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Compact embedding results of Sobolev spaces and positive solutions to an elliptic equation

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Dedicated to my little angel Jacquelyn and her mother, my dear wife, Jingbo.

Using a regular Borel measure $\mu \geqslant 0$ we derive a proper subspace $D^1_\mu(\mathbb{R}^N)$ of the commonly used Sobolev space $D^1(\mathbb{R}^N)$ when $N \geqslant 3$. The space $D^1_\mu(\mathbb{R}^N)$ resembles the standard Sobolev space $H^1(\Omega)$ when Ω is a bounded region with a compact Lipschitz boundary $\partial\Omega$. An equivalence characterization and an example are provided that guarantee that $D^1_\mu(\mathbb{R}^N)$ is compactly embedded into $L^1(\mathbb{R}^N)$. In addition, as an application we prove an existence result of positive solutions to an elliptic equation in $\mathbb{R}^N$ that involves the Laplace operator with the critical Sobolev nonlinearity, or with a general nonlinear term that has a subcritical and superlinear growth. We also briefly discuss the compact embedding of $W^{1,p}_\mu(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ when $N \geqslant 2$ and $2 \leqslant p \leqslant N$.

Keywords: Sobolev spaces; capacity; elliptic equations; positive solutions

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1. Introduction

When $N \geqslant 3$ and Ω is a bounded region in $\mathbb{R}^N$ that has a compact Lipschitz boundary $\partial\Omega$, we know that the standard Sobolev space $H^1(\Omega)$ is continuously embedded into the spaces $L^s(\Omega)$ for $1 \leqslant s \leqslant 2^* := 2N/(N-2)$, and this embedding is also compact for $1 \leqslant s < 2^*$. When $N = 2$ this embedding is compact for all $1 \leqslant s < \infty$. Many generalizations have been made that can be found, for example, in [1,13,26]. To recover the compact embedding results on $\mathbb{R}^N$ one usually uses spaces of special functions (for example, those that are radially symmetric) or introduces weights to the function spaces.

In this paper we investigate which proper subspace of $D^1(\mathbb{R}^N)$ for $N \geqslant 3$ (or $H^1(\mathbb{R}^N)$ for $N = 2$) can have the same embedding as $H^1(\Omega)$. We borrow an idea from Bucur and Buttazzo [8] to bring a Borel measure μ into the search of such good subspaces of $D^1(\mathbb{R}^N)$.

In fact, when μ is a non-negative regular Borel measure on $\mathbb{R}^N$, possibly infinite valued, that vanishes on all sets of capacity zero, we define $L^{\tilde{q}}_\mu(\mathbb{R}^N)$ as the linear space (or the family of equivalent classes) of μ-measurable functions $u \in D^1(\mathbb{R}^N)$

that satisfy

$$\int_{\mathbb{R}^N} |u|^{\tilde{q}} \, d\mu < \infty \quad \text{for some } \tilde{q} \in (1, \infty). \tag{1.1}$$

In this way we can obtain the desired subspace $D^1_\mu(\mathbb{R}^N) := D^1(\mathbb{R}^N) \cap L^2_\mu(\mathbb{R}^N)$ of $D^1(\mathbb{R}^N)$ that behaves like $H^1(\Omega)$, provided that μ enables the constant functions to be in the dual space of $D^1_\mu(\mathbb{R}^N)$. An equivalence condition on this is described in terms of the qualitative behaviour of a characteristic partial differential equation (PDE) that is discussed in § 3, including the case in which $N = 2$.

Recall that for problems involving the Laplace operator on $\mathbb{R}^N$, the space quite often used when $N \geqslant 3$, instead of $H^1(\mathbb{R}^N)$, is $D^1(\mathbb{R}^N)$. In this sense, $D^1_\mu(\mathbb{R}^N)$ is the *authentic* counterpart of $H^1(\Omega)$. Also, when $d\mu = V(x)\,dx$ for a measurable function $V(x) \geqslant 0$, $V^{-1} \in L^1(\mathbb{R}^N)$ guarantees this compact embedding. Note that no *a priori* assumption on μ is given to ensure an embedding of $D^1_\mu(\mathbb{R}^N)$ to $L^1(\mathbb{R}^N)$ or $L^2(\mathbb{R}^N)$, which seems to be a unique phenomenon only when $N \geqslant 3$.

On the other hand, to derive the same result when $N = 2$ we start with the space $H^1(\mathbb{R}^N)$. This has already been discussed in [8] and we will work a little bit more on it in § 3.

As an application we study the existence of (distributional) positive solutions of

$$-\Delta u + \alpha(x) u^{q-1} = \lambda u^r + u^{2^*-1} \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

This equation may be viewed as a combination of the equations discussed in [2,3,10]. For $q = 2$ and $r = 1$, (1.2) was also studied by Clapp and Ding [11] in a very different situation. It is worth mentioning here that the assumptions we impose on $\alpha(x)$ are different (see theorem 4.1).

We prove that problem (1.2) has a positive solution provided that $0 < r < 1$, $q \geqslant 2$ and $\lambda > 0$ is a sufficiently small constant, and this solution bifurcates from zero since it decays to zero when $\lambda \to 0^+$. On the other hand, if one replaces u^{2^*-1} by a general nonlinear term $f(x, u)$ (like in [4, 28]) that satisfies certain subcritical and superlinear growth conditions but doesn't satisfy the well-known Ambrosetti–Rabinowitz condition, then we can show the existence of a second positive solution to (1.2) from the *mountain pass theorem* of Cerami. These cases are discussed in §§ 4 and 5, where μ is generated through $\alpha(x)$.

Section 2 is devoted to detailed analyses of a Sobolev(-type) space $M^{q,p}(\mathbb{R}^N)$, where each function $u \in M^{q,p}(\mathbb{R}^N)$ satisfies $u \in L^q(\mathbb{R}^N)$ and $|\nabla u| \in L^p(\mathbb{R}^N)$ for $p, q \in [1, \infty]$.

A result concerning the compact embedding from $W^{1,p}_\mu(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ when $p \geqslant 2$, using an idea due to Maz'ya and Shubin [27], is also briefly described in § 6.

2. The function space $M^{q,p}(\mathbb{R}^N)$

In this section we describe a Sobolev(-type) space $M^{q,p}(\mathbb{R}^N)$ that may be viewed as a natural generalization of the standard Sobolev space $W^{1,p}(\mathbb{R}^N)$, $W^{1,p}(\mathbb{R}^N) = M^{p,p}(\mathbb{R}^N)$.

We mention that some of the results discussed here have already been derived, with details, in [17] when $N \geqslant 3$ and $1 \leqslant p < N$, so we shall be sketchy from time to time.

Let Ω be a region, i.e. it is open and connected. In this paper all functions are assumed to be real valued and (Borel) measurable. $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, denotes the space of pth integrable functions u on Ω, with norm written by $\|u\|_{p,\Omega}$. A function u is said to be in $L^1_{\mathrm{loc}}(\Omega)$, provided that $u \in L^1(K)$ for all compact subsets $K \Subset \Omega$. $\mathfrak{L}$ denotes the Lebesgue measure on $\mathbb{R}^N$, with $\mathrm{d}x$ ($\mathrm{d}y$) its associated volume element, and $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure.

Recall that a function u is said to be a *Sobolev function* provided that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. That is, u and its weak (distributional) derivatives $\mathrm{D}_j u$, with $j = 1, 2, \dots, N$, are in $L^1_{\mathrm{loc}}(\Omega)$.

$W^{1,p}(\Omega)$ denotes the standard Sobolev space of functions u on Ω such that u and $|\nabla u|$ are in $L^p(\Omega)$. It is a Banach space with respect to the usual $W^{1,p}$-norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{p,\Omega} + \||\nabla u|\|_{p,\Omega}. \tag{2.1}$$

In particular, when $p = 2$ the notation $H^1(\Omega)$ is commonly used instead of $W^{1,2}(\Omega)$.

$M^{q,p}(\Omega)$ (see [17]) is defined to be the space of functions u on Ω that are in $L^q(\Omega)$, while $|\nabla u|$ are in $L^p(\Omega)$. It is a Banach space with respect to the $M^{q,p}$-norm

$$\|u\|_{M^{q,p}(\Omega)} := \|u\|_{q,\Omega} + \||\nabla u|\|_{p,\Omega}. \tag{2.2}$$

Hereafter, $\nabla u := (\mathrm{D}_1 u, \mathrm{D}_2 u, \dots, \mathrm{D}_N u)$ denotes the weak gradient of u.

In addition, a function $u \colon \Omega \to \mathbb{R}$ is said to be *Hölder continuous* with *exponent* $\gamma \in (0, 1]$, provided that $|u(x) - u(y)| \leqslant C|x - y|^\gamma$ for a constant $C > 0$ that depends on γ, Ω. Write $C^{0,\gamma}(\bar{\Omega})$ for the associated space. Then it is a Banach space under the usual $C^{0,\gamma}$-norm

$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} := \sup_{x\in\Omega}\{|u(x)|\} + \sup_{y\neq z\in\Omega}\left\{\frac{|u(y) - u(z)|}{|y - z|^\gamma}\right\}. \tag{2.3}$$

2.1. Ω is a bounded region having a compact Lipschitz boundary $\partial\Omega$

Assume that $N \geqslant 2$. One easily sees that, when $1 \leqslant p, q < \infty$, $C^1(\bar{\Omega})$ is a dense subset of $M^{q,p}(\Omega)$.

When $p \in [1, N)$ we have

$$\left.\begin{aligned} M^{q,p}(\Omega) &= W^{1,p}(\Omega) \quad \text{for } 1 \leqslant q \leqslant p^*, \\ M^{q_1,p}(\Omega) &\subseteq M^{q_2,p}(\Omega) \quad \text{for } p^* \leqslant q_2 \leqslant q_1 \leqslant \infty. \end{aligned}\right\} \tag{2.4}$$

In fact, *Hölder's inequality* yields $W^{1,p}(\Omega) \subseteq M^{q,p}(\Omega)$ for $q \in [1, p]$, *Poincaré's* and *Minkowski's inequalities* yield $M^{1,p}(\Omega) \subseteq W^{1,p}(\Omega)$, and then the *Sobolev embedding theorem* yields (2.4). Also, when $q \in [1, \infty]$ there exists a constant $C_{p,q} > 0$, depending on p, q, Ω, such that

$$\|u\|_{W^{1,p}(\Omega)} \leqslant C_{p,q}\|u\|_{M^{q,p}(\Omega)} \quad \forall u \in M^{q,p}(\Omega). \tag{2.5}$$

Here, as usual, for $p \in [1, N)$ we write the critical Sobolev exponent as $p^* := pN/(N - p)$. Below we use '$\rightharpoonup$' to denote weak convergence, and use '$\hookrightarrow$' to denote compact embedding.

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When $p = N$ we can just replace the *Sobolev embedding theorem* by the (compact) embedding result $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$, for $1 \leqslant s < \infty$, to conclude that

$$M^{\infty,p}(\Omega) \subseteq M^{q,p}(\Omega) = W^{1,p}(\Omega) \quad \text{for } 1 \leqslant q < \infty. \tag{2.6}$$

Moreover, the embedding $\iota\colon M^{\infty,p}(\Omega) \to L^s(\Omega)$ is also continuous for $1 \leqslant s \leqslant \infty$.

When $p \in (N, \infty]$ we use *Morrey's inequality* to obtain that

$$M^{q,p}(\Omega) = W^{1,p}(\Omega) \quad \text{for } 1 \leqslant q \leqslant \infty. \tag{2.7}$$

Furthermore, the *Arzelà–Ascoli theorem*, Evans and Gariepy [16, p. 135, theorem 1], and Evans [15, theorem 5.6.5] state that the embedding $\iota\colon M^{q,p}(\Omega) \to C^{0,s}(\bar{\Omega})$ is continuous if $0 \leqslant s \leqslant \gamma$ and also compact if $0 \leqslant s < \gamma$ for $\gamma := 1 - N/p$. Note that u is Lipschitz continuous if and only if $u \in M^{q,\infty}(\Omega)$ by [15, theorem 5.8.4] (see also [16, p. 131, theorem 5]), since we only need $|\nabla u| \in L^\infty(\Omega)$.

All the preceding discussions imply that, among all $M^{q,p}(\Omega)$, $W^{1,p}(\Omega)$ is the largest space. On the other hand, when $p \in [1, N)$ we can prove the following embedding result.

Proposition 2.1. *When $1 \leqslant p < N$ and $1 \leqslant q \leqslant \infty$, the embedding $\iota\colon M^{q,p}(\Omega) \to L^s(\Omega)$ is continuous if $1 \leqslant s \leqslant \max\{q, p^*\}$ and also compact if $1 \leqslant s < \max\{q, p^*\}$.*

Proof. This result follows (almost) directly from the proof of [16, p. 144, theorem 1] in view of $r := \max\{q, p^*\} > 1$ and (2.5), and was described in [17, proposition 2.2]. For the sake of completeness, we present a slightly simpler proof.

