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GENERALIZED GAUSSIAN ESTIMATES AND RIESZ MEANS OF SCHRÖDINGER GROUPS

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Abstract

We show that generalized Gaussian estimates for selfadjoint semigroups $(e^{-tA})_{t\in\mathbb{R}_+}$ on L_2 imply L_p-boundedness of Riesz means and other regularizations of the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$. This generalizes results restricted to semigroups with a heat kernel, which are due to Sjöstrand, Alexopoulos and more recently Carron, Coulhon and Ouhabaz. This generalization is crucial for elliptic operators A that are of higher order or have singular lower order terms since, in general, their semigroups fail to have a heat kernel.

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Introduction

It is well known that the Schrödinger group $(e^{it\Delta})_{t\in\mathbb{R}}$ acts on $L_p(\mathbb{R}^D)$ only if $p = 2$ [18]. Various authors showed that suitable regularizations of the Schrödinger group such as the Riesz means

$$t^{-\alpha}\int_0^t (t-s)^{\alpha-1}e^{is\Delta}\,ds$$

act even on $L_p(\mathbb{R}^D)$ for $p \neq 2$; see, for example, the works of Lanconelli [20] on boundedness of $(I-\Delta)^{-\alpha}e^{it\Delta}$ and of Sjöstrand [26] on Riesz means. These results were extended by Alexopoulos [1] to Laplacians Δ on Lie groups and Riemannian manifolds where the heat semigroup satisfies Gaussian estimates.

Carron, Coulhon and Ouhabaz [10] generalized this approach to arbitrary self-adjoint operators A on measured metric spaces (Ω, μ, d) of some dimension D, that is, $|B(x, \lambda r)| \le C\lambda^D |B(x,r)|$ for all $x \in \Omega$, $r > 0$, $\lambda \ge 1$, where $B(x,r)$ denotes the ball around x of radius r and $|B(x,r)|$ its volume. They showed L_p-boundedness of suitable regularizations of the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$ provided A satisfies Gaussian estimates (GEs), that is, the e^{-tA} have integral kernels $k_t(x,y)$ satisfying

$$(1) \qquad |k_t(x,y)| \le |B(x,r_t)|^{-1} g\left(\frac{d(x,y)}{r_t}\right) \quad \text{for all } x, y \in \Omega,\ t > 0.$$

Here the r_t are suitable positive radii and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a suitable decay function. The central part of [10] was to deduce from the GE (1) the following $L_p \to L_p$-norm estimate for the semigroup:

$$(2) \qquad \left\| e^{-zA} \right\|_{p\to p} \le C_\varepsilon \left(\frac{|z|}{\operatorname{Re} z}\right)^{D|1/2-1/p|+\varepsilon} \quad \text{for all } p \in [1,\infty],\ z \in \mathbb{C}_+,$$

where $\mathbb{C}_+ := \{z \in \mathbb{C}; \operatorname{Re} z > 0\}$. Then one can apply directly the following result on Riesz means and regularized groups due to El-Mennaoui [23] and Boyadzhiev and de Laubenfels [9]. Recall that if X is a Banach space and $S \in \mathcal{L}(X)$ is injective, then a strongly continuous family $(W(t))_{t\in\mathbb{R}}$ in $\mathcal{L}(X)$ is called an *S-regularized group* if $W(0) = S$ and $W(s)W(t) = SW(s+t)$ for all $s, t \in \mathbb{R}$. Its generator B is defined by $B = S^{-1}W'(0)$ with maximal domain; see, for example, [13] for details.

PROPOSITION A. *Let (Ω, μ) be a measure space, A a non-negative selfadjoint operator on $L_2(\Omega)$, $p \in [1,\infty]$, and $\alpha > \beta \ge 0$ such that $\|e^{-zA}\|_{p\to p} \le C(|z|/\operatorname{Re} z)^\beta$ for all $z \in \mathbb{C}_+$.*

(a) *The following Riesz means $(I_\alpha(t))_{t\in\mathbb{R}}$ are uniformly bounded on $L_p(\Omega)$:*

$$I_\alpha(t) := \begin{cases} t^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{-isA}\, ds & t \ge 0, \\ I_\alpha(-t)^* & t < 0. \end{cases}$$

(b) *If (e^{-tA}) is bounded analytic of angle $\pi/2$ on $L_p(\Omega)$, then $W_\alpha(t) := (I+A)^{-\alpha} e^{itA}$ defines a $(I+A)^{-\alpha}$-regularized group on $L_p(\Omega)$ with generator iA satisfying*

$$\|W_\alpha(t)\|_{p\to p} \le C(1+|t|)^\alpha \quad \textit{for all } t \in \mathbb{R}.$$

Unfortunately, there are many important operators A which do not satisfy GEs (in particular, [10] cannot be applied!). This occurs, for example, for elliptic operators A that are of higher order or have singular lower order terms [12, 21]. However, in many of these cases A still satisfies so-called generalized Gaussian estimates (GGEs); see [11, 25]. By this we mean an estimate of the following type:

$$(3) \qquad \left\| \chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)} \right\|_{p_o \to p_o'} \le |B(x,r_t)|^{1/p_o' - 1/p_o} g\left(\frac{d(x,y)}{r_t}\right)$$

for all $x, y \in \Omega, t > 0$, and for some $p_o \in [1, 2)$. The GGE (3) for $p_o = 1$ is equivalent to the GE (1) [5, Proposition 2.9]. The central part of this paper is to deduce from the GGE (3) the following generalization and slight improvement of the $L_p \to L_p$-norm estimate (2) (Theorem 1.1 below), which improves a result of Davies [11]:

$$(4) \qquad \|e^{-zA}\|_{p\to p} \le C\left(\frac{|z|}{\operatorname{Re} z}\right)^{D|1/2-1/p|} \quad \text{for all } p \in [p_o, p_o'], z \in \mathbb{C}_+.$$

We obtain new results on regularizations of the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$, as before directly from Proposition A, and for the optimal range $p \in [p_o, p_o'], \alpha > D|1/2-1/p|$; see Theorem 1.3 below. We want to mention that, for the class of operators A satisfying the GGE (3), the interval $[p_o, p_o']$ is, in general, optimal for the existence of the semigroup $(e^{-tA})_{t\in\mathbb{R}_+}$ on L_p [12], and the $\|e^{-zA}\|_{p\to p}$-estimate (4) is optimal also [3].

