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Generalized Lagrange multiplier rule for non-convex vector optimization problems

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In this paper a non-convex vector optimization problem among infinite-dimensional spaces is presented. In particular, a generalized Lagrange multiplier rule is formulated as a necessary and sufficient optimality condition for weakly minimal solutions of a constrained vector optimization problem, without requiring that the ordering cone that defines the inequality constraints has non-empty interior. This paper extends the result of Donato (*J. Funct. Analysis* **261** (2011), 2083–2093) to the general setting of vector optimization by introducing a constraint qualification assumption that involves the Fréchet differentiability of the maps and the tangent cone to the image set. Moreover, the constraint qualification is a necessary and sufficient condition for the Lagrange multiplier rule to hold.

Keywords: Lagrange multiplier rule; vector optimization problems; tangent cone

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1. Introduction

In this paper we deal with the following constrained vector optimization problem having an objective function with values in a non-empty subset of a partially ordered linear space and both cone and equality constraints.

Let $(X, \|\cdot\|_X)$ and $(Z_2, \|\cdot\|_{Z_2})$ be real Banach spaces, let $(Y, \|\cdot\|_Y)$ and $(Z_1, \|\cdot\|_{Z_1})$ be partially ordered normed spaces, and let $C_Y \subset Y$ and $C_{Z_1} \subset Z_1$ be, respectively, the closed, convex and pointed cones in Y and Z_1, with $\operatorname{int}(C_Y) \neq \emptyset$, where $\operatorname{int}(C_Y)$ denotes the interior of the set C_Y. These cones induce a partial order relation $\leqslant_{C_Y}$, $\leqslant_{C_{Z_1}}$ on Y, Z_1, respectively, as follows:

$$\forall y_1, y_2 \in Y,\ y_1 \leqslant_{C_Y} y_2 \iff y_2 - y_1 \in C_Y,$$
$$\forall z_1, z_2 \in Z_1,\ z_1 \leqslant_{C_{Z_1}} z_2 \iff z_2 - z_1 \in C_{Z_1}.$$

Let Y^*, Z_1^*, Z_2^* be, respectively, the topological dual spaces of Y, Z_1, Z_2 and let

$$C_Y^* = \{u \in Y^* \colon u(y) \geqslant 0\ \forall y \in C_Y\}$$

be the dual cone of C_Y. Let S be a non-empty convex subset of X and let $f\colon X \to Y$, $g\colon X \to Z_1$, $h\colon X \to Z_2$ be three general maps. Fixing the constraint set as

$$K = \{x \in S\colon g(x) \in -C_{Z_1},\ h(x) = \theta_{Z_2}\}, \tag{1.1}$$

where θ_{Z_2} is the zero element in the space Z_2, we consider the following constrained vector optimization problem:

$$\text{find } \bar{x} \in K \text{ such that } f(\bar{x}) = \min_{x \in K} f(x). \tag{1.2}$$

This problem is interpreted in the following way: determine a minimal solution $\bar{x} \in K$, which is defined as the inverse image of a minimal element $f(\bar{x})$ of the image set $f(K)$.

The solution concepts for a vector optimization problem are as follows.

Definition 1.1.

(a) A point $\bar{x} \in K$ is called a minimal solution (or an efficient solution) of the problem (1.2) if $f(\bar{x})$ is a minimal element of the image set $f(K)$:

$$f(\bar{x}) \not\geq_{C_Y \setminus \{\theta_Y\}} f(x) \quad \forall x \in K,$$

where the inequality means that $f(\bar{x}) - f(x) \notin C_Y \setminus \{\theta_Y\}$ for all $x \in K$.

(b) A point $\bar{x} \in K$ is called a weakly minimal solution (or a weakly efficient solution) of problem (1.2) (when $\operatorname{int}(C_Y) \neq \emptyset$) if $f(\bar{x})$ is a weakly minimal element of the image set $f(K)$:

$$f(\bar{x}) \not\geq_{\operatorname{int}(C_Y)} f(x) \quad \forall x \in K,$$

where the inequality means that $f(\bar{x}) - f(x) \notin \operatorname{int}(C_Y)$ for all $x \in K$.

(c) A point $\bar{x} \in K$ is called a local weakly minimal solution (or a local weakly efficient solution) of problem (1.2) if there exists a neighbourhood U of $\bar{x}$ such that:

$$f(\bar{x}) \not\geq_{\operatorname{int}(C_Y)} f(x) \quad \forall x \in K \cap U,$$

where the inequality means that $f(\bar{x}) - f(x) \notin \operatorname{int}(C_Y)$ for all $x \in K \cap U$.

The following lemma shows that under suitable assumptions every minimal element of a set is also a weakly minimal element of the same set. Hence, a necessary condition for weakly minimal elements is a necessary condition for minimal elements as well.

Lemma 1.2. *Let N be a non-empty subset of a partially ordered linear space W with an ordering cone C for which $C \neq W$ and $\operatorname{int}(C) \neq \emptyset$. Then every minimal element of the set N is also a weakly minimal element of the set N.*

The aim of this paper is to present an extension of results in [6, 17] to a more general context: the vector optimization framework. More precisely, in [6] Donato provided a generalized Lagrange multiplier rule as necessary and sufficient optimality conditions for infinite-dimensional convex optimization problems by using a

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new constraint qualification called assumption S. This assumption was introduced in [3] to guarantee the strong duality between a convex optimization problem and its Lagrange dual in the scalar context (that is, when the objective functional is real-valued) without requiring that the ordering cone, which defines the inequality constraints, has non-empty interior. Moreover, in [17] Maugeri and Puglisi introduce a new constraint qualification condition, assumption S'. For other results in the literature regarding the infinite-dimensional duality theory for scalar convex optimization problems, see also [1–5, 7–10, 13, 18].