We keep the same notation as in [16, p. 144, theorem 1]. Suppose that $\{f_k\colon k \geqslant 1\}$ is a bounded sequence in $M^{q,p}(\Omega)$. Note then that (2.4) and (2.5) together imply that $\{f_k\colon k \geqslant 1\}$ is a bounded sequence in $W^{1,p}(\Omega)$ as well. So, we find a sequence of functions $\{\tilde{f}_k\colon k \geqslant 1\}$ in $W^{1,p}(\mathbb{R}^N)$, where each $\tilde{f}_k$ is an extension of f_k to $\mathbb{R}^N$. As a result, we observe that

$$\sup_{k\geqslant 1} \|\tilde{f}_k\|_{W^{1,p}(\mathbb{R}^N)} \leqslant C_\Omega \Big\{ \sup_{k\geqslant 1} \|f_k\|_{W^{1,p}(\Omega)} \Big\} \leqslant C'_{p,q} \Big\{ \sup_{k\geqslant 1} \|f_k\|_{M^{q,p}(\Omega)} \Big\}.$$

Here, $C_\Omega, C'_{p,q} > 0$ are some constants that depend on p, q and Ω. Hence, we can follow steps 1–6 in [16, p. 144, theorem 1] to obtain a function $f \in L^r(\Omega)$ such that $f_{k_j} \to f$ in $L^s(\Omega)$ when $1 \leqslant s \leqslant p$ and $f_{k_j} \rightharpoonup f$ in $L^r(\Omega)$. Take $s \in [p, r)$ and set $\theta := p(r-s)/s(r-p) \in (0, 1]$ to derive

$$\lim_{j\to\infty} \|f_{k_j} - f\|_{s,\Omega} \leqslant \lim_{j\to\infty} \{\|f_{k_j} - f\|^{\theta}_{p,\Omega} \|f_{k_j} - f\|^{1-\theta}_{r,\Omega}\} = 0. \tag{2.8}$$

Notice that when $p > 1$ we also have $f \in M^{q,p}(\Omega)$. (This is not true if $p = 1$.) □

2.2. Ω is $\mathbb{R}^N$

When $N \geqslant 3$ and $1 \leqslant p < N$ the notation $D^{1,p}(\mathbb{R}^N)$ is widely used to denote the space of functions u such that $u \in L^{p^*}(\mathbb{R}^N)$ and $|\nabla u| \in L^p(\mathbb{R}^N)$. By the *Gagliardo–Nirenberg–Sobolev inequality*, there is a sharp constant $C_{p,N} > 0$, depending on p, N, such that

$$\|u\|_{p^*,\mathbb{R}^N} \leqslant C_{p,N} \||\nabla u|\|_{p,\mathbb{R}^N} \quad \forall u \in D^{1,p}(\mathbb{R}^N). \tag{2.9}$$

Notice that when $p = 2$ the notation $D^1(\mathbb{R}^N)$ is commonly used in the literature.

When $1 < p < N$ Lieb and Loss gave us an equivalent definition: $D^{1,p}(\mathbb{R}^N)$ is the subspace of functions in $L^1_{\text{loc}}(\mathbb{R}^N)$ that vanish at infinity and have L^p-integrable gradients in $\mathbb{R}^N$. Here, a function $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ is said to *vanish at infinity* provided that $\mathfrak{L}(\{x \in \mathbb{R}^N : |u(x)| \geqslant c\}) < \infty$ for all constants $c > 0$. See *Sobolev's inequality for gradients* [23, §§ 8.2 and 8.3]. We can thus use *Chebyshev's inequality* and an interpolation inequality, like (2.8), to observe that

$$M^{q_1,p}(\mathbb{R}^N) \subseteq M^{q_2,p}(\mathbb{R}^N) \quad \text{when either } 1 \leqslant q_1 \leqslant q_2 \leqslant p^* \text{ or } p^* \leqslant q_2 \leqslant q_1 < \infty. \tag{2.10}$$

When $p = 1$ we have (2.10) only if $1 \leqslant q_1 \leqslant q_2 \leqslant 1^*$, directly via a density argument.

When $N = 2$ and $1 \leqslant p < N$ we again have $D^{1,p}(\mathbb{R}^N)$ and (2.9). In addition, as above, we also have (2.10) when $1 < p < N$, yet have (2.10) only if $1 \leqslant q_1 \leqslant q_2 \leqslant 1^*$ when $p = 1$.

Clearly, we notice that $W^{1,p}(\mathbb{R}^N) \subsetneq D^{1,p}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \subsetneq D^1(\mathbb{R}^N)$ by density.

It is perhaps helpful to stress the importance of Lieb and Loss's result: it is very easy to see that $C^1_c(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$ (for all p, N), and thus in $M^{q,p}(\mathbb{R}^N)$ when $N \geqslant 2$, $p \in [1, N)$ and $q \in [p, p^*]$, but from their result we also have the density of $C^1_c(\mathbb{R}^N)$ in $M^{q,p}(\mathbb{R}^N)$ when $N \geqslant 2$, $p \in (1, N)$ and $q \in [1, \infty)$. More details can be found in [17, § 2].

When $N \geqslant 2$ and $p = N$ we know that the embedding $W^{1,N}(\mathbb{R}^N) \to L^s(\mathbb{R}^N)$ is continuous for $N \leqslant s < \infty$. Let $M^{q,N}(\mathbb{R}^N)$ be the completion of $C^1_c(\mathbb{R}^N)$ with respect to (2.2) for now. We can prove a more general result that can be interpreted as saying that, like (2.6),

$$M^{q_1,p}(\mathbb{R}^N) \subseteq M^{q_2,p}(\mathbb{R}^N) \quad \text{when } 1 \leqslant q_1 \leqslant q_2 < \infty. \tag{2.11}$$

PROPOSITION 2.2. *When $N \geqslant 2$ and $q \in [1, \infty)$ the embedding $\iota\colon M^{q,N}(\mathbb{R}^N) \to L^s(\mathbb{R}^N)$ is continuous if $q \leqslant s < \infty$. When $q = \infty$ this embedding is continuous only for $s = \infty$.*

Proof. Recall that estimate (14) in [15, theorem 5.6.1] says, for all $u \in M^{q,N}(\mathbb{R}^N)$, that

$$\left(\int_{\mathbb{R}^N} |u|^{\kappa N/(N-1)}\,\mathrm{d}x\right)^{(N-1)/N} \leqslant C_{1,N}\int_{\mathbb{R}^N} |\nabla |u|^\kappa|\,\mathrm{d}x = \kappa C_{1,N}\int_{\mathbb{R}^N} |u|^{\kappa-1}|\nabla u|\,\mathrm{d}x.$$

Use *Hölder's* and *Young's inequalities* and set $\kappa_1 := 1 + q(N-1)/N > 1$ to derive that

$$\begin{aligned}
\|u\|^{\kappa_1}_{\kappa_1 N/(N-1),\mathbb{R}^N} &\leqslant \kappa_1 C_{1,N}\|u\|^{\kappa_1-1}_{q,\mathbb{R}^N}\||\nabla u|\|_{N,\mathbb{R}^N} \\
&\leqslant C_{\kappa_1}(\|u\|^{\kappa_1}_{q,\mathbb{R}^N} + \||\nabla u|\|^{\kappa_1}_{N,\mathbb{R}^N}) \\
&\leqslant C'_{\kappa_1}\|u\|^{\kappa_1}_{M^{q,N}(\mathbb{R}^N)}.
\end{aligned}$$

Here and below, $C_{\kappa_1}, C'_{\kappa_1}, C_{\kappa_2}, C'_{\kappa_2} > 0$ are absolute constants depending on q, N.

Since

$$\frac{\kappa_1 N}{N-1} = q + 1 + \frac{1}{N-1} > q,$$

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an application of an interpolation inequality, like (2.8), says that $u \in L^s(\mathbb{R}^N)$ for each $s \in [q, q+1+1/(N-1)]$. Next, set $\kappa_2 := 1+\kappa_1 > 2$ to observe that

$$\|u\|_{\kappa_2 N/(N-1),\mathbb{R}^N} \leqslant C_{\kappa_2}\|u\|_{M^{\kappa_1 N/(N-1),N}(\mathbb{R}^N)} \leqslant C'_{\kappa_2}\|u\|_{M^{q,N}(\mathbb{R}^N)}.$$

As

$$\frac{\kappa_2 N}{N-1} = \frac{\kappa_1 N}{N-1} + 1 + \frac{1}{N-1},$$

we thereby can extend s to be in $[q, q+2+2/(N-1)]$. Continuing like this to set $\kappa_m := m-1+\kappa_1 = m+q(N-1)/N > m$, we can likewise extend s to be in the interval $[q, q+m+m/(N-1)]$ for each $m \geqslant 1$. As a consequence, we finally arrive at $s \in [q, \infty)$. □

When $N \geqslant 2$, $p \in (N, \infty]$ and $q \in [1, \infty]$, we can simply adapt the proof of *Morrey's inequality* [15, theorem 5.6.4] to see that, for $\gamma := 1 - N/p$ and some constants $C_1, C'_1 > 0$,

$$\sup_{x\in\mathbb{R}^N}\{|u(x)|\} \leqslant C_1\|u\|_{M^{q,p}(\mathbb{R}^N)} \quad \text{and} \quad \|u\|_{C^{0,\gamma}(\mathbb{R}^N)} \leqslant C'_1\|u\|_{M^{q,p}(\mathbb{R}^N)}.$$

In particular, we have (2.11) when $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, as $M^{q,p}(\mathbb{R}^N) \subseteq L^\infty(\mathbb{R}^N)$.

All the foregoing discussions may be viewed as a certain complement to Lions [24, lemma I.1] for the (most important) case in which $p = 2$. (Lieb and Loss's result plays a key role.)

As a final remark, note that the space $L^q(\mathbb{R}^N)$ is often defined via the family of equivalent classes $[u]$ of functions u, and two functions $u_1, u_2 \in [u]$ are identified when $u_1 = u_2$ almost everywhere (a.e.) (that is, if we ignore a subset of $\mathbb{R}^N$ of Lebesgue measure zero). Define, for all $x \in \mathbb{R}^N$,

$$u^\star(x) := \begin{cases} \lim\limits_{R\to 0^+} \dfrac{1}{\mathfrak{L}(\boldsymbol{B}_R)} \displaystyle\int_{\boldsymbol{B}_R(x)} u(y)\,\mathrm{d}y & \text{if this limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Then $u^\star \in [u]$ and $u_1^\star \equiv u_2^\star$. Henceforth, we will use this *precise representative* $u^\star$ of $[u]$. Hereafter, $\boldsymbol{B}_R(x)$ denotes the ball of radius R centred at x, and $\boldsymbol{B}_R := \boldsymbol{B}_R(0)$.

3. The function space $M^{q,p}_\mu(\mathbb{R}^N)$

In this section we describe a compact embedding result that may be treated as a counterpart to proposition 2.1 on $\mathbb{R}^N$ when $N \geqslant 2$, $1 < p < N$ and $1 < q < \infty$. In particular, for $D^1_\mu(\mathbb{R}^N)$ defined as $D^1(\mathbb{R}^N) \cap L^2_\mu(\mathbb{R}^N)$ when $N \geqslant 3$, we see that $D^1_\mu(\mathbb{R}^N)$ behaves like $H^1(\Omega)$ on $\mathbb{R}^N$. We recall here that when $N \geqslant 2$ this has been described in [8] using the space $H^1(\mathbb{R}^N)$.

We first briefly review the concept of p-capacity (see [13, 16, 26] for more details). When $N \geqslant 2$ and $1 \leqslant p < N$ define the *p-capacity* of a subset A of $\mathbb{R}^N$ to be

$$\mathrm{Cap}_p(A) := \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x \colon u \in D^{1,p}(\mathbb{R}^N) \text{ and } A \subseteq \{u \geqslant 1\}^o\right\}. \tag{3.1}$$

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On the other hand, when $p = N$ define the N-*capacity* of A to be

$$\mathrm{Cap}_N(A) := \inf\{\|u\|^N_{W^{1,N}(\mathbb{R}^N)} \colon u \in W^{1,N}(\mathbb{R}^N) \text{ and } A \subseteq \{u \geqslant 1\}^o\}. \tag{3.2}$$

Note that A is required to lie entirely inside the interior of the set $\{u \geqslant 1\}$.

Capacity is used to characterize precisely some fine properties of Sobolev functions.

The result below describes some relations between capacity and Hausdorff measure.

PROPOSITION 3.1. *When* $1 < p < N$ *and* $\mathcal{H}^{N-p}(A) < \infty$, $\mathrm{Cap}_p(A) = 0$. *When* $1 \leqslant p < \infty$ *and* $\mathrm{Cap}_p(A) = 0$, $\mathcal{H}^s(A) = 0$ *for all* $s > N - p$. $\mathrm{Cap}_1(A) \leqslant \mathfrak{c}\mathcal{H}^{N-1}(A)$, *and* $\mathrm{Cap}_1(A) = 0$ *if and only if* $\mathcal{H}^{N-1}(A) = 0$ *when* A *is compact. Also,* $\mathrm{Cap}_p(\lambda A) = \lambda^{N-p}\,\mathrm{Cap}_p(A)$ *for all* $\lambda > 0$ *and* $\mathfrak{L}(A) \leqslant \mathfrak{c}_1[\mathrm{Cap}_p(A)]^{N/(N-p)}$ *when* $1 \leqslant p < N$. *Here,* $\mathfrak{c}, \mathfrak{c}_1 > 0$ *are absolute constants.*

For each $u \in D^{1,p}(\mathbb{R}^N)$ (or $W^{1,N}(\mathbb{R}^N)$) there is a Borel subset E of $\mathbb{R}^N$ with $\mathrm{Cap}_p(E) = 0$ such that the limit in (2.12) exists and is identically equal to $u^\star(x)$ when $x \in \mathbb{R}^N \setminus E$. Moreover, for all $\epsilon > 0$ there exists a continuous function $u_\epsilon \colon \mathbb{R}^N \to \mathbb{R}$ such that $\mathrm{Cap}_p(\{u_\epsilon \neq u^\star\}) \leqslant \epsilon$. That is, $u^\star$ is p-*quasi-continuous*. Remembering that we have identified the function $u \in [u]$ with its p-quasi-continuous precise representative $u^\star$, a pointwise condition can therefore be imposed on $u(x)$ for quasi-everywhere (q.e.) $x \in \mathbb{R}^N$ (that is, if we ignore a subset of $\mathbb{R}^N$ of p-capacity zero).

When $p \in [1, N)$ and Ω is as in § 2.1, we write $W^{1,p}_0(\Omega)$ to be the space of functions $u \in W^{1,p}(\Omega)$ that satisfy $u = 0$ q.e. in $\mathbb{R}^N \setminus \Omega$. $W^{1,p}_0(\Omega)$ is a subspace of $D^{1,p}(\mathbb{R}^N)$ in terms of the gradient L^p-norm. (Often there is no need to involve the norm $\|u\|_{p,\Omega}$.)

Below, unless otherwise specified, we assume that $N \geqslant 2$, $p \in (1, N)$ and $q \in (1, \infty)$.

Let $\mu \geqslant 0$ be a regular Borel measure on $\mathbb{R}^N$, possibly infinite-valued, that vanishes on each set of p-capacity zero. Use $D^{1,p}(\mathbb{R}^N)$ to define $L^{\tilde q}_\mu(\mathbb{R}^N)$, as in (1.1), and write $M^{q,p}_\mu(\mathbb{R}^N) := M^{q,p}(\mathbb{R}^N) \cap L^{\tilde q}_\mu(\mathbb{R}^N)$. It is a Banach subspace of $D^{1,p}(\mathbb{R}^N)$ under the norm

$$\|u\|_{M^{q,p}_\mu(\mathbb{R}^N)} := \|u\|_{M^{q,p}(\mathbb{R}^N)} + \|u\|_{\tilde q,\mathbb{R}^N_\mu}. \tag{3.3}$$

Here, we define $\|u\|_{\tilde q,\mathbb{R}^N_\mu}$ to be $(\int_{\mathbb{R}^N} |u|^{\tilde q}\,\mathrm{d}\mu)^{1/\tilde q}$ and always assume $1 < \tilde q < \infty$.

