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A sharp oscillation property involving the critical Sobolev exponent for a class of superlinear elliptic problems

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This paper studies the asymptotic behaviour as $\alpha := u(0) \uparrow \infty$ of the first zero $R(\alpha)$ of the radially symmetric solution of the semilinear equation

$$-\Delta u = |u|^{\beta-1}u + h$$

in $\mathbb{R}^n$, $n \geqslant 1$, where $h > 0$ and $\beta > 1$. We establish that $R(\alpha) = O(\alpha^{-(\beta-1)/2})$ if $n = 1, 2$ or $n \geqslant 3$ and $\beta < (n+2)/(n-2)$, and conjecture that $\liminf_{\alpha\to\infty} R(\alpha) > 0$ if $n \geqslant 3$ and $\beta > (n+2)/(n-2)$.

1. Introduction

This paper ascertains how fast the unique solution of the Cauchy problem

$$\left.\begin{aligned} -\psi''(r) - \frac{n-1}{r}\psi'(r) = c(|\psi(r)|^{\beta-1}\psi(r) + 1), \quad r > 0,\\ \psi(0) = \alpha, \quad \psi'(0) = 0, \end{aligned}\right\} \tag{1.1}$$

reaches its first zero, where

$$n \in \mathbb{N}, \quad n \geqslant 1, \qquad \beta, c \in \mathbb{R}, \quad \beta > 1,\ c > 0, \tag{1.2}$$

and $\alpha \geqslant 0$ is an arbitrary non-negative real number. This problem arises in a natural way by analysing the radially symmetric solutions of

$$-\Delta u = |u|^{\beta-1}u + h,$$

with $h > 0$ arbitrary and $u(0) > 0$. Indeed, if $\Psi(r)$ stands for any of these solutions, the rescaled function

$$\psi(r) = h^{-1/\beta}\Psi(r)$$

satisfies (1.1) with

$$c = h^{(\beta-1)/\beta}, \qquad \alpha := h^{-1/\beta}\Psi(0).$$

As $\beta > 1$, for every $\alpha \geqslant 0$, (1.1) has a unique (maximal) solution to the right of $r = 0$, denoted by

$$\psi := \psi(r, \alpha) = \psi(r).$$

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According to lemma 2.1, for every $\alpha > 0$ there exists $R(\alpha) > 0$ such that

$$\psi(R(\alpha), \alpha) = 0 \quad \text{and} \quad \psi'(r, \alpha) < 0 \ \forall r \in (0, R(\alpha)]. \tag{1.3}$$

The main result of this paper provides us with the exact behaviour of $R(\alpha)$ as $\alpha \uparrow \infty$. It can be stated precisely as follows.

THEOREM 1.1. *Suppose that* $n \in \{1, 2\}$, *or that*

$$n \geqslant 3 \quad \textit{and} \quad \beta < \frac{n+2}{n-2}. \tag{1.4}$$

Then,

$$R(\alpha) = O(\alpha^{-(\beta-1)/2}) \quad \textit{as } \alpha \uparrow \infty. \tag{1.5}$$

More precisely,

$$\lim_{\alpha \uparrow \infty} (\alpha^{(\beta-1)/2} R(\alpha)) = \sqrt{x_0}, \tag{1.6}$$

where $x_0 > 0$ *is the lowest positive zero of the (unique) regular solution of*

$$\left.\begin{aligned} 2xu''(x) + nu'(x) = -\tfrac{1}{2}c|u(x)|^{\beta-1}u(x), \quad x > 0, \\ u(0) = 1. \end{aligned}\right\} \tag{1.7}$$

Thus, $R(\alpha) \downarrow 0$ as $\alpha \uparrow \infty$, which is genuine superlinear behaviour. Rather strikingly, this behaviour seems to fail when $n \geqslant 3$ if $\beta > (n+2)/(n-2)$. Actually, in such a case, we conjecture that

$$\liminf_{\alpha \to \infty} R(\alpha) > 0. \tag{1.8}$$

Theorem 1.1 should have a number of applications in analysing the rapid oscillatory behaviour of the regular solutions of a wide class of superlinear boundary-value problems. Though this rapid oscillatory behaviour has been used to show the existence of an arbitrarily large number of periodic solutions for a wide class of nonlinear Hill equations (see, for example, [1]), and to establish some multiplicity results of large nodal and positive solutions in some classes of superlinear indefinite problems (see, for example, [3, 4]), this seems to be the first paper where the oscillatory behaviour of the solutions is measured.

The rest of this paper consists of the proof of theorem 1.1 and it is structured as follows. Section 2 contains some important preliminary results and it reduces the proof of theorem 1.1 to the proof of the existence of a positive zero for the solution of (1.7) through theorem 2.3. Section 3 proves theorem 2.3 in cases $n \in \{1, 2\}$ and $n \geqslant 3$ with $\beta \leqslant n/(n-2)$. Finally, § 4 completes the proof of theorem 2.3 when $n \geqslant 3$ and

$$\frac{n}{n-2} < \beta < \frac{n+2}{n-2}.$$

The nature of the proof is substantially different in each of these cases.

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2. Preliminaries

The next result establishes the existence of $R(\alpha)$.

LEMMA 2.1. *For every $\alpha > 0$, there exists $R(\alpha) > 0$ satisfying (1.3). Moreover,*

$$\lim_{\alpha \downarrow 0} R(\alpha) = 0.$$

Proof. For each $r > 0$, we have that

$$-(r^{n-1}\psi'(r))' = cr^{n-1}(|\psi(r)|^{\beta-1}\psi(r) + 1).$$

Thus, integrating in $(0, r)$, we find that

$$\begin{aligned} r^{n-1}\psi'(r) &= -c\int_0^r s^{n-1}(|\psi(s)|^{\beta-1}\psi(s) + 1)\,\mathrm{d}s \\ &= -c\int_0^r s^{n-1}|\psi(s)|^{\beta-1}\psi(s)\,\mathrm{d}s - \frac{c}{n}r^n, \end{aligned}$$

and hence

$$\psi'(r) = -c\int_0^r \left(\frac{s}{r}\right)^{n-1} |\psi(s)|^{\beta-1}\psi(s)\,\mathrm{d}s - \frac{c}{n}r. \tag{2.1}$$

According to (2.1), $\psi'(r) < 0$ for all $r > 0$ such that $\psi \geqslant 0$ in $[0, r]$, because

$$\psi'(r) \leqslant -\frac{c}{n}r < 0.$$

Moreover, integrating this inequality, we obtain that

$$\psi(r) \leqslant \alpha - \frac{c}{2n}r^2,$$

and therefore there exists

$$R(\alpha) \leqslant \sqrt{\frac{2n\alpha}{c}}$$

such that

$$\psi(R(\alpha), \alpha) = 0 \quad \text{and} \quad \psi'(r, \alpha) < 0 \ \forall r \in (0, R(\alpha)].$$

This concludes the proof. □

The next result provides us with the regularity of $R(\alpha)$ with respect to α.