A singular integral theory based on GGEs allows us to extend other L_2-properties of A (above the boundedness of regularizations of $(e^{itA})_{t\in\mathbb{R}}$ considered in this paper) to L_p for $p \in (p_o, p_o')$. We mention the properties of having maximal regularity [5], an H^∞ functional calculus [6] or Riesz transforms [7, 17]. In [4], this approach was applied to so-called 'spectral multipliers', which yields results of the type $F(A) \in \mathfrak{L}(L_p)$, $p \in (p_o, p_o')$ for more general functions F than the $F_{\alpha,t}(x) := t^{-\alpha}\int_0^t (t-s)^{\alpha-1}e^{-isx}\,ds$ corresponding to our Riesz means (that is, $F_{\alpha,t}(A) = I_\alpha(t)$). The advantage of the method in the present paper is that it allows us to include the cases $p = p_o, p_o'$ and gives a direct approach for Schrödinger groups avoiding singular integral theory.

1. Main results

We begin with some basic notation and assumptions. For the rest of this paper, (Ω, μ, d) is a metric measure space. By $B(x, r)$ we denote balls in Ω and by $|B(x, r)|$ or $v_r(x)$ their volume. For integral operators T, we denote their integral kernel by $k_T(x, y)$, that is, $Tf(x) = \int k_T(x, y) f(y)\, d\mu(y)$.

1.1. Optimal estimates for $\|e^{-zA}\|_{p\to p}$ In order to deduce from Proposition A optimal L_p-boundedness results for regularizations of Schrödinger groups $(e^{itA})_{t\in\mathbb{R}}$, one needs optimal $\|e^{-zA}\|_{p\to p}$-estimates. For the case of operators satisfying GEs, (almost) optimal $\|e^{-zA}\|_{p\to p}$-estimates are obtained in [10, Theorem 4.3] by making tricky use of the identity

$$\|k_T(\cdot, y)\|_2^2 = k_{T^*T}(y, y).$$

Here we optimize and generalize this method to GGEs. The right substitute of the above kernel identity can be seen in the elementary norm identity

$$(5) \qquad \|T\|_{p\to 2}^2 = \|T^*T\|_{p\to p'}.$$

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This allows us to prove the following $\|e^{-zA}\|_{p\to p}$-estimate. The proof of this estimate will be given at the end of this subsection.

THEOREM 1.1. *Let* (Ω, μ, d) *be a space of dimension* D *and* $p \in [1, 2)$. *Let* A *be a non-negative selfadjoint operator on* $L_2(\Omega)$ *such that*

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p\to p'} \le |B(x, r_t)|^{1/p'-1/p} g\left(\frac{d(x,y)}{r_t}\right)$$

for all $x, y \in \Omega$, $t \in \mathbb{R}_+$, *where* $r_t = t^{1/m}$ *and* $g(s) = C\exp(-bs^{m/(m-1)})$ *for some* $m \ge 2$, *and* $C, b > 0$. *Then* $(e^{-tA})_{t\in\mathbb{R}_+}$ *is bounded analytic of angle* $\pi/2$ *on* $L_p(\Omega)$, *and we have*

$$\|e^{-zA}\|_{p\to p} \le C\left(\frac{|z|}{\operatorname{Re} z}\right)^{D(1/p-1/2)} \quad \textit{for all } z \in \mathbb{C}_+.$$

REMARK 1.2. (a) The Laplacian $A = -\Delta$ on $\Omega = \mathbb{R}^D$ shows the optimality of our result [3].

(b) Duality and interpolation with $\|e^{-zA}\|_{2\to 2} \le 1$ yield that $(e^{-tA})_{t\in\mathbb{R}_+}$ is bounded analytic of angle $\pi/2$ on $L_q(\Omega)$ for all $q \in [p, p']$, $q \ne \infty$, and we have

$$\|e^{-zA}\|_{q\to q} \le C\left(\frac{|z|}{\operatorname{Re} z}\right)^{D|1/2-1/q|} \quad \text{for all } q \in [p, p'], z \in \mathbb{C}_+.$$

(c) In [11], Davies verified the hypothesis of Theorem 1.1 for elliptic operators A of order $m \in 2\mathbb{N}$ on $\Omega = \mathbb{R}^D$ and for $p := (2D/(D+m)) \vee 1$, but he only obtained the following weaker conclusion, see [11, Theorems 20 and 25]:

$$\|e^{-zA}\|_{p\to p} \le C\left(\frac{|z|}{\operatorname{Re} z}\right)^{2D(1/p-1/2)} \quad \text{for all } z \in \mathbb{C}_+.$$

(d) For the special case $p = 1$, our Theorem 1.1 is a slight improvement of [10, Theorem 4.3], where the following estimate is obtained for all $\varepsilon > 0$:

$$\|e^{-zA}\|_{1\to 1} \le C_\varepsilon\left(\frac{|z|}{\operatorname{Re} z}\right)^{D/2+\varepsilon} \quad \text{for all } z \in \mathbb{C}_+.$$

Deducing from the $L_p \to L_{p'}$ GGE in the hypothesis, an $L_p \to L_2$ GGE, and extending the latter to complex times are the main steps in the following short proof of Theorem 1.1.