For brevity, denote by $M^\star$ the space of linear functionals on $M^{q,p}_\mu(\mathbb{R}^N)$. Also, for $\varphi \in M^\star$, we say that $\varphi \geqslant 0$ provided that $\varphi(u) \geqslant 0$ for all $u \geqslant 0$ in $M^{q,p}_\mu(\mathbb{R}^N)$. Take $\varphi \geqslant 0$ in $M^\star$ and consider the characteristic PDE (clearly in the distributional sense) of the space $M^{q,p}_\mu(\mathbb{R}^N)$

$$-\Delta_p u + u^{q-1} + \mu u^{\tilde q-1} = \varphi. \tag{3.4}$$

We look for weak solutions $u \in M^{q,p}_\mu(\mathbb{R}^N)$ to (3.4) that satisfy

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{q-2}uv\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{\tilde q-2}uv\,\mathrm{d}\mu = \varphi(v) \tag{3.5}$$

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for all $v \in M^{q,p}_\mu(\mathbb{R}^N)$. Note that solutions u to (3.4) or (3.5), if they exist, are unique and can be found as critical points of the associated energy functional $\mathcal{F}\colon M^{q,p}_\mu(\mathbb{R}^N) \to \mathbb{R}$, defined by

$$\mathcal{F}(u) := \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |u|^q \,\mathrm{d}x + \frac{1}{\tilde{q}}\int_{\mathbb{R}^N} |u|^{\tilde{q}} \,\mathrm{d}\mu - \varphi(u).$$

Since $\varphi(u) \leqslant C_\varphi \|u\|_{M^{q,p}_\mu(\mathbb{R}^N)}$, $\mathcal{F}$ is coercive. Besides, $\mathcal{F}(u) \geqslant \mathcal{F}(u^+)$. Thus, we may, without loss of generality, seek the minimum of $\mathcal{F}$ in the cone of positive functions in $M^{q,p}_\mu(\mathbb{R}^N)$. Here, as usual, we write $u^+ := \max\{u, 0\}$. As we are only interested in finding out when the constant functions are in $M^\star$, we shall take $\varphi = 1$ from now on. (This does not mean that $1 \in M^\star$.)

We first show the following integral version of Damascelli [12, lemma 2.1].

Lemma 3.2. *Let Ω be a region in $\mathbb{R}^N$ for $N \geqslant 1$, and let f, g be two functions in $L^r(\Omega)$ for $1 < r < \infty$. Then there is a constant $C_r > 0$ depending on r, N such that*

$$\int_\Omega (|f|^{r-2}f - |g|^{r-2}g)(f-g)\,\mathrm{d}x \geqslant \begin{cases} C_r \|f-g\|^r_{r,\Omega} & \text{when } r \geqslant 2, \\ C_r \dfrac{\|f-g\|^2_{r,\Omega}}{(\|f\|_{r,\Omega} + \|g\|_{r,\Omega})^{2-r}} & \text{when } 1 < r < 2. \end{cases}$$

Proof. Clearly, we only need to show the second estimate via *Hölder's inequality*.

Precisely, when $r \geqslant 2$ we trivially have $(|f|^{r-2}f - |g|^{r-2}g)(f-g) \geqslant C_r|f-g|^r$ from estimate (2-6) in [12]; on the other hand, when $1 < r < 2$ estimate (2-2) in [12] says that $(|f|^{r-2}f - |g|^{r-2}g)(f-g) \geqslant C_r|f-g|^2/(|f|+|g|)^{2-r}$. Thus, for the latter, it follows that

$$\begin{aligned} \|f-g\|^r_{r,\Omega} &= \int_\Omega \frac{|f-g|^r}{(|f|+|g|)^{r(2-r)/2}} (|f|+|g|)^{r(2-r)/2}\,\mathrm{d}x \\ &\leqslant \biggl\{\int_\Omega \frac{|f-g|^2}{(|f|+|g|)^{2-r}}\,\mathrm{d}x\biggr\}^{r/2} \biggl\{\int_\Omega (|f|+|g|)^r\,\mathrm{d}x\biggr\}^{(2-r)/2} \\ &\leqslant \biggl\{C_r^{-1}\int_\Omega (|f|^{r-2}f - |g|^{r-2}g)(f-g)\,\mathrm{d}x\biggr\}^{r/2} \\ &\qquad \times \{\|f\|_{r,\Omega} + \|g\|_{r,\Omega}\}^{r(2-r)/2}, \end{aligned}$$

where *Minkowski's inequality* was used. This clearly leads to the desired estimate. □

For the sake of brevity, henceforth, we shall use the notation

$$\mathcal{D}^r_\Omega[f,g] := \int_\Omega (|f|^{r-2}f - |g|^{r-2}g)(f-g)\,\mathrm{d}x. \tag{3.6}$$

For each $B \subsetneq \mathbb{R}^N$ define

$$\infty_B(A) := \begin{cases} 0 & \text{when } \operatorname{Cap}_p(A \cap B) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Write

$$\mu\lceil \boldsymbol{B}_R := \mu + \infty_{\mathbb{R}^N \setminus \boldsymbol{B}_R}. \tag{3.7}$$

Then

$$\mu = \lim_{R\to\infty} \mu\lceil \boldsymbol{B}_R$$

(as measures), and $M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$ is a subspace of $M^{q,p}_0(\boldsymbol{B}_R)$, where each function $u \in M^{q,p}_0(\boldsymbol{B}_R)$ satisfies $u = 0$ q.e. in $\mathbb{R}^N \setminus \boldsymbol{B}_R$. We note that 1 is in the dual space of the space $M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$. Therefore, we can prove the following easy result.

PROPOSITION 3.3. *When $1 < p < N$ and $1 < q, \tilde{q} < \infty$, there exists a unique weak solution $\omega_{\mu_R} \geqslant 0$ of (3.4) in $M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$ associated with the linear functional $\varphi = 1$.*

Proof. Recall that $\mathcal{F}$ is coercive. Suppose that $\{u_k \geqslant 0 \colon k \geqslant 1\}$ is a minimizing sequence in $M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$ that clearly is bounded. Then we can find a function ω_{μ_R} and a subsequence $\{u_k \geqslant 0 \colon k \geqslant 1\}$, using the same notation, such that $u_k \rightharpoonup \omega_{\mu_R} \in M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$. (By reflexivity, $|\nabla u_k| \rightharpoonup |\nabla \omega_{\mu_R}|$ in the space $L^p(\mathbb{R}^N)$, and $u_k \rightharpoonup \omega_{\mu_R}$ in the spaces $L^q(\mathbb{R}^N)$ and $L^{\tilde{q}}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$.) Noting that $\mathcal{F}$ is differentiable, one has, for each $u \in M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$ and all $v \in M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$,

$$\mathcal{F}'(u)(v) = \int_{\boldsymbol{B}_R} |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\boldsymbol{B}_R} |u|^{q-2}uv \,\mathrm{d}x + \int_{\boldsymbol{B}_R} |u|^{\tilde{q}-2}uv \,\mathrm{d}\mu - \varphi(v).$$

As $\mathcal{F}'(u_k) \to 0$ and $\mathcal{F}'(\omega_{\mu_R})$ is a linear functional on $M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$, one sees that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left\{ \mathcal{F}'(u_k)(u_k - \omega_{\mu_R}) - \mathcal{F}'(\omega_{\mu_R})(u_k - \omega_{\mu_R}) + \int_{\boldsymbol{B}_R} (u_k - \omega_{\mu_R}) \,\mathrm{d}x \right\} \\ &= \lim_{k\to\infty} \left\{ \mathcal{D}^p_{\boldsymbol{B}_R}[|\nabla u_k|, |\nabla \omega_{\mu_R}|] + \mathcal{D}^q_{\boldsymbol{B}_R}[u_k, \omega_{\mu_R}] + \mathcal{D}^{\tilde{q}}_{\mathbb{R}^N_{\mu\lceil \boldsymbol{B}_R}}[u_k, \omega_{\mu_R}] \right\}. \end{aligned}$$

So, we can apply lemma 3.2 to derive that $u_k \to \omega_{\mu_R} \geqslant 0$ in $M^{q,p}_{\mu\lceil \boldsymbol{B}_R}(\mathbb{R}^N)$ when $k \to \infty$. □

Note that ω_{μ_R} is p-quasi-continuous and equals zero q.e. in $\mathbb{R}^N \setminus \boldsymbol{B}_R$. When $R_1 \leqslant R_2$ we have $\omega_{\mu_{R_1}} \equiv \omega_{\mu_{R_2}}$ on $\boldsymbol{B}_{R_1}$ by uniqueness. This observation enables us to define

$$\omega_\mu := \lim_{R\to\infty} \omega_{\mu_R} \geqslant 0 \quad \text{in } \mathbb{R}^N, \tag{3.8}$$

which formally solves (3.5) ($\varphi = 1$). To remove the word *formally*, we only need to see that $1 \in M^\star$. The next two results provide us with an equivalence condition and an example for this.

LEMMA 3.4. *Let $p \in (1, N)$ and let $q, \tilde{q} \in (1, \infty)$. Then $1 \in M^\star$ if and only if $\omega_\mu \in L^1(\mathbb{R}^N)$.*

Proof. When $1 \in M^\star$ we can solve (3.5) directly to give a weak solution $\omega_\mu \in M^{q,p}_\mu(\mathbb{R}^N)$ associated with (3.4). Also, we have $\omega_\mu \in L^1(\mathbb{R}^N)$ by letting $v = \omega_\mu$ in (3.5).

On the other hand, if $\omega_\mu \in L^1(\mathbb{R}^N)$, the *monotone convergence theorem* implies that

$$\||\nabla \omega_{\mu_R}|\|^p_{p,\boldsymbol{B}_R} + \|\omega_{\mu_R}\|^q_{q,\boldsymbol{B}_R} + \|\omega_{\mu_R}\|^{\tilde q}_{\tilde q,\mathbb{R}^N_{\mu\lceil \boldsymbol{B}_R}} = \|\omega_{\mu_R}\|_{1,\boldsymbol{B}_R} \leqslant \|\omega_\mu\|_{1,\mathbb{R}^N}. \tag{3.9}$$

So, from (3.8), $\{\omega_{\mu_R} \geqslant 0\colon R \geqslant 0\}$ is a bounded sequence that converges weakly to the function $\omega_\mu \in M^{q,p}_\mu(\mathbb{R}^N)$. As a result, taking test functions $v \in C^1_c(\mathbb{R}^N) \cap M^{q,p}_\mu(\mathbb{R}^N)$ in the equation satisfied by ω_{μ_R} and then letting $R \to \infty$ shows that ω_μ solves (3.5) for $\varphi = 1$ on $\mathbb{R}^N$ by a standard density argument about the functions v. As (3.9) implies that $\|\omega_\mu\|_{M^{q,p}_\mu(\mathbb{R}^N)} < \infty$, we have $1 \in M^\star$. □

We note the application above of the density results described in § 2.2.

EXAMPLE 3.5. Let $p \in (1, N)$, let $q, \tilde q \in (1, \infty)$ and let $V(x)\colon \mathbb{R}^N \to (0, \infty)$ be a measurable function with $V^{-\beta} \in L^1(\mathbb{R}^N)$ for some $\beta \in (0, 1/(\tilde q - 1)]$. Then, for $\mathrm{d}\mu := V(x)\,\mathrm{d}x$, one has $1 \in M^\star$.

Proof. Define

$$\mathfrak{x} := \frac{\beta(q-1)}{q+\beta(q-\tilde q)}, \quad \mathfrak{y} := \frac{1+\beta(1-\tilde q)}{q+\beta(q-\tilde q)} \quad \text{and} \quad \mathfrak{z} := \frac{q-1}{q+\beta(q-\tilde q)};$$

see table 1. Then, $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in (0, 1)$ and $\mathfrak{x} + \mathfrak{y} + \mathfrak{z} = 1$. We set $r_1 := \mathfrak{x}^{-1}$, $r_2 := \mathfrak{y}^{-1}$ and $r_3 := \mathfrak{z}^{-1}$ to derive

$$\begin{aligned}
\int_{\mathbb{R}^N} \omega_\mu \,\mathrm{d}x &= \int_{\mathbb{R}^N} (V\omega_\mu^{\tilde q})^{\mathfrak{x}} (\omega_\mu)^{1-\tilde q\mathfrak{x}} \frac{1}{V^{\mathfrak{x}}}\,\mathrm{d}x \\
&\leqslant \biggl(\int_{\mathbb{R}^N} (V\omega_\mu^{\tilde q})^{r_1\mathfrak{x}}\,\mathrm{d}x\biggr)^{1/r_1} \biggl(\int_{\mathbb{R}^N} (\omega_\mu)^{r_2(1-\tilde q\mathfrak{x})}\,\mathrm{d}x\biggr)^{1/r_2} \biggl(\int_{\mathbb{R}^N} \frac{1}{V^{r_3\mathfrak{x}}}\,\mathrm{d}x\biggr)^{1/r_3} \\
&\leqslant \biggl(\int_{\mathbb{R}^N} \omega_\mu^{\tilde q}\,\mathrm{d}\mu\biggr)^{1/r_1} \biggl(\int_{\mathbb{R}^N} \omega_\mu^{q}\,\mathrm{d}x\biggr)^{1/r_2} \biggl(\int_{\mathbb{R}^N} \frac{1}{V^{\beta}}\,\mathrm{d}x\biggr)^{1/r_3},
\end{aligned}$$

from which we easily deduce that $\|\omega_\mu\|_{1,\mathbb{R}^N} \leqslant \|V^{-\beta}\|_{1,\mathbb{R}^N} < \infty$. That is, $1 \in M^\star$. □

Finally, we shall describe a result that resembles proposition 2.1 on $\mathbb{R}^N$ when $N \geqslant 2$. Note that we assume that $1 < \tilde q < \infty$ and μ is such that 1 belongs to the dual space of $M^{q,p}_\mu(\mathbb{R}^N)$.