LEMMA 2.2. *For every $\alpha \geqslant 0$ the identity*

$$\alpha = c\int_0^{R(\alpha)} \int_0^r \left(\frac{s}{r}\right)^{n-1} \psi^\beta(s, \alpha)\,\mathrm{d}s\,\mathrm{d}r + \frac{c}{2n}R^2(\alpha) \tag{2.2}$$

holds. Consequently, $R \in \mathcal{C}^1[0, \infty)$.

Proof. Integrating (2.1) in $(0, \rho)$, it becomes apparent that

$$\psi(\rho, \alpha) - \psi(0, \alpha) = -c\int_0^\rho \int_0^r \left(\frac{s}{r}\right)^{n-1} \psi^\beta(s, \alpha)\,\mathrm{d}s\,\mathrm{d}r - \frac{c}{2n}\rho^2$$

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for all $\rho \in (0, R(\alpha)]$. Since

$$\psi(R(\alpha), \alpha) = 0 \quad \text{and} \quad \psi(0, \alpha) = \alpha,$$

(2.2) follows, by particularizing the previous identity at $\rho = R(\alpha)$.

Subsequently, we consider the function

$$H\colon (0, \infty) \times (0, \infty) \to \mathbb{R}$$

defined through

$$H(\alpha, \rho) := c \int_0^\rho \int_0^r \left(\frac{s}{r}\right)^{n-1} |\psi(s, \alpha)|^{\beta-1} \psi(s, \alpha) \, \mathrm{d}s \, \mathrm{d}r + \frac{c}{2n}\rho^2 - \alpha.$$

As, thanks to the Peano differentiation theorem (see, for example, [2]), ψ is a $\mathcal{C}^1$-function of α, H must be a $\mathcal{C}^1$-function of (α, ρ) such that

$$H(\alpha, R(\alpha)) = 0 \quad \text{for all } \alpha > 0,$$

by (2.2). Moreover,

$$\frac{\partial H}{\partial \rho}(\alpha, \rho) = c \int_0^\rho \left(\frac{s}{\rho}\right)^{n-1} |\psi(s, \alpha)|^{\beta-1} \psi(s, \alpha) \, \mathrm{d}s + \frac{c}{n}\rho > 0$$

for all $\rho \in (0, R(\alpha)]$. Thus, the fact that $R(\alpha)$ is of class $\mathcal{C}^1$ follows as an easy application of the implicit function theorem to H. □

Throughout the rest of this paper, for every $\alpha > 0$, we consider

$$\varphi = \varphi(s, \alpha) = \varphi(s), \quad s \geqslant 0,$$

the function defined through

$$\psi(r, \alpha) = \alpha\varphi(\alpha^{\beta-1} r^2, \alpha), \quad r \geqslant 0, \ \alpha > 0. \tag{2.3}$$

Then,

$$\begin{aligned} \psi'(r) &= \alpha\varphi'(\alpha^{\beta-1}r^2) 2r\alpha^{\beta-1}, \\ \psi''(r) &= \alpha\varphi''(\alpha^{\beta-1}r^2)(2r\alpha^{\beta-1})^2 + \alpha\varphi'(\alpha^{\beta-1}r^2) 2\alpha^{\beta-1}. \end{aligned}$$

Thus, substituting these identities in (1.1) and setting

$$x = \alpha^{\beta-1} r^2 \tag{2.4}$$

yields

$$\alpha\varphi''(x)(2r\alpha^{\beta-1})^2 + n\alpha\varphi'(x) 2\alpha^{\beta-1} = -c(\alpha^\beta |\varphi(x)|^{\beta-1}\varphi(x) + 1)$$

for all $r \geqslant 0$ and $\alpha > 0$. As dividing this identity by $2\alpha^\beta$ gives

$$2r^2\alpha^{\beta-1}\varphi''(x) + n\varphi'(x) = -\tfrac{1}{2}c(|\varphi(x)|^{\beta-1}\varphi(x) + \alpha^{-\beta}),$$

it becomes apparent that

$$2x\varphi''(x) + n\varphi'(x) = -\tfrac{1}{2}c(|\varphi(x)|^{\beta-1}\varphi(x) + \alpha^{-\beta}) \tag{2.5}$$

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for all $x \geqslant 0$ and $\alpha > 0$. Moreover,

$$\varphi(0, \alpha) = 1 \quad \text{for all } \alpha > 0, \tag{2.6}$$

and, setting

$$x_0(\alpha) = \alpha^{\beta-1} R^2(\alpha), \quad \alpha > 0, \tag{2.7}$$

it is easily seen that (1.3) implies that

$$\varphi(x_0(\alpha)) = 0 \quad \text{and} \quad \varphi'(x) < 0 \text{ for all } x \in (0, x_0(\alpha)]. \tag{2.8}$$

Throughout this paper, we consider the limiting problem as $\alpha \uparrow \infty$ of the Cauchy problem (2.5), (2.6), which is given by (1.7). Our proof of theorem 1.1 is based on the following result.

THEOREM 2.3. *Suppose that* $n \in \{1, 2\}$*, or else that (1.4) holds. There then exists* x_0 *such that the (unique) regular solution of (1.7) satisfies* $u(x) > 0$ *for all* $x \in [0, x_0)$ *and* $u(x_0) = 0$*. Moreover, the regular solution of (1.7) satisfies* $u(x) > 0$ *for all* $x > 0$ *if*

$$n \geqslant 3 \quad \textit{and} \quad \beta \geqslant \frac{n+2}{n-2}.$$

Sections 3 and 4 consist of the proof of theorem 2.3. To end this section we will use theorem 2.3 to complete the proof of theorem 1.1.

