}^{|\Gamma^*_n|} d_{nj}$. It is also straightforward that $E|\Gamma_{n+1}| = E|\Gamma^*_{n+1}| = \prod_{j=0}^{n} Ed_j$.

The following lemma is extracted from [8].

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LEMMA 4: *If the degree sequence* $\{d_{nk}\}$ *of* Γ^* *is uniformly bounded, then*

$$\frac{|\Gamma_n^*|}{E|\Gamma_n^*|} \longrightarrow W \text{ a.s.},$$

where $W > 0$ a.s.

The *shorting principle* decides that shorting together some nodes will not increase the effective resistance of the network. Nevertheless, the effective resistance remains unchanged if nodes of the same potential are shorted. For Γ and Γ^*, assign *one unit resistance* to each edge. Let R_n and R_n^* denote respectively the effective resistance between the roots of Γ and Γ^* and their respective nth levels, after being shorted in one vertex. The following lemma is represented in [5] and follows from the shorting principle.

LEMMA 5: *For* Γ, $R_n = \sum_{j=1}^{n} (1/|\Gamma_j|)$, *whereas for* Γ^*, $R_n^* \geq \sum_{j=1}^{n} (1/|\Gamma_j^*|)$.

The following result is presented in [4].

LEMMA 6: *For* Γ *and* Γ^*, $E(R_n) \geq E(R_n^*)$. *However, no stochastic domination between* R_n *and* R_n^* *exists.*

4. FRACTAL DIMENSIONS

THEOREM 7: *Consider a spherically symmetric tree with a degree sequence* $\{d_n\}$ *such that for* $n \geq 0$,

$$d_n = \begin{cases} 1 & \text{with probability } 1 - q_{n2} - q_{n3} - \cdots - q_{nk} \\ 2 & \text{with probability } q_{n2} \\ 3 & \text{with probability } q_{n3} \\ \cdot & \\ \cdot & \\ \cdot & \\ k & \text{with probability } q_{nk}, \end{cases} \tag{6}$$

where for $2 \leq j \leq k$, $0 < q_{nj} < 1$, $q_{nj} \downarrow 0$, *and* $\sum_n q_{nj} = \infty$. *Then,* $\mathrm{FD} = 1 + \lim_n nE \log d_n$.

PROOF: We can assume, without loss of generality, that $k = 3$. Hence,

$$\begin{aligned} E \log d_n &= (\log 2) q_{n2} + (\log 3) q_{n3} \\ &= \log (2^{q_{n2}})(3^{q_{n3}}). \end{aligned}$$

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Consequently,

$$\begin{aligned} E\log|\Gamma_n| &= E\sum_{j=0}^{n-1}\log d_j = \sum_{j=0}^{n-1}\log 2^{q_{j2}}3^{q_{j3}} \\ &= \log\prod_{j=0}^{n-1}2^{q_{j2}}3^{q_{j3}} \\ &= \log\ 2^{\sum_{j=0}^{n-1}q_{j2}}\ 3^{\sum_{j=0}^{n-1}q_{j3}} \end{aligned}$$

It follows from Lemma 1 that

$$t_n = \frac{\log|\Gamma_n|}{E\log|\Gamma_n|} \longrightarrow 1 \quad \text{a.s. as } n \longrightarrow \infty,$$

from which we obtain

$$\log|\Gamma_n| = t_n\left(\sum_{j=0}^{n-1}q_{j2}\right)\log 2 + t_n\left(\sum_{j=0}^{n-1}q_{j2}\right)\log 3.$$

This implies, using Lemma 2(i), that

$$\lim_n\frac{\log|\Gamma_n|}{\log n} = (\log 2)(\lim_n nq_{n2}) + (\log 3)(\lim_n nq_{n3}).$$

As such, Lemma 3(i) ensures that

$$\begin{aligned} \mathrm{FD} &= \lim_n\frac{\log b_n}{\log n} \\ &= 1 + (\log 2)(\lim_n nq_{n2}) + (\log 3)(\lim_n nq_{n3}) \\ &= 1 + \lim_n nE\log d_n. \end{aligned}$$

■

It is shown in [6] for *SSRT* that FD = RD. As such, we have the following result.

