New Deformations of Convolution Algebras and Fourier Algebras on Locally Compact Groups

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Abstract. In this paper we introduce a new way of deforming convolution algebras and Fourier algebras on locally compact groups. We demonstrate that this new deformation allows us to reveal some information about the underlying groups by examining Banach algebra properties of deformed algebras. More precisely, we focus on representability as an operator algebra of deformed convolution algebras on compact connected Lie groups with connection to the real dimension of the underlying group. Similarly, we investigate complete representability as an operator algebra of deformed Fourier algebras on some finitely generated discrete groups with connection to the growth rate of the group.

1 Introduction

For a locally compact group G it has long been a tradition to investigate its associated Banach algebras, namely the convolution algebra $L^1(G)$ and the Fourier algebra $A(G)$, in the hope that we could find connections between Banach algebraic properties of $L^1(G)$ (or $A(G)$) and the group properties of G. This line of research is based on the fundamental result of Wendel ([23]) (resp. Walter ([22])) saying that for two locally compact groups G and H, the algebras $L^1(G)$ and $L^1(H)$ (resp. $A(G)$ and $A(H)$) are isometrically isomorphic if and only if G and H are isomorphic as topological groups. Of course, making concrete connections between two objects is a completely different task, and this is one of the most successful examples of such connections. The celebrated results by B. E. Johnson [11] and Z.-J. Ruan [18] tell us that $L^1(G)$ is amenable as a Banach algebra if and only if G is amenable if and only if $A(G)$ is operator amenable as a completely contractive Banach algebra. Recall that G is called *amenable* if $L^\infty(G)$ has a left invariant mean, and a (completely contractive) Banach algebra $\mathcal{A}$ is called *(operator) amenable* if every (completely) bounded derivation $D: \mathcal{A} \to X^*$ for any (operator) $\mathcal{A}$-bimodule X is inner (*i.e.*, there is $\phi \in X^*$ such that $D(a) = \phi \cdot a - a \cdot \phi$, $a \in \mathcal{A}$).

The list of such connections continues, but at the same time there certainly are limitations. One possibility at this point would be to consider modified versions of

$L^1(G)$ and $A(G)$, expecting further connections between group properties and Banach algebraic properties. The construction of weighted convolution algebras begins with a choice of Borel measurable (or continuous) weight function $w: G \to (0, \infty)$ that is sub-multiplicative (*i.e.,* $w(xy) \le w(x)w(y)$, $x, y \in G$ a.e.). Now we consider the weighted L^1 space

$$L^1(G, w) := \left\{ f: G \to \mathbb{C} \mid \|f\|_{L^1(G,w)} := \int_G |f(x)| w(x) dx < \infty \right\}.$$

The sub-multiplicativity of w ensures that $L^1(G, w)$ is still a Banach algebra with respect to the convolution product. In other words, weighted convolution algebras are obtained by modifying the norm structure via multiplying by the weight function when we calculate the L^1-norm but essentially keeping the same algebra multiplication, which is convolution in this case. As is expected there are results establishing connections between weighted algebras and the groups. Recall that a Banach algebra $\mathcal{A}$ is called *representable as an operator algebra* if there is an operator algebra $\mathcal{B}$ (*i.e.,* a closed subalgebra of $B(H)$ for some Hilbert space H) and a bijective bounded isomorphism $T: \mathcal{A} \to \mathcal{B}$ with bounded inverse T^{-1}. We define *complete representability as an operator algebra* of a completely contractive Banach algebra similarly. In [14] it was proved that $\ell^1(G, \omega_\beta)$ is representable as an operator algebra if $\beta > \frac{k_0+1}{2}$, where G is a finitely generated discrete group with polynomial growth of order k_0. Note that weighted convolution algebras, in general, have been studied extensively; see [3, 8] and the references therein, for example.

The corresponding investigation for Fourier algebras was begun quite recently by Ludwig, Spronk, and Turowska [15] and Lee and Samei [13]. Weighted Fourier algebras follow the same philosophy of modification with more involved technicalities, and there are results connecting Banach algebraic properties and group properties. In [9] it is proved that for a compact connected Lie group G the weighted Fourier algebra $A(G, w)$ is completely representable as an operator algebra if w is a "polynomially growing weight" whose growth of order is strictly greater than $d(G)/2$, where $d(G)$ is the real dimension of the Lie group G. These results show us that group information of G such as polynomial growth rate or real Lie group dimension are reflected in weighted convolution (Fourier) algebras.

This paper deals with a different way of modifying $L^1(G)$ and $A(G)$. The main difference from weighted versions is that we would like to multiply a certain fixed "function" by the Fourier transform of the given function. Suppose for the moment that G is abelian and consider a Borel measurable $w: \widehat{G} \to (0, \infty)$, where $\widehat{G}$ is the dual group of G. Then we define the deformed L^1-norm by

$$\|\|f\|\| := \|\mathcal{F}^{-1}(w \cdot \mathcal{F}(f))\|_{L^1(G)},$$

for "nice" functions $f \in L^1(G)$, where $\mathcal{F}$ is the group Fourier transform on G. If we set $g = \mathcal{F}^{-1}(w \cdot \mathcal{F}(f))$, then $f = \mathcal{F}^{-1}(\frac{1}{w} \cdot \mathcal{F}(g))$. In other words we are looking at the map

$$\Phi: L^1(G) \longrightarrow L^1(G), \quad g \longmapsto \mathcal{F}^{-1}\left(\frac{1}{w} \cdot \mathcal{F}(g) \right)$$

and $\|\|f\|\| = \|g\|_{L^1(G)}$ with $f = \Phi(g)$. Of course, we want Φ to be well-defined and bounded, which means that Φ is nothing but a (Fourier) multiplier on $L^1(G)$. From

a classical result on multipliers on $L^1(G)$ we know that there must be a complex measure $\mu \in M(G) \cong ML^1(G)$ such that

$$\Phi(g) = \mu * g, \quad g \in L^1(G).$$

This informal observation suggests the deformed L^1-space $L^1_\mu(G)$ given by

$$L^1_\mu(G) = \mu * L^1(G)$$

with the norm $\|\mu * g\|_\mu = \|g\|_{L^1(G)}$, $g \in L^1(G)$. This definition can be easily extended to the case of general locally compact groups. The case of deformed Fourier algebras follows the same idea, so that we begin with an element in the multiplier algebra $M_{cb}A(G)$. See the detailed rigorous definitions in Section 3.

Given these new deformations we would like to focus again on (complete) representability as an operator algebra, expecting that we could extract similar information on the underlying groups. Indeed, we prove the following results in this paper. Let G be a compact connected Lie group and let ν_α be the probability measure whose Fourier coefficients are polynomially decreasing of order α on $\widehat{G}$ (see Section 4.2 for the precise definitions).

Theorem A   *Let G be a compact connected Lie group. The algebra $L^1_{\nu_\alpha}(G)$ is (completely) representable as an operator algebra if and only if $\alpha > d(G)/2$, where $d(G)$ is the real dimension of G.*

We also have a corresponding result for the dual setting. Let G be a finitely generated discrete group belonging to a certain class of groups (more precisely, G is either $\mathbb{Z}^n$, a Coxeter group, or a hyperbolic group). Let W_α and w_t be the functions in $M_{cb}A(G)$ decreasing polynomially of order α and exponentially, respectively, with respect to the canonical word length function. See Section 5.1 for the precise definition. Then we have the following result.

Theorem B   *Let G be a discrete group as described above.*

(i)   *Suppose G is of polynomial growth of order k_0. The algebra $A_{W_\alpha}(G)$ is completely representable as an operator algebra if and only if $\alpha > \frac{k_0}{2}$.*
(ii)   *Suppose that G is exponential growing with the growth rate λ. The algebra $A_{w_t}(G)$ is completely representable as an operator algebra if $t > \log \lambda/2$ and $A_{w_t}(G)$ is not completely representable as an operator algebra if $t < \log \lambda/2$.*

These results also show us that the polynomial (or exponential) growth rate of some finitely generated groups and the real dimension of compact connected Lie groups can be precisely detected by examining complete representability as an operator algebra of the corresponding deformed algebras.