2.1. Proof of theorem 1.1

Suppose that either $n \in \{1, 2\}$, or (1.4) holds. Then, by theorem 2.3, the solution of (1.7) admits a positive zero. Suppose that $u(x) > 0$ for all $x \in [0, x_0)$ and $u(x_0) = 0$. Then, by the uniqueness of the underlying Cauchy problem, necessarily,

$$u'(x_0) < 0,$$

as $u'(x_0) = u(x_0) = 0$ implies that $u = 0$. Moreover, according to the Peano differentiation theorem, for sufficiently small $\varepsilon > 0$,

$$\lim_{\alpha \uparrow \infty} \varphi(\cdot, \alpha) = u$$

uniformly in $[0, x_0 + \varepsilon]$, where u is the regular solution of (1.7). Consequently, for sufficiently large $\alpha > 0$, there exists $x_\alpha \sim x_0$ such that

$$\varphi(x_\alpha, \alpha) = 0, \qquad \varphi(x, \alpha) > 0 \quad \text{for all } x \in [0, x_\alpha)$$

and

$$\lim_{\alpha \uparrow \infty} x_\alpha = x_0.$$

As

$$\psi > 0 \quad \text{in } [0, R(\alpha)) \quad \text{and} \quad 0 = \psi(R(\alpha), \alpha) = \alpha \varphi(\alpha^{\beta-1} R^2(\alpha), \alpha),$$

necessarily,

$$\alpha^{\beta-1} R^2(\alpha) = x_\alpha,$$

and therefore

$$\lim_{\alpha \uparrow \infty} (\alpha^{(\beta-1)/2} R(\alpha)) = \lim_{\alpha \uparrow \infty} \sqrt{x_\alpha} = \sqrt{x_0}, \tag{2.9}$$

which ends the proof of theorem 1.1.

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3. Proof of theorem 2.3 when $n = 1, 2$ or $n \geqslant 3$ and $\beta \leqslant n/(n-2)$

Throughout this section no special restrictions will be imposed on the values of $n \geqslant 1$ and $\beta > 1$, unless strictly necessary. This observation is extremely important, as some of our findings here will be used in the next section to prove the theorem in the general case.

The proof of the theorem will proceed by contradiction. Suppose the solution of (1.7) does not admit a positive zero. By (1.7),

$$u'(0) = -\frac{c}{2n} < 0,$$

and hence $u'(x) < 0$ for sufficiently small $x > 0$. Suppose there exists $x_1 > 0$ such that

$$u'(x) < 0 \quad \text{for all } x \in [0, x_1) \quad \text{and} \quad u'(x_1) = 0.$$

Then, $u(x_1) > 0$, as $u(x_1) = u'(x_1) = 0$ implies that $u = 0$, and we find from (1.7) that

$$2x_1 u''(x_1) = 2x_1 u''(x_1) + nu'(x_1) = -\tfrac{1}{2} c u^{\beta}(x_1) < 0.$$

Hence, $u''(x_1) < 0$, which is impossible. Consequently,

$$u(x) > 0 \quad \text{and} \quad u'(x) < 0 \quad \text{for all } x \geqslant 0. \tag{3.1}$$

In particular, $u(x)$ is globally defined in $[0, \infty)$. Also, differentiating with respect to x, we find from (1.7) that

$$2u''(x) + 2xu'''(x) + nu''(x) = -\tfrac{1}{2} c\beta u^{\beta-1}(x) u'(x), \tag{3.2}$$

and therefore

$$u''(0) = -\frac{c\beta}{2(2+n)} u'(0) > 0.$$

Suppose that there exists $x_2 > 0$ such that

$$u''(x) > 0 \quad \text{for all } x \in [0, x_2) \quad \text{and} \quad u''(x_2) = 0.$$

Then, necessarily, $u'''(x_2) \leqslant 0$. But, owing to (3.1) and (3.2), we find that

$$2x_2 u'''(x_2) = -\tfrac{1}{2} c\beta u^{\beta-1}(x_2) u'(x_2) > 0$$

and, hence, $u'''(x_2) > 0$, which is impossible. Consequently,

$$u(x) > 0, \quad u'(x) < 0 \quad \text{and} \quad u''(x) > 0 \quad \text{for all } x \geqslant 0. \tag{3.3}$$

Note that (3.3) holds for every $n \geqslant 1$ and $\beta > 1$. According to (3.3), the limit

$$L := \lim_{x \uparrow \infty} u(x) \geqslant 0$$

is well defined. Moreover, as $u' < 0$ increases, because $u'' > 0$, we also have that

$$\tilde{L} := \lim_{x \uparrow \infty} u'(x) \leqslant 0$$

is well defined.

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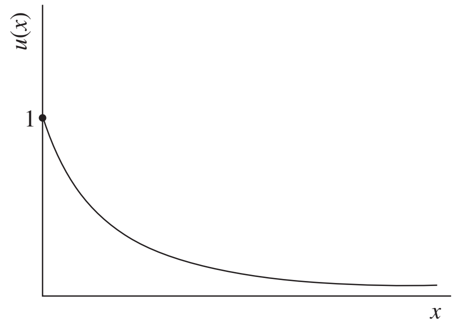

Figure 1. The profile of u when $u^{-1}(0) = \emptyset$.

Suppose that $\tilde{L} < 0$. Then, $u'(x) \leqslant \tilde{L} < 0$ for all $x \geqslant 0$, and, hence,

$$u(x) \leqslant u(0) + \tilde{L}x = 1 + \tilde{L}x, \quad x \geqslant 0,$$

which implies that $u(\tilde{x}) = 0$ for some $\tilde{x} \leqslant -\tilde{L}^{-1}$ and contradicts our assumption that u does not admit a positive zero. Thus, $\tilde{L} = 0$.