This paper intends to provide an extension of the aforementioned results, from scalar convex optimization problems to vector optimization problems, without requiring any convexity assumption on the objective function and by introducing assumption S'_V. This assumption, which involves the Fréchet derivatives of maps and uses the tangent cone in the image set, turns out to be necessary and sufficient for the infinite-dimensional Lagrange multiplier rule. Finally, it is worth comparing the main results of this paper (theorem 3.3 and theorem 3.5) with [14, theorem 7.4] for the necessary condition and with [14, theorem 7.20] for the sufficient condition. Let us observe that theorem 3.3 generalizes theorem 7.4 of [14]. In fact the requirement that the ordering cone, which defines the inequality constraints, has non-empty interior is removed and the regularity condition, in theorem 3.3, is replaced by the assumption S'_V. For other approaches concerning the Lagrange multiplier rule for vector optimization problems, see [11, 12, 15]. More precisely, in [11] Durea *et al*. study the non-convex vector optimization in infinite-dimensional spaces by using a very different scalarization approach to the ones used in the literature. In [12] Dutta and Tammer show the boundedness of the set of Lagrange multipliers for vector optimization problems in infinite-dimensional spaces. In [15] Jiménez *et al*. introduce a basic constraint qualification for non-convex infinite-dimensional vector optimization problems by assuming the Hadamard differentiability of maps.

2. Preliminaries and definitions

We devote this section to recalling some basic definitions and results that will play an important role in this paper. For the proofs and further details we refer the reader to [14, 16] and references therein.

Definition 2.1. Let X be a real linear space, let Y be a real topological linear space, let S be a non-empty subset of X, and let $f\colon S \to Y$ be a given map. If for $\bar{x} \in S$ and $h \in X$ the limit

$$f'(\bar{x})(h) := \lim_{\lambda \to 0^+} \frac{1}{\lambda}(f(\bar{x} + \lambda h) - f(\bar{x}))$$

exists, then $f'(\bar{x})(h)$ is called the directional derivative of f at $\bar{x}$ in the direction h. If the limit exists for all $h \in X$, then f is called directionally differentiable at $\bar{x}$.

The following definition represents a generalization of the directional derivative.

Definition 2.2. Let X and Y be real linear spaces, let S be a non-empty subset of X and let T be a non-empty subset of Y. Moreover, let a map $f\colon S \to Y$ and an element $\bar{x} \in S$ be given. A map $f'(\bar{x})\colon S - \{\bar{x}\} \to Y$ is called a directional variation of f at $\bar{x}$ with respect to T if the following holds: if there is an element $x \in S$ with $x \neq \bar{x}$ and $f'(\bar{x})(x-\bar{x}) \in T$, then there is a $\bar{\lambda} > 0$ with

$$\bar{x} + \lambda(x - \bar{x}) \in S \quad \forall \lambda \in (0, \bar{\lambda}]$$

and

$$\frac{1}{\lambda}(f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x})) \in T \quad \forall \lambda \in (0, \bar{\lambda}].$$

Remark 2.3. It is important to observe that when Y is a real topological space, if there is $\bar{\lambda} > 0$ with $\bar{x} + \lambda(x - \bar{x}) \in S$ for all $\lambda \in (0, \bar{\lambda}]$, and if $f'(\bar{x})$ is the directional derivative of f at $\bar{x}$ in the direction $x - \bar{x}$, then $f'(\bar{x})$ is a directional variation of f at $\bar{x}$ with respect to all non-empty open subsets of Y.

Definition 2.4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let S be a non-empty open subset of X, and let $f\colon S \to Y$ be a given map. Furthermore, let an element $\bar{x} \in S$ be given. If there is a continuous linear map $f'(\bar{x})\colon X \to Y$ with the property

$$\lim_{\|h\|_X \to 0} \frac{\|f(\bar{x} + h) - f(\bar{x}) - f'(\bar{x})(h)\|}{\|h\|_X} = 0,$$

then $f'(\bar{x})$ is called the Fréchet derivative of f at $\bar{x}$ and f is called Fréchet differentiable at $\bar{x}$.

Definition 2.5. Let X and Y be real linear spaces, let C_Y be a convex cone in Y and let S be a non-empty convex subset of X. A map $f\colon S \to Y$ is called convex (or C_Y-convex) if for all $x, y \in S$ and all $\lambda \in [0,1]$,

$$\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y) \in C_Y.$$

The following theorem gives a characterization of a convex Fréchet differentiable map.