There are a few advantages of new deformations compared to the weighted versions. First, some information on the groups can be precisely detected in the above theorems, whilst we only have partial results in the theory of weighted algebras ([9, 14]). Secondly, new deformations can be applied to convolution algebras and Fourier algebras on those groups with trivial weighted versions. For example, when G is compact the weighted convolution algebra $L^1(G, w)$ is isomorphic to $L^1(G)$ as Banach

algebras, since every weight function w is known to be ([12, Lemma 1.3.3]) bounded and bounded away from zero (in other words, equivalent to the constant 1 function). For the same reason weighted Fourier algebras on discrete groups have never been investigated. However, we focus on $L_\mu^1(G)$ with compact G and $A_w(G)$ with discrete G in this article.

This paper is organized as follows. In Section 2 we collect some preliminaries we need. In Section 3 we define deformed algebras $L_\mu^1(G)$ and $A_w(G)$ for a locally compact group G. In Section 4 we focus on $L_\mu^1(G)$ for a compact group G and prove that representability as an operator algebra is closely related to the square-integrability of the deformation measure μ. Moreover, we apply this to establish connections between representability as an operator algebra of $L_\mu^1(G)$ and the dimension of G when G is a compact connected Lie group. In Section 5 we turn our attention to the case of $A_w(G)$ for discrete groups. We also prove a general result saying that representability as an operator algebra is equivalent to the square-summability of the deformation function w. We apply this to a certain class of finitely generated groups and show that complete representability as an operator algebra of $A_w(G)$ is closely related to the growth rate of G.

2 Preliminaries

2.1 Operator Spaces

We will assume that the reader is familiar with standard operator space theory (see [6], for instance) including injective, projective, and Haagerup tensor products of operator spaces, which we denote by $\otimes_{\min}$, $\widehat{\otimes}$, and $\otimes_h$, respectively. We will also frequently use a dual version of a Haagerup tensor product, namely the extended Haagerup tensor product. The extended Haagerup tensor product of dual operator spaces E^* and F^* will be denoted by

$$E^* \otimes_{\mathrm{eh}} F^*,$$

and is given by $(E \otimes_h F)^*$ in [2]. There are several characterizations of $\otimes_{\mathrm{eh}}$, but we will only be using the following two aspects. First, for $X \in M_n(E^* \otimes_{\mathrm{eh}} F^*)$ we have

$$\|X\|_{\mathrm{eh}} = \min\big\{ \|A\| \|B\| \big\},$$

where the minimum runs over all possible factorization satisfying $X = A \odot B$ with $A \in M_{n,I}(E^*)$ and $B \in M_{I,n}(F^*)$ and $\odot$ is the Haagerup product given by

$$(A_1 \otimes A_2) \odot (B_1 \otimes B_2) = A_1 B_1 \otimes A_2 \otimes B_2$$

for $A_1 \in M_{n,I}, B_1 \in M_{I,n}$ and $A_2 \in E^*, B_2 \in F^*$. Note that the index set I could be arbitrary. See [19, Theorem 2.4]. Secondly, if $E^* \subseteq B(H)$ for some Hilbert space H, then we have a completely isometric embedding

$$E^* \otimes_{\mathrm{eh}} F^* \hookrightarrow CB^\sigma\big(B(H), B(H)\big), \quad A \otimes B \longmapsto T_{A,B},$$

where $CB^\sigma(B(H), B(H))$ refers to the space of all w^*-w^*-continuous completely bounded maps and $T_{A,B}(X) = AXB, X \in B(H)$.

A Banach algebra $\mathcal{A}$ with the algebra multiplication map

$$m\colon \mathcal{A} \otimes_\gamma \mathcal{A} \longrightarrow \mathcal{A},$$

where $\otimes_\gamma$ is the projective tensor product of Banach spaces, is called a *completely contractive Banach algebra* if $\mathcal{A}$ is endowed with an operator space structure and the map m extends to a complete contraction $m\colon \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$. Note that any operator algebra carries a natural operator space structure, which makes it a completely contractive Banach algebra. Operator algebras form a quite distinctive class of completely contractive Banach algebras. In the category of completely contractive Banach algebras we have the following characterization of operator algebras by Blecher ([1]).

Theorem 2.1   *Let $\mathcal{A}$ be a completely contractive Banach algebra with the algebra multiplication $m\colon \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$. Then $\mathcal{A}$ is completely representable as an operator algebra if and only if the multiplication map extends to a completely bounded map $m\colon \mathcal{A} \otimes_h \mathcal{A} \to \mathcal{A}$.*

2.2   Relevant Spaces in Abstract Harmonic Analysis

Let G be a locally compact group, and we denote the convolution algebra and the measure algebra of G with $L^1(G)$ and $M(G)$, respectively. It is well known that $L^1(G)$ is a 2-sided closed ideal in $M(G)$, so that for $\mu \in M(G)$ we have the (left) multiplier

$$M_\mu\colon L^1(G) \longrightarrow L^1(G), \quad f \longmapsto \mu * f.$$

Moreover, it is also known that the multiplier algebra $M(L^1(G))$ of $L^1(G)$ can be identified with $M(G)$. We note that $L^1(G)$ is equipped with the operator space structure as the predual of $L^\infty(G)$, and similarly $M(G)$ has a natural dual operator space structure, since we have $M(G) = (C_0(G))^*$, where $C_0(G)$ is the algebra of continuous functions on G vanishing at infinity.

Let $P(G)$ be the set of all continuous positive definite functions on G and let $B(G)$ be its linear span. The space $B(G)$ can be identified with the dual of the full group C^*-algebra $C^*(G)$, which is the completion of $L^1(G)$ under its largest C^*-norm. The space $B(G)$ with the pointwise multiplication and the dual norm is a commutative Banach algebra. The Fourier algebra $A(G)$ is the closure of $B(G) \cap C_c(G)$ in $B(G)$. It was shown in [5] that $A(G)$ is a commutative Banach algebra that is a 2-sided closed ideal in $B(G)$. Thus, any element $w \in B(G)$ gives rise to a multiplier on $A(G)$, but we actually have more multipliers. Recall that a function w on G is called a *multiplier* on $A(G)$ if $w \cdot A(G) \subseteq A(G)$. Then we have the multiplier

$$M_w\colon A(G) \longrightarrow A(G), \quad g \longmapsto w \cdot g,$$

which is automatically bounded. We denote the collection of all multipliers on $A(G)$ by $MA(G)$. Note that the space $A(G)$ is equipped with the operator space structure as the predual of $VN(G)$, the group von Neumann algebra of G generated by the translation operators λ_g, $g \in G$ on $L^2(G)$ given by $\lambda_g f(x) = f(g^{-1}x)$, $f \in L^2(G)$, $x \in G$. We define the space of all cb-multipliers $M_{\mathrm{cb}}A(G)$ by

$$M_{\mathrm{cb}}A(G) := \{w \in MA(G) : \|M_w\|_{\mathrm{cb}} < \infty\}.$$

Both of the spaces are clearly commutative Banach algebras with respect to pointwise multiplication and we have the following inclusions:

$$A(G) \subseteq B(G) \subseteq M_{\mathrm{cb}}A(G) \subseteq MA(G).$$

Let G be a compact group. Then any element $\mu \in M(G)$ can be understood through its Fourier coefficients $(\widehat{\mu}(\pi))_{\pi \in \widehat{G}}$, where $\widehat{G}$ is the equivalence class of irreducible unitary representations on G and

$$\widehat{\mu}(\pi) := \int_G \pi(g^{-1})^t d\mu(g).$$

We use the notation

$$\mu \sim \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\big(\widehat{\mu}(\pi)\pi^t\big),$$

which comes from the Fourier inversion formula stating that

$$f(x) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\big(\widehat{f}(\pi)\pi^t(x)\big)$$

for $f \in A(G) \cap L^1(G)$.