Now, suppose that $L > 0$. Then, letting $x \uparrow \infty$ in (1.7), it becomes apparent that

$$\lim_{x\uparrow\infty}(2xu''(x)) = -n\lim_{x\uparrow\infty} u'(x) - \tfrac{1}{2}c\lim_{x\uparrow\infty} u^{\beta}(x) = -n\tilde{L} - \tfrac{1}{2}cL^{\beta} = -\tfrac{1}{2}cL^{\beta} < 0,$$

which implies that $u''(x) < 0$ for sufficiently large x and contradicts (3.3). Therefore,

$$\lim_{x\uparrow\infty} u(x) = \lim_{x\uparrow\infty} u'(x) = 0. \tag{3.4}$$

Figure 1 shows the profile of $u(x)$ when it does not admit a positive zero, in the general case when $n \geqslant 1$ and $\beta > 1$.

Note that, due to (1.7) and (3.4), we also have that

$$\lim_{x\uparrow\infty}(xu''(x)) = 0.$$

Subsequently, we write the differential equation of (1.7) in the form

$$[2xu'(x) + (n-2)u(x)]' = -\tfrac{1}{2}cu^{\beta}(x), \quad x \geqslant 0. \tag{3.5}$$

Then, u solves (1.7) if and only if it solves the integral equation

$$2xu'(x) + (n-2)u(x) = n - 2 - \frac{c}{2}\int_0^x u^{\beta}, \quad x \geqslant 0. \tag{3.6}$$

As the function

$$x \mapsto n - 2 - \frac{c}{2}\int_0^x u^{\beta}$$

is decreasing, the limit

$$d := \frac{1}{2}\left(n - 2 - \frac{c}{2}\int_0^{\infty} u^{\beta}\right) \in [-\infty, \infty) \tag{3.7}$$

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is well defined. Moreover, letting $x \uparrow \infty$ in (3.6), we find from (3.4) that

$$\lim_{x\uparrow\infty}(xu(x))' = \lim_{x\uparrow\infty}(xu'(x) + u(x)) = \lim_{x\uparrow\infty}(xu'(x)) = d. \tag{3.8}$$

Suppose that $d = -\infty$. Then,

$$\int_0^\infty u^\beta = \infty \quad \text{and} \quad \lim_{x\uparrow\infty}(xu(x))' = -\infty.$$

In particular, there exists $x_1 > 0$ such that

$$(xu(x))' \leqslant -1 \quad \forall x \geqslant x_1.$$

Thus, integrating this inequality in (x_1, x), we find that

$$xu(x) \leqslant x_1 u(x_1) - (x - x_1) \quad \forall x \geqslant x_1$$

or, equivalently,

$$u(x) \leqslant \frac{x_1}{x}u(x_1) - 1 + \frac{x_1}{x} \quad \forall x \geqslant x_1.$$

Consequently, letting $x \uparrow \infty$ in this inequality shows that

$$\lim_{x\uparrow\infty} u(x) \leqslant -1,$$

which is impossible by (3.4). Therefore, $d \in \mathbb{R}$. Actually, $d = 0$. Indeed, according to (3.8), for every $\varepsilon > 0$ there exists $x_\varepsilon > 0$ such that

$$d - \varepsilon \leqslant (xu(x))' \leqslant d + \varepsilon \quad \forall x \geqslant x_\varepsilon.$$

Thus, integrating in $[x_\varepsilon, x]$ gives that

$$(d - \varepsilon)(x - x_\varepsilon) \leqslant xu(x) - x_\varepsilon u(x_\varepsilon) \leqslant (d + \varepsilon)(x - x_\varepsilon) \quad \forall x \geqslant x_\varepsilon.$$

Consequently, dividing these inequalities by $x > x_\varepsilon > 0$ and letting $x \uparrow \infty$, it becomes apparent that

$$d - \varepsilon \leqslant \lim_{x\uparrow\infty} u(x) = 0 \leqslant d + \varepsilon$$

for all $\varepsilon > 0$ and, therefore, $d = 0$. Equivalently,

$$\int_0^\infty u^\beta = \frac{2(n-2)}{c}. \tag{3.9}$$

Obviously, due to (3.3) and (3.9), we must have that $n \geqslant 3$. Consequently, when $n \in \{1, 2\}$, there must exist $x_0 > 0$ such that

$$u(x) > 0 \quad \forall x \in [0, x_0) \quad \text{and} \quad u(x_0) = 0.$$

Moreover, $u'(x_0) < 0$, because $u(x_0) = u'(x_0) = 0$ implies that $u = 0$, by the uniqueness of solution for the associated Cauchy problem, which is impossible. This ends the proof of theorem 2.3 in this case.

Subsequently, we will assume that $n \geqslant 3$ and that $u(x) > 0$ for all $x \geqslant 0$. According to (3.3), we have that

$$nu'(x) < 2xu''(x) + nu'(x) = -\tfrac{1}{2}cu^\beta(x)$$

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for all $x > 0$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{u^{-\beta+1}(x)}{-\beta+1} = u^{-\beta}(x)u'(x) < -\frac{c}{2n} \quad \forall x > 0.$$

As $\beta > 1$, we have that $-\beta + 1 < 0$ and so

$$(u^{-\beta+1}(x))' > \frac{c(\beta-1)}{2n} \quad \forall x > 0.$$

Consequently, integrating in $[0, x]$ shows that

$$u^{-\beta+1}(x) - 1 > \frac{c(\beta-1)}{2n}x$$

for all $x > 0$. Therefore,

$$u^{\beta-1}(x) < \frac{1}{1 + (c(\beta-1)/(2n))x} < \frac{2n}{c(\beta-1)}x^{-1}$$

and, hence,

$$u(x) < \left(\frac{2n}{c(\beta-1)}\right)^{1/(\beta-1)} x^{-1/(\beta-1)} \quad \forall x > 0. \tag{3.10}$$