Theorem 2.6. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, let S be a non-empty open convex subset of X, let C_Y be a closed convex cone in Y, and let a map $f\colon S \to Y$ be given that is Fréchet differentiable at every $x \in S$. Then the map f is convex if and only if*

$$f(x) - f(y) - f'(y)(x - y) \in C_Y \quad \forall x, y \in S.$$

Proposition 2.7. *Let C_Y be a pointed convex cone of the partially ordered normed space Y with $\operatorname{int}(C_Y) \neq \emptyset$. If $t \in C_Y^* \backslash \{\theta_{Y^*}\}$ and $y \in -\operatorname{int}(C_Y)$, then $t(y) < 0$.*

In order to obtain a necessary condition for weakly minimal solutions of the vector optimization problem we use the concept of the tangent cone. Given a point $x \in X$ and a subset C of X, the set

$$T(C, x) = \Big\{ u \in X\colon u = \lim_{n \to \infty} \lambda_n (x_n - x),\ \lambda_n \in \mathbb{R} \text{ and } \lambda_n > 0\ \forall n \in \mathbb{N},\\ x_n \in C\ \forall n \in \mathbb{N} \text{ and } \lim_{n \to \infty} x_n = x \Big\}$$

is called the tangent cone (or Bouligand tangent cone or contingent cone) to C at x. Of course, if $T(C,x) \neq \emptyset$, then $x \in \operatorname{cl} C$. For every $x \in \operatorname{cl} C$ we have

$$T(C,x) \subset \operatorname{cl}\operatorname{cone}(C - \{x\}).$$

When the set C is convex we have

$$T(C,x) = \operatorname{cl}\operatorname{cone}(C - \{x\}),$$

where

$$\operatorname{cone}(C) = \{\lambda x \colon x \in C,\ \lambda \in \mathbb{R},\ \lambda \geqslant 0\}$$

is the cone generated by C and $\operatorname{cl} C$ denotes the closure of a set C.

Furthermore, it is important to note that the tangent cone is a closed set and $\theta_X \in T(C,x)$. In addition, if $C \subset D \subset X$ and $x \in \operatorname{cl} C$, then $T(C,x) \subset T(D,x)$.

The following result gives a first-order necessary optimality condition for a vector optimization problem.

Theorem 2.8. *Let M be a non-empty convex subset of a real normed space X and let Y be a real normed space partially ordered with ordering cone $C_Y \neq Y$ with non-empty interior. Let $f \colon X \to Y$ be a Fréchet differentiable map at $\bar{x}$. If $\bar{x} \in M$ is a weakly minimal solution of the vector optimization problem*

$$\min_{x \in M} f(x),$$

then

$$f'(\bar{x})(u) \notin -\operatorname{int}(C_Y) \quad \forall u \in T(M, \bar{x}).$$

In the next section we obtain the necessary optimality condition for the weakly minimal solution by using the following strict separation theorem.

Theorem 2.9. *Let X be a real normed space and let $C \subseteq X$ be a closed cone. If $x_0 \in X \backslash C$, then there exists $x^* \in X^*$ with $x_0^* \neq \theta_{X^*}$ such that*

$$\langle x_0^*, x_0\rangle < 0 \leqslant \inf_{y \in C} \langle x_0^*, y\rangle.$$

Finally, the following generalized convexity concept will be useful in obtaining the sufficiency of the generalized multiplier rule.

Definition 2.10. Let S be a non-empty convex subset of a real linear space X and let Y be a partially ordered linear space with an ordering cone C_Y. A map $F \colon S \to Y$ is called quasi-convex if $x_1, x_2 \in S$ with $F(x_1) - F(x_2) \in C_Y$ implies that

$$F(x_1) - F(\lambda x_1 + (1-\lambda)x_2) \in C_Y \quad \forall \lambda \in [0,1].$$

The following concept extends the quasi-convexity of maps.

Definition 2.11. Let S be a non-empty subset of a real linear space X and let C be a non-empty subset of a real linear space Y. Let $\bar{x} \in S$ be a given element. A map $f \colon S \to Y$ is called C-quasi-convex at $\bar{x}$ if the following holds: if there is some $x \in S\backslash\{\bar{x}\}$ with $F(\bar{x}) - F(x) \in C$, then there is some $\tilde{x} \in S\backslash\{\bar{x}\}$ with

$$\lambda\tilde{x} + (1-\lambda)\bar{x} \in S \quad \forall \lambda \in (0,1],$$
$$F(\bar{x}) - F(\lambda\tilde{x} + (1-\lambda)\bar{x}) \in C \quad \forall \lambda \in (0,1].$$

Definition 2.12. Let S be a non-empty subset of a real linear space X and let C_1 and $C_2 \subset C_3$ be non-empty subsets of a real linear space Y. Moreover, let $\bar{x} \in S$ be a given element and let a map $F\colon S \to Y$ have a directional variation at $\bar{x}$ with respect to C_3. The map F is called differentiably C_1-C_2-quasi-convex at $\bar{x}$ if the following holds: if there is some $x \in S$ with $x \neq \bar{x}$ and $F(x) - F(\bar{x}) \in C_1$, then there is an $\tilde{x} \in S \backslash \{\bar{x}\}$ with $\lambda\tilde{x} + (1-\lambda)\bar{x} \in S$ for all $\lambda \in (0, 1]$ and $F'(\bar{x})(\tilde{x} - \bar{x}) \in C_2$.

In the case in which $C_1 = C_2 = C$, the map F is called differentiably C-quasi-convex at $\bar{x}$.

