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# CORNER BEHAVIOR OF SOLUTIONS OF SEMILINEAR DIRICHLET PROBLEMS

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**1. Introduction.** In recent years there has been considerable attention paid to the behavior of solutions of elliptic boundary value problems in domains with piecewise smooth boundary. In two dimensions the study concerns the behavior of a solution near a corner, and in three (or more) dimensions two cases have been given considerable attention: a conical vertex on the boundary, or an edge.

The solution of such a problem may be singular at the nonsmooth boundary points. The standard example in two dimensions is a solution in polar coordinates of the Dirichlet problem near a corner of interior angle $\pi\alpha; u = r^{1/\alpha} \sin \theta/\alpha$ is a function which is harmonic in the sector $0 < \theta < \pi\alpha$, has zero boundary values near the corner, and yet at the origin has unbounded derivatives of order $> 1/\alpha$ unless $1/\alpha$ is an integer.

In this paper we are interested in the asymptotic behavior, near a corner of a plane domain, of solutions of a certain class of semilinear boundary value problems. The solution is assumed to exist and be continuous in the closed domain, and is also assumed to have homogeneous Dirichlet boundary values near the corner.

A thorough survey paper on the subject of elliptic, parabolic and hyperbolic boundary value problems in nonsmooth domains in two or higher dimensions is that of Kondrat'ev and Oleinik [**9**]. The bibliography contains 275 references, and the authors review what seems to be all of the Russian literature on the subject as well as much of the rest.

A very general approach to the asymptotic behavior of solutions of second or higher order elliptic boundary value problems in two or more dimensions is given in [**9**] (see also [**8**]). One considers a linear elliptic boundary value problem in an angular ($n = 2$) or conical ($n \geqq 3$) region. One then changes to polar (spherical) coordinates with distance $r$ to the vertex, set $\tau = \log 1/r$ and finally one takes the Fourier transform with respect to $\tau$. This approach is very general and yields asymptotic expansions for the solution in terms of $r$, $\log r$ and the angular coordinates.

Another approach which is less general but yields stronger results, is that of Lewy [**14**], and later Lehman [**10, 11, 12**]. In this method complex

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variables are used, so that one is limited to two dimensions. The solution is expressed as an integral in terms of an appropriate Green's Function. The integral on the right hand side is then studied using two lemmas on singular integrals which are originally due to Lewy, and asymptotic expansions similar to those of Kondrat'ev are shown to exist.

The advantage of the Lewy-Lehman method is that the expansions are differentiable, and thus can be inserted into the differential equation and boundary conditions. This will yield the coefficients of the expansion in most, but not all, cases. Another advantage is that the worst term (lowest order) of an expansion, typically of the form $O(r^{1/\alpha})$, gives an immediate answer to questions concerning the belonging of the solution to various Sobolev spaces.

The asymptotic behavior is also of use to the numerical analyst in proving the order of convergence of a particular numerical scheme to the actual solution (see, e.g., **[16]** ). And recently the author has come across a method of calculating the coefficients of the asymptotic expansions in certain special cases; the asymptotic estimates are then used to gain improved numerical approximation to the solution. Results are, however, still preliminary.

For another kind of application, to the design of ship propellers, see **[5]**. For surface waves see **[10, 13]**, and for plane gravity flows see **[2, 3]**.

In this paper we shall extend the Lewy-Lehman technique to solutions of the semilinear equation

$$\Delta u = a_{ij}(x, u)u_{x_i}u_{x_j} + a_i(x, u)u_{x_i} + a(x, u).$$

Our results are based on a theorem of Dziuk **[4]** which proves that, given a certain semilinear elliptic inequality with homogeneous Dirichlet data, a continuous solution is in fact Hölder continuous (or better). This yields the initial hypothesis for our inductive step which, in turn, yields the asymptotic expansions.

For more nonlinear cases see **[7, 15]**.

**2. Notation.** Let $D$ be a domain in the $z = x + iy$ plane part of whose boundary consists of two open arcs $\Gamma_1$ and $\Gamma_2$ which meet at the origin and form there an interior angle $\pi\alpha > 0$. We assume that $\Gamma_1$ (resp. $\Gamma_2$) is tangent to the ray $\arg z = 0$ (resp. $= \pi\alpha$). We consider a solution $u \in C^2(D)$, $u \in C^0(D \cup \Gamma_1 \cup \Gamma_2 \cup \{0\})$ of the semilinear elliptic equation

$$(2.1)\quad \Delta u = a_{11}(z, u)u_x^2 + a_{12}(z, u)u_xu_y + a_{22}(z, u)u_y^2 + a_1(z, u)u_x + a_2(z, u)u_y + a(z, u)$$

for $z \in D$, together with the Dirichlet boundary condition

$$u = 0$$

for $z \in \Gamma_1 \cup \Gamma_2$. Our use of $z$ for the ordered pair $(x, y)$ is for convenience and is not meant to imply analyticity of any functions (except the conformal maps in Section 4).

It is assumed that $\bar{\Gamma}_i \in C^{N+\lambda_1}$ where $N$ is a positive integer $\geqq 2$ and $0 < \lambda_1 < 1$. In addition we shall require some regularity assumptions on the coefficients $a_{ij}$, $a_i$ and $a$, which will be given in Section 3 and which are (for $\alpha < 1$) somewhat weaker than belonging to $C^{N-2+\lambda}$ in their arguments. First, however, we shall define some norms, seminorms, and error terms.

Let $D_A$ be the truncated domain $D \cap \{ |z| < A \}$ where $A > 0$. Let $P, Q \in D_A \cup \Gamma_1 \cup \Gamma_2$ and let $d_P$, $d_Q$ denote their distances from the corner, and set

$$d_{P,Q} = \min(d_P, d_Q).$$

Let $\mu \in R$ and let $k$ be a nonnegative integer. For $0 < \lambda < 1$ we set

$$[F]_{k,0}^{(-\mu)} = [F]_k^{(-\mu)} = \sup d_P^{k-\mu} |D^k F(P)|$$

where the supremum is taken over all $P \in D_A \cup \Gamma_1 \cup \Gamma_2$ and all derivatives of order $k$. Next we set

$$[F]_{k,\lambda}^{(-\mu)} = \sup d_{P,Q}^{k+\lambda-\mu} \frac{|D^k F(P) - D^k F(Q)|}{|P - Q|^\lambda}$$

where the supremum is taken over all $P, Q \in D_A \cup \Gamma_1 \cup \Gamma_2$ such that

$$0 < 4|P - Q| < d_{P,Q}.$$

Finally we define

$$|F|_k^{(-\mu)} = \sum_{j=0}^{k} [F]_j^{(-\mu)},$$

$$|F|_{k,\lambda}^{(-\mu)} = |F|_k^{(-\mu)} + [F]_{k,\lambda}^{(-\mu)}.$$

The space of functions $F$ for which $|F|_{k,\lambda}^{(-\mu)}$ is finite shall be denoted

$$\hat{C}_\mu^{k+\lambda} = \hat{C}_\mu^{k+\lambda}(D_A \cup \Gamma_1 \cup \Gamma_2)$$

and we shall sometimes write

$$F(z) = O_{k+\lambda}(z^\mu)$$

as $z \to 0$, $z \in D_A \cup \Gamma_1 \cup \Gamma_2$. This notation is in agreement with the usual meaning of an error in the sense that

$$|D^j F(z)|/|z|^{\mu-j}$$

is bounded as $z \to 0$, $z \in D_A \cup \Gamma_1 \cup \Gamma_2$, for any derivative $D^j F(z)$ of order $j$, $0 \leqq j \leqq k$. In addition we have a bound on the Hölder constant for $D^k F(z)$.