PROOF OF THEOREM 1.1. We identify g and $\widetilde{g}$, where $\widetilde{g}(s) = \widetilde{C}\exp\left(-\widetilde{b}s^{m/(m-1)}\right)$. By Proposition 3.1 (i) below, the $L_p \to L_{p'}$ GGE in the hypothesis implies the following $L_p \to L_2$ GGE:

$$\left\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\right\|_{p\to 2} \le |B(x, r_t)|^{1/2-1/p} g\left(\frac{d(x,y)}{r_t}\right) \quad \text{for all } t \in \mathbb{R}_+.$$

By Theorem 2.1 below, the latter extends to complex times $z \in \mathbb{C}_+$ as follows:

$$\left\|\chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)}\right\|_{p\to 2} \le |B(x,r_z)|^{1/2-1/p} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(1/p-1/2)} g\left(\frac{d(x,y)}{r_z}\right)$$

for all $z \in \mathbb{C}_+$, where $r_z = (\operatorname{Re} z)^{1/m-1}|z|$. This implies by Proposition 3.1 (ii) below for $R = (|z|/\operatorname{Re} z)^{-D(1/p-1/2)} e^{-zA}$:

$$\left\|e^{-zA}\right\|_{p\to p} \le C_0 \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(1/p-1/2)} \quad \text{for all } z \in \mathbb{C}_+.$$

It remains to show that $(e^{-zA})_{z\in\mathbb{C}_+}$ is strongly continuous in L_p on all strict subsectors of $\mathbb{C}_+$. Arguing as in [24], but on L_p instead of L_1, one obtains the strong continuity on subsectors from the previous L_p–L_p-estimate. □

1.2. L_p-boundedness of regularizations of $(e^{itA})_{t\in\mathbb{R}}$ Theorem 1.1 allows us to apply our approach given in the introduction, that is, to verify the $L_p \to L_p$ norm estimate

$$\left\|e^{-zA}\right\|_{p\to p} \le C \left(\frac{|z|}{\operatorname{Re} z}\right)^{D|1/2-1/p|} \quad \text{for all } p \in [p_o, p_o'], z \in \mathbb{C}_+$$

and to obtain L_p-boundedness of Riesz means and $(I + A)^{-\alpha}$-regularizations of $(e^{itA})_{t\in\mathbb{R}}$ directly from Proposition A. This yields the following result.

THEOREM 1.3. *Let (Ω, μ, d) be a space of dimension D and $p_o \in [1, 2)$. Let A be a non-negative selfadjoint operator on $L_2(\Omega)$ such that*

$$\left\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\right\|_{p_o\to p_o'} \le |B(x,r_t)|^{1/p_o'-1/p_o} g\left(\frac{d(x,y)}{r_t}\right)$$

for all $x, y \in \Omega$, $t \in \mathbb{R}_+$, where $r_t = t^{1/m}$ and $g(s) = C\exp\left(-bs^{m/(m-1)}\right)$ for some $m \ge 2$, and $C, b > 0$. Then we have for all $p \in [p_o, p_o']$ and $\alpha > D|1/2-1/p|$:

(a) *The following Riesz means $(I_\alpha(t))_{t\in\mathbb{R}}$ are uniformly bounded on $L_p(\Omega)$:*

$$I_\alpha(t) := \begin{cases} t^{-\alpha}\int_0^t (t-s)^{\alpha-1} e^{isA}\,ds & t \ge 0,\\ I_\alpha(-t)^* & t < 0.\end{cases}$$

(b) *If $p \ne \infty$ then $W_\alpha(t) := (I + A)^{-\alpha} e^{itA}$ defines an $(I + A)^{-\alpha}$-regularized group $(W_\alpha(t))_{t\in\mathbb{R}}$ on $L_p(\Omega)$, with generator iA satisfying $\|W_\alpha(t)\|_{p\to p} \le C(1 + |t|)^\alpha$ for all $t \in \mathbb{R}$.*

REMARK 1.4. (a) For the special case $p_o = 1$ our Theorem 1.3 corresponds to Theorems 5.1–5.2 in [10].

(b) The Laplacian $A = -\Delta$ on $\Omega = \mathbb{R}^D$ shows the optimality of our result [3].
(c) In Remark 1.2 (c) above we already mentioned Davies' estimate [11]

$$\left\|e^{-zA}\right\|_{p\to p} \le C\left(\frac{|z|}{\operatorname{Re} z}\right)^{2D|1/2-1/p|} \quad \text{for all } p \in [p_o, p_o'], z \in \mathbb{C}_+$$

for elliptic operators A of order $m \in 2\mathbb{N}$ on $\Omega = \mathbb{R}^D$ and for $p_o = (2D/(D+m)) \vee 1$. By Proposition A, this yields the conclusion of Theorem 1.3 for the range $\alpha > 2D|1/2 - 1/p|$. Our result yielding the optimal range $\alpha > D|1/2 - 1/p|$ is new.

PROOF OF THEOREM 1.3. The assertions follow directly from Theorem 1.1 (in the form of Remark 1.2 (b)) and Proposition A for $\beta = D|1/2 - 1/p|$. □

1.3. Examples In this subsection, we give some examples of elliptic operators A for which our Theorem 1.3 on regularizations of the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$ applies, that is, for which the following GGE holds:

$$(6) \quad \left\|\chi_{B(x,r_t)}e^{-tA}\chi_{B(y,r_t)}\right\|_{p_o\to p_o'} \le |B(x,r_t)|^{1/p_o'-1/p_o} C \exp\left(-b\left(\frac{d(x,y)}{r_t}\right)^{m/(m-1)}\right)$$

for all $x, y \in \Omega$, $t \in \mathbb{R}_+$, $r_t = t^{1/m}$ and for some $m \ge 2$, $p_o \in [1, 2)$.