3. Lagrange multiplier rule under a new constraint qualification

The aim of this section is to give a necessary and sufficient optimality condition for the non-convex vector optimization problem (1.2) in order to ensure the existence of Lagrange multipliers. To this end we introduce a new constraint qualification condition that involves the Fréchet derivatives of maps and uses the tangent cone in the image space. This new condition turns out to be necessary and sufficient for the infinite-dimensional generalized Lagrange multiplier rule.

The constraint qualification assumption, called assumption S'_V, is the following.

Definition 3.1. Given three maps f, g, h, Fréchet differentiable at $\bar{x} \in K$, we say that $\bar{x}$ verifies assumption S'_V if and only if

$$T(\hat{M}, \theta_{Y\times Z_1\times Z_2}) \cap (-\operatorname{int}(C_Y) \times \{\theta_{Z_1}\} \times \{\theta_{Z_2}\}) = \emptyset,$$

where

$$\begin{aligned}\hat{M} = \{&(f'(\bar{x})(x-\bar{x}) + y, g(\bar{x}) + g'(\bar{x})(x-\bar{x}) + z_1, h'(\bar{x})(x-\bar{x}))\\ &\in Y \times Z_1 \times Z_2 \colon x \in S \setminus K,\ y \in C_Y,\ z_1 \in C_{Z_1}\}\end{aligned}$$

and the tangent cone to the subset $\hat{M}$ of the image space $Y \times Z_1 \times Z_2$ at $\theta_{Y\times Z_1\times Z_2}$ is

$$\begin{aligned}&T(\hat{M}, \theta_{Y\times Z_1\times Z_2})\\ &= \Big\{a = \lim_{n\to+\infty} \gamma_n[(f'(\bar{x})(x_n-\bar{x}) + y_n,\\ &\qquad g(\bar{x}) + g'(\bar{x})(x_n-\bar{x}) + z_{1n}, h'(\bar{x})(x_n-\bar{x})) - (\theta_Y, \theta_{Z_1}, \theta_{Z_2})]\colon\\ &\qquad \{\gamma_n\} \subset \mathbb{R},\ \gamma_n > 0\ \forall n \in \mathbb{N},\ x_n \in S\backslash K\ \forall n \in \mathbb{N},\ \{y_n\} \subseteq C_Y,\ \{z_{1n}\} \subseteq C_{Z_1},\\ &\qquad \lim_{n\to+\infty}(f'(\bar{x})(x_n-\bar{x}) + y_n) = \theta_Y,\ \lim_{n\to+\infty}(g(\bar{x}) + g'(\bar{x})(x_n-\bar{x}) + z_{1n}) = \theta_{Z_1},\\ &\qquad \lim_{n\to+\infty}(h'(\bar{x})(x_n-\bar{x})) = \theta_{Z_2}\Big\}.\end{aligned}$$

Remark 3.2. We can observe that if $T(\hat{M}, \theta_{Y\times Z_1\times Z_2}) = \emptyset$, then assumption S'_V is clearly verified. If $T(\hat{M}, \theta_{Y\times Z_1\times Z_2}) \neq \emptyset$, assumption S'_V is verified when for all $(r, \theta_{Z_1}, \theta_{Z_2}) \in T(\hat{M}, \theta_{Y\times Z_1\times Z_2})$ with $r = \lim_{n\to+\infty} \gamma_n(f'(\bar{x})(x_n-\bar{x}) + y_n)$ one has that $r \notin -\operatorname{int}(C_Y)$.

First, we formulate the Lagrange multiplier rule as a necessary optimality condition. It is important to emphasize that we require the convexity of the constraint set K but we do not assume any convexity assumption on the objective function f, and furthermore, thanks to assumption S'_V, we do not require that the ordering cone C_{Z_1} has non-empty interior.

THEOREM 3.3. *Let the vector optimization problem (1.2) be given and let $\bar{x} \in K$ be a weakly minimal solution of (1.2) with the set K convex. Let f, g and h be Fréchet differentiable at $\bar{x}$. If assumption S'_V is fulfilled at $\bar{x}$, then there exist continuous linear functionals $t \in C^*_Y \setminus \{\theta_{Y^*}\}$, $\lambda \in C^*_{Z_1}$ and $\mu \in Z^*_2$ such that*

$$t(f'(\bar{x})(x-\bar{x})) + \lambda(g'(\bar{x})(x-\bar{x})) + \mu(h'(\bar{x})(x-\bar{x})) \geqslant 0 \quad \forall x \in S, \tag{3.1}$$

$$\lambda(g(\bar{x})) = 0. \tag{3.2}$$

Proof. We define the set

$$\tilde{M} = \{(f'(\bar{x})(x-\bar{x}) + y, g(\bar{x}) + g'(\bar{x})(x-\bar{x}) + z_1, h'(\bar{x})(x-\bar{x})) \in Y \times Z_1 \times Z_2 \colon x \in S,\ y \in C_Y,\ z_1 \in C_{Z_1}\},$$

which can be written in the following way:

$$\tilde{M} = (f'(\bar{x}), g'(\bar{x}), h'(\bar{x}))(S - \{\bar{x}\}) + C_Y \times (\{g(\bar{x})\} + C_{Z_1}) \times \{\theta_{Z_2}\}.$$

Since $S - \{\bar{x}\}$ is a convex set, cones C_Y and C_{Z_1} are convex sets and $f'(\bar{x})$, $g'(\bar{x})$, $h'(\bar{x})$ are linear maps, we can see that $\tilde{M}$ is a non-empty and convex set.