We write

$$z = re^{i\theta},\ z^{1/\alpha} = r^{1/\alpha}e^{i\theta/\alpha} \quad \text{and} \quad \log z = \log r + i\theta$$

where $\theta = \arg z$, $r = |z|$. We shall say a function $F(z)$ defined for $z \in D_A \cup \Gamma_1 \cup \Gamma_2$ has an asymptotic expansion if there exists a polynomial

$$P(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}, \log z, \log \bar{z})$$

such that

$$F(z) = P(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}, \log z, \log \bar{z}) + O_{k+\lambda}(z^{\mu})$$

for some $\mu \geqq 0$, $k \geqq 0$, $0 < \lambda < 1$. This is, of course, equivalent to saying that

$$F - P \in \hat{C}_{\mu}^{k+\lambda}(D_A \cup \Gamma_1 \cup \Gamma_2).$$

It should be noted that if $F$ is real-valued then the polynomial can be rewritten as a polynomial in $x, y, r^{1/\alpha} \sin \theta/\alpha, r^{1/\alpha} \cos \theta/\alpha, \log r$ and $\theta$, and we may assume that each monomial tends to zero slower than $|z|^{\mu}$ as $z \to 0$ (other terms can be absorbed into the error term). It follows from standard results on asymptotic series that $P$ will be real-valued if $F$ is.

Two kinds of polynomials shall be encountered. The first is the kind studied in [**11**, **12**, **13**, **14**, **17**, **18**, **19**]. We say that $F$ has an asymptotic expansion of type I if

(i) for irrational $\alpha$, no logarithmic terms appear in the polynomial $P$, so that for some polynomial $P_1$

$$F(z) = P_1(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}) + O_{k+\lambda}(z^{\mu});$$

(ii) if $\alpha = p/q$, a reduced fraction, then the logarithmic terms are damped in the sense that there exists a polynomial $P_1$ such that

$$F(z) = P_1(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}, z^q \log z, \bar{z}^q \log \bar{z}) + O_{k+\lambda}(z^{\mu}).$$

The other type of expansion, which we shall call an expansion of type II, only occurs when the quadratic terms in (2.1) actually appear, i.e., $a_{ij} \not\equiv 0$. We say $F(z)$ has an expansion of type II when there exists a polynomial $P_1$ such that

$$F(z) = P_1(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}, z^{1/\alpha} \log z, \bar{z}^{1/\alpha} \log \bar{z}) + O_{k+\lambda}(z^{\mu}).$$

We have the following properties, proved in [**19**]:

1. The expansions can be differentiated formally, i.e., if

$$F(z) = P + O_{k+\lambda}(z^{\mu})$$

then

$$D^j F(z) = D^j P + O_{k-j+\lambda}(z^{\mu-j})$$

for any partial derivative $D^j$ with $0 \leqq j \leqq k$.

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2. The expansions are unique modulo $\hat{C}_{\mu}^{k+\lambda}$, i.e., if

$$F = P_i + O_{k+\lambda}(z^{\mu})$$

for $i = 1, 2$, then $P_1 - P_2 \in \hat{C}_{\mu}^{k+\lambda}$. This means the coefficient of monomials of $P_1 - P_2$ which vanish slower than $|z|^{\mu}$ as $z \to 0$ must equal zero.

3. If $F(z) = O_{k+\lambda}(z^{\mu})$ and $G(z) = O_{k+\lambda}(z^{\nu})$ then

$$F(z)G(z) = O_{k+\lambda}(z^{\mu+\nu}).$$

In particular this means that products of expansions are also expansions.

4. Let $D'_A$ be a domain satisfying the same hypotheses as $D_A$ and let $G(z):D_A \to D'_A$. Let

$$G(z) \in \hat{C}_{\mu}^{k+\lambda}(D_A \cup \Gamma_1 \cup \Gamma_2) \quad \text{and}$$
$$F(z) \in \hat{C}_{\nu}^{k+\lambda}(D'_A \cup \Gamma'_1 \cup \Gamma'_2).$$

If $\mu > 0$ and if $\exists \delta > 0$ such that

$$|G(z)| \geqq \delta |z|^{\mu} \quad \text{for } z \in D_A \cup \Gamma_1 \cup \Gamma_2$$

then

$$H(z) = F(G(z)) \in \hat{C}_{\mu\nu}^{k+\lambda^2}(D_A \cup \Gamma_1 \cup \Gamma_2).$$

Thus sums, products and compositions of expansions also have expansions.

We shall frequently make use of the fact that the spaces $\hat{C}_{\mu}^{k+\lambda}$ are decreasing as $\mu$, $k$ or $\lambda$ increases.

**3. Principal results.** We shall now state the hypotheses on the coefficients and give the principal theorems.

First, assume that (2.1) is linear in $u_x$ and $u_y$, i.e., that $a_{ij} \equiv 0$ for $i, j = 1, 2$. Then for some integers $l \geqq 0$ and $N \geqq 2$ and some $\lambda$, $0 < \lambda < \lambda_1^2 < 1$, there shall exist type I polynomials $P_k^i$ $P_k$, $Q_k$ and $R_k$ ($i = 1, 2$, $0 \leqq k \leqq l$) such that for any $v \in \hat{C}_{\gamma}^{N+\lambda}(D_A \cup \Gamma_1 \cup \Gamma_2)$, where $\gamma = \min(1, 1/\alpha)$,

$$a_1(z, v(z)) = \sum_{k=0}^{l} P_k^i v(z)^k + O_{N-1+\lambda}(z^{N-1+\lambda-\gamma})$$

$$\text{(3.1)} \quad a(z, v(z)) = z^{1/\alpha-1} \sum_{k=0}^{l} P_k v(z)^k + \bar{z}^{1/\alpha-1} \sum_{k=0}^{l} Q_k v(z)^k + \sum_{k=0}^{l} R_k v(z)^k + O_{N-1+\lambda}(z^{N-2+\lambda}).$$

For $\alpha < 1$ these requirements are met if $a_{ij}(z, v)$ and $a(z, v)$ belong to

$$C^{N-2+\lambda}(\bar{D}_A \times [-\epsilon, \epsilon]) \quad \text{for some } \epsilon > 0.$$

If $\alpha \geqq 1$ then more regularity of $a_i(z, v)$ and $a(z, v)$ in the variable $v$ may be required. The terms $z^{1/\alpha - 1}$ and $\bar{z}^{1/\alpha - 1}$ in the second equation are included to accommodate a proof of the existence of expansions for solutions of (2.1) with nonhomogeneous boundary conditions.