When G is a discrete group we use a similar notation. Let $\tau(\cdot)$ be the vacuum state on $VN(G)$ given by $\tau(\cdot) = \langle \cdot \delta_e, \delta_e \rangle$, where δ_e is the point mass function on the identity e of G and $L^2(VN(G))$ the associated L^2-space, which is the completion of $VN(G)$ with respect to the inner product $\langle \lambda_{g_1}, \lambda_{g_2} \rangle := \tau(\lambda_{g_2}^* \lambda_{g_1})$. Note that we have $\ell^2(G) \cong L^2(VN(G))$ via the identification $\delta_g \mapsto \lambda_g$. Then clearly we have $VN(G) \subseteq L^2(VN(G))$ and any $T \in VN(G)$ is associated with a uniquely determined sequence $(\alpha_g)_{g \in G} \in \ell^2(G)$. In this case we write

$$T \sim \sum_{g \in G} \alpha_g \lambda_g.$$

We will use the same notation for elements in

$$M_n(M(G)), \quad M_n(VN(G)), \quad \text{and} \quad M_n(VN(G \times G)).$$

3 Construction of Deformed Convolution Algebras and Fourier Algebras

Definition 3.1 Let $\mu \in M(G)$ and $w \in M_{\mathrm{cb}}A(G)$ be norm 1 elements such that the corresponding multipliers M_μ and M_w are injective with dense range. We define the *deformed spaces* $L^1_\mu(G)$ and $A_w(G)$ as follows:

$$L^1_\mu(G) := \mu * L^1(G), \quad A_w(G) := w \cdot A(G),$$

with the norms $\|\cdot\|_\mu$ and $\|\cdot\|_w$ given by

$$\|\mu * f\|_\mu := \|f\|_{L^1(G)} \quad \text{and} \quad \|w \cdot g\|_w := \|g\|_{A(G)}, \quad f \in L^1(G), \quad g \in A(G).$$

In this case we call μ and w the *deformation measure* for $L^1_\mu(G)$ and the *deformation function* for $A_w(G)$, respectively.

Remark 3.2 (i) The injectivity of the associated multipliers implies that the above norm formulas are well defined.

(ii) In this case we have natural onto isometries:

$$\Phi: L^1(G) \longrightarrow L^1_\mu(G), \quad f \longmapsto \mu * f \quad \text{and} \quad \Psi: A(G) \longrightarrow A_w(G), \quad g \longmapsto w \cdot g.$$

(iii) The natural operator space structures on $L^1_\mu(G)$ and $A_w(G)$ are the ones making Φ and Ψ complete isometries, respectively.

Proposition 3.3 *The following maps are completely contractive:*

$$m_\mu: L^1_\mu(G) \widehat{\otimes} L^1_\mu(G) \longrightarrow L^1_\mu(G), \qquad f \otimes g \longmapsto f * g,$$
$$m_w: A_w(G) \widehat{\otimes} A_w(G) \longrightarrow A_w(G), \qquad f \otimes g \longmapsto f \cdot g.$$

Proof By applying the complete isometry Φ we get

$$\widetilde{m_\mu} = \Phi^{-1} \circ m_\mu \circ (\Phi \otimes \Phi): L^1(G) \widehat{\otimes} L^1(G) \longrightarrow L^1(G), \quad f * \mu * g,$$

which is clearly a complete contraction, so that we know m_μ is also a complete contraction. The explanation for m_w is the same.   ∎

Definition 3.4 The completely contractive Banach algebra $(L^1_\mu(G), m_\mu)$ is called *a deformed convolution algebra on G*. The completely contractive Banach algebra $(A_w(G), m_w)$ is called *a deformed Fourier algebra on G*.

We record here the deformed multiplication maps:

$$\widetilde{m_\mu} = \Phi^{-1} \circ m_\mu \circ (\Phi \otimes \Phi): L^1(G) \widehat{\otimes} L^1(G) \longrightarrow L^1(G), \quad f * \mu * g =: f *_\mu g,$$
$$\widetilde{m_w} = \Psi^{-1} \circ m_w \circ (\Psi \otimes \Psi): A(G) \widehat{\otimes} A(G) \longrightarrow A(G), \quad f \cdot w \cdot g =: f \cdot_w g.$$

Remark 3.5   (i) We can identify the following algebras as completely contractive Banach algebras:

$$\left(L^1_\mu(G), m_\mu \right) \cong \left(L^1(G), \widetilde{m_\mu} \right), \quad \left(A_w(G), m_w \right) \cong \left(A(G), \widetilde{m_w} \right).$$

(ii) Definition 3.1 can easily be extended to the case of a general Banach algebra $\mathcal{A}$. We fix a norm 1 element $a \in M(\mathcal{A})$ with the left multiplication map $L_a: \mathcal{A} \to \mathcal{A}$, $b \mapsto a \cdot b$ being injective and $\mathrm{ran} L_a$ being dense. Then we define $\mathcal{A}_a := a \cdot \mathcal{A}$ with the norm $\|a \cdot b\|_{\mathcal{A}_a} := \|b\|_{\mathcal{A}}$ for $b \in \mathcal{A}$. Then Proposition 3.3 still holds so that the space $\mathcal{A}_a$ becomes a subalgebra of $\mathcal{A}$. However, we would like to focus on the specific case of $L^1_\mu(G)$ and $A_w(G)$ in this paper.

(iii) Note that the above construction produces a trivial object when the algebra $\mathcal{A}$ is unital. Indeed, for a unital $\mathcal{A}$ the set of left multipliers of $\mathcal{A}$, $L(\mathcal{A})$, coincides with $\mathcal{A}$ itself via the map $a \mapsto L_a$. The injectivity of L_a and the density of $\mathrm{ran} L_a$ imply that there is $b \in \mathcal{A}$ such that

$$\|L_a(b) - 1_{\mathcal{A}}\|_{\mathcal{A}} < 1,$$

which implies that $L_a(b) = a \cdot b$ is invertible, and so a is right invertible. This means that L_a is surjective and consequently an isomorphism. Thus, the two norms $\|\cdot\|_{\mathcal{A}_a}$ and $\|\cdot\|_{\mathcal{A}}$ are equivalent on the $\mathcal{A}_a$ so that $\mathcal{A}_a$ is isomorphic to $\mathcal{A}$.

The above observation tells us that we need a non-discrete G for $L^1_\mu(G)$ and a non-compact G for $A_w(G)$.

We end this section by describing the duality for the spaces $L^1_\mu(G)$ and $A_w(G)$. We only consider the $L^1_\mu(G)$ case, since the other case is similar. Our understanding of the space $L^1_\mu(G)$ is based on the natural inclusion $L^1_\mu(G) \subseteq L^1(G)$. Recall that $L^1_\mu(G)$

is dense in $L^1(G)$, which is equivalent to the fact that the following restriction map is injective:

$$\iota : L^\infty(G) \cong (L^1(G))^* \hookrightarrow (L^1_\mu(G))^*, \quad \varphi \longmapsto \varphi|_{L^1_\mu(G)}.$$

Note that $L^\infty(G)$ (or more precisely $\iota(L^\infty(G))$) is clearly a w^*-dense subspace in $(L^1_\mu(G))^*$. Thus, we can say that the dual space $(L^1_\mu(G))^*$ is "not too big" in the sense that it contains a w^*-dense subspace $L^\infty(G)$, which is a concrete function space. This is why we require M_μ to have dense range. Note also that this condition is redundant in the cases we are focusing on in Sections 4 and 5 (see Lemmas 4.1 and 5.1).

Remark 3.6   (i) Note that $L^1_\mu(G)$ is, in general, not closed in $L^1(G)$ with respect to the L^1-norm.

    (ii) Recall the onto isometry $\Phi : L^1(G) \to L^1_\mu(G)$ gives us again an onto isometry

$$\Phi^* : (L^1_\mu(G))^* \longrightarrow L^\infty(G).$$

Now we can readily check that

$$(\Phi^* \circ \iota)(g) = \check{\mu} * g, \quad g \in L^\infty(G),$$

where $\check{\mu}$ is the measure given by

$$\check{\mu}(E) = \mu(E^{-1}).$$

4 Representability of the Deformed Convolution Algebras on Compact Groups as Operator Algebras

In this section G is always a compact group.