THEOREM 3.6. *Let $p \in (1, N)$ and let $q \in (1, \infty)$. Then the embedding*

$$\iota\colon M^{q,p}_\mu(\mathbb{R}^N) \to L^s(\mathbb{R}^N)$$

is continuous if $1 \leqslant s \leqslant \max\{q, p^*\}$ and also compact if $1 \leqslant s < \max\{q, p^*\}$.

Proof. This result follows easily from proposition 2.1 and (2.10) if we can prove it for the case in which $s = 1$. Since $1 \in M^\star$, one has $\omega_\mu \in L^1(\mathbb{R}^N) \cap M^{q,p}_\mu(\mathbb{R}^N)$, which satisfies

$$\int_{\mathbb{R}^N} |\nabla \omega_\mu|^{p-2} \nabla\omega_\mu \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} \omega_\mu^{q-1} v\,\mathrm{d}x + \int_{\mathbb{R}^N} \omega_\mu^{\tilde q-1} v\,\mathrm{d}\mu = \int_{\mathbb{R}^N} v\,\mathrm{d}x, \tag{3.10}$$

Table 1. *Table showing how to choose the $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ in example 3.5, and giving the best choice of the upper bound ℓ of $\beta \in (0, \ell]$ when $q, \tilde{q} \in (1, \infty)$.*

ℓ	$\mathfrak{x} > 0$	$\mathfrak{x} < 1$	$\mathfrak{y} > 0$	$\mathfrak{y} < 1$	$\mathfrak{z} > 0$	$\mathfrak{z} < 1$	best choice
$1 < \tilde{q} \leqslant q$	any ℓ	$\ell \leqslant \dfrac{q}{\tilde{q}-1}$	$\ell \leqslant \dfrac{1}{\tilde{q}-1}$	any ℓ	any ℓ	any ℓ	$\ell \leqslant \dfrac{1}{\tilde{q}-1}$
$q < \tilde{q} < \infty$	$\ell \leqslant \dfrac{q}{\tilde{q}-q}$	$\ell \leqslant \dfrac{q}{\tilde{q}-1}$	$\ell \leqslant \dfrac{1}{\tilde{q}-1}$	$\ell \leqslant \dfrac{q}{\tilde{q}-q}$	$\ell \leqslant \dfrac{q}{\tilde{q}-q}$	$\ell \leqslant \dfrac{1}{\tilde{q}-q}$	$\ell \leqslant \dfrac{1}{\tilde{q}-1}$

so that $\|v\|_{1,\mathbb{R}^N} \leqslant C_{\omega_\mu}\|v\|_{M_\mu^{q,p}(\mathbb{R}^N)} < \infty$ for each $v \in M_\mu^{q,p}(\mathbb{R}^N)$. Here, $C_{\omega_\mu} > 0$ is a constant that depends on ω_μ. That is, $M_\mu^{q,p}(\mathbb{R}^N)$ is continuously embedded into $L^1(\mathbb{R}^N)$.

Now, given a sequence $\{u_k \colon k \geqslant 1\}$ of functions in $M_\mu^{q,p}(\mathbb{R}^N)$ with $u_k \rightharpoonup 0$, we may assume that it is bounded. Notice that $u_k \to 0$ in $L^1(\boldsymbol{B}_R)$ for all $R > 0$. So, as measures, we have

$$\lim_{k\to\infty} |u_k| \rightharpoonup \mathrm{d}\upsilon := \upsilon_\infty \delta_\infty, \quad \text{with } \upsilon_\infty := \lim_{R\to\infty} \limsup_{k\to\infty} \int_{|x|>R} |u_k| \,\mathrm{d}x \geqslant 0,$$

where an idea from [7, proposition 2] was used.

We see that $\upsilon_\infty = 0$. Actually, there is a decreasing function $\chi_R \in C_c^\infty(\mathbb{R}^N)$ such that $\chi_R \equiv 1$ on $|x| \leqslant R$ and $\chi_R \equiv 0$ on $|x| \geqslant R^2$, which satisfies $1/R^2 \leqslant \||\nabla\chi_R|\|_{\infty,\mathbb{R}^N} \leqslant 6/R^2$ whenever $R > 2$ (see [17, appendix (i)]). Set $\theta_R := 1 - \chi_R$. Then, in view of (3.10), it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} |u_k\theta_R| \,\mathrm{d}x \leqslant \||\nabla\omega_\mu|\|_{p,\mathbb{R}^N\setminus\boldsymbol{B}_R}^{p-1} \biggl(\int_{\mathbb{R}^N} |u_k\nabla\theta_R + \theta_R\nabla u_k|^p \,\mathrm{d}x\biggr)^{1/p} \\ + \|\omega_\mu\|_{q,\mathbb{R}^N\setminus\boldsymbol{B}_R}^{q-1} \|u_k\|_{q,\mathbb{R}^N} + \|\omega_\mu\|_{\tilde{q},\mathbb{R}^N_{\mu\lceil\mathbb{R}^N\setminus\boldsymbol{B}_R}}^{\tilde{q}-1} \|u_k\|_{\tilde{q},\mathbb{R}^N_\mu},\end{aligned}$$

which clearly tends to zero as $R \to \infty$, by virtue of the estimate

$$\begin{aligned}\||\nabla(u_k\theta_R)|\|_{p,\mathbb{R}^N} &\leqslant \||\nabla u_k|\|_{p,\mathbb{R}^N} + \||\nabla\theta_R|\|_{\infty,\mathbb{R}^N} \|u_k\|_{p,\boldsymbol{B}_{R^2}\setminus\boldsymbol{B}_R} \\ &\leqslant \||\nabla u_k|\|_{p,\mathbb{R}^N} + \||\nabla\theta_R|\|_{\infty,\mathbb{R}^N} [\mathfrak{L}(\boldsymbol{B}_{R^2})]^{1/N} \|u_k\|_{p^*,\mathbb{R}^N}\end{aligned}$$

and (2.9). As a consequence, $M_\mu^{q,p}(\mathbb{R}^N)$ is compactly embedded into $L^1(\mathbb{R}^N)$. □

3.1. $q \in [1, p^*]$

In this case it is superfluous to get $L^q(\mathbb{R}^N)$ involved via (2.10). So, we define $N_\mu^{q,p}(\mathbb{R}^N) := D^{1,p}(\mathbb{R}^N) \cap L_\mu^q(\mathbb{R}^N)$ instead. It is a Banach space under the norm

$$\|u\|_{N_\mu^{q,p}(\mathbb{R}^N)} := \|u\|_{q,\mathbb{R}^N_\mu} + \||\nabla u|\|_{p,\mathbb{R}^N}. \tag{3.11}$$

In particular, we will use the notation $D_\mu^{1,p}(\mathbb{R}^N)$ to denote $D^{1,p}(\mathbb{R}^N) \cap L_\mu^p(\mathbb{R}^N)$.

We note that $D_\mu^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space under the norm (3.11).

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Let $\varphi \geqslant 0$ be a linear functional on $N_\mu^{q,p}(\mathbb{R}^N)$ and consider the characteristic PDE

$$-\Delta_p u + \mu u^{q-1} = \varphi. \tag{3.12}$$

We look for the unique solution $\omega_\mu \geqslant 0$ in $N_\mu^{q,p}(\mathbb{R}^N)$ to (3.12) for $\varphi = 1$ that satisfies

$$\int_{\mathbb{R}^N} |\nabla\omega_\mu|^{p-2}\nabla\omega_\mu \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} \omega_\mu^{q-1} v \,\mathrm{d}\mu = \int_{\mathbb{R}^N} v \,\mathrm{d}x \quad \forall v \in N_\mu^{q,p}(\mathbb{R}^N). \tag{3.13}$$

Then, assuming that μ enables $\omega_\mu \in L^1(\mathbb{R}^N)$, we have the following parallel result.

Corollary 3.7. *Let $p \in (1, N)$ and let $q \in (1, \infty)$. Then the embedding*

$$\iota\colon N_\mu^{q,p}(\mathbb{R}^N) \to L^s(\mathbb{R}^N)$$

is continuous if $1 \leqslant s \leqslant p^*$ and also compact if $1 \leqslant s < p^*$.

When $N \geqslant 3$ and $p = 2$ we simply denote $D^1(\mathbb{R}^N) \cap L_\mu^2(\mathbb{R}^N)$ by $D_\mu^1(\mathbb{R}^N)$. It is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{D_\mu^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} uv \,\mathrm{d}\mu + \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \,\mathrm{d}x. \tag{3.14}$$

This space behaves like $H^1(\Omega)$ on $\mathbb{R}^N$ if we assume that μ enables $\omega_\mu \in L^1(\mathbb{R}^N)$.

Next, let $V(x)\colon \mathbb{R}^N \to (0, \infty)$ be a measurable function with $V^{-\beta} \in L^1(\mathbb{R}^N)$. Applying a similar calculation as before, we can introduce $\omega_\mu^{p^*}$ and use (2.9) to observe that when $\beta \in (0, 1/(q-1)]$ we have $\omega_\mu \in L^1(\mathbb{R}^N)$ with $\mathrm{d}\mu := V(x)\,\mathrm{d}x$.

3.2. $p = N$

We recall that this has been discussed in [8]. The interesting case to us is when $N = 2$, where $H_\mu^1(\mathbb{R}^N)$ behaves like $H^1(\Omega)$ on $\mathbb{R}^N$, provided that μ enables $\omega_\mu \in L^1(\mathbb{R}^N)$.

When $q \in [N, \infty)$ one can define $M^{q,N}(\mathbb{R}^N)$ directly (see the descriptions above (2.2) and above (2.11)) as the space of $u \in L^q(\mathbb{R}^N)$ with $|\nabla u| \in L^N(\mathbb{R}^N)$ by the density of $C_c^1(\mathbb{R}^N)$ in $M^{q,N}(\mathbb{R}^N)$. Actually, to see this fact, for $u \in M^{q,N}(\mathbb{R}^N)$ set $u_R := u\theta_R$ to derive

$$\|u - u_R\|_{q,\mathbb{R}^N} \leqslant \|u\|_{q,\mathbb{R}^N\setminus \boldsymbol{B}_R}$$

and

$$\begin{aligned} \||\nabla(u-u_R)|\|_{N,\mathbb{R}^N} &\leqslant \||\nabla u|\|_{N,\mathbb{R}^N\setminus\boldsymbol{B}_R} + \||\nabla\theta_R|\|_{\infty,\mathbb{R}^N}\|u\|_{N,\boldsymbol{B}_{R^2}\setminus\boldsymbol{B}_R} \\ &\leqslant \||\nabla u|\|_{N,\mathbb{R}^N\setminus\boldsymbol{B}_R} \\ &\quad + \||\nabla\theta_R|\|_{\infty,\mathbb{R}^N}[\mathfrak{L}(\boldsymbol{B}_{R^2})]^{(q-N)/qN}\|u\|_{q,\mathbb{R}^N\setminus\boldsymbol{B}_R}. \end{aligned}$$

As u_R has compact support, the density result on bounded regions confirms the answer.

On the other hand, if we simply select a measure μ, with density $V(x) \geqslant 0$, that is absolutely continuous with respect to the Lebesgue measure, one actually can modify the preceding proofs very slightly to observe, in view of proposition 2.2, the following embedding result.

COROLLARY 3.8. *Assume that* $N \geqslant 2$, *that* $q, \tilde{q} \in (1, \infty)$ *and that* $\mathrm{d}\mu := V(x)\,\mathrm{d}x$ *with* $V^{-\beta} \in L^1(\mathbb{R}^N)$ *for some* $\beta \in (0, 1/(\tilde{q}-1)]$. *Then the embedding*

$$\iota\colon M_\mu^{q,N}(\mathbb{R}^N) \to L^s(\mathbb{R}^N)$$

is compact if $1 \leqslant s < \infty$.

Here, we write $M_\mu^{q,N}(\mathbb{R}^N) := M^{q,N}(\mathbb{R}^N) \cap L_\mu^{\tilde{q}}(\mathbb{R}^N)$ with norm given by (3.3).

Finally, let $V(x)$ be a measurable function in $\mathbb{R}^N$ such that $V \geqslant 0$ and $V^{-1} \in L^1(\mathbb{R}^N)$ when $N \geqslant 3$ and such that $V \geqslant 1$ and $(V-1)^{-1} \in L^1(\mathbb{R}^2)$. Define the Hilbert space $\mathfrak{H} := D_\mu^1(\mathbb{R}^N)$ when $N \geqslant 3$ and $\mathfrak{H} := H_\mu^1(\mathbb{R}^2)$ through V, where we write $H_\mu^1(\mathbb{R}^2) := H^1(\mathbb{R}^2) \cap L_\mu^2(\mathbb{R}^2)$ as in [8]. Define $\mathtt{L}u := -\operatorname{div}(\nabla u) + Vu$ on $\mathbb{R}^N$. Concerning the eigenvalue problem

$$\mathtt{L}w = \lambda w \quad \text{in } \mathbb{R}^N, \tag{3.15}$$

we can simply modify [15, theorems 6.5.1 and 6.5.2] to prove the following result.

THEOREM 3.9. *There exists a sequence of increasing eigenvalues* $\{\lambda_k > 0\colon k \geqslant 1\}$ *for problem (3.15) such that* $\lim_{k\to\infty} \lambda_k \to \infty$, *and a family of associated eigenfunctions* $\{w_k\colon k \geqslant 1\}$ *in* $\mathfrak{H}$ *that provides an orthonormal basis for* $\mathfrak{H}$. *In addition, the first eigenvalue* λ_1 *is simple and isolated, and every associated eigenfunction of it does not change sign in* $\mathbb{R}^N$.

The inner product on $H_\mu^1(\mathbb{R}^2)$, with $\mathrm{d}\mu := (V-1)\,\mathrm{d}x$ for $V(x) \geqslant 1$, is given by

$$\langle u, v\rangle_{H_\mu^1(\mathbb{R}^2)} := \int_{\mathbb{R}^2} [\nabla u \cdot \nabla v + uv]\,\mathrm{d}x + \int_{\mathbb{R}^2} uv\,\mathrm{d}\mu. \tag{3.16}$$

Proof. Define $\mathtt{S} := \mathtt{L}^{-1}$. Then $\mathtt{S}$ is a compact linear symmetric operator, and the existence of $\{\lambda_k > 0\colon k \geqslant 1\}$, with $\lim_{k\to\infty} \lambda_k \to \infty$, and $\{w_k\colon k \geqslant 1\}$ follows immediately from [15, theorem D.6.7]; moreover, $\{w_k\colon k \geqslant 1\}$ in fact provides an orthonormal basis to $L^2(\mathbb{R}^N)$.