In the general quadratic case we require the existence of type I polynomials $P_k^{ij}$, $P_k^i$, $Q_k^i$, $R_k^i$ $P_k$, $Q_k$, $R_k$ and $S_k$ ($i, j = 1, 2, 0 \leqq k \leqq l$) such that for $v \in \hat{C}_\gamma^{N+\lambda}$ ($\gamma = \min(1, 1/\alpha)$),

$$a_{ij}(z, v(z)) = \sum_{k=0}^{l} P_k^{ij} v(z)^k + O_{N-1+\lambda}(z^{N-2\gamma+\lambda})$$

$$\begin{aligned} a_i(z, v(z)) &= z^{1/\alpha - 1} \sum_{k=0}^{l} P_k^i v(z)^k + \bar{z}^{1/\alpha - 1} \sum_{k=0}^{l} Q_k^i v(z)^k \\ &\quad + \sum_{k=0}^{l} R_k^i v(z)^k + O_{N-1+\lambda}(z^{N-1+\lambda-\gamma}) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} a(z, v(z)) &= z^{2/\alpha - 2} \sum_{k=0}^{l} P_k v(z)^k + z^{1/\alpha - 1} \bar{z}^{1/\alpha - 1} \sum_{k=0}^{l} Q_k v(z)^k \\ &\quad + \bar{z}^{2/\alpha - 2} \sum_{k=0}^{l} R_k v(z)^k + \sum_{k=0}^{l} S_k v(z)^k + O_{N-1+\lambda}(z^{N-2+\lambda}). \end{aligned}$$

Again, these hypotheses will be satisfied if

$$a, a_i, a_{ij} \in \hat{C}^{N-2+\lambda}(D_A \times [-\epsilon, \epsilon]) \text{ for some } \epsilon > 0.$$

We now state the main results. Throughout the paper the Hölder coefficient $\lambda$ will have (finitely many) restrictions put on it; to avoid the use of subscripts we shall use the same symbol $\lambda$ to denote any one of the various values.

THEOREM. *A solution $u \in C^2(\bar{D}_A) \cap C(D_A \cup \Gamma_1 \cup \Gamma_2)$ of ) in $D_A$ satisfying homogeneous boundary values $u = 0$ on $\Gamma_1$ and $\Gamma_2$ has an asymptotic expansion*

$$u(z) = P(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}, \log z, \log \bar{z}) + O_{N+\lambda}(z^{N+\lambda}),$$

*where $P$ is a polynomial of type* II. *If in addition* (2.1) *is linear in $u_x$ and $u_y$ then $P$ is of type* I.

It should be noted that in a specific example where the coefficients are specially given there may be much more information on the polynomial $P$; see Section 9 for some examples.

**4. Some lemmas.** We now state some lemmas which will be needed later.

LEMMA 4.1. *Let $\zeta = G(z)$ be a conformal map of $D$ onto the sector $0 < |\zeta| < \infty$, $0 < \arg \zeta < \pi\alpha$ such that $G(0) = 0$, $\arg G(z) = 0$ if $z \in \Gamma_1$, and $\arg G(z) = \pi\alpha$ if $z \in \Gamma_2$. Then $G(z)$ has an asymptotic expansion*

$$G(z) = zN + O_{N+\lambda}(z^{N+\lambda})$$

*where $N$ is a polynomial in $z$ and $z^{1/\alpha}$ if $\alpha$ is irrational and in $z$, $z^{1/\alpha}$ and $z^q \log z$ if $\alpha = p/q$, a reduced fraction. In either case the constant term of $N$ is nonzero. A similar expansion (only the coefficients are different) holds for $z = G^{-1}(\zeta)$ which is defined on the sector.*

*Proof.* This lemma is the main theorem of [**19**], although the conclusion was stated there in a weaker form. However, the hypothesis $\overline{\Gamma}_i \in C^{N+\lambda'}$ implies the existence of an error term of the form $O_{N+\lambda}(z^{N+\lambda})$ for $\lambda < (\lambda')^2$ and the proof follows from the arguments used in [**19**].

LEMMA 4.2. *Let $F(z)$ have an asymptotic expansion*

$$F(z) = N + O_{k+\lambda}(z^{\mu})$$

*where $N$ is a polynomial of type* I *or type* II *with nonvanishing constant term. If $\gamma$ is a real number then $F(z)^{\gamma}$ and $\log F(z)$ also have asymptotic expansions of the form $P + O_{k+\lambda}(z^{\mu})$ where $P$ is of the same type (i.e., type* I *or* II*) as $N$.*

*Proof.* This follows directly from the Taylor series for $h(\zeta) = (1 + \zeta)^{\gamma}$ and $h(\zeta) = \log(1 + \zeta)$.

LEMMA 4.3. *Let $F(z)$ have an asymptotic expansion*

$$F(z) = P + O_{k+\lambda}(z^{\mu})$$

*of type* I *or type* II. *Then $f(\zeta) = F(G^{-1}(\zeta))$ also has an expansion*

$$f(\zeta) = P_1 + O_{k+\lambda}(\zeta^{\mu})$$

*where $P_1$ is a polynomial of the same type as $P$. The converse is also true, i.e., if $f$ has an expansion so does $F(z) = f(G(z))$.*

**5. Some preliminary growth estimates at the corner.** In [4] Dziuk has shown that a function $u \in C^2(D_A) \cap C(\overline{D}_A)$ satisfying

$$|\Delta u| \leqq a|\nabla u|^2 + b \quad \text{in } D_A,$$
$$u = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2$$

where $a$ and $b$ are constants, satisfies

$$u \in C^{1,\mu}(\bar{D}_A) \text{ for any } \mu \in (0, 1), \text{ if } 0 < \alpha \leqq 1/2,$$
$$(5.1) \quad u \in C^{1,1/\alpha-1}(\bar{D}_A) \text{ if } 1/2 < \alpha < 1, \text{ and}$$
$$u \in C^{0,1/\alpha}(\bar{D}_A) \text{ if } 1 \leqq \alpha \leqq 2.$$

In addition, if $1/2 < \alpha \leqq 2$, then

$$(5.2) \quad |\nabla u(z)| = O(|z|^{1/\alpha-1})$$

as $z \to 0$, $z \in D_A \cup \Gamma_1 \cup \Gamma_2$.

We wish to conclude that, for any $\lambda$ with $0 < \lambda < 1$,

$$(5.3) \quad \begin{aligned} u(z) &= O_{1+\lambda}(z) \text{ if } 0 < \alpha < 1, \text{ and} \\ u(z) &= O_{1+\lambda}(z^{1/\alpha}) \text{ if } 1 \leqq \alpha \leqq 2. \end{aligned}$$

For $1 < \alpha \leqq 2$ the coefficients are unbounded. However, the transformation $\zeta = z^{1/\alpha}$ maps $D_A$ onto a domain with angle $\alpha = 1$, and transforms (2.1) to an equation with bounded coefficients and we get, with $U(\zeta) = u(z)$,

$$U(\zeta) \in C^{0,1} \quad \text{and} \quad |\nabla U| = 0(1).$$

It follows directly that for $0 < \lambda < 1$

$$(5.4) \quad u(z) = O_\lambda(z^{1/\alpha})$$

for $1 < \alpha \leqq 2$.

The estimate (5.3) follows directly from (5.1) if $0 < \alpha \leqq 1/2$. For $1/2 < \alpha \leqq 2$ we use (5.1) and (5.4) coupled with Lemma 5 of [**4**] and its corollary together with Dziuk's argument (although Dziuk used his Lemma 5 only for $0 < \alpha < 1$).

At this point we could improve the estimates (5.3) to

$$u \in \hat{C}_\gamma^{N+\lambda}(D_A \cup \Gamma_1 \cup \Gamma_2)$$

(where $\gamma = \min(1, 1/\alpha)$) by straightening out the boundary using its parameterizations and then applying the boundary estimates of [**1**]. We shall, however, use a conformal map for the straightening. This makes the calculations somewhat simpler and does not cause any loss of generalization as the methods of Section 8 do not generalize to higher dimensions.

**6. Transformation to a sector.** Let $\zeta = G(z)$ be the conformal map discussed in Section 4. We define

$$U(\zeta) = U(G(z)) = u(z)$$

and observe that $U(\zeta) = 0$ for $\arg \zeta = 0, \pi\alpha$ and $0 < |\zeta| < B$ where $B$ is a small positive constant. Moreover, on $0 < \arg \zeta < \pi\alpha$, $0 < |\zeta| < B$,

$$\Delta U(\zeta) = A_{11}U_\xi^2 + A_{12}U_\xi U_\eta + A_{22}U_\eta^2 + A_1 U_\xi + A_2 U\eta + A,$$

and we shall show that the coefficients satisfy the same hypotheses as those of (2.1).