4.1 The General Case

Lemma 4.1   *Let $\mu \in M(G)$ with norm 1. Then the associated multiplier M_μ is injective if and only if $\widehat{\mu}(\pi)$ is invertible for any $\pi \in \widehat{G}$. Moreover, in this case M_μ has dense range automatically.*

Proof   Note that L_μ is also decomposed into a direct sum of operators acting on finite dimensional spaces. Then this is trivial. ∎

In this section we will provide a characterization of representability of the deformed algebra $L^1_\mu(G)$ as an operator algebra. For that purpose we focus on the equivalent algebra $(L^1(G), \widetilde{m_\mu})$ with the dual perspective. Indeed, we can easily see that $L^1_\mu(G)$ is completely representable as an operator algebra if and only if the map

$$(\widetilde{m_\mu})^* : L^\infty(G) \longrightarrow L^\infty(G) \overline{\otimes} L^\infty(G)$$

extends to a completely bounded map $(\widetilde{m_\mu})^* : L^\infty(G) \to L^\infty(G) \otimes_{\mathrm{eh}} L^\infty(G)$. Here $\overline{\otimes}$ is the spatial tensor product of von Neumann algebras. We call the map $(\widetilde{m_\mu})^*$, *the deformed co-multiplication*. We first need to know how the deformed co-multiplication $(\widetilde{m_\mu})^*$ acts on concrete elements of $L^\infty(G)$.

Proposition 4.2   *For $f \in L^\infty(G)$ we have*

$$(\widetilde{m_\mu})^*(f)(x, y) = \int_G f(xzy) d\mu(z), \quad x, y \in G.$$

In particular for $f = \pi_{ij}$ we have

$$(4.1) \qquad (\widetilde{m_\mu})^*(\pi_{ij}) = \sum_{k=1}^{d_\pi} \pi_{ik} \otimes [\widehat{\mu}(\overline{\pi})\pi]_{kj} = \sum_{k=1}^{d_\pi} \pi_{ik} \otimes \Big(\sum_{l=1}^{d_\pi} \widehat{\mu}(\overline{\pi})_{kl} \pi_{lj} \Big),$$

where $\overline{\pi}$ is the conjugate representation of π and $\overline{A}$ is the matrix complex conjugate given by $[\overline{A}]_{ij} = \overline{a_{ij}}$.

Proof   For $g, h \in L^1(G)$ we have

$$\langle (\widetilde{m_\mu})^*(f), g \otimes h \rangle = \langle f, g * \mu * h \rangle$$
$$= \int_G \int_G \int_G g(x) h(z^{-1} x^{-1} y) f(y) d\mu(z) dx dy$$
$$= \int_G \int_G \int_G g(x) h(y) f(xzy) d\mu(z) dx dy$$
$$= \int_G \int_G \Big(\int_G f(xzy) d\mu(z) \Big) g(x) h(y) dx dy.$$

In particular, for $f = \pi_{ij}$ we have

$$(\widetilde{m_\mu})^*(\pi_{ij}) = \int_G \pi_{ij}(xzy) d\mu(z) = \Big[\int_G \pi(xzy) d\mu(z) \Big]_{ij}$$
$$= \Big[\pi(x) \Big(\int_G \pi(z) d\mu(z) \Big) \pi(y) \Big]_{ij},$$

which gives us the desired conclusion.   ∎

We also need the following theorem by S. Helgason, which is a compact group generalization of a Littlewood's theorem.

Theorem 4.3 (Helgason [10])   *Let $A = (A^\pi)_{\pi \in \widehat{G}}$ be any family of matrices with $A^\pi \in M_{d_\pi}$. Suppose that the formal series*

$$\sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}(A^\pi U^\pi \pi^t)$$

belong to $M(G)$ for any choice of unitary $U = (U^\pi)_{\pi \in \widehat{G}}$, $U^\pi \in \mathcal{U}(d_\pi)$, where $\mathcal{U}(n)$ refers to the set of all $n \times n$ unitary matrices. Then we have

$$\sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}((A^\pi)^* A^\pi) < \infty.$$

Here is the main result of this section.

Theorem 4.4   *The deformed algebra $L^1_\mu(G)$ is completely representable as an operator algebra if $\mu \in L^2(G)$. The converse is true when μ is central, i.e., $\widehat{\mu}(\pi) = c_\pi I_\pi$ for some c_π for all $\pi \in \widehat{G}$. Moreover, the same holds in the category of Banach spaces.*

Proof First we suppose that $\mu \in L^2(G)$. If we pick $F = [f_{kl}]_{kl} \in M_n(L^\infty(G))$, then we have

$$F \sim \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \widehat{F}(\pi)_{ij} \otimes \pi_{ij},$$

where $\widehat{F}(\pi)_{ij} = [(\widehat{f_{kl}})_{ij}]_{kl} \in M_n$. Then by (4.1) we have

$$(I_n \otimes (\widetilde{m_\mu})^*)(F) \sim \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j,k=1}^{d_\pi} \widehat{F}(\pi)_{ij} \otimes \pi_{ik} \otimes [\widehat{\mu}(\overline{\pi})\pi]_{kj}$$

$$= \sum_{\pi \in \widehat{G}} \sum_{j,k=1}^{d_\pi} \Big(\sum_{i=1}^{d_\pi} \sqrt{d_\pi}\widehat{F}(\pi)_{ij} \otimes \pi_{ik} \Big) \odot \Big(\sqrt{d_\pi}I_n \otimes [\widehat{\mu}(\overline{\pi})\pi]_{kj} \Big)$$

$$= A \odot B,$$

where A and B are row and column matrices, respectively, given as follows. For the index set $J = \{(\pi, j, k) : \pi \in \widehat{G}, 1 \le j, k \le d_\pi\}$, the matrix A is the $M_n(L^\infty(G))$-valued row matrix in $M_{1,J}$ whose (π, j, k)-th entry is $\sum_{i=1}^{d_\pi} \sqrt{d_\pi}\widehat{F}(\pi)_{ij} \otimes \pi_{ik} \in M_n(L^\infty(G))$. Similarly, the matrix B is the $M_n(L^\infty(G))$-valued column matrix in $M_{J,1}$ whose (π, j, k)-th entry is $\sqrt{d_\pi}I_n \otimes [\widehat{\mu}(\overline{\pi})\pi]_{kj} \in M_n(L^\infty(G))$. Moreover, we have

$$\|A\|^2_{M_{1,J}(M_n(L^\infty(G)))} = \Big\| \sum_{\pi \in \widehat{G}} \sum_{j,k=1}^{d_\pi} \Big(\sum_{i,i'=1}^{d_\pi} d_\pi \widehat{F}(\pi)_{ij}\widehat{F}(\pi)^*_{i'j} \otimes \pi_{ik}\overline{\pi_{i'k}} \Big) \Big\|_{M_n(L^\infty(G))}$$

$$= \Big\| \sum_{\pi \in \widehat{G}} \sum_{j,i,i'=1}^{d_\pi} d_\pi \widehat{F}(\pi)_{ij}\widehat{F}(\pi)^*_{i'j}\delta_{i,i'} \otimes 1_G \Big\|_{M_n(L^\infty(G))}$$

$$= \Big\| \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi \widehat{F}(\pi)_{ij}\widehat{F}(\pi)^*_{ij} \Big\|_{M_n}$$

$$= \Big\| \int_G F(x)F^*(x)dx \Big\|_{M_n} \le \|F\|^2_{M_n(L^\infty(G))}.$$

For B we have

$$\|B\|^2_{M_{J,1}(M_n(L^\infty(G)))} = \Big\| \sum_{\pi \in \widehat{G}} \sum_{j,k=1}^{d_\pi} d_\pi (I_n)^* I_n \otimes \overline{[\widehat{\mu}(\overline{\pi})\pi]_{kj}}[\widehat{\mu}(\overline{\pi})\pi]_{kj} \Big\|_{M_n(L^\infty(G))}$$

$$= \Big\| I_n \otimes \sum_{\pi \in \widehat{G}} d_\pi \sum_{j,k=1}^{d_\pi} |[\widehat{\mu}(\overline{\pi})]_{kj}|^2 \cdot 1_G \Big\|_{M_n(L^\infty(G))}$$

$$= \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{\mu}(\overline{\pi})\|^2_{S^2_{d_\pi}} = \|\mu\|^2_{L^2(G)}.$$

Note that we use unitarity of π for the second equality. Now combining the above two we get

$$\|(I_n \otimes (\widetilde{m_\mu})^*)(F)\|_{\mathrm{eh}} \le \|F\|_{M_n(L^\infty(G))} \cdot \|\mu\|_{L^2(G)},$$

which implies that $(\widetilde{m_\mu})^*$ is completely bounded with cb-norm $\le \|\mu\|_{L^2(G)}$.