1.3.1. Higher order operators with bounded coefficients on $\mathbb{R}^D$ These operators A are given by forms $\mathfrak{a} : H^k(\mathbb{R}^D) \times H^k(\mathbb{R}^D) \to \mathbb{C}$ of the type

$$\mathfrak{a}(u, v) = \int_{\mathbb{R}^D} \sum_{|\alpha|=|\beta|=k} a_{\alpha\beta}\partial^\alpha u\, \overline{\partial^\beta v}\, dx,$$

where we assume $a_{\alpha,\beta} = \overline{a_{\beta,\alpha}} \in L_\infty(\mathbb{R}^D)$ for all α, β and Garding's inequality

$$\mathfrak{a}(u, u) \ge \delta\|\nabla^k u\|_2^2 \quad \text{for all } u \in H^k(\mathbb{R}^D),$$

for some $\delta > 0$ and $\|\nabla^k u\|_2^2 := \sum_{|\alpha|=k}\|\partial^\alpha u\|_2^2$. Then $\mathfrak{a}$ is a closed symmetric form The associated operator A is given by $u \in D(A)$, and $Au = g$ if and only if $u \in H^k$ and $\int g\bar{v}\, dx = \mathfrak{a}(u, v)$ for all $v \in H^k$.

In this situation, the GGE (6) holds for $p_o := (2D/(m+D)) \vee 1$ and $m := 2k$; see, for example, [11] and [2, Section 1.7]. Hence the conclusion of Theorem 1.3 holds for all $p \in [p_o, p_o']$ and $\alpha > D|1/2 - 1/p|$.

1.3.2. Schrödinger operators with singular potentials on $\mathbb{R}^D$ Now we study Schrödinger operators $A = -\Delta + V$ on $\mathbb{R}^D$, $D \ge 3$, where $V = V_+ - V_-$, $V_\pm \ge 0$

are locally integrable, and V_+ is bounded for simplicity (for the general case, see for example [25]). We assume the following form bound:

$$\int (V_- u)\bar{u}\,dx \le \gamma\left(\|\nabla u\|_2^2 + \langle V_+ u, u\rangle\right) + c(\gamma)\|u\|_2^2 \quad \text{for all } u \in H^1(\mathbb{R}^D)$$

and some $\gamma \in (0,1)$. Then the form sum $A := -\Delta + V \dotplus (-\Delta + V_+) - V_-$ is defined and the associated form is closed and symmetric with form domain $H^1(\mathbb{R}^D)$. By standard arguments using ellipticity and the Sobolev inequality, the GGE (6) holds for $p_o = 2D/(D+2)$ and $m = 2$ (after replacing A by $A + c(\gamma)$). Due to [21], $(e^{-tA})_{t\in\mathbb{R}_+}$ is bounded on $L_q(\mathbb{R}^D)$ for all $q \in (p_\gamma, p'_\gamma)$ and

$$p_\gamma := \frac{2D}{D(1+\sqrt{1-\gamma}) + 2(1-\sqrt{1-\gamma})} < \frac{2D}{D+2}.$$

Hence, by interpolation, one obtains the GGE (6) even for all $p_o \in (p_\gamma, 2)$. Thus, the conclusion of Theorem 1.3 holds for all $p \in (p_\gamma, p'_\gamma)$ and $\alpha > D|1/2 - 1/p|$.

1.3.3. Elliptic operators on Riemannian manifolds Let $A = -\Delta$ be the Laplacian on a Riemannian manifold Ω. Let d be the geodesic distance and μ the Riemannian measure. Assume that Ω satisfies the so-called volume doubling property and that the heat kernel $k_t(x, y)$ satisfies

$$k_t(x,x) \le C|B(x,\sqrt{t})|^{-1} \quad \text{for all } x \in \Omega, t > 0.$$

Then $(e^{t\Delta})_{t\in\mathbb{R}_+}$ satisfies GEs [16] or, equivalently, the GGE (6) for $p_o = 1$ and $m = 2$. Hence the results we obtain on regularizations of $(e^{-it\Delta})_{t\in\mathbb{R}}$ are contained in [10].

2. Extension of GGEs for $(e^{-tA})_{t\in\mathbb{R}_+}$ to $(e^{-zA})_{z\in\mathbb{C}_+}$

Theorem 1.3 on regularizations of Schrödinger groups is based on GGEs of the type

$$\left\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\right\|_{p\to q} \le |B(x,r_t)|^{1/q-1/p} g\left(\frac{d(x,y)}{r_t}\right). \tag{7}$$

Here we show, for analytic semigroups, how the latter estimate for real times t extends to an estimate for complex times z of the type

$$\left\|\chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)}\right\|_{p\to q} \le |B(x,r_z)|^{1/q-1/p} C(z) g\left(\frac{d(x,y)}{r_z}\right). \tag{8}$$

This is important for the proof of our $\|e^{-zA}\|_{p\to p}$-estimate in Theorem 1.1. Moreover, many other applications of GGEs require such complex time estimates, for example, the H^∞ functional calculus; see, for example, [6, 14].

In our setting, we have to extend (7) to the whole right halfplane $\mathbb{C}_+$. This is crucial in order to obtain optimal results on Riesz means or more general so-called 'spectral multipliers' of selfadjoint semigroups; see [4, 10, 15].

For extensions of (7) to strict subsectors of the sector of analyticity of the semigroup, one can usually choose $r_z := r_{|z|}$ and $C(z) = 1$ in (8); see [19, 21]. The latter choice is impossible on the whole of $\mathbb{C}_+$, as can be seen by fixing $x \neq y$ and by letting $|\arg z| \to \pi/2$.