First we have, with $\zeta = \xi + i\eta = M(z) + iN(z) = G(z)$,

$$A_{11}(\zeta, U(\zeta)) = |G'(z)|^{-2}\{a_{11}(z, U)M_x^2 + a_{12}(z, U)M_xM_y + a_{22}(z, U)M_y^2\}.$$

By property 4 of Section 2, and the results of Section 5,

$$U(\zeta) = O_{1+\lambda}(\zeta^\gamma), \quad \gamma = \min(1, 1/\alpha).$$

By the lemmas of Section 4 it follows that $A_{11}(\zeta, U(\zeta))$ satisfies, as a function of $\zeta$, the same hypotheses as were imposed in Section 3 on $a_{11}(z, u(z))$. Similarly $A_{12}$ and $A_{22}$ satisfy the same hypotheses as do $a_{12}$ and $a_{22}$.

Next, we have

$$A_1(\zeta, U(\zeta)) = |G'(z)|^{-2}\{a_1(z, U)M_x + a_2(z, U)M_y\}.$$

Since

$$z = G^{-1}(\zeta) = \zeta N + O_{N+\lambda}(z^{N+\lambda})$$

where $N$ is a type I polynomial with nonvanishing constant term, the lemmas of Section 4 show that $A_1(\zeta, U(\zeta))$ satisfies the same hypotheses as does $a_1(z, u(z))$. A similar argument holds for $A_2(\zeta, U(\zeta))$.

Finally,

$$A(\zeta, U(\zeta)) = |G'(z)|^{-2}a(z, u(z))$$

and thus $A(\zeta, U(\zeta))$ satisfies the same hypotheses as does $a(z, u(z))$.

Thus the problem for $U(\zeta)$ in the sector $0 < \arg \zeta < \pi\alpha$, $0 < |\zeta| < B$, is the same as the original problem for $u(z)$ in $D_A$. Moreover, by Lemma 4.3, if the theorem is true for $U(\zeta)$ then it will also be true for $u(z)$. Thus for the remainder of this paper we shall assume that $\Gamma_1$ and $\Gamma_2$ are segments of straight lines.

**7. More estimates at the corner.** With $\gamma = \min(1, 1/\alpha)$, we have, from Section 5,

$$u(z) = O_{1+\lambda}(z^\gamma).$$

We wish to improve this to get

$$u(z) = O_{N+\lambda}(z^\gamma).$$

For this purpose we shall use a boundary estimate of Agmon, Douglis and Nirenberg [**1**]. Let $\Sigma_R$ be the semidisc

$$(x - x_0)^2 + y^2 \leqq R^2, \quad y \geqq 0.$$

Let $f$ be defined on $\Sigma_R$. Let $l$ be an integer $\geqq 2$ and $p$ an integer $\geqq -l$. We define

$$[f]_{p,l} = \sup d_P^{p+l}|D^l f(P)|$$

where $d_P$ denotes the distance from $P$ to the spherical part of the boundary of $\Sigma_R$, namely $(x - x_0)^2 + y^2 = R^2$, $y > 0$. The supremum is taken over all $P \in \Sigma_R$ and all $l$-th order derivatives. We define

$$|f|_{p,l} = \sum_{j=0}^{l} [f]_{p,j},$$

and

$$[f]_{p,l+\lambda} = \sup d_P^{p+l+\lambda} \frac{|D^l f(P) - D^l f(Q)|}{|P - Q|^\lambda}$$

where the supremum is taken over $P, Q \in \Sigma_R$, $0 < 4|P - Q| < d_P, d_Q$ and all $l$-th order derivatives. Finally, we set

$$|f|_{p,l+\lambda} = |f|_{p,l} + [f]_{p,l+\lambda}.$$

LEMMA 7.1. *Let $u \in C^{2+\lambda}(\Sigma_R)$ and $F \in C^{l-2+\lambda}(\Sigma_R)$, and suppose*

$$\Delta u(z) = F(z),\ z \in \Sigma_R,$$

*and*

$$u(x, 0) = 0,\ x_0 - R < x < x_0 + R.$$

*Then $u \in C^{l+\lambda}(\Sigma_R)$ and there exists a constant $C$ depending only on $l$ and $\lambda$ such that*

$$|u|_{0,l+\lambda} \leqq C(|F|_{2,l-2+\lambda} + \sup_{P \in \Sigma_R} |u(P)|).$$

We shall apply this estimate to the semidiscs $\Sigma_R:(x - x_0)^2 + y^2 \leqq R^2$, $y \geqq 0$, where $R = x_0 \tan \delta$, $0 < x_0 < A - R$, and $\delta$ is a small fixed positive number; these semidiscs cover the sector $0 < \arg z < \delta$ for small $|z|$. We set

$$F(z) = a_{11}u_x^2 + a_{12}u_x u_y + a_{22}u_y^2 + a_1 u_x + a_2 u_y + a. \tag{7.1}$$

We apply the inequality of Lemma 7.1 by considering the suprema of the left hand side of the inequality only for $P, Q \in \Sigma_{R/2}$. For such $P$ and $Q$ we have

$$R/2 \leqq d_P, d_Q \leqq R = x_0 \tan \delta.$$

Thus the quantities $x_0$, $R$, and $d_P$, $d_Q$ are all proportional to one another.

For $l = 2$ we have, from (7.1),

$$F(z) = O_\lambda(z^{2\gamma-2}).$$

Thus, from Lemma 7.1 we have

$$u(z) = O_{2+\lambda}(z^{\gamma})$$

for $0 \leqq \arg z < \delta_1$ where $R/2 = x_0 \tan \delta_1$. It follows from (7.1) that

$$F(z) = O_{1+\lambda}(z^{2\gamma-2})$$

for $0 \leqq \arg z < \delta_1$, and again by the lemma with $R/2$ replacing $R$, that (provided $N \geqq 3$)

$$u(z) = 0_{3+\lambda}(z^{\gamma})$$

for $0 \leqq \arg z < \delta_2$ where $R/4 = x_0 \tan \delta_2$. Continuing, we obtain finally

$$u(z) = O_{N+\lambda}(z^{\gamma})$$

for $0 \leqq \arg z < \delta_{N-2}$.

A similar argument yields

$$u(z) = O_{N+\lambda}(z^{\gamma})$$

for $\pi\alpha - \delta_{N-2} < \arg z \leqq \pi\alpha$. To obtain the estimate on the interior

$$\frac{1}{2}\delta_{N-2} \leqq \arg z \leqq \pi\alpha - \frac{1}{2}\delta_{N-2}$$

we apply the following interior estimate (e.g. Theorem 4.6 of [6]):

LEMMA 7.2. *Let $\Omega$ be a domain in $R^2$ and let $u \in C^2(\Omega)$, $F \in C^{\lambda}(\Omega)$ satisfy $\Delta u = F$ in $\Omega$. Then $u \in C^{2+\lambda}(\Omega)$ and for any two concentric discs*

$$B_1 = \{ |z - z_0| < R \} \text{ and } B_2 = \{ |z - z_0| < 2R \}$$

*which lie at a positive distance from $\partial\Omega$ we have*

$$\sum_{j=0}^{2} R^j \sup |D^2u(P)| + R^{2+\lambda} \sup \frac{|D^2u(P) - D^2u(Q)|}{|P - Q|^{\lambda}}$$

$$\leqq C\left(\sup |U(P)| + \sup |F(P)| + 4R^{\lambda} \sup \frac{|F(P) - F(Q)|}{|P - Q|^{\lambda}}\right)$$

*where $C$ depends only on $\lambda$, the suprema on the left are taken over all second derivatives of $u$ and $P$, $Q \in B_1$, $P \neq Q$, and the suprema on the right are taken over all $P$, $Q \in B_2$, $P \neq Q$.*

We apply this lemma to $u, Du, \ldots, D^{l-2}u$, and $f, Df, \ldots, D^{l-2}f$ and use methods similar to those used above on the semidiscs, and finally conclude that

$$u(z) = O_{N+\lambda}(z^{\gamma})$$

for $0 \leqq \arg z \leqq \pi\alpha$.