For the converse direction we assume that $L^1_\mu(G)$ is completely representable as an operator algebra and μ is central with $\widehat{\mu}(\pi) = c_\pi I_\pi$, $\pi \in \widehat{G}$. If we take any $f \in L^\infty(G)$,

then we have

$$f \sim \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{ij} \pi_{ij}.$$

By (4.1) we also have

$$(\widetilde{m_\mu})^*(f) \sim \sum_{\pi \in \widehat{G}} c_{\overline{\pi}} d_\pi \sum_{i,j,k=1}^{d_\pi} \widehat{f}(\pi)_{ij} \pi_{ik} \otimes \pi_{kj}.$$

Now we will use the embedding $L^\infty(G) \otimes_{eh} L^\infty(G) \hookrightarrow CB^\sigma(B(L^2(G)))$. For any choice of unitary $U = (U^\pi)_{\pi \in \widehat{G}}$, $U^\pi \in \mathcal{U}(d_\pi)$, we define $T_U \in B(L^2(G))$ by

$$T_U(\pi_{ij}) := [\pi^* U^\pi]_{ij} = \sum_{r=1}^{d_\pi} \overline{\pi}_{ri} U_{rj}^\pi.$$

We can actually check that T_U is a linear isometry. Indeed, for any $\pi \in \widehat{G}$ we have

$$\langle T_U(\pi_{ij}), T_U(\sigma_{kl}) \rangle = \sum_{r,s=1}^{d_\pi} \langle \overline{\pi}_{ri} U_{rj}^\pi, \overline{\sigma}_{sk} U_{sl}^\pi \rangle = \sum_{r,s=1}^{d_\pi} \frac{1}{d_\pi} \delta_{\pi\sigma} \delta_{rs} \delta_{ik} U_{rj}^\pi \overline{U_{sl}^\pi}$$

$$= \frac{\delta_{\pi\sigma} \delta_{jl} \delta_{ik}}{d_\pi} = \langle \pi_{ij}, \sigma_{kl} \rangle.$$

Now we recall the embedding $L^\infty(G) \hookrightarrow B(L^2(G))$, $f \mapsto M_f$, where M_f is the multiplication operator with respect to f, so that we have

$$(\widetilde{m_\mu})^*(f)(T_U)(1_G) = \sum_{\pi \in \widehat{G}} c_{\overline{\pi}} d_\pi \sum_{i,j,k=1}^{d_\pi} \widehat{f}(\pi)_{ij} M_{\pi_{ik}} \circ T_U \circ M_{\pi_{kj}}(1_G)$$

$$= \sum_{\pi \in \widehat{G}} c_{\overline{\pi}} d_\pi \sum_{i,j,k,l=1}^{d_\pi} \widehat{f}(\pi)_{ij} \pi_{ik} \overline{\pi}_{lk} U_{lj}^\pi$$

$$= \sum_{\pi \in \widehat{G}} c_{\overline{\pi}} d_\pi \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{ij} [\pi \pi^* U^\pi]_{ij}$$

$$= \sum_{\pi \in \widehat{G}} c_{\overline{\pi}} d_\pi \operatorname{Tr}(\widehat{f}(\pi)[U^\pi]^t) \cdot 1_G$$

$$= \langle (\widehat{f}(\pi))_{\pi \in \widehat{G}}, (c_{\overline{\pi}}[U^\pi]^t)_{\pi \in \widehat{G}} \rangle \cdot 1_G$$

for $f \in \operatorname{Pol}(G) := \operatorname{span}\{\pi_{ij} : \pi \in \widehat{G}, 1 \le i, j \le d_\pi\}$. Since $\operatorname{Pol}(G)$ is dense in $C(G)$, we have

$$\|(\widetilde{m_\mu})^*\| \ge \sup\{\|(\widetilde{m_\mu})^*(f)(T_U)(1_G)\|_{L^2(G)} : \|f\|_{C(G)} \le 1, f \in \operatorname{Pol}(G)\}$$

$$= \sup\{|\langle (\widehat{f}(\pi))_{\pi \in \widehat{G}}, (c_{\overline{\pi}}[U^\pi]^t)_{\pi \in \widehat{G}} \rangle| : \|f\|_{C(G)} \le 1, f \in \operatorname{Pol}(G)\}$$

$$= \Big\| \sum_{\pi \in \widehat{G}} c_{\overline{\pi}} d_\pi \operatorname{Tr}([U^\pi]^t \pi^t) \Big\|_{M(G)}.$$

Now we appeal to Theorem 4.3 to get the conclusion we wanted.

The result in the Banach space category follows easily from the above calculation and [14, Theorem 2.8]. Note that [14, Theorem 2.8] deals with discrete groups, but the same proof works for general locally compact groups. ∎

Remark 4.5 In the case of compact connected Lie groups we have a replacement of Theorem 4.3 in the proof of the negative direction of Theorem 4.4 as follows. Suppose that $L^1_\mu(G)$ is completely representable as an operator algebra and μ is central with $\widehat{\mu}(\pi) = c_\pi I_\pi$, $\pi \in \widehat{G}$. Recall that For any choice of unitary $U = (U^\pi)_{\pi \in \widehat{G}}$, $U^\pi \in \mathcal{U}(d_\pi)$, we have

$$\|(\widetilde{m_\mu})^*\| \geq \Big\| \sum_{\pi \in \widehat{G}} c_\pi d_\pi \operatorname{Tr}([U^\pi]^t \pi^t) \Big\|_{M(G)}.$$

Now we put $U^\pi = r_\pi I_\pi$, where $(r_\pi)_{\pi \in \widehat{G}}$ is an I.I.D. family of Bernoulli variables. We use cotype 2 condition of $M(G)$ to get

$$\mathbb{E} \Big\| \sum_{\pi \in \widehat{G}} c_\pi d_\pi r_\pi \operatorname{Tr}(\pi^t) \Big\|_{M(G)} \geq C \Big(\sum_{\pi \in \widehat{G}} d_\pi^2 |c_\pi|^2 \|\chi_\pi\|_1^2 \Big)^{\frac{1}{2}}$$

for some constant $C > 0$, where $\chi_\pi = \operatorname{Tr}(\pi) = \operatorname{Tr}(\pi^t)$ is the character function associated to π. Recall that a Banach space X is said to be of cotype 2 if there is a constant $D > 0$ such that

$$\mathbb{E} \Big\| \sum_i r_i x_i \Big\|_X \geq D \Big(\sum_i \|x_i\|_X^2 \Big)^{\frac{1}{2}}$$

for any $(x_i) \subseteq X$ and an I.I.D. family of Bernoulli variables (r_i). Note that the dual of a C^*-algebra is known to be of cotype 2 ([7]) and $M(G) \cong C_0(G)^*$. We finally note that that there is a constant $C' > 0$ such that $\|\chi_\pi\|_1 \geq C'$ for any $\pi \in \widehat{G}$([17]), which leads us to the conclusion we wanted.

4.2 The Case of Compact Connected Lie Groups and their Real Dimensions

In this subsection we apply Theorem 4.4 in the case of compact connected Lie groups with the deformation measures coming from the Laplacian on the group. We will demonstrate that the representability of $L^1_\mu(G)$ can precisely detect the dimension of the group G.