We only prove the remaining parts for $D_\mu^1(\mathbb{R}^N)$. Note that $\langle w_{k_1}, w_{k_2}\rangle_{D_\mu^1(\mathbb{R}^N)} = 0$ if and only if $\langle w_{k_1}, w_{k_2}\rangle_{2,\mathbb{R}^N} = 0$. Assume that $\{w_k/\sqrt{\lambda_k}\colon k \geqslant 1\}$ and $\{w_k\colon k \geqslant 1\}$ are orthonormal bases of $D_\mu^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$, respectively. For each $w \in D_\mu^1(\mathbb{R}^N)$ write $w = w^+ - w^-$. Then ∇w^+, ∇w^- are well defined. Follow steps 1–6 of [15, theorem 6.5.2] to derive

$$\lambda_1 = \min_{\substack{w \in D_\mu^1(\mathbb{R}^N),\\ w \not\equiv 0}} \frac{\|w\|^2_{D_\mu^1(\mathbb{R}^N)}}{\|w\|^2_{2,\mathbb{R}^N}}$$

and $\mathtt{L}w_1^\pm = \lambda_1 w_1^\pm$. We can simply take $w_1 \equiv w_1^+ > 0$ by [30, theorem 1].

Finally, let w_1^* be an eigenfunction associated with λ_1 in $D_\mu^1(\mathbb{R}^N)$. Assume that $\|w_1^*\|_{2,\mathbb{R}^N} = 1$. Consider $\bar{w}_1 := w_1 - w_1^*$ and suppose that $\bar{w}_1 \not\equiv 0$. Then we have, say, $\bar{w}_1 > 0$ on $\mathbb{R}^N$ as well since $\mathtt{L}\bar{w}_1 = \lambda_1 \bar{w}_1$. However, noticing that $\langle \bar{w}_1, w_1 + w_1^*\rangle_{2,\mathbb{R}^N} = 0$, a contradiction follows. □

4. Existence results for (1.2)

In this section we study (1.2) when $N \geqslant 3$. Precisely, we shall prove the following result.

THEOREM 4.1. *Let $r \in (0,1)$, let $q \in [2,\infty)$ and let $\beta = 1/(q-1)$. Let $\alpha(x) \geqslant 0$ be a measurable function such that $\alpha^{-\beta} \in L^1(\mathbb{R}^N)$ if $q \in [2,2^*)$, while $\alpha \geqslant 1$ and $(\alpha-1)^{-\beta} \in L^1(\mathbb{R}^N)$ if $q \in [2^*,\infty)$. Let $\lambda > 0$ be a constant. Then there is a $\lambda_0 > 0$ such that problem (1.2) has a solution $u_\lambda > 0$ for each $0 < \lambda \leqslant \lambda_0$, and $u_\lambda \to 0$ when $\lambda \to 0^+$.*

These solutions are sought in the space $D^1_\mu(\mathbb{R}^N)$ when $q = 2$, in $N^{q,2}_\mu(\mathbb{R}^N)$ when $q \in (2,2^*]$, and in $M^{q,2}_\mu(\mathbb{R}^N)$ when $q \in (2^*,\infty)$, where we shall take $\tilde{q} = q$ from now on.

Below, we assume that $\mathrm{d}\mu = V\,\mathrm{d}x$ for $V := \alpha$ when $q \in [2,2^*)$ and use (3.11) to define $\|u\|_{N^{q,2}_\mu(\mathbb{R}^N)}$ with

$$\|u\|^q_{q,\mathbb{R}^N_V} := \int_{\mathbb{R}^N} V|u|^q\,\mathrm{d}x;$$

when $q \in (2^*,\infty)$ we set $V := \alpha - 1$ and define

$$\|u\|_{M^{q,2}_\mu(\mathbb{R}^N)} := \|u\|_{q,\mathbb{R}^N_V} + \||\nabla u|\|_{2,\mathbb{R}^N} \tag{4.1}$$

with

$$\|u\|^q_{q,\mathbb{R}^N_V} := \int_{\mathbb{R}^N} (1+V)|u|^q\,\mathrm{d}x = \int_{\mathbb{R}^N} \alpha|u|^q\,\mathrm{d}x,$$

slightly different from (3.3); when $q = 2^*$ we apply (3.11) to define $\|u\|_{N^{2^*,2}_\mu(\mathbb{R}^N)}$ with $V := \alpha - 1$. We note that all these assumptions are imposed to guarantee the conclusions of theorem 3.6 as well as of corollary 3.7.

When $q \neq 2^*$ these solutions are found as critical points of the associated energy functional $\mathcal{J}\colon N^{q,2}_\mu(\mathbb{R}^N) \to \mathbb{R}$ if $q \in [2,2^*)$ or $\mathcal{J}\colon M^{q,2}_\mu(\mathbb{R}^N) \to \mathbb{R}$ if $q \in (2^*,\infty)$, which is defined by

$$\mathcal{J}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} \alpha|u|^q\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} (u^+)^{2^*}\,\mathrm{d}x - \frac{\lambda}{r+1}\int_{\mathbb{R}^N} (u^+)^{r+1}\,\mathrm{d}x.$$

Before we proceed to the proof of theorem 4.1, we derive the following result.

PROPOSITION 4.2. *When $q \in [2,2^*)$ and $c < S^{N/2}/N - K\lambda^{q/(q-1-r)}$, the functional $\mathcal{J}$ satisfies the $(PS)_c$-condition. Here, $K = K(N,\alpha,q,r) > 0$ is an absolute constant.*

Recall that a sequence $\{u_k\colon k \geqslant 1\}$ of functions in $N^{q,2}_\mu(\mathbb{R}^N)$ is said to be a *Palais–Smale sequence* of $\mathcal{J}$ at level c, a *$(PS)_c$-sequence*, provided that $\mathcal{J}(u_k) \to c$ and $\mathcal{J}'(u_k) \to 0$ as $k \to \infty$. $\mathcal{J}$ is said to satisfy the *Palais–Smale condition* at level c, the *$(PS)_c$-condition*, provided that each $(\mathrm{PS})_c$-sequence of $\mathcal{J}$ admits a strongly convergent subsequence in $N^{q,2}_\mu(\mathbb{R}^N)$.

Here, $S := C^{-2}_{2,N}$ by (2.9) and the norm of $D^1_\mu(\mathbb{R}^N)$ is defined via (3.14).

Proof. First, given a $(\mathrm{PS})_c$-sequence $\{u_k\colon k \geqslant 1\}$ of $\mathcal{J}$ in $N^{q,2}_\mu(\mathbb{R}^N)$, one observes easily that it is bounded. Actually, a routine calculation of $\mathcal{J}(u_k) - (1/2^*)\mathcal{J}'(u_k)(u_k)$

will do. Hence, we may without loss of generality assume that there is a function $u \in N_\mu^{q,2}(\mathbb{R}^N)$ such that $u_k \rightharpoonup u$ in $L_V^q(\mathbb{R}^N)$ and $L^{2^*}(\mathbb{R}^N)$, $|\nabla u_k| \rightharpoonup |\nabla u|$ in $L^2(\mathbb{R}^N)$, whereas $u_k \to u$ in $L^{r+1}(\mathbb{R}^N)$. Here, we write $L_V^q(\mathbb{R}^N)$ as the subspace of functions $u \in D^1(\mathbb{R}^N)$ that satisfy $\|u\|_{q,\mathbb{R}_V^N} < \infty$.

From the classical result of Lions [25, lemma I.1] and [7, proposition 2], we have

$$\left.\begin{aligned} |\nabla u_k|^2 \rightharpoonup \mathrm{d}\mu \geqslant |\nabla u|^2 + \sum_{i=1}^{\infty} \mu_i \delta_{x_i} + \mu_\infty \delta_\infty, \\ |u_k|^{2^*} \rightharpoonup \mathrm{d}\nu = |u|^{2^*} + \sum_{i=1}^{\infty} \nu_i \delta_{x_i} + \nu_\infty \delta_\infty, \end{aligned}\right\} \tag{4.2}$$

in the sense of measures, as $k \to \infty$. Here, δ_{x_i} is the Dirac delta function at $x_i \in \mathbb{R}^N$, and μ, ν are the generated measures. Also, $\mu_i, \mu_\infty, \nu_i, \nu_\infty \geqslant 0$ satisfy $S\nu_i^{2/2^*} \leqslant \mu_i$ and $S\nu_\infty^{2/2^*} \leqslant \mu_\infty$.

Recall that $\theta_R \in C^\infty(\mathbb{R}^N)$ and $\||\nabla\theta_R|\|_{\infty,\mathbb{R}^N} = O(1/R^2)$. We have $u_k\theta_R \in N_\mu^{q,2}(\mathbb{R}^N)$ and

$$\begin{aligned} 0 &= \lim_{k\to\infty} \mathcal{J}'(u_k)(u_k\theta_R) \\ &= \lim_{k\to\infty} \left\{ \int_{\mathbb{R}^N} |\nabla u_k|^2 \theta_R \,\mathrm{d}x - \int_{\mathbb{R}^N} (u_k^+)^{2^*} \theta_R \,\mathrm{d}x \right\} \\ &\quad + \lim_{k\to\infty} \left\{ \int_{\mathbb{R}^N} \nabla u_k \cdot \nabla\theta_R u_k \,\mathrm{d}x + \int_{\mathbb{R}^N} V|u_k|^q \theta_R \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} (u_k^+)^{r+1} \theta_R \,\mathrm{d}x \right\}. \end{aligned}$$

Note that $V = \alpha$. By the definition of θ_R and (2.9) it follows that, for every $k \geqslant 1$,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_k \cdot \nabla\theta_R u_k| \,\mathrm{d}x &\leqslant \||\nabla\theta_R|\|_{\infty,\mathbb{R}^N} [\mathfrak{L}(\boldsymbol{B}_{R^2})]^{1/N} \||\nabla u_k|\|_{2,\mathbb{R}^N\setminus\boldsymbol{B}_R} \|u_k\|_{2^*,\mathbb{R}^N\setminus\boldsymbol{B}_R} \\ &\to 0 \end{aligned}$$

when $R \to \infty$, by virtue of [3, lemma 2.2]. Next, define

$$\tilde{v}_\infty := \lim_{R\to\infty} \limsup_{k\to\infty} \int_{|x|>R} V|u_k|^q \,\mathrm{d}x \geqslant 0.$$

Then we have $\mu_\infty + \tilde{v}_\infty = \nu_\infty$, so that either $\mu_\infty = \tilde{v}_\infty = \nu_\infty = 0$ or $\nu_\infty \geqslant \mu_\infty \geqslant S^{N/2} > 0$ by a straightforward computation. However, when the latter holds we have

$$\begin{aligned} c &= \lim_{k\to\infty} \left\{ \mathcal{J}(u_k) - \frac{1}{2^*}\mathcal{J}'(u_k)(u_k) \right\} \\ &\geqslant \frac{\mu_\infty}{N} + \frac{1}{N}\||\nabla u|\|_{2,\mathbb{R}^N}^2 + \left(\frac{1}{q} - \frac{1}{2^*}\right)\tilde{v}_\infty + \left(\frac{1}{q} - \frac{1}{2^*}\right)\|u\|_{q,\mathbb{R}_V^N}^q \\ &\quad - C_2\left(\frac{\lambda}{r+1} - \frac{\lambda}{2^*}\right)(\||\nabla u|\|_{2,\mathbb{R}^N}^{r+1} + \|u\|_{q,\mathbb{R}_V^N}^{r+1}) \\ &\geqslant \frac{\mu_\infty}{N} - K\lambda^{q/(q-1-r)} \geqslant \frac{S^{N/2}}{N} - K\lambda^{q/(q-1-r)} \\ &> 0, \end{aligned}$$

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provided that $\lambda > 0$ is sufficiently small. Here, $C_2 > 0$ is a constant that depends on the embedding constant of $\iota\colon N_\mu^{q,2}(\mathbb{R}^N) \hookrightarrow L^{r+1}(\mathbb{R}^N)$. So, we must have $\mu_\infty = \tilde{v}_\infty = \nu_\infty = 0$.