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**8. An integral representation; obtaining the expansions.** We apply Green's theorem to $u(z)$ and the Green's function (see **[18]**)

$$G(z, \zeta) = -\frac{1}{2\pi}\{\log |z^{1/\alpha} - \zeta^{1/\alpha}| - \log |z^{1/\alpha} - \bar{\zeta}^{1/\alpha}|\}$$

in the domain $\delta < \arg \zeta < \pi\alpha$, $0 < |\delta| < A$ to obtain

$$u(z) = \int_0^{\pi\alpha}\int_\delta^A G(z, \zeta)\Delta u(\zeta)\rho d\rho d\rho + \int_{|\zeta|=A}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)ds$$
$$- \int_{|\zeta|=\delta}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)ds$$

where $\zeta = \rho e^{i\phi}$. The integral over $|\zeta| = A$ is a difference $p(z^{1/\alpha}) - p(\bar{z}^{1/\alpha})$ of power series where $p(z^{1/\alpha})$ converges for $|z| < A$. The integral over $|\zeta| = \delta$ tends to zero as $\delta \to 0$, since for fixed $z$ and $|\zeta| = \delta$.

$$|G(z, \zeta)| = O(\delta^{1/\alpha}), \quad \left|\frac{\partial}{\partial n}(G(z, \zeta)\right| = O(\delta^{1/\alpha - 1})$$

and $u(z) = O_{1+\lambda}(z^\gamma)$, $\gamma = \min(1, 1/\alpha)$. Thus

$$(8.1) \quad u(z) = \int_0^{\pi\alpha}\int_0^A G(z, \zeta)F(\zeta)\rho d\rho d\phi + P(z^{1/\alpha}) - p(\bar{z}^{1/\alpha})$$

where $F(\zeta)$ is the right hand side of (2.1) with $z$ replaced by $\zeta$.

To obtain the expansions we introduce the following notation. Let $h(z)$ be continuous on $D \cup \Gamma_1 \cup \Gamma_2$ and satisfy the estimate

$$h(z) = O(z^\mu)$$

as $z \to 0$, $z \in D \cup \Gamma_1 \cup \Gamma_2$, where $\mu > -2$. We define

$$\Lambda(h, z) = \int_0^{\pi\alpha}\int_0^A G(z, \zeta)h(\zeta)\rho d\rho d\phi.$$

The following lemmas are originally due to Lewy **[14]**.

LEMMA 8.1. *Let $\beta$ and $\gamma$ be real numbers $> -1$ and let $m$ and $n$ be nonnegative integers. Let*

$$h(z) = z^\beta \bar{z}^\gamma (\log z)^m (\log \bar{z})^n.$$

*Then*

$$\Lambda(h, z) = z^{\beta+\gamma+2}P_1(\log z) + \bar{z}^{\beta+\gamma+2}P_2(\log \bar{z})$$
$$+ z^{\beta+1}\bar{z}^{\gamma+1} P_3(\log z, \log \bar{z}) + p_1(z^{1/\alpha}) + p_2(\bar{z}^{1/\alpha})$$

*for $0 \leqq \arg z \leqq \pi\alpha$, where $P_1$ and $P_2$ are polynomials of degree $\leqq m + n$ if $(\beta + \gamma + 2)\alpha$ is not an integer and of degree $\leqq m + n + 1$ if $(\beta + \gamma + 2)\alpha$ is an integer; $P_3$ is a polynomial of degree $\leqq m$ in $\log z$ and of degree $\leqq n$ in $\log \bar{z}$; and $p_1$ and $p_2$ are power series in their arguments which converge for $|z| < A$.*

*Proof*. This is essentially Lemma 8.1 of [**18**].

LEMMA 8.2. *Let $\mu$ be a real number $> -2$ which is not an integer and let $l$ be a nonnegative integer. Suppose*

$$h(z) = O_{l+\lambda}(z^{\mu}).$$

*Then*

$$\Lambda(h, z) = q_1(z^{1/\alpha}) + q_2(\bar{z}^{1/\alpha}) + O_{l+1+\lambda}(z^{\mu+2})$$

*for $0 \leqq \arg z \leqq \pi\alpha$, where $q_1$ and $q_2$ are polynomials in their arguments.*

*Proof*. See Lemma 8.2 of [**18**], which is essentially Lewy's original lemma, and Lemma 5 of [**19**], which allows for the extra derivative in the error term of $\Lambda(h, z)$.

*Proof of the main theorem.* We set

$$(8.1)\quad F(z) = a_{11}(z, u)u_x^2 + a_{12}(z, u)u_x u_y + a_{22}(z, u)u_y^2 + a_1(z, u)u_x + a_2(z, u)u_y + a(z, u).$$

We insert the estimates

$$u(z) = O_{N+\lambda}(z^{\gamma}), \quad \nabla u = O_{N-1+\lambda}(z^{\gamma-1})$$

into the right hand side of (8.1). It follows that

$$(8.2)\quad F(z) = O_{N-1+\lambda}(z^{2\gamma-2}).$$

From Lemma 8.2 we get (assuming $\gamma \neq 1, 1/2$)

$$u(z) = p_1(z^{1/\alpha}) + p_2(\bar{z}^{1/\alpha}) + O_{N+\lambda}(z^{2\gamma})$$

where $p_1$ and $p_2$ are polynomials. If $2\gamma$ is an integer then the exponent $2\gamma - 2$ of (8.2) can be replaced by $2\gamma - 2 - \epsilon$ and (8.3) then has exponent $2\gamma - \epsilon$.

We break the rest of the proof into three cases.

*Case* I. Let $a_{ij} \equiv 0$ for $i, j = 1, 2$ and suppose $\alpha$ is irrational. Then the expansions are supposed to contain only terms in $z$, $\bar{z}$, $z^{1/\alpha}$ and $\bar{z}^{1/\alpha}$. For $k \geqq 2$ assume that we have, for some polynomial $P$,

$$u(z) = P(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}) + O_{N+\lambda}(z^{k\gamma-\epsilon}).$$

Then

$$u_x = z^{1/\alpha-1}P_1 + \bar{z}^{1/\alpha-1}P_2 + P_3 + O_{N-1+\lambda}(z^{k\gamma-1-\epsilon})$$

where $P_1$ and $P_2$ are polynomials in $z$, $\bar{z}$, $z^{1/\alpha}$ and $\bar{z}^{1/\alpha}$. A similar expansion holds for $u_y$. Inserting these estimates into (8.1) and using (3.1) we see that

$$\Delta u = F(z) = z^{1/\alpha - 1}P_4 + z^{1/\alpha - 1}P_5 + P_6$$
$$+ O_{N-1+\lambda}(z^{k\gamma + 1/\alpha - 2 - \epsilon}) + O_{N-1+\lambda}(z^{N-2+\lambda-\gamma+1/\alpha-\epsilon})$$
$$+ O_{N-1+\lambda}(z^{k\gamma-1}).$$