Example 4.6 Let $\Omega_\pi = -\kappa_\pi I_\pi$ be the Casimir operator (in other words, Laplacian on G) for $\pi \in \widehat{G}$ with $\kappa_\pi \geq 0$. Then there is a family of probability measures μ_t, $t > 0$ on G such that $\widehat{\mu_t}(\pi) = e^{-t\kappa_\pi} I_\pi$. Using μ_t we could find probability measures with polynomially decreasing Fourier coefficients by a standard argument. For $\alpha > 0$ we recall the formula

$$(1 + n)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} e^{-tn} dt,$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the Gamma function. We define

$$\nu_\alpha := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha/2-1} e^{-t} \mu_t \, dt$$

Then ν_α is clearly a probability measure with

$$(4.2) \qquad \widehat{\nu_\alpha}(\pi) = \frac{1}{(1 + \kappa_\pi)^{\alpha/2}} I_\pi, \quad \pi \in \widehat{G}.$$

Now we need some standard Lie theory. See [21] or [15, section 5] for the details. Let $\mathfrak{g}$ be the Lie algebra of G with the decomposition $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}_1$, where $\mathfrak{z}$ is the center of

$\mathfrak{g}$ and $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}_1$ and $T = \exp \mathfrak{t}$. Then there are fundamental weights $\lambda_1, \ldots, \lambda_r, \Lambda_1, \ldots, \Lambda_l \in \mathfrak{g}^*$ with $r = \dim \mathfrak{z}$ and $l = \dim \mathfrak{t}$ such that any $\pi \in \widehat{G}$ is in one-to-one correspondence with its associated highest weight $\Lambda_\pi = \sum_{i=1}^r a_i \lambda_i + \sum_{j=1}^l b_j \Lambda_j$ with $(a_i)_{i=1}^r \in \mathbb{Z}^r$ and $(b_j)_{j=1}^l \in \mathbb{Z}_+^l$. The 1-norm $\|\pi\|_1$ of π is given by

$$\|\pi\|_1 := \sum_{i=1}^r |a_i| + \sum_{j=1}^l b_j.$$

This 1-norm is known to be equivalent to $\sqrt{\kappa_\pi}$ from the Casimir operator. More precisely, there are positive constants c_1 and c_2 independent of π such that ([21, Lemma 5.6.6])

$$c_1 \|\pi\|_1^2 \le \kappa_\pi \le c_2 \|\pi\|_1^2.$$

Moreover, the following summability condition is known.

Proposition 4.7 ([4, Lemma 3.1])   *For $\alpha > 0$ we have*

$$\sum_{\pi \in \widehat{G}} \frac{d_\pi^2}{(1 + \|\pi\|_1)^{2\alpha}} < \infty$$

if and only if $\alpha > \frac{d(G)}{2}$, where $d(G)$ is the dimension of G as a real Lie group.

By combining Theorem 4.4 and Proposition 4.7 we get the following theorem.

Theorem 4.8   *Let ν_α be the probability measure from (4.2). Then $L^1_{\nu_\alpha}(G)$ is (completely) representable as an operator algebra if and only if $\alpha > d(G)/2$.*

Remark 4.9   (i) The above theorem tells us that the representability of $L^1_{\nu_\alpha}(G)$ as an operator algebra precisely detects the dimension of the underlying group.

(ii) The 1-norm is also known to be equivalent to the length function τ on $\widehat{G}$, which is given as follows. Let χ_i be the character of G associated with the highest weight λ_i and let π_j be the irreducible representation associated with the weight Λ_j. It is well known that $S = \{\pm\chi_i, \pi_j : 1 \le i \le r, 1 \le j \le l\}$ generates $\widehat{G}$, *i.e.*, $\bigcup_{k \ge 1} S^{\otimes k} = \widehat{G}$, where $S^{\otimes k} = \{\pi \in \widehat{G} : \pi \subset \sigma_1 \otimes \cdots \otimes \sigma_k \text{ where } \sigma_1, \ldots, \sigma_k \in S \cup \{1\}\}$, $k \ge 1$. The $\tau \colon \widehat{G} \to \mathbb{N} \cup \{0\}$ is given by $\tau(\pi) := k$, if $\pi \in S^{\otimes k} \backslash S^{\otimes(k-1)}$. Then we have some constant $C > 0$ such that (see the proof of [15, Theorem 5.4])

$$\|\pi\|_1 \le C\tau(\pi) \le C\|\pi\|_1, \quad \pi \in \widehat{G}.$$

This justifies the statement that the quantity $\sqrt{\kappa_\pi}$ describes the growth rate of $\widehat{G}$.

(iii) When $G = \mathbb{T}^n$, the algebra $L^1(\mathbb{T}^n)_{\nu_\alpha}$ is one of the equivalent form of the Sobolev space $H^{\alpha,1}(\mathbb{T}^n)$.

5 Representability of the Deformed Fourier Algebras on Discrete Groups as Operator Algebras

In this section G is always a discrete group.

Lemma 5.1 *Let $w \in M_{cb}A(G)$ with norm 1. Then the associated multiplier L_w is injective if and only if w has no zero value. In this case, L_w always has dense range.*

Proof The first statement is straightforward to check, and the second statement follows from the fact that the image of L_w contains $\{\delta_x : x \in G\}$, which is dense in $A(G)$. ∎

In this section we will provide a characterization of complete representability of the deformed algebra $A_w(G)$ as an operator algebra. As before, we focus on the equivalent algebra $(A(G), \widetilde{m_w})$ with the dual perspective. Indeed, we can easily see that $A_w(G)$ is completely representable as an operator algebra if and only if the map

$$(\widetilde{m_w})^* : VN(G) \to VN(G) \overline{\otimes} VN(G)$$

extends to a completely bounded map $(\widetilde{m_w})^* : VN(G) \to VN(G) \otimes_{eh} VN(G)$. We call the map $(\widetilde{m_w})^*$, the *deformed co-multiplication* as before. This time it is quite straightforward to see how $(\widetilde{m_w})^*$ acts on concrete elements of $VN(G)$, so we just record it without proof.

Proposition 5.2 *For $T \sim \sum_{g \in G} \alpha_g \lambda_g \in VN(G)$ we have*

$$(\widetilde{m_w})^*(T) \sim \sum_{g \in G} w(g) \alpha_g \lambda_g \otimes \lambda_g.$$

For the negative direction of our main result we need the following Lemma, which is a direct consequence of Lust-Piquard's non-commutative version of Kahane, Katznelson, and de Leeuw's coefficient problem.

Lemma 5.3 *Let $\tau(\,\cdot\,) = \langle \cdot \delta_e, \delta_e \rangle$ be the vacuum state on $VN(G)$. There is a constant $K > 0$ such that for any $(c_g) \in \ell^2(G)$ with norm ≤ 1 there exist $T \in VN(G)$ with norm $\leq K$ such that $|\tau(T \lambda_g^*)| \geq |c_g|$, $g \in G$.*

Proof Fix a sequence $(c_g)_{g \in G}$ with $\sum_g |c_g|^2 \leq 1$ and set $A_g = c_g \lambda_g$. Here, we can assume that the index set of sequence is countable. Then the sequence (A_g) satisfies the conditions (i) and (ii) in [16, Theorem 4]. More precisely, for $g_1, g_2, g_3 (g_2 \neq g_3)$, we have

$$\text{Re}\big(\tau(A_{g_1}^* A_{g_1} A_{g_2}^* A_{g_3}) \big) = |c_{g_1}|^2 \, \text{Re}\big(\tau(\overline{c_{g_2}} c_{g_3} \lambda_{g_2^{-1} g_3}) \big) = 0$$

and

$$\text{Re}\big(\tau(A_{g_1} A_{g_1}^* A_{g_2} A_{g_3}^*) \big) = |c_{g_1}|^2 \, \text{Re}\big(\tau(c_{g_2} \overline{c_{g_3}} \lambda_{g_2 g_3^{-1}}) \big) = 0.$$

Moreover, we have

$$\Big\| \sum_g \frac{A_g^* A_g + A_g A_g^*}{2} \Big\| = \sum_g |c_g|^2 \leq 1.$$

By [16, Theorem 4] there is $T \in VN(G)$ with norm $\leq K$ such that

$$|c_g| |\tau(T^* \lambda_g)| = |\tau(T^* A_g)| \geq \tau(A_g^* A_g) = |c_g|^2, \ g \in G,$$

which is the conclusion we wanted. ∎

Here is the main result of this section.