The remaining case in which the points x_i are involved can be discussed similarly. Thus, $u_k \to u$ in $L^{2^*}(\mathbb{R}^N)$ and we have

$$\begin{aligned}\lim_{k\to\infty}\biggl\{\int_{\mathbb{R}^N}|\nabla u_k-\nabla u|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}V(|u_k|^{q-2}u_k-|u|^{q-2}u)(u_k-u)\,\mathrm{d}x\biggr\}\\ =\lim_{k\to\infty}\{(\mathcal{J}'(u_k)-\mathcal{J}'(u))(u_k-u)+\mathcal{D}^{2^*}_{\mathbb{R}^N}[u_k^+,u^+]+\lambda\mathcal{D}^{r+1}_{\mathbb{R}^N}[u_k^+,u^+]\}\\ =0. \qquad (4.3)\end{aligned}$$

As a consequence, we can apply lemma 3.2 to show that $u_k \to u$ in $N_\mu^{q,2}(\mathbb{R}^N)$. □

Proof of theorem 4.1. First we assume that $q \in [2, 2^*)$. Using (2.9), we see that

$$\begin{aligned}\mathcal{J}(u) \geqslant \frac{1}{2}|||\nabla u|||^2_{2,\mathbb{R}^N}\biggl\{1-\frac{2}{2^*}C^{2^*}_{2,N}|||\nabla u|||^{2^*-2}_{2,\mathbb{R}^N}-\lambda\mathfrak{c}_2|||\nabla u|||^{r-1}_{2,\mathbb{R}^N}\biggr\}\\ +\frac{1}{q}\|u\|^q_{q,\mathbb{R}^N_V}\{1-\lambda\mathfrak{c}_2'\|u\|^{r+1-q}_{q,\mathbb{R}^N_V}\}.\end{aligned}$$

Here, $\mathfrak{c}_2, \mathfrak{c}_2' > 0$ are some constants depending on C_2. Select a $\varrho \in (0, 1)$ with

$$1-\frac{2}{2^*}C^{2^*}_{2,N}\varrho^{2^*-2}\geqslant\frac{1}{2}>0.$$

Then a $\lambda_0 := \min\{\varrho^{1-r}/4\mathfrak{c}_2, \varrho^{q-1-r}/2\mathfrak{c}_2'\} > 0$ exists such that

$$\mathcal{J}(u)\geqslant\min\{\varrho^2/32,\varrho^q/2q2^q\}>0$$

for each $0 < \lambda \leqslant \lambda_0$ if $\varrho/2 \leqslant |||\nabla u|||_{2,\mathbb{R}^N}, \|u\|_{q,\mathbb{R}^N_V} \leqslant \varrho$. Fix this $\lambda_0 > 0$. Let $u_0 \geqslant 0$ in $N_\mu^{q,2}(\mathbb{R}^N)$ satisfy $|||\nabla u_0|||_{2,\mathbb{R}^N} = \|u_0\|_{q,\mathbb{R}^N_V} = \varrho$. As $\mathcal{J}(tu_0) < 0$ for sufficiently small $t > 0$, one has

$$-\infty<\imath(\lambda):=\inf_{\|u\|_{N_\mu^{q,2}(\mathbb{R}^N)}\leqslant\varrho}\mathcal{J}(u)<0. \qquad (4.4)$$

Take λ_0 smaller if necessary so that $S^{N/2}/N - K\lambda^{q/(q-1-r)} > 0$ when $0 < \lambda \leqslant \lambda_0$. Apply *Ekeland's variational principle* to get a minimizing sequence $\{u_k \geqslant 0\colon k \geqslant 1\}$ such that $\|u_k\|_{N_\mu^{q,2}(\mathbb{R}^N)} < \varrho$, while $\mathcal{J}(u_k) \to \imath(\lambda)$ and $\mathcal{J}'(u_k) \to 0$ as $k \to \infty$. Note that the $(\mathrm{PS})_{\imath(\lambda)}$-condition is satisfied and $\mathcal{J}(u^+) \leqslant \mathcal{J}(u)$. It is easy to see that $\mathcal{J}$ achieves a local minimum $u_\lambda \geqslant 0$ in $N_\mu^{q,2}(\mathbb{R}^N)$ with $\|u_\lambda\|_{N_\mu^{q,2}(\mathbb{R}^N)} < \varrho$. Since $q \geqslant 2$ and $u_\lambda \not\equiv 0$, [30, theorem 1] then says that $u_\lambda > 0$ in $\mathbb{R}^N$.

Because we are only interested in the bifurcation phenomenon near zero, we select a $\varrho_1 \in (0, \varrho)$ to be such that $\varrho_1^2 - C^{2^*}_{2,N}\varrho_1^{2^*} \geqslant \frac{1}{2}\varrho_1^2$. As $\mathcal{J}'(u_\lambda) = 0$, using (2.9), we observe that

$$\|u_\lambda\|_{N_\mu^{q,2}(\mathbb{R}^N)}\leqslant\{2\lambda C_2\}^{1/(q-1-r)}\to0\quad\text{as }\lambda\to0^+. \qquad (4.5)$$

Next, when $q = 2^*$ we define the energy functional $\tilde{\mathcal{J}}\colon N_\mu^{2^*,2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\tilde{\mathcal{J}}(u):=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x+\frac{1}{2^*}\int_{\mathbb{R}^N}(\alpha-1)|u|^{2^*}\,\mathrm{d}x-\frac{\lambda}{r+1}\int_{\mathbb{R}^N}(u^+)^{r+1}\,\mathrm{d}x.$$

Clearly, $\tilde{\mathcal{J}}$ is coercive. Follow the proof of proposition 3.3 to derive a minimum $u_\lambda > 0$ of $\tilde{\mathcal{J}}$ in $N_\mu^{2^*,2}(\mathbb{R}^N)$ for each $\lambda > 0$. Also, assuming that $\|u_\lambda\|_{N_\mu^{2^*,2}(\mathbb{R}^N)} \leqslant 1$, we similarly have

$$\|u_\lambda\|_{N_\mu^{2^*,2}(\mathbb{R}^N)} \leqslant \{\lambda C_2\}^{1/(2^*-1-r)} \to 0 \quad \text{as } \lambda \to 0^+. \tag{4.6}$$

Finally, for $q \in (2^*, \infty)$ we first recall the compact embedding $\iota\colon M_\mu^{q,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Furthermore, we follow the discussion for $q \in [2, 2^*)$ to find a minimizing sequence $\{\tilde{u}_k \geqslant 0\colon k \geqslant 1\}$ with $\tilde{u}_k \rightharpoonup u_\lambda$, and repeat (4.3) and apply lemma 3.2 to obtain $\tilde{u}_k \to u_\lambda$ in $M_\mu^{q,2}(\mathbb{R}^N)$. So, u_λ is a local minimum of $\mathcal{J}$ and $u_\lambda > 0$ in $\mathbb{R}^N$. As $\mathcal{J}'(u_\lambda) = 0$, we have (4.5) again.

Combining the preceding three situations, we thus finish the proof completely. □

5. Some variants of (1.2)

In this section we study some variants of (1.2) when $N \geqslant 2$, and we start with

$$-\Delta u + \alpha(x)u^{q-1} = \lambda u^r + f(x, u) \quad \text{in } \mathbb{R}^N, \tag{5.1}$$

where we assume that $q \geqslant 2$, $r \in (0, 1)$, $\lambda > 0$ is a constant and $f(x, u)$ satisfies the following conditions.

(f1) $f(x, u) \in C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R})$ and $\lim_{u\to 0} f(x, u)/u^{q-1} = 0$ uniformly in x.

(f2) $f(x, u) \geqslant 0$ when $u \geqslant 0$ and $\lim_{u\to+\infty} f(x, u)/u^{q-1} = \infty$ uniformly in x.

(f3) We have

$$\lim_{u\to+\infty} \frac{f(x, u)}{u^{2^*-1}} = 0$$

uniformly in x when $N \geqslant 3$, and

$$\lim_{u\to+\infty} \frac{f(x, u)}{e^{\xi u^2} - 1} = 0$$

uniformly in x when $N = 2$ for each $\xi > 0$.

(f4) There exist constants $s > q$ and $\vartheta \geqslant 1$ such that, for $H(x, u) := uf(x, u) - sF(x, u)$ with $F(x, u) := \int_0^u f(x, v)\,\mathrm{d}v$, we have $H(x, v) \leqslant \vartheta H(x, u)$ uniformly in x when $0 < v < u$.

Recall that problem (5.1) was studied in [4, 28] without the sublinear term λu^r. Here we study it with a more general nonlinearity $f(x, u)$ that does not satisfy the Ambrosetti–Rabinowitz condition (see condition (f4′)), following the idea from Lam and Lu [22] via the *mountain pass theorem* of Cerami [9]. Note that condition (f4) was introduced by Jeanjean [19].

Now, under the above hypotheses, we can prove the following existence result.

Theorem 5.1. *Assume that $q \in [2, 2^*)$ when $N \geqslant 3$, and that $q \in [2, \infty)$ when $N = 2$. Let $r \in (0, 1)$ and let $\beta = 1/(q - 1)$. Let $\alpha \geqslant 0$ satisfy $\alpha^{-\beta} \in L^1(\mathbb{R}^N)$ when $N \geqslant 3$, and let $\alpha \geqslant 1$ and $(\alpha - 1)^{-\beta} \in L^1(\mathbb{R}^2)$. Let $f(x, u)$ satisfy conditions (f1)–(f4), and let $\lambda > 0$ be a constant. Then there exists a $\lambda_1 > 0$ such that problem (5.1) has two solutions $u_\lambda, \tilde{u}_\lambda > 0$ for each $0 < \lambda \leqslant \lambda_1$.*

These solutions are sought in $M^{q,2}_\mu(\mathbb{R}^2)$ (in $H^1_\mu(\mathbb{R}^2)$ if $q=2$) and in $N^{q,2}_\mu(\mathbb{R}^N)$ (in $D^1_\mu(\mathbb{R}^N)$ if $q=2$) when $N\geqslant 3$, where we shall again take $\tilde{q}=q$ from now on. In addition, the norms on $D^1_\mu(\mathbb{R}^N)$ and $H^1_\mu(\mathbb{R}^2)$ are generated through (3.14) and (3.16), respectively.

Since we are only interested in obtaining positive solutions, we define

$$f_+(x,u)\in C(\mathbb{R}^N\times\mathbb{R};\mathbb{R})\quad\text{to be}\quad f_+(x,u):=\begin{cases} f(x,u) & \text{when } u\geqslant 0,\\ 0 & \text{when } u\leqslant 0.\end{cases}$$

Just like (1.2), these solutions are found to be critical points of the associated energy functional $\mathcal{G}\colon M^{q,2}_\mu(\mathbb{R}^2)\to\mathbb{R}$, or $\mathcal{G}\colon N^{q,2}_\mu(\mathbb{R}^N)\to\mathbb{R}$ when $N\geqslant 3$, which is defined by

$$\mathcal{G}(u):=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}\alpha|u|^q\,\mathrm{d}x-\int_{\mathbb{R}^N}F_+(x,u)\,\mathrm{d}x-\frac{\lambda}{r+1}\int_{\mathbb{R}^N}(u^+)^{r+1}\,\mathrm{d}x.$$

Here, $F_+(x,u):=\int_0^u f_+(x,v)\,\mathrm{d}v$ denotes the primitive of the function $f_+(x,u)\geqslant 0$.

Proposition 5.2. *$\mathcal{G}\colon N^{q,2}_\mu(\mathbb{R}^N)\to\mathbb{R}$ satisfies the $(C)_c$-condition for all $c\in\mathbb{R}$.*

Recall that a sequence $\{u_k\colon k\geqslant 1\}$ of functions in $N^{q,2}_\mu(\mathbb{R}^N)$ is said to be a *Cerami sequence* of $\mathcal{G}$ at level c, a *$(C)_c$-sequence*, provided that $\mathcal{G}(u_k)\to c$ while $(1+\|u_k\|_{N^{q,2}_\mu(\mathbb{R}^N)})\mathcal{G}'(u_k)\to 0$ when $k\to\infty$. Then we say that $\mathcal{G}$ satisfies the *Cerami condition* at level c, the *$(C)_c$-condition*, provided that each $(\mathrm{C})_c$-sequence of $\mathcal{G}$ admits a strongly convergent subsequence in $N^{q,2}_\mu(\mathbb{R}^N)$.

Proof. Since we are looking for positive solutions, without loss of generality let $\{u_k\geqslant 0\colon k\geqslant 1\}$ be a $(\mathrm{C})_c$-sequence of $\mathcal{G}$ in $N^{q,2}_\mu(\mathbb{R}^N)$. Thus, for $V:=\alpha$ and large k, one has

$$\||\nabla u_k|\|^2_{2,\mathbb{R}^N}+\|u_k\|^q_{q,\mathbb{R}^N_V}=\int_{\mathbb{R}^N}f_+(x,u_k)u_k\,\mathrm{d}x+\lambda\|u_k\|^{r+1}_{r+1,\mathbb{R}^N}+o(1).\tag{5.2}$$

First, we prove that $\{u_k\geqslant 0\colon k\geqslant 1\}$ is bounded. On the contrary, suppose that it is unbounded. Then, as $0<r<1$ and $q\geqslant 2$, for sufficiently large k it follows from (5.2) that

$$\frac{1}{2}(\||\nabla u_k|\|^2_{2,\mathbb{R}^N}+\|u_k\|^2_{q,\mathbb{R}^N_V})\leqslant\int_{\mathbb{R}^N}f_+(x,u_k)u_k\,\mathrm{d}x\leqslant\||\nabla u_k|\|^q_{2,\mathbb{R}^N}+\|u_k\|^q_{q,\mathbb{R}^N_V}.\tag{5.3}$$

Set $w_k:=u_k/\|u_k\|_{N^{q,2}_\mu(\mathbb{R}^N)}$ for all $k\geqslant 1$ with $\|w_k\|_{N^{q,2}_\mu(\mathbb{R}^N)}=1$. Then we may assume that $w_k\to w$ in $L^p(\mathbb{R}^N)$ and $w_k(x)\to w(x)$ a.e. in $\mathbb{R}^N$ for all $1\leqslant p<2^*$ and some $w\in N^{q,2}_\mu(\mathbb{R}^N)$.