THEOREM 8: *For SSRT, defined by Eq. (6),* $\mathrm{RD} = 1 + \lim_n nE\log d_n$.

THEOREM 9: *The random walk on SSRT is transient if* $\lim_n nE\log d_n > 1$ *and recurrent if* $\lim_n nE\log d_n < 1$ *and it could be of either type if* $\lim_n nE\log d_n = 1$.

PROOF: The result follows from Theorem 8 and Note 1. Two examples are presented in [3] showing that the random walk could be transient or recurrent at the critical value. ■

We now turn our attention to branching process in random environments tree.

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THEOREM 10: *Consider a BPRET* Γ^* *with a degree sequence* $\{d_{nj}\}$ *such that for all* n, $d_{nj} \stackrel{D}{=} d_n$; *That is,*

$$d_{ni} = \begin{cases} 1 & \text{with probability } 1 - q_{n2} - q_{n3} - \cdots - q_{nk} \\ 2 & \text{with probability } q_{n2} \\ 3 & \text{with probability } q_{n3} \\ \cdot & \\ \cdot & \\ \cdot & \\ k & \text{with probability } q_{nk}, \end{cases} \tag{7}$$

where $k \geq 2, i = 1, 2, \ldots, |\Gamma^*_{n-1}|$, *and* $0 < q_{nj} < 1$. *In addition, let* $q_{nj} \downarrow 0$ *as* $n \to \infty$, *and* $\sum_n q_{nj} = \infty$ *for all* j. *Then* $\mathrm{FD} = 1 + \lim_n n(Ed_{nj} - 1)$.

PROOF: It follows from the definition of d_{nj} that

$$E|\Gamma^*_n| = E|\Gamma_n| = \prod_{j=0}^{n-1} Ed_j = \prod_{j=0}^{n-1} (1 + q_{j2} + 2q_{j3} + \cdots + (k-1)q_{jk})$$

and, hence,

$$\log E|\Gamma^*_n| = \sum_{j=0}^{n-1} \log\,(1 + q_{j2} + 2q_{j3} + \cdots + (k-1)q_{jk}).$$

This entails, using Lemmas 4 and 2(i), that

$$\begin{aligned} \lim_n \frac{\log|\Gamma^*_n|}{\log_n} &= \lim_n n(q_{n2} + 2q_{n3} + \cdots + (k-1)q_{nk}) \\ &= \lim_n n(Ed_{ni} - 1). \end{aligned} \tag{8}$$

Hence, the result follows from Lemma 3(i). ■

The following lemma is in [8].

LEMMA 11: *The random walk on* Γ^* *is recurrent if and only if* $\sum_n (1/E|\Gamma^*_n|) = \infty$.

The following lemma is straightforward.

LEMMA 12: *For a positive sequence* $\{a_n, n \geq 1\}$ *of real numbers,* $\sum_n (1/\prod_{k=1}^n a_k) = \infty$ *if and only if* $\lim_n n(a_n - 1) \leq 1$.

As a consequence of Lemmas 11 and 12, we have the following lemma.

LEMMA 13: *The random walk on* Γ^* *is recurrent if and only if* $\lim_n n(Ed_{ni} - 1) \leq 1$.

THEOREM 14: *Consider a tree* Γ^* *defined by Eq. (7). Then* $\mathrm{RD}^* \le 1 + \lim_n nE(d_{ni} - 1)$ *a.s. Moreover, if the random walk is recurrent, then* $E(\mathrm{RD}^*) \ge 1 + \lim_n n(1 - E(1/d_{ni}))$*. The same last inequality holds true for transient random walk provided that*

$$\lim_n n\left(1 - E\frac{1}{d_{nj}}\right) > 1. \tag{9}$$

PROOF: We first consider the case that the random walk on Γ^* is recurrent. Then from Lemma 13,

$$\lim_n n(Ed_{nj} - 1) \le 1. \tag{10}$$

It follows from Eq. (8) and inequality (10) that

$$\lim_n \frac{\log \frac{1}{|\Gamma_n^*|}}{\log n} = -\lim_n n(Ed_{ni} - 1) \ge -1. \tag{11}$$