The right candidate for the constant $C(z)$ in (8) is $C(z) = R(z)^{D(1/p-1/q)}$, where $R(z) := r_z/r_{\mathrm{Re}\, z}$. This is suggested by Proposition 3.1, Remark 3.2 below and confirmed by our following extension result. Its proof is given in Section 3.

We consider the standard case $r_t = t^{1/m}$ and $g(s) = Ce^{-bs^\omega}$, which corresponds (for $\omega = m/(m-1)$) to the typical estimates for elliptic operators A of order m; see [2, 11].

THEOREM 2.1. *Let (Ω, μ, d) be a space of dimension D and $1 \le p \le p_o \le q \le \infty$. Let $(e^{-tA})_{t\in\mathbb{R}_+}$ be a semigroup of linear operators on $L_{p_o}(\Omega)$, which has a bounded and analytic extension to $\mathbb{C}_+$ and satisfies, for all $t \in \mathbb{R}_+$,*

$$\left\| \chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)} \right\|_{p\to q} \le |B(x,r_t)|^{1/q-1/p} C e^{-b(d(x,y)/r_t)^\omega},$$

where $r_t = t^{1/m}$ for some $m > 0$ and $\omega \in (0, 2m)$. Then, for all $z \in \mathbb{C}_+$,

$$\left\| \chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)} \right\|_{p\to q} \le |B(x,r_z)|^{1/q-1/p} R(z)^{D(1/p-1/q)} C' e^{-b'(d(x,y)/r_z)^\omega},$$

where $r_z := \cos(\arg z)^{-1/\omega}|z|^{1/m}$ and $R(z) := r_z/r_{\mathrm{Re}\, z} = \cos(\arg z)^{-(1/\omega+1/m)}$.

REMARK 2.2. (a) For the classical case $\omega = m/(m-1)$, $m \ge 2$, $(p,q) = (1,\infty)$ of GEs, our Theorem 2.1 corresponds to [10, Proposition 4.1].

(b) In [11], Davies verified the hypothesis of Theorem 2.1 for elliptic operators A of order $m \in 2\mathbb{N}$ on $\Omega = \mathbb{R}^D$ and for $p := (2D/(D+m)) \vee 1$, $q := p'$, $\omega = m/(m-1)$. He obtained precisely the conclusion of our Theorem 2.1; see [11, line (11) and Lemma 24].

A direct application of Theorem 2.1 is the following.

COROLLARY 2.3. *Let (Ω, μ, d) be a space of some dimension D and $p \in [1, 2)$. Let A be a non-negative selfadjoint operator on $L_2(\Omega)$ such that*

$$\left\| \chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)} \right\|_{p\to p'} \le |B(x,r_t)|^{1/p'-1/p} C \exp\left(-b \left(\frac{d(x,y)}{r_t} \right)^\omega \right),$$

where $r_t = t^{1/m}$ for some $m > 1$ and $\omega \in (1, m]$. Then A has a bounded H^∞ functional calculus on $L_q(\Omega)$ for all $q \in (p, p')$.

Recall that A has a *bounded H^∞ functional calculus* if we have an estimate

$$\|f(A)\| \le C\|f\|_{H^\infty(\Sigma_\delta)} \quad \text{for all } f \in H^\infty(\Sigma_\delta)$$

and some $\delta \in (0, \pi/2)$. Here $H^\infty(\Sigma_\delta)$ denotes the space of all bounded holomorphic functions on the sector $\Sigma_\delta := \{z; |\arg(z)| < \delta\}$. This calculus was introduced by McIntosh; its construction and applications can be found in [22, 14, 6].

PROOF OF COROLLARY 2.3. The conclusion of Theorem 2.1 holds for $q = p'$ and, obviously, A has a bounded H^∞ functional calculus on $L_2(\Omega)$. Hence the assertion follows from [6, Theorem 1.2]. $\square$

3. Proof of Theorem 2.1

We use the symbols $\preceq$ and $\succeq$ to indicate domination up to constants independent of the relevant parameters. Let $\sim$ indicate the validity of $\preceq$ and $\succeq$. While the GGEs considered in our main results enlighten the analogy to GEs, they are not convenient from a technical point of view. For this purpose, we use an equivalent type of GGE, which is provided by the following characterization [8, Proposition 2.1].

PROPOSITION 3.1. *Let (Ω, μ, d) be a space of some dimension D and $1 \le p \le q \le \infty$. Let $g(s) := Ce^{-bs^\omega}$ for some $\omega > 1, b, C > 0$. Let R be a linear operator and $r > 0$.*

(i) *The following are equivalent*:
   (a) $\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{p\to q} \le v_r(x)^{1/q-1/p} g(d(x,y)/r)$, *for all $x, y \in \Omega$.*
   (b) $\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{p\to v} \le v_r(x)^{1/v-1/p} g(d(x,y)/r)$, *for all $x, y \in \Omega$ and $v \in [p, q]$.*
   (c) $\|\chi_{B_1} v_r^\alpha R v_r^\beta \chi_{B_2}\|_{p\to q} \le g(d(B_1, B_2)/r)$, *for all balls $B_1, B_2 \subset \Omega$ and $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1/p - 1/q$.*

(ii) *If* (a) *holds, then $\|v_r^\alpha R v_r^\beta\|_{u\to v} \le C_0$, for all $p \le u \le v \le q$ and all $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1/u - 1/v$. Here C_0 is independent of u, v, α, β and R, r.*

The statement (i) is written modulo identification of g and $\widetilde{g}$, where $\widetilde{g}(s) := cg(s)^a$ for some constants $c, a > 0$ independent of R and r.