We have $w\equiv 0$. In fact, if $\mathfrak{L}(\Omega)>0$ for $\Omega:=\{x\in\mathbb{R}^N\colon w(x)>0\}$, we take an $x\in\Omega$ to see that

$$\lim_{k\to\infty}u_k(x)=\lim_{k\to\infty}w_k(x)\|u_k\|_{N^{q,2}_\mu(\mathbb{R}^N)}=\infty.$$

Thus, $\lim_{k\to\infty} f_+(x, u_k(x))/u_k^{q-1}(x) = \infty$ by condition (f2), which, together with (5.3), (3.11) and *Fatou's lemma*, leads to the contradiction

$$\infty = \int_\Omega \liminf_{k\to\infty}\left[\frac{f_+(x,u_k(x))u_k(x)}{u_k^q(x)} w_k^q(x)\right] \mathrm{d}x \leqslant \liminf_{k\to\infty}\left[\frac{\int_{\mathbb{R}^N} f_+(x,u_k)u_k\,\mathrm{d}x}{\|u_k\|^q_{N_\mu^{q,2}(\mathbb{R}^N)}}\right] \leqslant O(1).$$

Now, take a subsequence $\{w_k \geqslant 0\colon k \geqslant 1\}$, using the same notation, with $\||\nabla w_k|\|_{2,\mathbb{R}^N} \geqslant \frac{1}{2}$. (It is similar if we select $\|w_k\|_{q,\mathbb{R}^N_V} \geqslant \frac{1}{2}$ instead.) Let $t_k \in [0,1]$ satisfy $\mathcal{G}(t_k u_k) = \max_{t\in[0,1]} \mathcal{G}(tu_k)$. In view of conditions (f1) and (f3),

$$f_+(x,u) \leqslant a_K|u|^{q-1} + (2^*/C_{2,N}^{2^*}K^{2^*})|u|^{2^*-1}$$

for any $K > 0$ with a sufficiently large $a_K > 0$. Noting that $\|u_k\|_{N_\mu^{q,2}(\mathbb{R}^N)} \geqslant K$ when k is large, we have

$$\mathcal{G}(t_k u_k) \geqslant \mathcal{G}(Kw_k) \geqslant \frac{K^2}{8} - \frac{\lambda K^{r+1}}{r+1}\|w_k\|^{r+1}_{r+1,\mathbb{R}^N} - \frac{a_K K^q}{q}\|w_k\|^q_{q,\mathbb{R}^N} - 1 \to \frac{K^2}{8} - 1$$

as $k \to \infty$. The arbitrariness of K leads to $\lim_{k\to\infty} \mathcal{G}(t_k u_k) = \infty$, so that $t_k \in (0,1)$ and

$$t_k^2\||\nabla u_k|\|^2_{2,\mathbb{R}^N} + t_k^q\|u_k\|^q_{q,\mathbb{R}^N_V} = \int_{\mathbb{R}^N} f_+(x,t_k u_k)(t_k u_k)\,\mathrm{d}x + \lambda t_k^{r+1}\|u_k\|^{r+1}_{r+1,\mathbb{R}^N}. \quad (5.4)$$

By the definition of $\mathcal{G}(u_k)$ and $\mathcal{G}(t_k u_k)$, (5.2), (5.4) and condition (f4), we verify that

$$\begin{aligned}\mathcal{G}(t_k u_k) &= \mathcal{G}(t_k u_k) - \frac{1}{s}\mathcal{G}'(t_k u_k)(t_k u_k)\\ &= \left(\frac{1}{2} - \frac{1}{s}\right)t_k^2\||\nabla u_k|\|^2_{2,\mathbb{R}^N} + \left(\frac{1}{q} - \frac{1}{s}\right)t_k^q\|u_k\|^q_{q,\mathbb{R}^N_V}\\ &\quad - \lambda\left(\frac{1}{r+1} - \frac{1}{s}\right)t_k^{r+1}\|u_k\|^{r+1}_{r+1,\mathbb{R}^N} + \frac{1}{s}\int_{\mathbb{R}^N} H_+(x,t_k u_k)\,\mathrm{d}x\end{aligned}$$

for $H_+(x,u) := uf_+(x,u) - sF_+(x,u)$, with $H_+(x,t_k u_k) \leqslant \vartheta H_+(x,u_k)$, as well as

$$\begin{aligned}\frac{\vartheta}{s}\int_{\mathbb{R}^N} H_+(x,u_k)\,\mathrm{d}x &\leqslant \vartheta\left(\frac{1}{s} - \frac{1}{2}\right)\||\nabla u_k|\|^2_{2,\mathbb{R}^N} + \vartheta\left(\frac{1}{s} - \frac{1}{q}\right)\|u_k\|^q_{q,\mathbb{R}^N_V}\\ &\quad - \lambda\vartheta\left(\frac{1}{s} - \frac{1}{r+1}\right)\|u_k\|^{r+1}_{r+1,\mathbb{R}^N} + O(1).\end{aligned}$$

Hence, noting that $t_k^{r+1} > t_k^2, t_k^q$ and $\vartheta \geqslant 1$, we combine the two previous estimates to derive

$$\begin{aligned}&\mathcal{G}(t_k u_k) + \left(\frac{1}{2} - \frac{1}{s}\right)(\vartheta - t_k^{r+1})\||\nabla u_k|\|^2_{2,\mathbb{R}^N} + \left(\frac{1}{q} - \frac{1}{s}\right)(\vartheta - t_k^{r+1})\|u_k\|^q_{q,\mathbb{R}^N_V}\\ &\qquad \leqslant C_2\left(\frac{\lambda}{r+1} - \frac{\lambda}{s}\right)(\vartheta - t_k^{r+1})(\||\nabla u_k|\|^{r+1}_{2,\mathbb{R}^N} + \|u_k\|^{r+1}_{q,\mathbb{R}^N_V}) + O(1), \quad (5.5)\end{aligned}$$

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from which we have a contradiction with the assumption that $\{u_k\colon k \geqslant 1\}$ is unbounded.[1]

As a result, $\{u_k \geqslant 0\colon k \geqslant 1\}$ is bounded. So, we may simply assume that there exists a function $u \in N_\mu^{q,2}(\mathbb{R}^N)$ such that $u_k \rightharpoonup u$ in $L_V^q(\mathbb{R}^N)$ and $L^{2^*}(\mathbb{R}^N)$, $|\nabla u_k| \rightharpoonup |\nabla u|$ in $L^2(\mathbb{R}^N)$, while $u_k \to u$ in $L^p(\mathbb{R}^N)$ for each $1 \leqslant p < 2^*$. Using (4.2), for $u_k\theta_R \in N_\mu^{q,2}(\mathbb{R}^N)$ we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} \mathcal{G}'(u_k)(u_k\theta_R) \\ &= \lim_{k\to\infty} \biggl\{ \int_{\mathbb{R}^N} |\nabla u_k|^2\theta_R \,\mathrm{d}x - \int_{\mathbb{R}^N} f_+(x,u_k)(u_k\theta_R)\,\mathrm{d}x \biggr\} \\ &\quad + \lim_{k\to\infty} \biggl\{ \int_{\mathbb{R}^N} \nabla u_k \cdot \nabla\theta_R u_k \,\mathrm{d}x + \int_{\mathbb{R}^N} V|u_k|^q\theta_R\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} (u_k^+)^{r+1}\theta_R\,\mathrm{d}x \biggr\}. \end{aligned}$$

From conditions (f1) and (f3), we have $f_+(x,u) \leqslant a_\varepsilon|u|^{q-1} + \varepsilon|u|^{2^*-1}$ for all $\varepsilon > 0$ and some sufficiently large constant $a_\varepsilon > 0$ that depends only on ε. Thus, it follows that

$$0 \leqslant \int_{\mathbb{R}^N} f_+(x,u_k)(u_k\theta_R)\,\mathrm{d}x \leqslant a_\varepsilon \|u_k\|_{q,\mathbb{R}^N\setminus \boldsymbol{B}_R}^q + \varepsilon\|u_k\|_{2^*,\mathbb{R}^N\setminus \boldsymbol{B}_R}^{2^*} \to \varepsilon\nu_\infty \tag{5.6}$$

when $k, R \to \infty$. Hence, we observe that $\mu_\infty + \tilde{v}_\infty \leqslant \varepsilon\nu_\infty$, and then $S\nu_\infty^{2/2^*} \leqslant \mu_\infty \leqslant \varepsilon\nu_\infty$; that is, $\nu_\infty = 0$ since $S > 0$ via the arbitrariness of ε. As a consequence, we have $\mu_\infty = \tilde{v}_\infty = 0$. The remaining case in which the points x_i are involved can be discussed similarly. So, $u_k \to u$ in $L^{2^*}(\mathbb{R}^N)$ and therefore in $N_\mu^{q,2}(\mathbb{R}^N)$. □

Analogously, we can prove the following result.

PROPOSITION 5.3. *$\mathcal{G}\colon M_\mu^{q,2}(\mathbb{R}^2) \to \mathbb{R}$ satisfies the $(C)_c$-condition for all $c \in \mathbb{R}$.*

Here, we define the *$(C)_c$-sequence* and the *$(C)_c$-condition* for $\mathcal{G}\colon M_\mu^{q,2}(\mathbb{R}^2) \to \mathbb{R}$ similarly to before; also, we set $V := \alpha - 1$ and use (4.1) to define $\|u\|_{M_\mu^{q,2}(\mathbb{R}^2)}$ for $\|u\|_{q,\mathbb{R}_V^2}^q := \int_{\mathbb{R}^2} \alpha|u|^q\,\mathrm{d}x$.

Before we proceed to the proof of proposition 5.3, we recall Ruf [29, proposition 2.1], which is related to the *Trudinger–Moser inequality*.

LEMMA 5.4. *When $u \in H^1(\mathbb{R}^2)$ satisfies $\|u\|_{H^1(\mathbb{R}^2)} \leqslant 1$, we have*

$$\int_{\mathbb{R}^2} (\mathrm{e}^{\xi u^2} - 1)\,\mathrm{d}x \leqslant C_\xi \tag{5.7}$$

provided that $0 \leqslant \xi \leqslant 4\pi$, where $C_\xi > 0$ is a constant that depends only on ξ.

Proof of proposition 5.3. Let $\{u_k \geqslant 0\colon k \geqslant 1\}$ be a $(C)_c$-sequence of $\mathcal{G}$ in $M_\mu^{q,2}(\mathbb{R}^2)$ and suppose that it is unbounded. Then for $w_k := u_k/\|u_k\|_{M_\mu^{q,2}(\mathbb{R}^2)}$ we have $w_k \to 0$ in $L^p(\mathbb{R}^2)$ for every $1 \leqslant p < \infty$, using condition (f2). Choose a subsequence $\{w_k \geqslant 0\colon k \geqslant 1\}$ (using the same notation) with $\||\nabla w_k|\|_{2,\mathbb{R}^2} \geqslant \frac{1}{2}$, and let $t_k \in [0,1]$ satisfy

[1] From the proof, we know that we can replace s by q when $\lambda = 0$, or use a $(\mathrm{PS})_c$-sequence when $\vartheta > 1$.

$\mathcal{G}(t_k u_k) = \max_{t\in[0,1]} \mathcal{G}(tu_k)$. Via conditions (f1) and (f3), for each $K > 0$ with a sufficiently large $a_K > 0$, it yields that

$$f_+(x,u) \leqslant a_K|u|^{q-1} + \frac{1}{K\sqrt{C_{2\pi}}}(\mathrm{e}^{(\pi/(1+C_3)K^2)u^2} - 1).$$

Here, $C_3 > 0$ is a constant that depends on the embedding constant of

$$\iota\colon M_\mu^{q,2}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$$

by corollary 3.8. So, for Kw_k with $\|w_k\|_{M_\mu^{q,2}(\mathbb{R}^2)} = 1$, we apply lemma 5.4 to see that

$$\begin{aligned}
\frac{1}{K\sqrt{C_{2\pi}}}&\int_{\mathbb{R}^2}\int_0^{Kw_k} (\mathrm{e}^{\pi v^2/(1+C_3)K^2} - 1)\,\mathrm{d}v\,\mathrm{d}x \\
&\leqslant \frac{1}{\sqrt{C_{2\pi}}}\int_{\mathbb{R}^2} w_k(\mathrm{e}^{\pi w_k^2/(1+C_3)} - 1)\,\mathrm{d}x \\
&\leqslant \frac{1}{\sqrt{C_{2\pi}}}\|w_k\|_{2,\mathbb{R}^2}\biggl\{\int_{\mathbb{R}^2}\biggl[\exp\biggl(2\pi\frac{w_k^2}{\|w_k\|_{H^1(\mathbb{R}^2)}^2}\biggr) - 1\biggr]\mathrm{d}x\biggr\}^{1/2} \\
&\leqslant \|w_k\|_{2,\mathbb{R}^2} \to 0 \qquad (5.8)
\end{aligned}$$

as $k \to \infty$. Here, *Hölder's inequality* and the elementary estimate $(x-y)^2 \leqslant x^2 - y^2$ with $x \geqslant y \geqslant 0$ are used. So, we have $\lim_{k\to\infty} \mathcal{G}(t_k u_k) = \infty$ and $t_k \in (0,1)$. Therefore, we may continue as before and apply condition (f4) to see that $\{u_k \geqslant 0\colon k \geqslant 1\}$ is bounded in $M_\mu^{q,2}(\mathbb{R}^2)$.

Thus, there exists a function $u \in M_\mu^{q,2}(\mathbb{R}^2)$ such that $u_k \rightharpoonup u$ in $L_V^q(\mathbb{R}^2)$, $|\nabla u_k| \rightharpoonup |\nabla u|$ in $L^2(\mathbb{R}^2)$ and $u_k \to u$ in $L^p(\mathbb{R}^2)$ for all $1 \leqslant p < \infty$. Suppose that $\|u_k\|_{H^1(\mathbb{R}^2)} \leqslant K$ for every $k \geqslant 1$, and $\|u\|_{H^1(\mathbb{R}^2)} \leqslant K$. From conditions (f1) and (f3),

$$f_+(x,u) \leqslant a_K|u|^{q-1} + \frac{1}{\sqrt{C_{2\pi}}}(\mathrm{e}^{(\pi/K^2)u^2} - 1)$$

for some sufficiently large constant $a_K > 0$. In view of (5.7) and (5.8), one has

$$\int_{\mathbb{R}^2} f_+(x,u_k)|u_k - u|\,\mathrm{d}x \leqslant a_K\|u_k\|_{q,\mathbb{R}^2}^{q-1}\|u_k - u\|_{q,\mathbb{R}^2} + \|u_k - u\|_{2,\mathbb{R}^2} \to 0 \qquad (5.9)$$

as $k \to \infty$. Similarly, $\int_{\mathbb{R}^2} f_+(x,u)|u_k - u|\,\mathrm{d}x \to 0$ as $k \to \infty$. Hence, we derive

$$\int_{\mathbb{R}^2} (f_+(x,u_k) - f_+(x,u))(u_k - u)\,\mathrm{d}x \to 0 \qquad (5.10)$$

when $k \to \infty$, so that we can apply lemma 3.2 to observe that $u_k \to u \in M_\mu^{q,2}(\mathbb{R}^2)$. □

Proof of theorem 5.1. We only detail the proof for $N \geqslant 3$ since that for $N = 2$ is similar.