Now, we prove that

$$T(\tilde{M}, \theta_{Y\times Z_1\times Z_2}) \cap (-\operatorname{int}(C_Y) \times \{\theta_{Z_1}\} \times \{\theta_{Z_2}\}) = \emptyset. \tag{3.3}$$

We consider the element $(r, \theta_{Z_1}, \theta_{Z_2}) \in T(\tilde{M}, \theta_{Y\times Z_1\times Z_2})$. By definition of the tangent cone, we have that there exist, for all $n \in \mathbb{N}$, $\gamma_n > 0$, $x_n \in S$ and $y_n \in C_Y$ such that

$$\lim_{n\to+\infty} (f'(\bar{x})(x_n - \bar{x}) + y_n) = \theta_Y \quad \text{and} \quad r = \lim_{n\to+\infty} \gamma_n (f'(\bar{x})(x_n - \bar{x}) + y_n).$$

To prove condition (3.3), we must have that $r \notin -\operatorname{int}(C_Y)$.

We consider the sequence $x_n \in S$ for all $n \in \mathbb{N}$.

- If $x_n \in S \setminus K$ for all $n \in \mathbb{N}$, then the element $(r, \theta_{Z_1}, \theta_{Z_2}) \in T(\hat{M}, \theta_{Y\times Z_1\times Z_2})$. Hence, by assumption S'_V, we have $r \notin -\operatorname{int}(C_Y)$.

- If $x_n \in K \subseteq S$ for all $n \in \mathbb{N}$, by assumption, $\bar{x}$ is a weakly minimal point of f on K, and thus

$$f'(\bar{x})(u) \notin -\operatorname{int}(C_Y) \quad \forall u \in T(K, \bar{x}). \tag{3.4}$$

  To prove that $r \notin -\operatorname{int}(C_Y)$, we proceed by contradiction. We assume that $r = \lim_{n\to+\infty} \gamma_n (f'(\bar{x})(x_n - \bar{x}) + y_n) \in -\operatorname{int}(C_Y)$. For sufficiently large $n \in \mathbb{N}$ one has $\gamma_n (f'(\bar{x})(x_n - \bar{x}) + y_n) \in -\operatorname{int}(C_Y)$, which implies that

$$f'(\bar{x})(x_n - \bar{x}) \in -C_Y - \operatorname{int}(C_Y) = -\operatorname{int}(C_Y)$$

  for sufficiently large $n \in \mathbb{N}$. But this contradicts condition (3.4). In fact, by the convexity of K, we find that for sufficiently large $n \in \mathbb{N}$ there exists a feasible direction $u_n = x_n - \bar{x} \in T(K, \bar{x})$ such that $f'(\bar{x})(x_n - \bar{x}) \in -\operatorname{int}(C_Y)$. Hence, $r \notin -\operatorname{int}(C_Y)$.

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- If $x_n \in S \setminus K$ for a finite number of indexes n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ definitely belongs to K and the conclusion of the second case holds.

- If $x_n \in S \setminus K$ for an infinite number of indexes n, then we can consider a subsequence $x_{n_k} \in S \setminus K$ and we proceed as in the first case.

Given the above, we have that

$$(-\operatorname{int}(C_Y) \times \{\theta_{Z_1}\} \times \{\theta_{Z_2}\}) \nsubseteq T(\tilde{M}, \theta_{Y\times Z_1\times Z_2}),$$

that is, for all $s \in -\operatorname{int}(C_Y)$ the element $(s, \theta_{Z_1}, \theta_{Z_2}) \notin T(\tilde{M}, \theta_{Y\times Z_1\times Z_2})$. Furthermore, since $\tilde{M}$ is a convex set, the set $T(\tilde{M}, \theta_{Y\times Z_1\times Z_2}) = \operatorname{cl}\operatorname{cone}(\tilde{M} - \{\theta_{Y\times Z_1\times Z_2}\})$ is a closed and convex cone. Thus, by virtue of the strict separation theorem (theorem 2.9), it follows that there exists $(t, \lambda, \mu) \in Y^* \times Z_1^* \times Z_2^*$ with $(t, \lambda, \mu) \neq \theta_{Y^*\times Z_1^*\times Z_2^*}$ such that

$$t(s) < 0 \leqslant t(f'(\bar{x})(x-\bar{x}) + y) + \lambda(g(\bar{x}) + g'(\bar{x})(x-\bar{x}) + z_1) + \mu(h'(\bar{x})(x-\bar{x})) \quad \forall x \in S,\ y \in C_Y,\ z_1 \in C_{Z_1}.$$

From the above inequality and the linearity of t, it follows that $t(-s) > 0$ for all $s \in -\operatorname{int}(C_Y)$, that is, $t \in C_Y^* \setminus \{\theta_{Y^*}\}$. Hence,

$$t(f'(\bar{x})(x-\bar{x}) + y) + \lambda(g(\bar{x}) + g'(\bar{x})(x-\bar{x}) + z_1) + \mu(h'(\bar{x})(x-\bar{x})) \geqslant 0 \quad \forall x \in S,\ y \in C_Y,\ z_1 \in C_{Z_1}. \tag{3.5}$$