Subsequently, we introduce the new variable $v(x)$ defined by

$$u(x) = x^{-(n-2)/2}v(x), \quad x > 0. \tag{3.11}$$

Then, substituting (3.11) in (3.6), we obtain that

$$\begin{aligned} n - 2 - \frac{c}{2}\int_0^x u^\beta &= 2xu'(x) + (n-2)u(x) \\ &= 2x\left(-\frac{n-2}{2}x^{-(n-2)/2-1}v(x) + x^{-(n-2)/2}v'(x)\right) \\ &\qquad\qquad\qquad\qquad + (n-2)x^{-(n-2)/2}v(x) \\ &= 2x^{-(n-2)/2+1}v'(x) \end{aligned}$$

for all $x > 0$. On the other hand, according to (3.3) and (3.9), we have that

$$\frac{c}{2}\int_0^x u^\beta < \frac{c}{2}\int_0^\infty u^\beta = n - 2,$$

and hence it follows from the previous identity that

$$v'(x) > 0 \quad \forall x > 0. \tag{3.12}$$

Moreover, differentiating it with respect to x, it becomes apparent that

$$2(x^{-(n-2)/2+1}v'(x))' = -\tfrac{1}{2}cu^\beta(x) = -\tfrac{1}{2}cx^{-\beta(n-2)/2}v^\beta(x) \quad \forall x > 0. \tag{3.13}$$

Also, owing to (3.11) and (3.10), we have that

$$v(x) = x^{(n-2)/2}u(x) < \left(\frac{2n}{c(\beta-1)}\right)^{1/(\beta-1)} x^{(n-2)/2-1/(\beta-1)} \tag{3.14}$$

for all $x > 0$.

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Now, besides $n \geqslant 3$, suppose that

$$\beta < \frac{n}{n-2}, \tag{3.15}$$

or, equivalently, that

$$\frac{n-2}{2} - \frac{1}{\beta - 1} < 0.$$

Then, (3.14) implies that

$$\lim_{x \uparrow \infty} v(x) = 0,$$

which contradicts (3.12). Therefore, much like in the case $n \in \{1, 2\}$, under (3.15), $u(x)$ must have a positive zero, which completes the proof of theorem 2.3 in the case of (3.15).

Finally, we consider the more delicate case when $n \geqslant 3$ and

$$\beta = \frac{n}{n-2}.$$

Then, (3.14) provides us with the estimate

$$v(x) = x^{(n-2)/2} u(x) < \left(\frac{2n}{c(\beta - 1)} \right)^{1/(\beta - 1)}, \quad x > 0,$$

and, hence, by (3.12), the limit

$$\ell := \lim_{x \uparrow \infty} v(x) = \lim_{x \uparrow \infty} (x^{(n-2)/2} u(x)) > 0 \tag{3.16}$$

is well defined. Moreover, (3.13) can equivalently be written as

$$2(x^{(4-n)/2} v'(x))' = -\tfrac{1}{2} c x^{-n/2} v^{\beta}(x) \quad \forall x > 0. \tag{3.17}$$

Due to (3.16), for every $\varepsilon > 0$ there exists $x_{\varepsilon} > 0$ such that

$$\ell^{\beta} - \varepsilon \leqslant v^{\beta}(x) \leqslant \ell^{\beta} + \varepsilon \quad \forall x \geqslant x_{\varepsilon}.$$

Consequently, substituting the lower estimate in (3.17) shows that

$$2(x^{(4-n)/2} v'(x))' \leqslant -\tfrac{1}{2} c(\ell^{\beta} - \varepsilon) x^{-n/2}, \quad x \geqslant x_{\varepsilon}.$$

Let $x \geqslant y \geqslant x_{\varepsilon}$ be arbitrary. Integrating in $[y, x]$, we find that

$$\begin{aligned} x^{(4-n)/2} v'(x) - y^{(4-n)/2} v'(y) &\leqslant -\frac{c}{4}(\ell^{\beta} - \varepsilon) \int_y^x s^{-n/2} \, \mathrm{d}s \\ &= \frac{c}{2(n-2)} (\ell^{\beta} - \varepsilon)(x^{-(n-2)/2} - y^{-(n-2)/2}). \end{aligned}$$

Thus,

$$0 < x^{(4-n)/2} v'(x) \leqslant y^{(4-n)/2} v'(y) + \frac{c}{2(n-2)} (\ell^{\beta} - \varepsilon)(x^{-(n-2)/2} - y^{-(n-2)/2})$$

and, letting $x \uparrow \infty$, it becomes apparent that

$$0 \leqslant y^{(4-n)/2} v'(y) - \frac{c(\ell^{\beta} - \varepsilon)}{2(n-2)} y^{-(n-2)/2}, \quad y \geqslant x_{\varepsilon}.$$

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Equivalently,

$$v'(y) \geqslant \frac{c(\ell^\beta - \varepsilon)}{2(n-2)} y^{-1}, \quad y \geqslant x_\varepsilon,$$

and integrating this inequality yields that

$$v(y) - v(x_\varepsilon) \geqslant \frac{c(\ell^\beta - \varepsilon)}{2(n-2)} \log \frac{y}{x_\varepsilon}$$

for all $y \geqslant x_\varepsilon$. Letting $y \uparrow \infty$ shows that

$$\lim_{y \uparrow \infty} v(y) = \infty,$$

which contradicts (3.16). This contradiction completes the proof of theorem 2.3 if $n \geqslant 3$ and $\beta \leqslant n/(n-2)$.

4. Proof of theorem 2.3 when $n \geqslant 3$ and $n/(n-2) < \beta < (n+2)/(n-2)$

Throughout this section we assume that

$$n \geqslant 3, \qquad \beta > \frac{n}{n-2}. \tag{4.1}$$

The next result then holds; the proof, being straightforward, will be omitted.

LEMMA 4.1. *Suppose (4.1) holds. Then, the function u_s defined by*

$$u_s(x) := \omega x^{-1/(\beta-1)}, \quad x > 0, \tag{4.2}$$

where

$$\omega := \left(\frac{2}{c} \frac{(n-2)\beta - n}{(\beta-1)^2} \right)^{1/(\beta-1)}, \tag{4.3}$$

provides us with a singular positive solution of

$$2xu''(x) + nu'(x) = -\tfrac{1}{2} c u^\beta(x), \quad x > 0. \tag{4.4}$$

Actually, the singular solution u_s exists if and only if (4.1) holds.