Theorem 5.4   *The deformed algebra $A_w(G)$ is completely representable as an operator algebra if and only if $w \in \ell^2(G)$.*

Proof   We first check the positive direction. Suppose that

$$T \sim \sum_{g \in G} A_g \otimes \lambda_g \in M_n(VN(G))$$

with $A_g \in M_n$. Then we have

$$(I_n \otimes m_w)^*(T) \sim \sum_{g \in G} w(g) A_g \otimes \lambda_g \otimes \lambda_g$$

$$= \sum_{g \in G} (A_g \otimes \lambda_g) \odot (w(g) \,\mathrm{id}_n \otimes \lambda_g)$$

$$= \begin{bmatrix} \cdots & A_g \otimes \lambda_g & \cdots \end{bmatrix} \odot \begin{bmatrix} \vdots \\ w(g) \,\mathrm{id}_n \otimes \lambda_g \\ \vdots \end{bmatrix} = A \odot B.$$

Now we have $A \in M_{1,|G|}(M_n(VN(G)))$ and

$$\|A\|^2 = \left\| \sum_{g \in G} (A_g \otimes \lambda_g)(A_g \otimes \lambda_g)^* \right\| = \left\| \sum_{g \in G} A_g A_g^* \right\|_{M_n}.$$

Moreover, we have $B \in M_{|G|,1}(M_n(VN(G)))$ and

$$\|B\|^2 = \left\| \sum_{g \in G} (w(g) \,\mathrm{id}_n \otimes \lambda_g)^*(w(g) \,\mathrm{id}_n \otimes \lambda_g) \right\| = \sum_{g \in G} |w(g)|^2.$$

Finally, we observe that

$$\left\| \sum_{g \in G} A_g A_g^* \right\|_{M_n} \le \|TT^*\|.$$

Indeed, let $\tau(\cdot) = \langle \cdot \, \delta_e, \delta_e \rangle$ be the vacuum state. Then we have

$$(\mathrm{id}_n \otimes \tau)(TT^*) = \sum_{g \in G} A_g A_g^*.$$

Combining all the above we get $\|\widetilde{m_w}\|_{\mathrm{cb}} \le \|w\|_{\ell^2(G)}$. For the converse direction we let $T \sim \sum_{g \in G} \alpha_g \lambda_g \in VN(G)$. Moreover, we set $X_r \in B(\ell^2(G))$ by

$$X_r(\delta_g) = r_g \delta_{g^{-1}}, \quad g \in G,$$

where $r_g \in \mathbb{C}$ with $|r_g| = 1$, $g \in G$. Then we have

$$(\widetilde{m_w})^*(T)(X_r)\delta_e = \left(\sum_{g \in G} w(g) \alpha_g r_g \right) \delta_e.$$

This allows us the following estimate:

$$(5.1) \qquad \|(\widetilde{m_w})^*\| \ge \frac{\left| \sum_{g \in G} w(g) \alpha_g r_g \right|}{\|T\|_{VN(G)}}.$$

Now we fix $(c_g) \in \ell^2(G)$ with norm ≤ 1. Then there is $T \in VN(G)$ (depending on (c_g)) with norm $\le K$ such that $|\tau(T\lambda_g^*)| \ge |c_g|$, $g \in G$ by Lemma 5.3. Then we choose

r_g (depending on T) so that $w(g)\alpha_g r_g = |w(g)\alpha_g| = |w(g)\tau(T\lambda_g^*)|$, $g \in G$. For these choices of T and (r_g) we get

$$\|(\widetilde{m_w})^*\| \geq \frac{1}{K} \sum_{g \in G} |w(g)\alpha_g| \geq \frac{1}{K} \sum_{g \in G} |w(g)c_g|.$$

The choice of (c_g) is arbitrary so that we get

$$\|(\widetilde{m_w})^*\| \geq \frac{1}{K} \|w\|_{\ell^2(G)}. \qquad \blacksquare$$

Remark 5.5 We also have a different route for the negative direction of the above theorem with a little bit of probabilistic flavor as in Remark 4.9, which allows us to avoid Lemma 5.3. We consider an I.I.D. family of Bernoulli variables $(r_g)_{g \in G}$. Then (5.1) tells us that

$$\|(\widetilde{m_w})^*\| \geq \mathbb{E}\| \sum_{g \in G} w(g)\lambda_g r_g \|_{(C_r^*(G))^*}$$

$$\geq C\Big(\sum_{g \in G} \|w(g)\lambda_g\|_{(C_r^*(G))^*}^2 \Big)^{\frac{1}{2}} = C \cdot \|w\|_{\ell^2(G)}$$

for some constant $C > 0$. Here we used the fact that $(C_r^*(G))^*$ is of cotype 2 and λ_g is understood as an element of $(C_r^*(G))^*$ given by $a \mapsto \tau(\lambda_g a)$.

5.1 The Case of Finitely Generated Groups and the Growth Order

In this subsection we apply Theorem 5.4 in the case of certain finitely generated groups with the deformation functions coming from the word length function. We will demonstrate that the complete representability of $A_w(G)$ can precisely detect the growth rate of the group G.

Example 5.6 Let G be a finitely generated group with a generating set S. Let $|\cdot|$ be the word length associated to S. We consider the function given by

$$w_t(g) = e^{-t|g|}, \quad g \in G.$$

If G is either $\mathbb{Z}^n$ or a Coxeter group with the canonical generating set S, then w_t is known to be a positive definite function. If G is a hyperbolic group with the canonical generating set, then it is known that $w_t \in M_{\mathrm{cb}}A(G)$ with

$$M := \sup_{t>0} \|w_t\|_{M_{\mathrm{cb}}A(G)} < \infty.$$

Note that $M = 1$ in the previous case. Using w_t we could find polynomially decreasing functions in $M_{\mathrm{cb}}A(G)$ as before. For $\alpha > 0$ we consider the function

$$W_\alpha(g) := \frac{1}{M(1+|g|)^\alpha}, \quad g \in G.$$

Since we have

$$W_\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha/2-1} e^{-t} w_t \, dt,$$

we have

$$\|W_\alpha\|_{M_{cb}A(G)} \leq \frac{1}{M\Gamma(\alpha)} \int_0^\infty t^{\alpha/2-1} e^{-t} \|w_t\|_{M_{cb}A(G)} \, dt \leq 1.$$

Recall that a finitely generated group with a fixed generating set S is said to be *polynomially growing* if there is a constant $C > 0$ and $k > 0$ such that

$$|B(n)| \leq C(n^k + 1), \quad n \geq 0,$$

where $B(n) = \{g \in G : |g| \leq n\}$ is the n-ball. Recall also that the infimum k_0 of such k is called the order of the polynomial growth of G or the growth rate of G. Moreover, G is said to be *exponentially growing* if there is a constant $a > 1$ such that

$$|B(n)| \geq a^n, \quad n \geq 0.$$

The exponential growth rate of G with respect to S is defined by

$$\lambda = \lambda(G, S) := \lim_{n \to \infty} |B(n)|^{\frac{1}{n}}.$$

Remark 5.7 (i) For the group G of polynomial growth, it is well known that k_0 has to be a natural number and

$$|B(n)| \sim n^{k_0}, \quad n \geq 0;$$

i.e., there are constants $C_1, C_2 > 0$ such that

$$C_1 n^{k_0} \leq |B(n)| \leq C_2 n^{k_0}, \quad n \geq 0.$$

(ii) The polynomial growth rate k_0 of G is known to be independent of the choice of the generating set S.