By inequality (3.5), for $x = \bar{x}$ and $y = \theta_Y$ we get

$$\lambda(g(\bar{x}) + z_1) \geqslant 0 \quad \forall z_1 \in C_{Z_1}.$$

In particular, since C_{Z_1} is a convex cone, by choosing $z_1 = \bar{z}_1 - g(\bar{x}) \in C_{Z_1}$ one has

$$0 \leqslant \lambda(g(\bar{x}) + z_1) = \lambda(\bar{z}_1) \quad \forall \bar{z}_1 \in C_{Z_1},$$

which implies that $\lambda \in C_{Z_1^*}$. Furthermore, by assuming that $z_1 = \theta_{Z_1} \in C_{Z_1}$, it follows that $\lambda(g(\bar{x})) \geqslant 0$. By assumption, $g(\bar{x}) \in -C_{Z_1}$, and then $\lambda(g(\bar{x})) \leqslant 0$. Consequently, the thesis (3.2) holds:

$$\lambda(g(\bar{x})) = 0.$$

Finally, by inequality (3.5), for $y = \theta_Y$ and $z_1 = -g(\bar{x})$ we get

$$t(f'(\bar{x})(x-\bar{x})) + \lambda(g'(\bar{x})(x-\bar{x})) + \mu(h'(\bar{x})(x-\bar{x})) \geqslant 0 \quad \forall x \in S.$$

□

Corollary 3.4. *Let the same assumptions of theorem 3.3 be satisfied and in addition let $S = X$. If $\bar{x} \in K$ is a weakly minimal solution of problem (1.2), then there exist continuous linear functionals $t \in C_Y^* \setminus \{\theta_{Y^*}\}$, $\lambda \in C_{Z_1}^*$ and $\mu \in Z_2^*$ such that*

$$t(f'(\bar{x})) + \lambda(g'(\bar{x})) + \mu(h'(\bar{x})) = \theta_{X^*}.$$

Proof. In this case condition (3.1) becomes

$$t(f'(\bar{x})(x-\bar{x})) + \lambda(g'(\bar{x})(x-\bar{x})) + \mu(h'(\bar{x})(x-\bar{x})) \geqslant 0 \quad \forall x \in X,$$

which implies, because of the linearity of the considered mappings,

$$t(f'(\bar{x})) + \lambda(g'(\bar{x})) + \mu(h'(\bar{x})) = \theta_{X^*}.$$

□

To achieve the sufficiency of the generalized Lagrange multiplier rule we have to consider additional assumptions on maps. To this end we make a generalized convexity assumption on the objective functional f, a convexity assumption on the map g and an affine-linearity assumption on the map h. It is important to observe that by making these assumptions on maps g and h, the constraint set K is convex.

THEOREM 3.5. *Let X be a linear topological space and let $S \subseteq X$ be non-empty and convex. Let Y and Z_1 be real normed spaces ordered by closed, convex and pointed cones C_Y and C_{Z_1}, respectively, with $\operatorname{int}(C_Y) \neq \emptyset$, and let Z_2 be a real normed space. Let us suppose that there exists a non-empty open subset G_0 with $-\operatorname{int}(C_Y) \subset G_0 \subset Y$. Let $f\colon S \to Y$ be a map with directional derivative at $\bar{x} \in K$ in every direction $x - \bar{x}$, with arbitrary $x \in S$. Assume that f is differentiably $-\operatorname{int}(C_Y)$-quasi-convex at $\bar{x}$. Let $g\colon X \to Z_1$ be a Fréchet differentiable at $\bar{x}$ and C_{Z_1}-convex map. Let $h\colon X \to Z_2$ be a Fréchet differentiable and affine-linear map. If there exist continuous linear functionals $t \in C_Y^* \setminus \{\theta_{Y^*}\}$, $\lambda \in C_{Z_1}^*$, $\mu \in Z_2^*$ such that*

$$t(f'(\bar{x})(x-\bar{x})) + \lambda(g'(\bar{x})(x-\bar{x})) + \mu(h'(\bar{x})(x-\bar{x})) \geqslant 0 \quad \forall x \in S, \tag{3.6}$$

$$\lambda(g(\bar{x})) = 0. \tag{3.7}$$

Then $\bar{x} \in K$ is a weakly minimal solution to the vector optimization problem (1.2). Moreover, if the map f is also Fréchet differentiable at $\bar{x}$, then assumption S'_V is fulfilled at $\bar{x}$.

Proof. We suppose by contradiction that $\bar{x} \in K$ is not a weakly minimal solution to problem (1.2), and thus there exists $x \in K$ with $x \neq \bar{x}$ such that

$$f(x) - f(\bar{x}) \in -\operatorname{int}(C_Y).$$

Since f has directional derivative at $\bar{x}$ in every direction $x - \bar{x}$, it follows, from remark 2.3, that $f'(\bar{x})$ is a directional variation of f at $\bar{x}$ with respect to all non-empty open subsets of Y. Hence, by the differentiable $-\operatorname{int}(C_Y)$-quasi-convexity of the map f at $\bar{x}$, there exists $\tilde{x} \in K$, with $\tilde{x} \neq \bar{x}$, such that $\gamma\tilde{x} + (1-\gamma)\bar{x} \in K$ for all $\gamma \in (0, 1]$ and

$$f'(\bar{x})(\tilde{x} - \bar{x}) \in -\operatorname{int}(C_Y).$$

Taking into account that $t \in C_Y^* \setminus \{\theta_{Y^*}\}$, it follows that

$$t(f(\bar{x})(\tilde{x} - \bar{x})) < 0. \tag{3.8}$$

If the map g is Fréchet differentiable at $\bar{x}$ and C_{Z_1}-convex, one has

$$g(\bar{x}) - g(x) + g'(\bar{x})(x - \bar{x}) \in -C_{Z_1} \quad \forall x \in S.$$