Due to lemma 4.1, it is natural to introduce the auxiliary function $v(x)$ defined by

$$u(x) = x^{-1/(\beta-1)} v(x), \quad x > 0, \tag{4.5}$$

where $u(x)$ is the unique regular solution of (1.7). As $u(0) = 1$, necessarily $v(0) = 0$. However, this does not necessarily imply that $u(0) = 1$. Substituting (4.5) into (4.4), rearranging terms and multiplying the resulting identity by $x^{\beta/(\beta-1)}$ yields

$$2x^2 v''(x) + \left(n - \frac{4}{\beta - 1} \right) x v'(x) - \frac{(n-2)\beta - n}{(\beta-1)^2} v(x) = -\frac{c}{2} v^\beta(x), \tag{4.6}$$

which is a semilinear Euler equation. Consequently, to study (4.6) it is natural to perform the change of variable

$$x := \mathrm{e}^t, \qquad w(t) := v(x). \tag{4.7}$$

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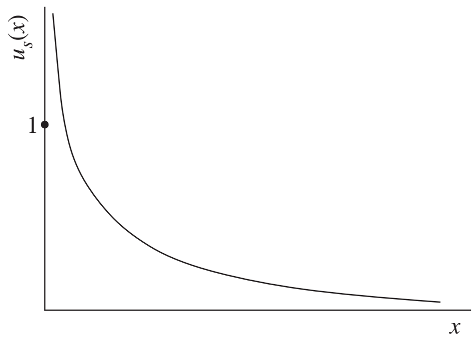

Figure 2. The profile of $u_s(x)$.

Then,

$$w(-\infty) = v(0) = 0,$$
$$\dot{w}(t) := \frac{\mathrm{d}w}{\mathrm{d}t}(t) = xv'(x), \qquad \ddot{w}(t) := \frac{\mathrm{d}^2 w}{\mathrm{d}t^2}(t) = x^2 v''(x) + \dot{w}(t),$$
$$x^2 v''(x) = \ddot{w}(t) - \dot{w}(t)$$

and, substituting these identities in (4.6) and rearranging terms, it becomes apparent that

$$2\ddot{w}(t) + \left(n - 2 - \frac{4}{\beta - 1}\right)\dot{w}(t) = \left(\frac{(n-2)\beta - n}{(\beta - 1)^2} - \frac{c}{2} w^{\beta - 1}(t)\right) w(t) \tag{4.8}$$

for all $t \in \mathbb{R}$. Besides zero, the value of ω given by (4.3) provides us with an equilibrium of (4.8). Actually, these are the unique equilibria of (4.8). Multiplying (4.8) by $\dot{w}(t)$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{w}^2 - \frac{(n-2)\beta - n}{2(\beta - 1)^2} w^2 + \frac{c}{2(\beta + 1)} w^{\beta + 1}\right) = -\left(n - 2 - \frac{4}{\beta - 1}\right)\dot{w}^2(t) \tag{4.9}$$

for all $t \in \mathbb{R}$. Consequently, the dynamics of (4.4) should depend on the size of the parameter $\beta > n/(n-2) > 1$.

Suppose that β equals the critical Sobolev exponent, i.e. that

$$\beta = \frac{n+2}{n-2}. \tag{4.10}$$

Then, (4.9) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{w}^2 - \frac{(n-2)\beta - n}{2(\beta - 1)^2} w^2 + \frac{c}{2(\beta + 1)} w^{\beta + 1}\right) = 0 \tag{4.11}$$

and, consequently, (4.8) provides us with a *conservative* system with *potential energy*

$$P(w) := \frac{c}{2(\beta + 1)} w^{\beta + 1} - \frac{(n-2)\beta - n}{2(\beta - 1)^2} w^2, \quad w > 0. \tag{4.12}$$

Note that $P'(w) = 0$ if and only if $w \in \{0, \omega\}$. Naturally, in the case of (4.10), the system associated with (4.8) has the phase portrait sketched in figure 4.

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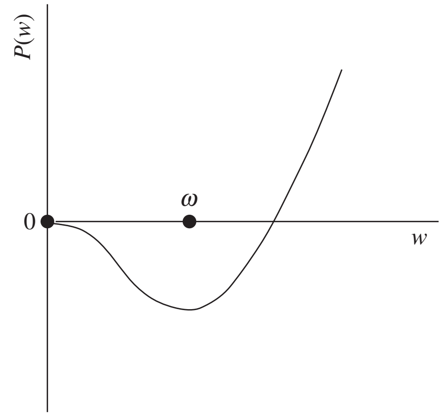

Figure 3. The potential energy $P(w)$.

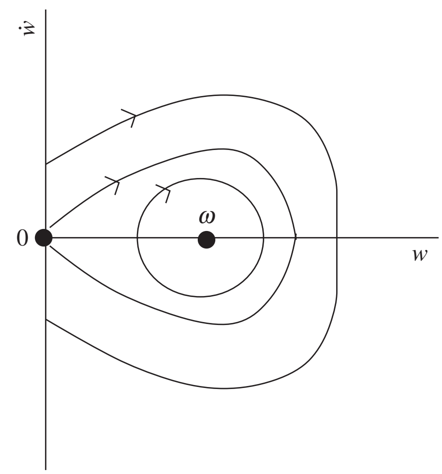

Figure 4. The phase portrait for $\beta = (n+2)/(n-2)$.

As we are interested in the solutions w such that $w(-\infty) = 0$, we should focus our attention on the homoclinic orbit of the origin. As w leaves zero at $t = -\infty$, it increases, reaching its maximum value at the unique positive zero of $P(w)$, and, then, decreases, reaching zero as $t \uparrow \infty$, in such a way that the profile of the associated

$$v(x) = w(t), \quad x = \mathrm{e}^t,$$

looks like that in figure 5. Consequently, under condition (4.10), the solution of (1.7) given through (4.5) crosses the singular solution

$$u_s(x) = \omega x^{-1/(\beta-1)}$$

at two single points $0 < x_1 < x_2$ in such a way that $u(x) < u_s(x)$ if $x < x_1$ or $x > x_2$, while $u(x) > u_s(x)$ if $x_1 < x < x_2$. In particular, $u(x) > 0$ for all $x > 0$, as claimed by the second assertion of theorem 2.3.

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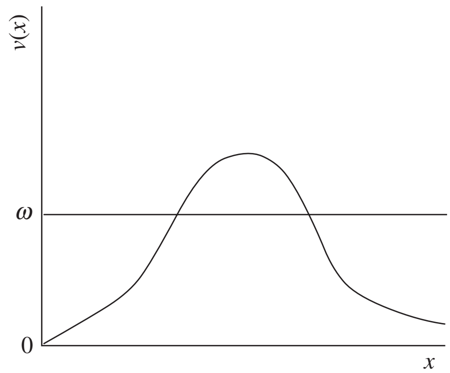

Figure 5. The graph of $v(x)$ for $\beta = (n+2)/(n-2)$.