Recalling that $\gamma = \min(1, 1/\alpha)$ and that, in any error term, the exponent can be replaced by any smaller exponent, we have

$$\Delta u = z^{1/\alpha - 1}P_4 + \bar{z}^{1/\alpha - 1}P_5 + P_6 + O_{N-1+\lambda}(z^{(k+1)\gamma - 2 - \epsilon})$$
$$+ O_{N-1+\lambda}(z^{N-2+\lambda}).$$

The monomials on the right are of the form

$$cz^{j-1+l/\alpha}\bar{z}^{m+n/\alpha}, \quad cz^{j+l/\alpha}\bar{z}^{m-1+n/\alpha}$$

where $j, l, m, n \geqq 0$. Using (8.1) and Lemma 8.1 we see that we must check to see if $(j + 1 + l/\alpha + m + n/\alpha)\alpha$ is an integer. Since $\alpha$ is irrational this is impossible, and thus no logarithmic terms are introduced. It follows that

$$(8.3) \quad u(z) = P_7(z, z, z^{1/\alpha}, \bar{z}^{1/\alpha}) + O_{N+\lambda}(z^{(k+1)\gamma - \epsilon}) + O_{N+\lambda}(z^{N+\lambda})$$

(if $N - 2 + \lambda$ is an integer we cannot apply Lemma 8.2 directly; but we can replace $\lambda$ with $\lambda - \epsilon$ and (8.3) is valid for this smaller value of $\lambda$). If

$$(k + 1)\gamma - \epsilon > N + \lambda$$

then the theorem is proved; otherwise we repeat the inductive step.

*Case* II. Let $a_{ij} \equiv 0$ and $\alpha = p/q$, a reduced fraction. Assume that for some $k$ and $\epsilon > 0$

$$u(z) = P(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}, z^q \log z, \bar{z}^q \log \bar{z}) + O_{N+\lambda}(z^{k\gamma - \epsilon}).$$

Then

$$u_x = z^{-1}P_1 + \bar{z}^{-1}P_2 + P_3 + O_{N-1+\lambda}(z^{k\gamma - 1 - \epsilon})$$

where $P_1, \ldots, P_4$ have the same arguments as $P$. A similar expansion holds for $u_y$. Inserting these expansions into (8.1) and using (3.1) we get

$$\Delta u(z) = z^{-1}P_4 + \bar{z}^{-1}P_5 + P_6 + O_{N-1+\lambda}(z^{k\gamma-1-\epsilon})$$
$$+ O_{N-1+\lambda}(z^{N-2+\lambda+1/\alpha-\gamma})$$

or

$$\Delta u(z) = z^{-1}P_3 + \bar{z}^{-1}P_4 + O_{N-1+\lambda}(z^{k\gamma-1-\epsilon})$$
$$+ O_{N-1+\lambda}(z^{N-2+\lambda+1/\alpha-\gamma}).$$

The monomials on the right are of the form

$$cz^{j-1+l/\alpha}(\log z)^m \bar{z}^{n+r/\alpha}(\log \bar{z})^s$$

(or similar terms with $z$ and $\bar{z}$ interchanged) where $l$, $m$, $r$, $s \geqq 0$ and $j \geqq mq$, $n \geqq sq$. If $(j + 1 + n)\alpha$ is an integer then we must show that

$$j + n + 1 \geqq q(m + s + 1).$$

We have $j + n \geqq (m + s)q$ and we know that $q$ divides $j + n + 1$ since $(p, q) = 1$. Thus $j + n + 1 \geqq q(m + s + 1)$. Thus any new, higher powers of the logarithms get multiplied by an appropriate power of $z^q$ (or $\bar{z}^q$). Finally, if either of the exponents $k\gamma - 1 - \epsilon$ or $N - 2 + \lambda + 1/\alpha - \gamma$ of the error terms is an integer, we can increase $\epsilon$ or decrease $\lambda$ a small amount. If

$$k\gamma - 1 - \epsilon \geqq N - 2 + \lambda,$$

the theorem is proved; otherwise we get the inductive hypothesis with $k$ increased to $k + 1$.

*Case* III. In this case we make no assumptions about the $a_{ij}$ (other than (3.2) ) or the arithmetic nature of $\alpha$. Assume for some $k \geqq 2$ and $\epsilon > 0$ that

$$u(z) = P(z, \bar{z}, z^{1/\alpha}, \bar{z}^{1/\alpha}, z^{1/\alpha} \log z, \bar{z}^{1/\alpha} \log \bar{z}) + O_{N+\lambda}(z^{k\gamma-\epsilon}).$$

Then

$$u_x(z) = z^{1/\alpha-1}P_1 + \bar{z}^{1/\alpha-1}P_2 + P_3 + O_{N-1+\lambda}(z^{k\gamma-1-\epsilon})$$

and similarly for $u_y$. Inserting these estimates into (8.1) and using (3.2) we obtain

$$\text{(8.4)}\quad \begin{aligned}\Delta u(z) &= z^{2/\alpha-2}P_4 + z^{1/\alpha-1}\bar{z}^{1/\alpha-1}P_5 + \bar{z}^{2/\alpha-2}P_6 + z^{1/\alpha-1}P_7\\ &\quad + \bar{z}^{1/\alpha-1}P_8 + O_{N-1+\lambda}(z^{k\gamma+1/\alpha-2-\epsilon})\\ &\quad + O_{N-1+\lambda}(z^{N-2+2\lambda}).\end{aligned}$$

Monomials on the right are of the form (up to conjugates)

$$cz^{k+l/\alpha}(\log z)^m \bar{z}^{n+r/\alpha}(\log \bar{z})^s.$$

From Lemma 8.2, new logarithmic terms can occur if $k = -2$ and $n \geqq 0$; $k = n = -1$; or $k \geqq 0$, $n = -2$. In these cases, however, we see from (8.4) that $l \geqq m + 1$ or $r \geqq s + 1$, so the new powers will be multiplied by the appropriate powers of $z^{1/\alpha}$ or $\bar{z}^{1/\alpha}$. The induction step on the error terms follows exactly as before.

## 9. Some examples.

*Example* 1. Let $\alpha = 1/2$ and consider $\Delta u = 1/4e^{\beta u}$ on $0 < \theta < \pi/2$, $u = 0$ on $\theta = 0, \pi/2$, where $\beta$ is a real constant. We write the differential equation as

$$u_{z\bar{z}} = e^{\beta u} = 1 + \beta u + \frac{(\beta u)^2}{2!} + \ldots$$

From $u(z) = o(1)$ we get

$$u_{z\bar{z}} = 1 + o(1)$$

and from Lemmas 8.1 and 8.2

$$(9.1) \quad u(z) = a_1 z^2 + a_2 z\bar{z} + a_3 \bar{z}^2 + b_1 z^2 \log z + b_2 \bar{z}^2 \log \bar{z} + o(z^2).$$

The differential equation yields

$$a_2 = e^{\beta u} + o(1), a_2 = 1 + o(1),$$

so $a_2 = 1$. The boundary condition for $\theta = 0$ implies

$$0 = (a_1 + a_2 + a_3)r^2 + (b_1 + b_2)r^2 \log r + o(r^2),$$

so $a_1 + a_3 = -1$, $b_2 = -b_1$. On $\theta = \pi/2$ we obtain

$$0 = \left(-a_1 + a_2 - a_3 + \frac{\pi i}{2}(-b_1 + b_2)\right)r^2$$
$$- (b_1 + b_2)r^2 \log r + o(r^2)$$

and thus $b_1 = -2i/\pi$, $b_2 = 2i/\pi$.