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Since $\lambda \in C^*_{Z_1}$, one has

$$\lambda(g'(\bar{x})(x-\bar{x})) \leqslant \lambda(g(x)-g(\bar{x})) = \lambda(g(x)) \leqslant 0 \quad \forall x \in K,$$

and hence

$$\lambda(g'(\bar{x})(x-\bar{x})) \leqslant 0 \quad \forall x \in K. \tag{3.9}$$

Moreover, h being Fréchet differentiable at $\bar{x}$, we have that $h'(\bar{x})(d) = \theta_{Z_2}$ for all $d \in T(K,\bar{x})$. In fact, for fixed $d \in T(K,\bar{x})$, there exist, for all $n \in \mathbb{N}$, $x_n \in K$, $\alpha_n > 0$ such that $\lim_{n\to+\infty} x_n = \bar{x}$ and $\lim_{n\to+\infty} \alpha_n(x_n-\bar{x}) = d$, where $\alpha_n(x_n-\bar{x}) = d_n$ for all $n \in \mathbb{N}$. Thus, by the definition of the Fréchet derivative we obtain

$$\begin{aligned} h'(\bar{x})(d) &= h'(\bar{x})\Big(\lim_{n\to+\infty} \alpha_n h'(\bar{x})(x_n-\bar{x})\Big) \\ &= -\lim_{n\to+\infty} \alpha_n(h(x_n)-h(\bar{x})-h'(\bar{x})(x_n-\bar{x})) \\ &= -\lim_{n\to+\infty} \|d_n\| \frac{(h(x_n)-h(\bar{x})-h'(\bar{x})(x_n-\bar{x}))}{\|x_n-\bar{x}\|} \\ &= \theta_{Z_2}. \end{aligned}$$

Moreover, the set K being convex, one has

$$h'(\bar{x})(x-\bar{x}) = \theta_{Z_2} \quad \forall x \in K. \tag{3.10}$$

Consequently, taking into account (3.8)–(3.10), it follows that

$$t(f'(\bar{x})(\tilde{x}-\bar{x})) + \lambda(g'(\bar{x})(\tilde{x}-\bar{x})) + \mu(h'(\bar{x})(\tilde{x}-\bar{x})) < 0,$$

which contradicts the statement (3.6). Then $\bar{x} \in K$ is a weakly minimal solution to (1.2).

Now, let us show that assumption S'_V holds true. From conditions (3.6) and (3.7) one has that the affine hyperplane

$$H = \{(\tilde{y},\tilde{z_1},\tilde{z_2}) \in Y \times Z_1 \times Z_2 \colon t(\tilde{y}) + \lambda(\tilde{z_1}) + \mu(\tilde{z_2}) = 0\}$$

separates the sets

$$(f'(\bar{x}), g'(\bar{x}), h'(\bar{x}))(S - \{\bar{x}\}) + (\{\theta_Y\} \times \{g(\bar{x})\} \times \{\theta_{Z_2}\}) = A$$

and

$$-\operatorname{int}(C_Y) \times (-C_{Z_1}) \times \{\theta_{Z_2}\} = B.$$

Hence,

$$A \subseteq H^+, \qquad B \subseteq H^-,$$

where H^+ and H^- denote the half-spaces

$$\begin{aligned} H^+ &= \{(\tilde{y},\tilde{z_1},\tilde{z_2}) \in Y \times Z_1 \times Z_2 \colon t(\tilde{y}) + \lambda(\tilde{z_1}) + \mu(\tilde{z_2}) \geqslant 0\}, \\ H^- &= \{(\tilde{y},\tilde{z_1},\tilde{z_2}) \in Y \times Z_1 \times Z_2 \colon t(\tilde{y}) + \lambda(\tilde{z_1}) + \mu(\tilde{z_2}) \leqslant 0\}. \end{aligned}$$

Clearly,

$$\tilde{M} = (f'(\bar{x}), g'(\bar{x}), h'(\bar{x}))(S - \{\bar{x}\}) + C_Y \times (\{g(\bar{x})\} + C_{Z_1}) \times \{\theta_{Z_2}\} \subseteq H^+;$$

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in fact, if there exists $\tilde{m} = (\widetilde{m_1}, \widetilde{m_2}, \widetilde{m_3}) \in \tilde{M}$ such that

$$t(\widetilde{m_1}) + \lambda(\widetilde{m_2}) + \mu(\widetilde{m_3}) < 0,$$

where $\tilde{m} = \bar{m} - \hat{m}$, $\bar{m} = (\overline{m_1}, \overline{m_2}, \overline{m_3}) \in A$ and $\hat{m} = (\widehat{m_1}, \widehat{m_2}, \widehat{m_3}) \in (-C_Y) \times (-C_{Z_1}) \times \{\theta_{Z_2}\}$, it results that

$$t(\widetilde{m_1}) + \lambda(\widetilde{m_2}) + \mu(\widetilde{m_3}) = t(\overline{m_1}) + \lambda(\overline{m_2}) + \mu(\overline{m_3}) - t(\widehat{m_1}) - \lambda(\widehat{m_2}) - \mu(\widehat{m_3}) < 0.$$

Namely,

$$0 \leqslant t(\overline{m_1}) + \lambda(\overline{m_2}) + \mu(\overline{m_3}) < t(\widehat{m_1}) + \lambda(\widehat{m_2}) + \mu(\widehat{m_3}) \leqslant 0,$$

which is impossible.