The preceding proposition and the following remark suggest that the right candidate for the constant $C(z)$ in complex time GGEs of the type (8) is $C(z) = R(z)^{D(1/p-1/q)}$, where $R(z) := r_z/r_{\operatorname{Re} z}$.

REMARK 3.2. Let (Ω, μ, d) be a space of dimension $D > 0$ and $p \in [1, \infty]$, $\alpha \ge 0$. Let A be a non-negative selfadjoint operator on $L_2(\Omega)$ and $(r_z)_{z\in\mathbb{C}_+}$ a family in $\mathbb{R}_+$ such that $r_z/r_{2\operatorname{Re} z} \geq \varepsilon > 0$ for all $z \in \mathbb{C}_+$. Suppose that $\|v^\alpha_{r_t} e^{-tA} v^\alpha_{r_t}\|_{p\to p'} \leq C$ for all $t \in \mathbb{R}_+$. Then $\|v^\alpha_{r_z} e^{-2zA} v^\alpha_{r_z}\|_{p\to p'} \leq C'(r_z/r_{2\operatorname{Re} z})^{2D\alpha}$ for all $z \in \mathbb{C}_+$.

PROOF. Using the norm identity $\|T\|^2_{p\to 2} = \|T^*T\|_{p\to p'}$ in the second step, the assertion is seen as follows:

$$\begin{aligned}
\|v^\alpha_{r_z} e^{-2zA} v^\alpha_{r_z}\|_{p\to p'} &\leq \left\|v^\alpha_{r_z} e^{-zA}\right\|_{2\to p'} \|e^{-zA} v^\alpha_{r_z}\|_{p\to 2} \\
&= \left\|v^\alpha_{r_z} e^{-(2\operatorname{Re} z)A} v^\alpha_{r_z}\right\|_{p\to p'} \qquad (e^{-zA} \text{ is normal, (5)}) \\
&\preceq \left(\frac{r_z}{r_{2\operatorname{Re} z}}\right)^{2D\alpha} \left\|v^\alpha_{r_{2\operatorname{Re} z}} e^{(-2\operatorname{Re} z)A} v^\alpha_{r_{2\operatorname{Re} z}}\right\|_{p\to p'} \qquad (\dim\Omega = D) \\
&\preceq \left(\frac{r_z}{r_{2\operatorname{Re} z}}\right)^{2D\alpha} \qquad \text{(by hypothesis)}. \qquad \square
\end{aligned}$$

The last preparation for the proof of Theorem 2.1 on the extension of GGEs to complex times is the following application of the three lines lemma. The proof is a straightforward modification of [11, Lemma 9].

LEMMA 3.3. *Let $\omega, m \in \mathbb{R}_+$ such that $0 < \omega < 2m$ and $r_z := \cos(\arg z)^{-1/\omega}|z|^{1/m}$ for all $z \in \mathbb{C}_+$. Let $\delta \geq 0$ and $C_0, b > 0$. Let X be a Banach space and $F : \mathbb{C}_+ \to X$ an analytic function satisfying*

$$\begin{aligned}
\|F(z)\| &\leq C_0 \cos(\arg z)^{-\delta} \quad \textit{for all } z \in \mathbb{C}_+, \\
\|F(t)\| &\leq C_0 \exp(-b r_t^{-\omega}) \quad \textit{for all } t \in \mathbb{R}_+.
\end{aligned}$$

Then we have $\|F(z)\| \leq C_0 2^\delta \cos(\arg z)^{-\delta} e^{-(\omega/2m) b r_z^{-\omega}}$ for all $z \in \mathbb{C}_+$.

PROOF. We define for all $\gamma \in (0, \pi/2)$:

$$\begin{aligned}
g^\pm_\gamma(z) &:= C_0 \exp\left(-b \frac{e^{\mp i(\pi/2 - \omega\gamma/m)}}{\sin(\omega\gamma/m)} z^{-\omega/m}\right), \qquad \pm \arg z \in [0, \gamma], \\
F^\pm_\gamma(z) &:= F(z) g^\pm_\gamma(z)^{-1}, \qquad \pm \arg z \in [0, \gamma].
\end{aligned}$$

By hypothesis, we have for all $\gamma \in (0, \pi/2)$ and $t > 0$:

$$\begin{aligned}
\|F^\pm_\gamma(t)\| &= \|F(t)\| C_0^{-1} \exp(b t^{-\omega/m}) \leq 1, \\
\|F^\pm_\gamma(e^{\pm i\gamma} t)\| &= \|F(e^{\pm i\gamma} t)\| C_0^{-1} \leq \cos(\gamma)^{-\delta}.
\end{aligned}$$

Hence the three lines lemma yields, for all $\gamma \in (0, \pi/2)$ and $\theta \in [0, 1]$,

$$\|F^\pm_\gamma(e^{\pm i\gamma\theta} t)\| \leq 1^{1-\theta} \cos(\gamma)^{-\delta\theta} \leq \cos(\gamma)^{-\delta}.$$

In other words, we have for all $\phi \in [0, \pi/2)$, $t > 0$ and $\gamma \in [\phi, \pi/2)$,

$$\begin{aligned} \|F(e^{\pm i\phi}t)\| &= \|F_\gamma^\pm(e^{\pm i\phi}t)\| |g_\gamma^\pm(e^{\pm i\phi}t)| \\ &\le \cos(\gamma)^{-\delta} C_0 \exp\left(-b \sin\left(\frac{\omega}{m}(\gamma-\phi)\right) t^{-\omega/m}\right). \end{aligned}$$