Next, we have, from (9.1)

$$u_{z\bar{z}} = e^{\beta u} = 1 + \beta(a_1 z^2 + a_2 z\bar{z} + \ldots) + o(z^2)$$

and from Lemmas 8.1 and 8.2 it follows that

$$\begin{aligned} u(z) = {} & a_1 z^2 + a_2 z\bar{z} + a_3 \bar{z}^2 + b_1 z^2 \log z + b_2 \bar{z}^2 \log \bar{z} + c_1 z^4 \\ & + c_2 z^3 \bar{z} + c_3 z^2 \bar{z}^2 + c_4 z \bar{z}^3 + c_5 \bar{z}^4 + d_1 z^4 \log z \\ & + d_2 z^3 \bar{z} \log z \\ & + d_3 z^2 \bar{z}^2 \log z + d_4 z^2 \bar{z}^2 \log \bar{z} + d_5 z \bar{z}^3 \log \bar{z} \\ & + d_6 \bar{z}^4 \log \bar{z} + k_1 z^4 \log^2 z + k_2 \bar{z}^4 \log^2 \bar{z} + o(z^4). \end{aligned}$$

From the differential equation and (9.1) we obtain

$$\begin{aligned} & a_2 + 3c_2 z^2 + 4c_3 z\bar{z}^2 + 3c_4 \bar{z}^2 + d_2 z^2 (3 \log z + 1) \\ & + 2d_3 z\bar{z}(2 \log z + 1) + 2d_4 z\bar{z}(2 \log \bar{z} + 1) + d_5 \bar{z}^2 (3 \log \bar{z} \\ & + 1) \\ & = 1 + \beta a_1 z^2 + \beta a_2 z\bar{z} + \beta a_3 \bar{z}^2 + \beta b_1 z^2 \log z \\ & + \beta b_2 \bar{z}^2 \log \bar{z} + o(z^2) \end{aligned}$$

from which we get

$$3c_2 + d_2 = \beta a_1, 4c_3 + 2d_3 + 2d_4 = \beta a_2, 3c_4 + d_5 = \beta a_3,$$
$$3d_2 = \beta b_1, 4d_3 = 4d_4 = 0, \text{ and } 3d_5 = \beta b_2.$$

Thus

$$d_3 = d_4 = 0, c_3 = \beta a_2/4 = \beta/4,$$
$$d_2 = \beta b_1/3 = -2\beta i/3\pi, d_5 = 2\beta i/3\pi,$$

and hence

$$c_2 = \frac{1}{3}(\beta a_1 - d_2) = \frac{1}{3}\beta a_1 + 2\beta i/9\pi, c_4 = \frac{1}{3}\beta a_3 - 2\beta i/9\pi,$$

together with

$$c_2 + c_4 = \frac{1}{3}\beta(a_1 + a_3) = -\frac{1}{3}\beta.$$

From $u = 0$ on $\theta = 0$ we obtain

$$c_1 + c_2 + c_3 + c_4 + c_5 = 0,$$
$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0,$$
$$k_1 + k_2 = 0$$

and thus

$$k_2 = -k_1, \quad d_1 + d_6 = -(d_2 + d_5) = 0$$

so $d_6 = -d_1$, and

$$c_1 + c_5 = -c_3 - (c_2 + c_4) = -\beta/4 + \beta/3 = \beta/12.$$

From $\theta = \pi$ we get

$$k_1 + k_2 = 0$$
$$d_1 - d_2 + d_3 + d_4 - d_5 + d_6 + i\pi(k_1 - k_2) = 0$$
$$c_1 - c_2 + c_3 - c_4 + c_5 + \frac{i\pi}{2}(d_1 - d_2 + d_3 - d_4 + d_5 - d_6)$$
$$- \frac{\pi^2}{4}(k_1 + k_2) = 0.$$

Thus

$$k_1 - k_2 = \frac{1}{\pi i}( (d_2 + d_5) - (d_1 + d_6) ) = 0,$$

so $k_1 = k_2 = 0$. Moreover

$$c_2 + c_4 = c_1 + c_3 + c_5 + \frac{i\pi}{2}(2d_1 - 2d_2)$$

$$= \beta/12 + \beta/4 + i\pi(d_1 - d_2).$$

Thus

$$d_1 - d_2 = \frac{1}{\pi i}(c_2 + c_4 - \beta/3) = \frac{2\beta i}{3\pi}.$$

Thus

$$-d_6 = d_1 = d_2 + \frac{2\beta i}{3\pi} = 0.$$

Hence

$$u(z) = a_1 z^2 + z\bar{z} + a_3\bar{z}^2 - \frac{2i}{\pi}(z^2 \log z - \bar{z}^2 \log \bar{z})$$

$$+ c_1 z^4 + c_5\bar{z}^4 + \left(\frac{1}{3}\beta a_1 + \frac{2\beta i}{9\pi}\right)z^3\bar{z} + \left(\frac{1}{3}\beta a_3 - \frac{2\beta i}{9\pi}\right)z\bar{z}^3$$

$$+ (\beta/4)z^2\bar{z}^2 - \frac{2\beta i}{3\pi}(z^3\bar{z} \log z - z\bar{z}^3 \log \bar{z}) + o(z^4)$$

where $a_1 + a_3 = -1$, $c_1 + c_5 = \beta/12$. If in addition we assume that $u(z)$ is real, then $a_1$ and $a_3$ are conjugates as are $c_1$ and $c_5$. Writing

$$a_1 = -\frac{1}{2} + ai, \quad a_3 = -\frac{1}{2} - ai,$$

$$c_1 = \beta/24 + ci \quad \text{and} \quad c_3 = \beta/24 - ci$$

we see that the expansion with error term $o(z^4)$ has two unknown real parameters $a$ and $c$:

$$u(z) = -\frac{1}{2}(z^2 + \bar{z}^2) + ai(z^2 - \bar{z}^2) + z\bar{z}$$

$$- \frac{2i}{\pi}(z^2 \log z - \bar{z}^2 \log \bar{z}) + (\beta/24)(z^4 + \bar{z}^4) + ci(z^4 - \bar{z}^4)$$

$$- \frac{\beta}{6}(z^3\bar{z} + z\bar{z}^3) - \left(\frac{a\beta}{3} + \frac{2\beta}{9\pi}\right)i(z^3\bar{z} - z\bar{z}^3) + (\beta/4)z^2\bar{z}^2$$

$$- \frac{2\beta i}{3\pi}(z^3\bar{z} \log z - z\bar{z}^3 \log \bar{z}) + o(z^4).$$

Any information on $a$ and $c$ would have to come from global considerations of the original boundary value problem.