Then by applying Lemmas 5 and 3(i) we get

$$\begin{aligned}
\mathrm{RD}^* &= 2 - \lim_n \frac{\log R_n^*}{\log n} \le 2 - \lim_n \frac{\log \sum_{j=0}^{n} \frac{1}{|\Gamma_j^*|}}{\log n} \\
&= 2 - (1 - \lim_n n(Ed_{ni} - 1)) \\
&= 1 + \lim_n n(Ed_{ni} - 1) \quad \text{a.s.}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E(\mathrm{RD}^*) &= 2 - E\left(\lim_n \frac{\log R_n^*}{\log n}\right) \\
&\ge 2 - \lim_n E\left(\frac{\log R_n^*}{\log n}\right) \\
&\ge 2 - \lim_n \frac{\log E(R_n^*)}{\log n} \\
&\ge 2 - \lim_n \frac{\log E(R_n)}{\log n},
\end{aligned} \tag{12}$$

where the last three inequalities follow respectively from Fatou's lemma, Jensen's inequality, and Theorem 6. For SSRT,

$$E\frac{1}{|\Gamma_n|} = \prod_{j=0}^{n-1}\left[1 - \left(\frac{1}{2}q_{j2} + \frac{2}{3}q_{j3} + \cdots + \left(1 - \frac{1}{k}\right)q_{jk}\right)\right].$$

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It follows from Lemma 2(ii) that

$$\lim_n \frac{\log E\left(\frac{1}{\Gamma_n}\right)}{\log n} = \lim_n \frac{\sum_{j=0}^{n-1} \log\left[1 - \left(\frac{1}{2}q_{j2} + \frac{2}{3}q_{j3} + \cdots + \left(1 - \frac{1}{k}\right)\right)q_{jk}\right]}{\log n}$$
$$= -\lim_n n\left(\frac{1}{2}q_{n2} + \frac{2}{3}q_{n3} + \cdots + \left(1 - \frac{1}{k}\right)q_{nk}\right)$$
$$= -\lim_n n\left(1 - E\frac{1}{d_{nj}}\right). \qquad \textbf{(13)}$$

Since $d_{nj} + 1/d_{nj} \geq 2$ a.s., then $E(d_{nj}) + E(1/d_{nj}) \geq 2$. As such, $1 - E(1/d_{nj}) \leq Ed_{nj} - 1$. This implies, using inequality (10), that $\lim_n n(1 - E(1/d_{nj})) \leq 1$. Now, Eq. (13) entails that

$$\lim_n \frac{\log E\left(\frac{1}{|\Gamma_n|}\right)}{\log n} \geq -1.$$

From Lemma 5, $R_n = \sum_{j=1}^{n}(1/|\Gamma_j|)$ a.s. Thus, Lemma 3(i) and Eq. (13) assure that

$$\lim_n \frac{\log E(R_n)}{\log n} = \lim_n \frac{\log \sum_{j=1}^{n} E\left(1/|\Gamma_j|\right)}{\log n} = 1 - \lim_n n\left(1 - E\frac{1}{d_{nj}}\right).$$

This ensures that

$$E(\mathrm{RD}^*) = 2 - E\left(\lim_n \frac{\log R_n}{\log n}\right)$$
$$\geq 2 - \lim_n \frac{\log E(R_n)}{\log n}$$
$$= 1 + \lim_n n\left(1 - E\frac{1}{d_{nj}}\right).$$

This completes the proof in the case that the random walk is recurrent.