Subsequently, we will assume that

$$\frac{n}{n-2} < \beta < \frac{n+2}{n-2}. \tag{4.13}$$

Then,

$$n - 2 - \frac{4}{\beta - 1} < 0,$$

and hence it follows from (4.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{w}^2 - \frac{(n-2)\beta - n}{2(\beta-1)^2} w^2 + \frac{c}{2(\beta+1)} w^{\beta+1}\right) > 0. \tag{4.14}$$

Consequently, the total energy increases along the trajectories of the positive solutions of (4.8). Note that, according to (3.10) and (4.5), it follows that

$$v(x) = x^{1/(\beta-1)} u(x) < \left(\frac{2n}{c(\beta-1)}\right)^{1/(\beta-1)} \quad \forall x > 0,$$

and hence

$$w(t) < C_0 := \left(\frac{2n}{c(\beta-1)}\right)^{1/(\beta-1)} > \omega \tag{4.15}$$

for all $t \in \mathbb{R}$ such that $w(s) > 0$ if $s \leqslant t$. Subsequently, we set

$$f(\xi) := \frac{(n-2)\beta - n}{(\beta-1)^2}\xi - \frac{c}{2}\xi^{\beta}, \quad \xi \geqslant 0,$$

and consider the first-order system associated to (4.8)

$$\left.\begin{aligned} \dot{w} &= z, \\ \dot{z} &= -\frac{1}{2}\left(n - 2 - \frac{4}{\beta-1}\right) z + \frac{1}{2} f(w). \end{aligned}\right\} \tag{4.16}$$

In terms of $z = \dot{w}$, (4.8) reads as follows:

$$2\dot{z}(t) + \left(n - 2 - \frac{4}{\beta - 1}\right) z(t) = f(w(t)).$$

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Hence, by the formula of variation of the constants, we obtain that

$$\dot{w}(t) = \exp\left(-\frac{1}{2}\left(n-2-\frac{4}{\beta-1}\right)(t-t_0)\right)\dot{w}(t_0) \\ + \frac{1}{2}\int_{t_0}^{t} \exp\left(-\frac{1}{2}\left(n-2-\frac{4}{\beta-1}\right)(t-s)\right) f(w(s))\,\mathrm{d}s$$

for all $t_0, t \in \mathbb{R}$ such that $w(s) > 0$ if $s \leqslant \max\{t_0, t\}$. Consequently, by (4.15), it is apparent that the solution $(w(t), z(t))$ cannot blow up within the set of times t for which $w > 0$ in $(-\infty, t]$. Therefore, one of the following alternatives occurs:

(A) $(w(t), z(t))$ is globally defined in time and $w(t) > 0$ for all $t \in \mathbb{R}$;

(B) there exists $t_0 \in \mathbb{R}$ such that $w(t) > 0$ for all $t < t_0$ and $w(t_0) = 0$, as illustrated by figure 6.

In order to prove theorem 2.3 we show that (B) occurs. The proof of this will proceed by contradiction assuming that, instead of (B), (A) holds. The contradiction will be reached by analysing the dynamics of (4.16).

The matrix of the linearization of (4.16) at the equilibrium $(0, 0)$ is given by

$$M_0 := \begin{pmatrix} 0 & 1 \\ \dfrac{(n-2)\beta - n}{2(\beta-1)^2} & -\dfrac{1}{2}\left(n-2-\dfrac{4}{\beta-1}\right) \end{pmatrix},$$

whose characteristic equation is

$$z^2 + \frac{1}{2}\left(n-2-\frac{4}{\beta-1}\right)z - \frac{(n-2)\beta-n}{2(\beta-1)^2} = 0.$$

As the characteristic roots are

$$\frac{1}{2}\left[-\frac{1}{2}\left(n-2-\frac{4}{\beta-1}\right) \pm \sqrt{\frac{1}{4}\left(n-2-\frac{4}{\beta-1}\right)^2 + 2\frac{(n-2)\beta-n}{(\beta-1)^2}}\,\right],$$

it follows from (4.13) that one of them is positive, while the other is negative. Thus, $(0, 0)$ is a saddle point, by the Grobman–Hartman theorem (see [2], if necessary). Note that, by construction, the unstable manifold of $(0, 0)$ is the trajectory of the solution $(w(t), \dot{w}(t))$, which will be denoted by Γ.

Similarly, the matrix of the linearization of (4.16) at $(\omega, 0)$ is given by

$$M_\omega := \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix},$$

where

$$p := \tfrac{1}{2}f'(\omega) = \frac{1}{2}\left(\frac{(n-2)\beta-n}{(\beta-1)^2} - \frac{1}{2}c\beta\omega^{\beta-1}\right) = -\frac{1}{2}\frac{(n-2)\beta-n}{\beta-1} < 0$$

and

$$q = -\frac{1}{2}\left(n-2-\frac{4}{\beta-1}\right) > 0,$$

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by (4.13). Therefore, the characteristic values of M_ω are

$$\tfrac{1}{2}(q \pm \sqrt{q^2+4p}).$$

Under (4.13), these roots always have positive real parts. Consequently, $(\omega, 0)$ is an unstable focus if $q^2+4p<0$, while it is an unstable node if $q^2+4p \geqslant 0$.

On the other hand, as the divergence of the planar vector field of (4.16) equals

$$-\frac{1}{2}\left(n-2-\frac{4}{\beta-1}\right)>0,$$

(4.16) cannot admit a non-trivial periodic solution, or a homoclinic connection of the origin, in the region $w>0$, by the Bendixson negative criterion.