Moreover, one has that

$$T(\tilde{M}, \theta_{Y \times Z_1 \times Z_2}) = \operatorname{cl}\operatorname{cone}(\tilde{M} - \{\theta_{Y \times Z_1 \times Z_2}\}) \subseteq \operatorname{cl}\operatorname{cone}(H^+) = H^+,$$
$$-\operatorname{int}(C_Y) \times \{\theta_{Z_1}\} \times \{\theta_{Z_2}\} \subseteq B \subseteq H^-.$$

Hence, H separates $T(\tilde{M}, \theta_{Y \times Z_1 \times Z_2})$ and $-\operatorname{int}(C_Y) \times \{\theta_{Z_1}\} \times \{\theta_{Z_2}\}$, so we can conclude that

$$T(\hat{M}, \theta_{Y \times Z_1 \times Z_2}) \cap -\operatorname{int}(C_Y) \times \{\theta_{Z_1}\} \times \{\theta_{Z_2}\} = \emptyset. \quad (3.11)$$

In fact, by properties of the tangent cone, since $\hat{M} \subset \tilde{M}$, one has

$$T(\hat{M}, \theta_{Y \times Z_1 \times Z_2}) \subset T(\tilde{M}, \theta_{Y \times Z_1 \times Z_2}),$$

and thus we get (3.11). Hence, assumption S'_V is fulfilled. □

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References

1 A. Barbagallo and A. Maugeri. Duality theory for a dynamic oligopolistic market equilibrium problem. *Optim.* **60** (2011), 29–52.

2 A. Barbagallo, P. Daniele, S. Giuffrè and A. Maugeri. Variational approach for a general financial equilibrium problem: the deficit formula, the balance law and the liability formula. A path to the economy recovery. *Eur. J. Oper. Res.* **237** (2014), 231–244.

3 P. Daniele and S. Giuffré. General infinite dimensional duality and applications to evolutionary network equilibrium problems. *Optim. Lett.* **1** (2007), 227–243.

4 P. Daniele, S. Giuffré and A. Maugeri. Remarks on general infinite dimensional duality with cone and equality constraints. *Commun. Appl. Analysis* **13** (2009), 567–578.

5 P. Daniele, S. Giuffré, A. Maugeri and F. Raciti. Duality theory and applications to unilateral problems. *J. Optim. Theory Applic.* **162** (2014), 718–734.

6 M. B. Donato. The infinite dimensional Lagrange multiplier rule for convex optimization problems. *J. Funct. Analysis* **261** (2011), 2083–2093.

7 M. B. Donato and M. Milasi. Lagrangean variables in infinite dimensional spaces for a dynamic economic equilibrium problem. *Nonlin. Analysis* **74** (2011), 5048–5056.

8 M. B. Donato, M. Milasi and C. Vitanza. Duality theory for a Walrasian equilibrium problem. *J. Nonlin. Conv. Analysis* **7** (2006), 393–404.

9 M. B. Donato, A. Maugeri, M. Milasi and C. Vitanza. Duality theory for a dynamic Walrasian pure exchange economy. *Pac. J. Optim.* **4** (2008), 537–547.

10 M. B. Donato, M. Milasi and L. Scrimali. Walrasian equilibrium problem with memory term. *J. Optim. Theory Applic.* **151** (2011), 64–80.

11 M. Durea, J. Dutta and C. Tammer. Bounded sets of Lagrange multipliers for vector optimization problems in infinite dimension. *J. Math. Analysis Applic.* **348** (2008), 589–606.

12 J. Dutta and C. Tammer. Lagrangian conditions for vector optimization in Banach spaces. *Math. Meth. Oper. Res.* **64** (2006), 521–540.

13 S. Giuffrè and A. Maugeri. A measure-type Lagrange multiplier for the elastic-plastic torsion. *Nonlin. Analysis* **102** (2014), 23–29.

14 J. Jahn. *Vector optimization* (Springer, 2011).

15 B. Jiménez, V. Novo and M. Sama. An extension of the basic constraint qualification to nonconvex vector optimization problems. *J. Global Optim.* **56** (2013), 1755–1771.

16 T. D. Luc. *Theory of vector optimization*, Lecture Notes in Economics and Mathematical Systems, vol. 319 (Springer, 1989).

17 A. Maugeri and D. Puglisi. A new necessary and sufficient condition for the strong duality and the infinite dimensional Lagrange multiplier rule. *J. Math. Analysis Applic.* **415** (2014), 661–676.

18 A. Maugeri and F. Raciti. On general infinite dimensional complementarity problems. *Optim. Lett.* **2** (2008), 71–90.