It remains to choose $\gamma = \pi/4 + \phi/2$ and to deduce from the concavity of sin on $[0, \pi/2]$:

$$\cos(\gamma) = \sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \ge \frac{1}{2}\sin\left(\frac{\pi}{2} - \phi\right) = \frac{1}{2}\cos(\phi),$$

$$\sin\left(\frac{\omega}{m}\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right) \ge \frac{\omega}{2m}\sin\left(\frac{\pi}{2} - \phi\right) = \frac{\omega}{2m}\cos(\phi) = \frac{\omega}{2m} r_{e^{\pm i\phi}t}^{-\omega} t^{\omega/m}. \qquad \square$$

PROOF OF THEOREM 2.1. We recall the basic volume estimate

$$v_s(x) \le C(1 + s/r)^D v_r(x) \quad \text{for all } x \in \Omega,\ r, s > 0. \tag{9}$$

Now fix $\alpha := 1/p_o - 1/q$ and $\beta := 1/p - 1/p_o$. By Proposition 3.1 (ii), we obtain

$$\|v_{r_t}^\alpha e^{-tA}\|_{p_o \to q},\ \|e^{-tA} v_{r_t}^\beta\|_{p \to p_o} \le C \quad \text{for all } t > 0.$$

This allows for all $z \in \mathbb{C}_+$, $t \in (0, \operatorname{Re} z/2)$ and $s > 0$ the estimate

$$\begin{aligned} \|v_s^\alpha e^{-zA} v_s^\beta\|_{p\to q} &\preceq \|v_s^\alpha e^{-tA}\|_{p_o\to q} \|e^{-(z-2t)A}\|_{p_o\to p_o} \|e^{-tA} v_s^\beta\|_{p\to p_o} \\ &\preceq \left(1 + \frac{s}{r_t}\right)^{D\alpha} \|v_{r_t}^\alpha e^{-tA}\|_{p_o\to q} \left(1 + \frac{s}{r_t}\right)^{D\beta} \|e^{-tA} v_{r_t}^\beta\|_{p\to p_o} \quad \text{(by (9))} \\ &\preceq \left(1 + \frac{s}{r_t}\right)^{D(1/p-1/q)}, \quad \text{since } \alpha + \beta = 1/p - 1/q. \end{aligned}$$

Denoting $z = e^{i\theta} r$ for $z \in \mathbb{C}_+$, we note that

$$\begin{aligned} 1 + \frac{s}{(\operatorname{Re} z/2)^{1/m}} &\le (2/\cos\theta)^{1/m} \left(1 + \frac{s}{r^{1/m}}\right) \\ &\le (2/\cos\theta)^{1/m} \sqrt{2} \left|1 + \frac{s}{z^{1/m}}\right| \\ &= C \cos(\arg z)^{-1/m} \left|1 + \frac{s}{z^{1/m}}\right| \quad \text{for all } z \in \mathbb{C}_+,\ s > 0, \end{aligned}$$

which combines with the preceding estimate for $t \nearrow \operatorname{Re} z/2$ to give

$$\|v_s^\alpha e^{-zA} v_s^\beta\|_{p\to q} \preceq \cos(\arg z)^{-(D/m)(1/p-1/q)} \left|1 + \frac{s}{z^{1/m}}\right|^{D(1/p-1/q)}.$$

Denote $g(u) := Ce^{-bu^{\omega}}$ and fix balls $B_1, B_2 \subset \Omega$. We have for all $s, t > 0$,

$$\begin{aligned}\|\chi_{B_1} v_s^{\alpha} e^{-tA} v_s^{\beta} \chi_{B_2}\|_{p\to q} &\preceq \left(1+\frac{s}{r_t}\right)^{D(\alpha+\beta)} \|\chi_{B_1} v_{r_t}^{\alpha} e^{-tA} v_{r_t}^{\beta} \chi_{B_2}\|_{p\to q} \quad \text{(by (9))}\\ &\preceq \left(1+\frac{s}{r_t}\right)^{D(1/p-1/q)} g(d(B_1, B_2) r_t^{-1}),\end{aligned}$$

since $\alpha + \beta = 1/p - 1/q$. Here we used the hypothesis, and in the last step, Proposition 3.1 (i). For fixed $s > 0$, we consider the analytic function $F : \mathbb{C}_+ \to \mathfrak{L}(L_p(\Omega), L_q(\Omega))$ defined by

$$F(z) := \left(1+\frac{s}{z^{1/m}}\right)^{-D(1/p-1/q)} \chi_{B_1} v_s^{\alpha} e^{-zA} v_s^{\beta} \chi_{B_2}.$$

We have shown the following two bounds:

$$\begin{aligned}\|F(t)\|_{p\to q} &\preceq g(d(B_1, B_2) t^{-1/m}) && \text{for all } t > 0,\\ \|F(z)\|_{p\to q} &\preceq \cos(\arg z)^{-(D/m)(1/p-1/q)} && \text{for all } z \in \mathbb{C}_+.\end{aligned}$$

We apply Lemma 3.3 (to $g(d(B_1, B_2)\cdot)$ instead of g) and obtain

$$\|F(z)\|_{p\to q} \preceq \cos(\arg z)^{-\frac{D}{m}(1/p-1/q)} g(d(B_1, B_2) r_z^{-1}) \quad \text{for all } z \in \mathbb{C}_+,$$

where $r_z = \cos(\arg z)^{-1/\omega}|z|^{1/m}$. This means for all $z \in \mathbb{C}_+$ and $s > 0$,

$$\begin{aligned}&\left\|\chi_{B_1} v_s^{\alpha} e^{-zA} v_s^{\beta} \chi_{B_2}\right\|_{p\to q}\\ &\quad\preceq \left|1+\frac{s}{z^{1/m}}\right|^{D(1/p-1/q)} \cos(\arg z)^{-(D/m)(1/p-1/q)} g(d(B_1, B_2) r_z^{-1}).\end{aligned}$$