*Example* 2. Let $\alpha$ be arbitrary and let

$$\Delta u = |\nabla u|^2 = u_x^2 + u_y^2$$

on $0 < \theta < \pi\alpha$, $u = 0$ for $\theta = 0, \pi\alpha$. We rewrite the differential equation as

$$u_{z\bar{z}} = u_z u_{\bar{z}}.$$

We start with the expansion

$$u(z) = a_1 z^{1/\alpha} + a_2 \bar{z}^{1/\alpha} + o(z^{1/\alpha}).$$

From $u = 0$ on $\theta = 0$ we have $a_2 = -a_1$, and

$$\begin{aligned} u_z &= (a_1/\alpha) z^{1/\alpha - 1} + o(z^{1/\alpha - 1}) \\ u_{\bar{z}} &= -(a_2/\alpha) \bar{z}^{1/\alpha - 1} + o(z^{1/\alpha - 1}) \end{aligned} \tag{9.2}$$

and thus

$$u_{z\bar{z}} = -\frac{a_1 a_2}{\alpha^2} z^{1/\alpha - 1} \, \bar{z}^{1/\alpha - 1} + o(z^{2/\alpha - 2}).$$

From the lemmas,

$$\begin{aligned} u(z) = {} & a_1(z^{1/\alpha} - \bar{z}^{1/\alpha}) + b_1 z^{2/\alpha} + b_2 z^{1/\alpha} \, \bar{z}^{1/\alpha} + b_3 \bar{z}^{2/\alpha} \\ & + c_1 z^{2/\alpha} \log z + c_2 \bar{z}^{2/\alpha} \log \bar{z} + o(z^{2/\alpha}). \end{aligned}$$

From (9.2) and the differential equation we have

$$\frac{b_2}{\alpha^2} z^{1/\alpha - 1} \bar{z}^{1/\alpha - 1} = -\frac{a_1^2}{\alpha^2} z^{1/\alpha - 1} \bar{z}^{1/\alpha - 1} + o(z^{2/\alpha - 2})$$

and thus $b_2 = -a_1^2$. From $\theta = 0$ we see

$$0 = (b_1 + b_2 + b_3) r^{2/\alpha} + (c_1 + c_2) r^{2/\alpha} \log r + o(r^{2/\alpha}),$$

whence $c_2 = -c_1$, $b_1 + b_3 = a_1^2$, and from $\theta = \pi\alpha$ we obtain

$$\begin{aligned} 0 = {} & (b_1 + b_2 + b_3 + i\pi\alpha(c_1 - c_2)) r^{2/\alpha} \\ & + (c_1 + c_2) r^{2/\alpha} \log r + o(r^{2/\alpha}), \end{aligned}$$

so $c_1 + c_2 = 0$, $c_1 - c_2 = 0$, and thus

$$\begin{aligned} u(z) = {} & a_1(z^{1/\alpha} - \bar{z}^{1/\alpha}) + b_1 z^{2/\alpha} - a_1^2 z^{1/\alpha} \bar{z}^{1/\alpha} \\ & + b_3 \bar{z}^{2/\alpha} + o(z^{2/\alpha}). \end{aligned}$$

Let us carry out one more step in the expansion. We have

$$
(9.3)\quad
\begin{aligned}
u_z &= (a_1/\alpha)z^{1/\alpha - 1} + 2(b_1/\alpha)z^{2/\alpha - 1} \\
&\quad - (a_1^2/\alpha)z^{1/\alpha - 1}\bar{z}^{1/\alpha - 1} + o(z^{2/\alpha - 1}), \\
u_z &= -(a_1/\alpha)\bar{z}^{1/\alpha - 1} + 2(b_3/\alpha)\bar{z}^{2/\alpha - 1} \\
&\quad - (a_1^2/\alpha)z^{1/\alpha}\bar{z}^{1/\alpha - 1} + o(z^{2/\alpha - 1}),
\end{aligned}
$$

and thus

$$
\begin{aligned}
u_{z\bar{z}} &= -(a_1/\alpha)^2 z^{1/\alpha - 1}\bar{z}^{1/\alpha - 1} - \frac{2a_1b_1}{\alpha^2}z^{2/\alpha - 1}\bar{z}^{1/\alpha - 1} \\
&\quad + \frac{a_1^3}{\alpha^2}z^{1/\alpha - 1}\bar{z}^{2/\alpha - 1} + \frac{2a_1b_3}{\alpha^2}z^{1/\alpha - 1}\bar{z}^{2/\alpha - 1} \\
&\quad - \frac{a_1^3}{\alpha^2}z^{2/\alpha - 1}\bar{z}^{1/\alpha - 1} + o(z^{3/\alpha - 2}).
\end{aligned}
$$

By the lemmas

$$
(9.4)\quad
\begin{aligned}
u(z) &= a_1(z^{1/\alpha} - \bar{z}^{1/\alpha}) + b_1 z^{2/\alpha} - a_1^2 z^{1/\alpha}\bar{z}^{1/\alpha} \\
&\quad + b_3\bar{z}^{2/\alpha} + c_1 z^{3/\alpha} \\
&\quad + c_2 z^{2/\alpha}\bar{z}^{1/\alpha} + c_3 z^{1/\alpha}\bar{z}^{2/\alpha} \\
&\quad + c_4\bar{z}^{3/\alpha} + d_1 z^{3/\alpha}\log z \\
&\quad + d_2\bar{z}^{3/\alpha}\log\bar{z} + o(z^{3/\alpha}).
\end{aligned}
$$

The boundary conditions yield

$$
\begin{aligned}
&c_1 + c_2 + c_3 + c_4 = 0,\ d_1 + d_2 = 0 \\
&-c_1 - c_2 - c_3 - c_4 - i\pi\alpha(d_1 - d_2) = 0,
\end{aligned}
$$

and thus $d_1 = d_2 = 0$. Using (9.3), (9.4) and the differential equation we have

$$
\begin{aligned}
&-\frac{a_1^2}{\alpha^2}z^{1/\alpha - 1}\bar{z}^{1/\alpha - 1} + \frac{2c_2}{\alpha^2}z^{2/\alpha - 1}\bar{z}^{1/\alpha - 1} + \frac{2c_3}{\alpha^2}z^{1/\alpha - 1}\bar{z}^{2/\alpha - 1} \\
&= -\frac{a_1^2}{\alpha^2}z^{1/\alpha - 1}\bar{z}^{1/\alpha - 1} + z^{2/\alpha - 1}\bar{z}^{1/\alpha - 1}\left(-\frac{2a_1b_1}{\alpha^2} - \frac{a_1^3}{\alpha^2}\right) \\
&+ z^{1/\alpha - 1}\bar{z}^{2/\alpha - 1}\left(\frac{2a_1b_3}{\alpha^2} + \frac{a_1^3}{\alpha^2}\right) + o(z^{3/\alpha - 2})
\end{aligned}
$$

and

$$
2c_2 = -a_1(a_1^2 + 2b_1),\ 2c_3 = a_1(a_1^2 + 2b_3).
$$

Finally, setting

$$a_1 = A,\ b_1 = A^2/2 + B,\ b_3 = A^2/2 - B,$$

$$c_1 = AB - C,\ c_4 = AB + C,$$

we obtain

$$\begin{aligned} u(z) = {} & A(z^{1/\alpha} - \bar{z}^{1/\alpha}) + \left(\frac{A^2}{2} + B\right)z^{2/\alpha} - A^2 z^{1/\alpha}\bar{z}^{1/\alpha} \\ & + \left(\frac{A^2}{2} + B\right)\bar{z}^{2/\alpha} \\ & + (AB - C)z^{3/\alpha} - (A^3 + AB)z^{2/\alpha}\bar{z}^{1/\alpha} \\ & + (A^3 - AB)z^{1/\alpha}\bar{z}^{2/\alpha} + (AB + C)z^{3/\alpha} + O(z^{3/\alpha}). \end{aligned}$$

It may be observed that one can show inductively that logarithmic terms will not appear in the expansion for any error term; in particular this shows that $u$ is infinitely differentiable at the origin if $1/\alpha$ is an integer. The last result could of course also be obtained by extending $u$ to $R^2 \setminus \{0\}$ by reflection and then removing the singularity.

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