We claim that there exists $t_1>0$ such that $z(t)=\dot{w}(t)>0$ for all $t<t_1$ and $z(t_1)=\dot{w}(t_1)=0$. Indeed, if $\dot{w}(t)>0$ for all $t \in \mathbb{R}$, then $w(t)$ must approximate some value $L_1 \in (0, C_0]$, by (4.15), and, in such a case,

$$\lim_{t\uparrow\infty} w(t)=L_1 \quad \text{and} \quad \lim_{t\uparrow\infty} \dot{w}(t)=0.$$

As $(L_1, 0)$ must be an equilibrium, necessarily $L_1=\omega$ and $(w(t), z(t))$ is a heteroclinic connection between $(0,0)$ and $(\omega, 0)$, which contradicts the fact that $(\omega, 0)$ must be either an unstable node or an unstable focus and proves the previous claim. Note that, necessarily, $w(t_1)>\omega$. Indeed, if $w(t_1)=\omega$, then $(w(t_1), z(t_1))=(\omega, 0)$, and hence $(w(t), z(t))=(\omega, 0)$ for all t, by uniqueness, which contradicts that $w(-\infty)=0$. Thus, $w(t_1) \neq \omega$. Moreover,

$$(\dot{w}(t_1), \dot{z}(t_1))=(0, \tfrac{1}{2} f(w(t_1)))$$

and $\dot{z}(t_1) \leqslant 0$, because $z(t)>0$ if $t<t_1$ and $z(t_1)=0$. Therefore, $f(w(t_1))<0$ and, consequently, $w(t_1)>\omega$. Moreover, since $\dot{z}(t_1)<0$, we find that

$$z(t)=\dot{w}(t)<0 \quad \text{for } t>t_1,\ t \sim t_1.$$

Suppose that $\dot{w}(t)<0$ for all $t>t_1$. Then, again by monotonicity, there exists $L_2 \in \{0, \omega\}$ such that $(w(t), \dot{w}(t))$ connects $(0,0)$, at $t=-\infty$, with $(L_2, 0)$, at $t=\infty$. This is impossible, because $(0,0)$ cannot admit a homoclinic connection and $(\omega, 0)$ is either an unstable node or an unstable focus. Therefore, there exists $t_2>t_1$ such that $z(t)<0$ for all $t \in (t_1, t_2)$ and $z(t_2)=\dot{w}(t_2)=0$. Arguing as above, it is easy to see that $w(t_2) \in (0, \omega)$ and, hence, Γ must spiral around $(\omega, 0)$. As $(\omega, 0)$ is a repeller, we find from the Poincaré–Bendixson theorem applied to (4.16) that the ω-limit set of Γ is a non-trivial period orbit. But this is impossible, because (4.16) cannot admit a non-trivial periodic solution. Consequently, alternative (B) occurs. Figure 6 sketches the global behaviour of Γ. Consequently, the solution $u(x)$ of (1.7) crosses the singular solution $u_s(x)$ twice before reaching 0 at some x_0, which completes the proof of the first claim of theorem 2.3.

Finally, suppose that

$$n \geqslant 3 \quad \text{and} \quad \beta>\frac{n+2}{n-2}.$$

Then,

$$n-2-\frac{4}{\beta-1}>0$$

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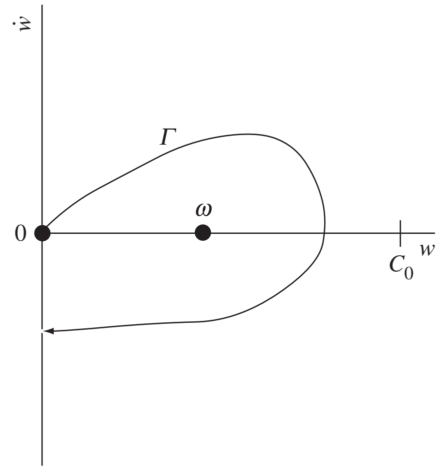

Figure 6. The trajectory Γ of $(w, \dot{w})$.

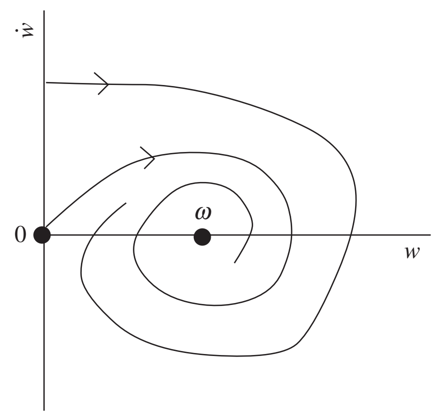

Figure 7. The phase portrait for $\beta > (n + 2)/(n - 2)$.

and, hence, it follows from (4.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{w}^2 - \frac{(n-2)\beta - n}{2(\beta - 1)^2} w^2 + \frac{c}{2(\beta + 1)} w^{\beta+1}\right) < 0.$$

In particular, (4.8) is dissipative, and therefore the trajectory Γ should exhibit one of the following two behaviours. Either it spirals towards $(\omega, 0)$, as illustrated by Figure 7, or it spirals towards a non-trivial periodic orbit surrounding $(\omega, 0)$. According to the Bendixson negative criterion, the second option cannot occur.

Therefore, the solution of (1.7) crosses the singular solution $u_s(x)$ an infinite number of times and remains positive for all $x > 0$. The proof of theorem 2.3 is complete. Figure 8 shows the profile of the corresponding $v(x)$.

Remark 4.2. The fact that $u(x)$ crosses the singular solution $u_s(x)$ an infinite number of times if $n \geqslant 3$ and $\beta > (n + 2)/(n - 2)$ is a really striking phenomenon, for, as u_s provides us with a (singular) supersolution of (1.7) such that $u_s(\infty) = 0$,

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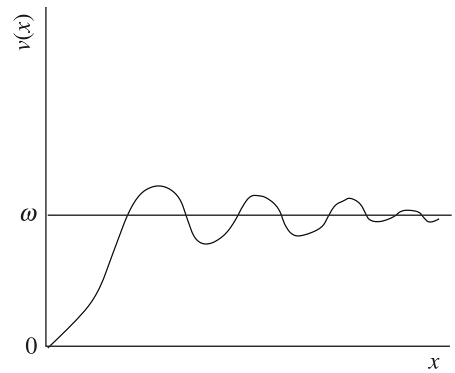

Figure 8. The graph of $v(x)$ for $\beta > (n+2)/(n-2)$.

one might be tempted to infer that, consequently, $u(x) < u_s(x)$ for all $x > 0$, which is false.

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