Choosing $s = r_z$ yields

$$\begin{aligned}&\left\|\chi_{B_1} v_{r_z}^{\alpha} e^{-zA} v_{r_z}^{\beta} \chi_{B_2}\right\|_{p\to q}\\ &\quad\preceq \left|1+\frac{r_z}{z^{1/m}}\right|^{D(1/p-1/q)} \cos(\arg z)^{-(D/m)(1/p-1/q)} g(d(B_1, B_2) r_z^{-1})\\ &\quad\preceq \cos(\arg z)^{-D(1/\omega+1/m)(1/p-1/q)} g(d(B_1, B_2) r_z^{-1}).\end{aligned}$$

By Proposition 3.1 (i) and $\alpha + \beta = 1/p - 1/q$, the above estimate is equivalent to

$$\begin{aligned}&\left\|\chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)}\right\|_{p\to q}\\ &\quad\preceq v_{r_z}(x)^{1/q-1/p} \cos(\arg z)^{-D(1/\omega+1/m)(1/p-1/q)} g\left(\frac{d(x,y)}{r_z}\right).\end{aligned}$$

□

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References

[1] G. Alexopoulos, 'L^p bounds for spectral multipliers from Gaussian estimates on the heat kernel', preprint.

[2] P. Auscher an Ph. Tchamitchian, 'Square root problem for divergence operators and related topics', *Astérisque* **249** (1998).

[3] W. Arendt, O. El Mennaoui and M. Hieber, 'Boundary values of holomorphic semigroups', *Proc. Amer. Math. Soc.* **125** (1997), 635–647.

[4] S. Blunck, 'A Hörmander-type spectral multiplier theorem for operators without heat kernel', *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **2** (2003), 449–459.

[5] S. Blunck and P. C. Kunstmann, 'Weighted norm estimates and maximal regularity', *Adv. Differential Equations* **7** (2002), 1513–1532.

[6] ———, 'Calderon-Zygmund theory for non-integral operators and the H^∞ functional calculus', *Rev. Mat. Iberoamericana* **19** (2003), 919–942.

[7] ———, 'Weak type (p, p) estimates for Riesz transforms', *Math. Z.* **247** (2004), 137–148.

[8] ———, 'Generalized Gaussian estimates and the Legendre transform', *J. Operator Theory* **53** (2005), 351–365 .

[9] K. Boyadzhiev and R. de Laubenfels, 'Boundary values of holomorphic semigroups', *Proc. Amer. Math. Soc.* **118** (1993), 113–118.

[10] G. Carron, T. Coulhon and E. M. Ouhabaz, 'Gaussian estimates and L_p-boundedness of Riesz means', *J. Evol. Equ.* **2** (2002), 299–317.

[11] E. B. Davies, 'Uniformly elliptic operators with measurable coefficients', *J. Funct. Anal.* **132** (1995), 141–169.

[12] ———, 'Limits on L^p-regularity of self-adjoint elliptic operators', *J. Differential Equations* **135** (1997), 83–102.

[13] R. de Laubenfels, 'Integrated semigroups, C-semigroups and the abstract Cauchy problem', *Semigroup Forum* **41** (1990), 83–95.

[14] X. T. Duong and A. McIntosh, 'Singular integral operators with non-smooth kernels on irregular domains', *Rev. Mat. Iberoamericana* **15** (1999), 233–265.

[15] X. T. Duong, A. Sikora and E. M. Ouhabaz, 'Plancherel type estimates and sharp spectral multipliers', *J. Funct. Anal.* **196** (2002), 443–485.

[16] A. Grigor'yan, 'Gaussian upper bounds for the heat kernel on an arbitrary manifolds', *J. Differential Geom.* **45** (1997), 33–52.

[17] S. Hofmann and J. M. Martell, 'L^p bounds for Riesz transforms and square roots associated to second order elliptic operators', *Publ. Mat.* **47** (2003), 497–515.

[18] L. Hörmander, 'Estimates for translation invariant operators in L^p spaces', *Acta Math.* **104** (1960), 93–140.

[19] P. C. Kunstmann, *L_p-spectral properties of elliptic differential operators* Habilitationsschrift, (Karlsruhe, 2002).

[20] E. Lanconelli, 'Valutazioni in $L^p(\mathbb{R}^n)$ della soluzione del problema di Cauchy per l'equazione di Schrödinger', *Boll. Un. Mat. Ital. (4)* **1** (1968), 591–607.

[21] V. Liskevich, Z. Sobol and H. Vogt, 'On L_p-theory of C_0-semigroups associated with second order elliptic operators II', *J. Funct. Anal.* **193** (2002), 55–76.

[22] A. McIntosh, 'Operators which have an H^∞ functional calculus', in: *Miniconference on operator theory and partial differential equations (North Ryde)*, Proc. Center Math. Appl. Austral. Nat. Univ. 14 (Australian National Univ., Canberra, 1986) pp. 210–231.

[23] O. El Mennaoui, *Trace des semi-groupes holomorphes singuliers à l'origine et comportement asymptotique* (Ph.D. Thesis, University of Franche-Comté, France, 1992).

[24] E. M. Ouhabaz, 'Gaussian estimates and holomorphy of semigroups', *Proc. Amer. Math. Soc.* **123** (1995), 1465–1474.

[25] G. Schreieck and J. Voigt, 'Stability of the L_p-spectrum of generalized Schrödinger operators with form small negative part of the potential', in: *Functional Analysis* (eds. ?. Pietsch, ?. Ruess and H. Vogt), *Proc. Essen 1991, Bierstedt* (Marcel-Dekker, New York, 1994).

[26] S. Sjöstrand, 'On the Riesz means of the solutions of the Schrödinger equation', *Ann. Scuola Norm. Sup. Pisa (3)* **24** (1970), 331–348.

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