(iii) Every finitely generated group has at most exponential growth. In other words, there is $b > 0$ such that $|B(n)| \leq b^n$ for all n. Thus, the above limit $\lambda(G, S)$ always exists by Fekete's subadditivity lemma.

(iv) The condition $\lambda(G, S) > 1$ implies that $\lambda(G, S') > 1$ for any other symmetric generating set S'. However, $\inf_S \lambda(G, S)$ could be equal to 1.

Proposition 5.8 *Let G be a finitely generated group with a fixed generating set S.*

(i) *Suppose G is of polynomial growth of order k_0. For $\alpha > 0$, we have*

$$\sum_{g \in G} \frac{1}{(1 + |g|)^{2\alpha}} < \infty$$

if and only if $\alpha > \frac{k_0}{2}$.

(ii) *Suppose that G is exponential growing with the growth rate λ. For $t > 0$ we have $\sum_{g \in G} \frac{1}{e^{2t|g|}} < \infty$ if $t > \frac{\log \lambda}{2}$ and $\sum_{g \in G} \frac{1}{e^{2t|g|}} = \infty$ if $t < \frac{\log \lambda}{2}$.*

Proof We only consider case (i), since the proof for (ii) is essentially the same. Put $C(n) := \{g \in G : |g| = n\}$, the n-sphere, $c_n := |C(n)|$ and $b_n := |B(n)|$. Then we have

$$\sum_{g \in G} \frac{1}{(1 + |g|)^{2\alpha}} = \sum_{n \geq 0} \frac{c_n}{(1 + n)^{2\alpha}}.$$

By summation by parts we have

$$\sum_{n\geq 0}^{N} \frac{c_n}{(1+n)^{2\alpha}} = \frac{b_N}{(1+N)^{2\alpha}} + \sum_{n\geq 0}^{N-1} b_n \Big(\frac{1}{(n+1)^{2\alpha}} - \frac{1}{(n+2)^{2\alpha}} \Big).$$

Recall that $b_n \sim n^{k_0}$, $n \geq 0$, and by the mean value theorem we have

$$\frac{1}{(n+1)^{2\alpha}} - \frac{1}{(n+2)^{2\alpha}} \sim \frac{2\alpha}{(n+1)^{2\alpha+1}}, \quad n \geq 0.$$

Then a standard summability criterion gives us the conclusion we wanted. ∎

Theorem 5.9 *Let G be one of $\mathbb{Z}^n$, a Coxeter group or a hyperbolic group, and let w_t and W_α be the functions from Example 5.6 with the canonical generating set S.*

(i) *Suppose G is of polynomial growth of order k_0. The algebra $A_{W_\alpha}(G)$ is completely representable as an operator algebra if and only if $\alpha > k_0/2$.*

(ii) *Suppose that G is exponential growing with the growth rate λ. The algebra $A_{w_t}(G)$ is completely representable as an operator algebra if $t > \log \lambda/2$ and $A_{w_t}(G)$ is not completely representable as an operator algebra if $t < \log \lambda/2$.*

Remark 5.10 (i) The above theorem tells us that the complete representability of $A_w(G)$ can precisely detect the growth rate of the underlying group when it is polynomially growing or exponentially growing.

(ii) We excluded the case of exponentially growing deformation measures in Section 4.2, since the dual of compact connected Lie groups are always of polynomial growth ([20]).

References

[1] D. P. Blecher, *A completely bounded characterization of operator algebras.* Math. Ann. **303**(1995), no. 2, 227–239.   http://dx.doi.org/10.1007/BF01460988

[2] D. P. Blecher and R. R. Smith, *The dual of the Haagerup tensor product.* J. Lond. Math. Soc. (2) **45**(1992), no. 1, 126–144.   http://dx.doi.org/10.1112/jlms/s2-45.1.126

[3] H. G. Dales, A. T.-M. Lau, *The second duals of Beurling algebras.* Mem. Amer. Math. Soc. **177**(2005), no. 836.   http://dx.doi.org/10.1090/memo/0836

[4] A. Dasgupta and M. Ruzhansky, *Gevrey functions and ultradistributions on compact Lie groups and homogeneous spaces.* Bull. Sci. Math. **138**(2014), no. 6, 756–782. http://dx.doi.org/10.1016/j.bulsci.2013.12.001

[5] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact.* Bull. Soc. Math. France **92**(1964), 181–236.

[6] E. G. Effros and Z.-J. Ruan, *Operator spaces.* London Mathematical Society Monographs, New Series, 23, The Clarendopn Press, Oxford University Press, New York, 2000.

[7] T. Fack, *Type and cotype inequalities for noncommutative L_p-spaces.* J. Operator Theory **17**(1987), no. 2, 255–279.

[8] N. Grønbæk, *Amenability of weighted convolution algebras on locally compact groups.* Trans. Amer. Math. Soc. **319**(1990), no. 2, 765–775. http://dx.doi.org/10.1090/S0002-9947-1990-0962282-5

[9] M. Ghandehari, H. H. Lee, E. Samei, and N. Spronk, *Some Beurling-Fourier algebras are operator algebras.* Trans. Amer. Math. Soc. **367**(2015), no. 10, 7029–7059. http://dx.doi.org/10.1090/tran6653

[10] S. Helgason, *Topologies of group algebras and a theorem of Littlewood.* Trans. Amer. Math. Soc. **86**(1957), 269–283.	http://dx.doi.org/10.1090/S0002-9947-1957-0095428-5

[11] B. E. Johnson, *Cohomology in Banach algebras.* Memoirs of the American Mathematical Society, 127, American Mathematical Society, Providence, RI, 1972.

[12] E. Kaniuth, *A course in commutative Banach algebras.* Graduate Texts in Mathematics, 246, Springer, New York, 2009.	http://dx.doi.org/10.1007/978-0-387-72476-8

[13] H. H. Lee and E. Samei, *Beurling-Fourier algebras, operator amenability and Arens regularity.* J. Funct. Anal. **262**(2012), no. 1, 167–209.	http://dx.doi.org/10.1016/j.jfa.2011.09.008

[14] H. H. Lee, E. Samei, and N. Spronk, *Some weighted group algebras are operator algebras.* Proc. Edinb. Math. Soc. (2) **58**(2015), no. 2, 499–519.	http://dx.doi.org/10.1017/S0013091514000212

[15] J. Ludwig, N. Spronk, and L. Turowska, *Beurling-Fourier algebras of compact groups.* J. Funct. Anal. **262**(2012), no. 2, 463–499.	http://dx.doi.org/10.1016/j.jfa.2011.09.017

[16] F. Lust-Piquard, *On the coefficient problem: a version of the Kahane-Katznelson-de Leeuw theorem for spaces of matrices.* J. Funct. Anal. **149**(1997), no. 2, 352–376. http://dx.doi.org/10.1006/jfan.1997.3111

[17] J. F. Price, *On local central lacunary sets for compact Lie groups.* Monatsh. Math. **80**(1975), no. 3, 201–204.	http://dx.doi.org/10.1007/BF01319915

[18] Z.-J. Ruan. *The operator amenability of $A(G)$.* Amer. J. Math. **117**(1995), no. 6, 1449–1474. http://dx.doi.org/10.2307/2375026

[19] N. Spronk, *Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras.* Proc. London Math. Soc. **89**(2004), no. 1, 161–192. http://dx.doi.org/10.1112/S0024611504014650

[20] R. Vergnioux, *The property of rapid decay for discrete quantum groups.* J. Operator Theory **57**(2007), no. 2, 303–324.

[21] N. R. Wallach, *Harmonic analysis on homogeneous spaces.* Pure and Applied Mathematics, 19, Marcel Dekker, Inc., New York, 1973.

[22] M. E. Walter, *W^*-algebras and nonabelian harmonic analysis.* J. Functional Analysis **11**(1972), 17–38.	http://dx.doi.org/10.1016/0022-1236(72)90077-8

[23] J. G. Wendel, *On isometric isomorphism of group algebras.* Pacific J. Math. **1**(1951), 305–311. http://dx.doi.org/10.2140/pjm.1951.1.305

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