We now consider the case where the random walk is transient. Hence, $\lim_n R_n^* = R^* < \infty$ and from Lemma 13,

$$\lim_n n(Ed_{nj} - 1) > 1, \qquad \textbf{(14)}$$

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in which case,

$$\mathrm{RD}^* = 2 - \lim_n \frac{\log (R^* - R_n^*)}{\log n} \leq 2 - \lim_n \frac{\log \sum_{j=n+1}^{\infty} \left(\frac{1}{|\Gamma_j^*|}\right)}{\log n}. \tag{15}$$

It follows from Eq.(8) and Lemma 3(ii) that

$$\lim_n \frac{\log \sum_{j=n+1}^{\infty} \left(\frac{1}{|\Gamma_j^*|}\right)}{\log n} = 1 - \lim_n n(Ed_{nj} - 1).$$

Hence, inequality (15) implies that

$$\mathrm{RD}^* \leq 1 + \lim_n n(Ed_{nj} - 1) \quad \text{a.s.}$$

On the other hand,

$$\begin{aligned} E(\mathrm{RD}^*) &= 2 - E\left(\lim_n \frac{\log (R^* - R_n^*)}{\log n}\right) \\ &\geq 2 - \lim_n E\left(\frac{\log (R^* - R_n^*)}{\log n}\right) \\ &\geq 2 - \lim_n \frac{\log E(R^* - R_n^*)}{\log n} \\ &\geq 2 - \lim_n \frac{\log E(R - R_n)}{\log n} \\ &= 2 - \lim_n \frac{\log \sum_{k=n+1}^{\infty} E\left(\frac{1}{|\Gamma_k|}\right)}{\log n} \\ &= 1 - \lim_n \frac{\log E\left(\frac{1}{|\Gamma_n|}\right)}{\log n} \qquad (16) \\ &= 1 + \lim_n n\left(1 - E\frac{1}{d_{nj}}\right), \qquad (17) \end{aligned}$$

where Eq. (16) follows from inequality (9) and Lemma 3(ii), whereas Eq. (17) follows from Eq. (13). ■

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References

1. Berman, K. & Konsowa, M. (1989). Random paths and cuts, electrical networks, and reversible Markov chains. *SIAM Journal of Discrete Mathematics* 3(3): 311–319.
2. Doyle, P. & Snell, L. (1984). *Random walks and electric networks*. The Carus Mathematical Monographs. Washington, DC: Mathematical Association of America.
3. Konsowa, M. (2005). Random walks and dimensions of random trees. Submitted.
4. Konsowa, M.H. (2003). A note on the conductivity of random trees. *Probability in the Engineering and Informational Sciences* 17: 571–576.
5. Konsowa, M.H. & Mitro, J. (1991). The type problem for random walks on trees. *Journal of Theoretical Probability* 4(3): 535–550.
6. Konsowa, M.H. & Oraby, T.F. (2003). Dimensions of random trees. *Statistics and Probability Letters* 62: 49–60.
7. Konsowa, M.H. & Oraby, T.F. (2003). Fractal dimensions and random walks on random trees. *Journal of Statistical Planning and Inference* 116: 333–342.
8. Lyons, R. (1992). Random walks, capacity, and percolation on trees. *Annals of Probability* 20: 2043–2088.
9. Lyons, R. & Peres, Y. (2004). *Probability on trees and networks*. Cambridge: Cambridge University Press. Available from http//php.indiana.edu/∼rdlyons.
10. Lyons, T. (1983). A simple criterion for transience of a reversible Markov chain. *Annals of Probability* 11: 393–402.
11. Nash-Williams, C.st. J.A. (1959). Random walks and electric currents in networks. *Proceeding of Cambridge Philosophical Society* 55: 181–194.
12. Peres, Y. (1999). *Probability on trees*. Lectures Notes in Mathematics Vol. 1717. Berlin: Springer, pp. 193–280.
13. Telcs, A. (1989). Random walks on graphs, electric networks, and fractals. *Probability Theory and Related Fields* 82: 435–449.
14. Zhou, X.Y. (1993). Resistance dimension, random walk dimension, and fractal dimension. *Journal of Theoretical Probability* 6(4): 635–652.