In view of conditions (f1) and (f3), we have $f_+(x,u) \leqslant q\varepsilon|u|^{q-1} + 2^* b_\varepsilon|u|^{2^*-1}$ for all $\varepsilon > 0$ and some sufficiently large constant $b_\varepsilon > 0$ depending only on ε. Thus, we

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obtain

$$\mathcal{G}(u) \geqslant \frac{1}{2}|||\nabla u|||^2_{2,\mathbb{R}^N}\{1 - 2\varepsilon C_4|||\nabla u|||^{q-2}_{2,\mathbb{R}^N} - 2b_\varepsilon C^{2^*}_{2,N}|||\nabla u|||^{2^*-2}_{2,\mathbb{R}^N} - \lambda\mathfrak{c}_2|||\nabla u|||^{r-1}_{2,\mathbb{R}^N}\} + \frac{1}{q}\|u\|^q_{q,\mathbb{R}^N_V}\{1 - q\varepsilon C_4 - \lambda\mathfrak{c}_2'\|u\|^{r+1-q}_{q,\mathbb{R}^N_V}\}.$$

Here, $C_4 > 0$ is a constant depending on the embedding constant of $\iota\colon N^{q,2}_\mu(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. Take an $0 < \varepsilon \leqslant 1/2qC_4$ and select a $\varrho \in (0,1)$ with $1 - 2\varepsilon C_4\varrho^{q-2} - 2b_\varepsilon C^{2^*}_{2,N}\varrho^{2^*-2} \geqslant 1/3 > 0$. Then, for $\lambda_1 := \min\{\varrho^{1-r}/6\mathfrak{c}_2, \varrho^{q-r-1}/4\mathfrak{c}_2'\} > 0$, we observe that $\mathcal{G}(u) \geqslant 2\varrho_0 := \min\{\varrho^2/48, \varrho^q/4q2^q\} > 0$ for each $0 < \lambda \leqslant \lambda_1$, provided that $\varrho/2 \leqslant |||\nabla u|||_{2,\mathbb{R}^N}, \|u\|_{q,\mathbb{R}^N_V} \leqslant \varrho$. Fix this $\lambda_1 > 0$.

Let $u_0 \geqslant 0$ in $N^{q,2}_\mu(\mathbb{R}^N)$ satisfy $|||\nabla u_0|||_{2,\mathbb{R}^N} = \|u_0\|_{q,\mathbb{R}^N_V} = \varrho$. Then (4.4) holds like before for $\mathcal{G}$, since $F_+(x,u) \geqslant 0$. We use *Ekeland's variational principle* to see that $\mathcal{G}$ achieves a local minimum $u_\lambda > 0$ in $N^{q,2}_\mu(\mathbb{R}^N)$ with $\mathcal{G}(u_\lambda) < 0$. (Note the minimizing sequence is bounded.)

On the other hand, we can take $\tilde{\lambda}_1 := \lambda_1^{2^*}$, for example, and therefore have

$$\mathcal{G}(u) \geqslant \frac{1}{6}|||\nabla u|||^2_{2,\mathbb{R}^N} - \lambda\mathfrak{c}_3|||\nabla u|||^{r+1}_{2,\mathbb{R}^N} + \frac{1}{2q}\|u\|^q_{q,\mathbb{R}^N_V} - \lambda\mathfrak{c}_3'\|u\|^{r+1}_{q,\mathbb{R}^N_V},$$

where $\mathfrak{c}_3 = \mathfrak{c}_2/2, \mathfrak{c}_3' = \mathfrak{c}_2'/q > 0$ are constants and $0 < \lambda \leqslant \tilde{\lambda}_1$. Note that

$$g_1(x) = \tfrac{1}{6}x^2 - \lambda\mathfrak{c}_3x^{r+1} \geqslant 0$$

when $x \geqslant (6\mathfrak{c}_3\lambda)^{1/(1-r)}$ and

$$\min_{x\geqslant 0} g_1(x) = \frac{r-1}{6(r+1)}\{3\lambda(r+1)\mathfrak{c}_3\}^{2/(1-r)}$$

at $x = \{3\lambda(r+1)\mathfrak{c}_3\}^{1/(1-r)}$, while

$$g_2(y) = \frac{1}{2q}y^q - \lambda\mathfrak{c}_3'y^{r+1} \geqslant 0$$

when $y \geqslant (2q\mathfrak{c}_3'\lambda)^{1/(q-r-1)}$ and

$$\min_{y\geqslant 0} g_2(y) = \frac{r+1-q}{2q(r+1)}\{2\lambda(r+1)\mathfrak{c}_3'\}^{q/(q-r-1)}$$

at $y = \{2\lambda(r+1)\mathfrak{c}_3'\}^{1/(q-r-1)}$, both in the range of $x, y \leqslant O(\varrho^{2^*})$. Take λ_1 smaller if necessary to see that

$$\min_{x+y=\varrho}\{g(x,y) := g_1(x) + g_2(y)\} \geqslant \varrho_0 > 0.$$

That is, $\inf_{\|u\|_{N^{q,2}_\mu(\mathbb{R}^N)}=\varrho} \mathcal{G}(u) \geqslant \varrho_0 > 0$. In addition, because condition (f2) implies that

$$F_+(x,u) \geqslant \frac{|||\nabla u_0|||^2_{2,\mathbb{R}^N} + \|u_0\|^q_{q,\mathbb{R}^N_V}}{\|u_0\|^q_{q,\mathbb{R}^N}}|u|^q - K'|u|$$

for some sufficiently large constant $K' > 0$ uniformly in x, we easily see that $\mathcal{G}(tu_0) \to -\infty$ when $t \to \infty$.

Hence, the *mountain pass theorem* of Cerami provides the existence of a critical point of $\mathcal{G}$ in

$$\mathtt{C} := \{h \in C([0,1]; N_\mu^{q,2}(\mathbb{R}^N)) \colon h(0) = 0,\ h(1) = w_0\}$$

with $c := \inf_{h\in\mathtt{C}} \max_{z\in[0,1]} \mathcal{G}(h(z)) \geqslant \varrho_0$. Here, $w_0 := t_0 u_0$ for some sufficiently large $t_0 > 0$ such that $\mathcal{G}(w_0) < 0$. Denote $\tilde{u}_\lambda$ to be this critical point of $\mathcal{G}$. Then $\mathcal{G}(\tilde{u}_\lambda) > 0$ and $\tilde{u}_\lambda > 0$. This finishes our proof for $N \geqslant 3$.

On the other hand, when $N = 2$, by conditions (f1) and (f3), one has

$$f_+(x,u) \leqslant q\varepsilon |u|^{q-1} + \frac{b'_\varepsilon}{\sqrt{C_{2\pi}}} |u|^{2q-1}(\mathrm{e}^{\pi u^2} - 1)$$

for all $\varepsilon > 0$ and some sufficiently large constant $b'_\varepsilon > 0$ depending on ε. Without loss of generality, assume that $u \geqslant 0$ and $\|u\|_{H^1(\mathbb{R}^2)} \leqslant 1$. Then we have

$$\begin{aligned} \frac{1}{\sqrt{C_{2\pi}}} \int_{\mathbb{R}^2} \int_0^u v^{2q-1}(\mathrm{e}^{\pi v^2} - 1)\,\mathrm{d}v\,\mathrm{d}x &\leqslant \frac{1}{\sqrt{C_{2\pi}}} \int_{\mathbb{R}^2} u^{2q}(\mathrm{e}^{\pi u^2} - 1)\,\mathrm{d}x \\ &\leqslant \|u\|_{4q,\mathbb{R}^2}^{2q} \\ &\leqslant C'_4(\||\nabla u|\|_{2,\mathbb{R}^2}^{2q} + \|u\|_{q,\mathbb{R}^2_V}^{2q}). \end{aligned} \tag{5.11}$$

Here, $C'_4 > 0$ is a constant depending on the embedding constant of $\iota\colon M_\mu^{q,2}(\mathbb{R}^2) \hookrightarrow L^{4q}(\mathbb{R}^2)$. Thus, when $\|u\|_{M_\mu^{q,2}(\mathbb{R}^2)}$ is sufficiently small that $\|u\|_{H^1(\mathbb{R}^2)} \leqslant 1$, we observe that

$$\begin{aligned} \mathcal{G}(u) \geqslant{}& \frac{1}{2}\||\nabla u|\|_{2,\mathbb{R}^2}^2 \{1 - 2\varepsilon C_4 \||\nabla u|\|_{2,\mathbb{R}^2}^{q-2} - 2b'_\varepsilon C'_4 \||\nabla u|\|_{2,\mathbb{R}^2}^{2q-2} - \lambda \mathfrak{c}_2 \||\nabla u|\|_{2,\mathbb{R}^2}^{r-1}\} \\ &+ \frac{1}{q}\|u\|_{q,\mathbb{R}^2_V}^q \{1 - q\varepsilon C_4 - q b'_\varepsilon C'_4 \|u\|_{q,\mathbb{R}^2_V}^q - \lambda \mathfrak{c}'_2 \|u\|_{q,\mathbb{R}^2_V}^{r+1-q}\}. \end{aligned}$$

Let $\varepsilon > 0$ and $\varrho \in (0,1)$ satisfy

$$1 - 2\varepsilon C_4 \varrho^{q-2} - 2b'_\varepsilon C'_4 \varrho^{2q-2} \geqslant \tfrac{1}{3} \quad \text{and} \quad 1 - q\varepsilon C_4 - q b'_\varepsilon C'_4 \varrho^q \geqslant \tfrac{1}{4}.$$

Then, for $\lambda'_1 := \min\{\varrho^{1-r}/6\mathfrak{c}_2, \varrho^{q-r-1}/8\mathfrak{c}'_2\} > 0$, we see that $\mathcal{G}(u) \geqslant 2\varrho'_0 := \min\{\varrho^2/48, \varrho^q/8q2^q\} > 0$ for all $0 < \lambda \leqslant \lambda'_1$, provided that $\varrho/2 \leqslant \||\nabla u|\|_{2,\mathbb{R}^2}$, $\|u\|_{q,\mathbb{R}^2_V} \leqslant \varrho$. Fix this $\lambda'_1 > 0$. Since $2q > q$, we can proceed exactly as before to finish the $N = 2$ case and so our proof completely. □

On the other hand, we can consider the problem

$$-\Delta u + V(x)u = \lambda |u|^{q-2}u + f(x,u) \quad \text{in } \mathbb{R}^N \tag{5.12}$$

as well, where $1 < q < 2$, λ is a constant, and $f(x,u)$ satisfies condition (f1) and the following conditions.

(f3′) There exist some constants $\mathfrak{c}_4, \mathfrak{c}'_4 > 0$ such that $|f(x,u)| \leqslant \mathfrak{c}_4 + \mathfrak{c}'_4 |u|^{\tilde{s}-1}$ with $\tilde{s} \in (2,\infty)$ when $N = 2$, and $\tilde{s} \in (2, 2^*)$ when $N \geqslant 3$ for every $u \in \mathbb{R}$, uniformly in x.

(f4′) There are constants $K > 0$ and $s > 2$ such that $uf(x,u) \geqslant sF(x,u) > 0$ when $|u| \geqslant K$, uniformly in x, where $F(x,u) := \int_0^u f(x,v)\,\mathrm{d}v$.

(f5) $f(x,u)$ is odd in u, that is, $f(x,-u) = -f(x,u)$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$.

Problem (5.12) was originally studied on bounded regions in [2] and extended in [5] using the so-called *fountain theorems*. In fact, the latter results are very general, and the only requirements are a decomposition or direct sum of the Hilbert space and associated compact embedding. In view of the results of § 3 and theorem 3.9, the result below is easily proved.

THEOREM 5.5. *Define $\mathfrak{H} := H^1_\mu(\mathbb{R}^2)$ or $\mathfrak{H} := D^1_\mu(\mathbb{R}^N)$ when $N \geqslant 3$ through $V(x)$. Suppose that $f(x,u)$ satisfies conditions (f1), (f3′), (f4′) and (f5). Then, for all $\lambda \in \mathbb{R}$, problem (5.12) has a sequence $\{\mathfrak{u}_k\}$ of solutions with $\tilde{\mathcal{G}}(\mathfrak{u}_k) > 0$ and $\|\mathfrak{u}_k\|_{\mathfrak{H}} \to \infty$ as $k \to \infty$; moreover, when $\lambda > 0$, it also has a sequence $\{\tilde{\mathfrak{u}}_k\}$ of solutions with $\tilde{\mathcal{G}}(\tilde{\mathfrak{u}}_k) < 0$ and $\tilde{\mathfrak{u}}_k \to 0$ as $k \to \infty$.*

Here, $\tilde{\mathcal{G}} \colon \mathfrak{H} \to \mathbb{R}$ is the energy functional defined by

$$\tilde{\mathcal{G}}(u) := \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u|^2 + V|u|^2]\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\,\mathrm{d}x.$$

The proof of theorem 5.5 follows easily from [31, theorems 3.7 and 3.20] via a minor modification. Note that [20, theorem 1] (see also [18, lemma 3.2 and remark 3.3]) was applied in deriving that $\tilde{\mathfrak{u}}_k \to 0$ as $k \to \infty$, which was also used in [17, § 5.1].

As a final remark, applying the variants of the *fountain theorems* of [32], combinations of the preceding conditions can be made to prove theorem 5.5 with broader nonlinearities.

6. Embedding results for $L^p(\mathbb{R}^N)$

When $N \geqslant 2$ and $p \in [1, N]$, Bartsch and Wang [4, theorem 2.1] derived a compact embedding $W^{1,p}_b(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, provided that $b(x)$ satisfies $\inf_{x\in\mathbb{R}^N} b(x) \geqslant b_0 > 0$ and $b^{-1}(x)$ vanishes at infinity in the sense of Lieb and Loss. Here,

$$W^{1,p}_b(\mathbb{R}^N) := \left\{u \in W^{1,p}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} b|u|^p\,\mathrm{d}x < \infty\right\}$$

is a reflexive Banach space with respect to the norm

$$\|u\|^p_{W^{1,p}_b(\mathbb{R}^N)} := \int_{\mathbb{R}^N} [|\nabla u|^p + b|u|^p]\,\mathrm{d}x. \tag{6.1}$$

We in fact can weaken the condition 'b^{-1} vanishes at infinity' (see § 2) to

$$\lim_{|x|\to\infty} \mathfrak{L}(\boldsymbol{B}_d(x) \cap \boldsymbol{V}_M) = 0 \tag{6.2}$$

for every $M > 0$, where $d \in (0,1]$ is the radius of the ball $\boldsymbol{B}_d(x)$. Actually, we have

$$\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x = \int_{\boldsymbol{V}_M} |u|^p\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus\boldsymbol{V}_M} |u|^p\,\mathrm{d}x \leqslant \int_{\widetilde{\boldsymbol{V}}_M} |u|^p\,\mathrm{d}x + \frac{1}{M}\|u\|^p_{W^{1,p}_b(\mathbb{R}^N)}.$$

Here, $\boldsymbol{V}_M := \{x \in \mathbb{R}^N \colon b(x) \leqslant M\}$, while $\widetilde{\boldsymbol{V}}_M (\supseteq \boldsymbol{V}_M)$ is an open set that again satisfies (6.2). By virtue of Edmunds and Evans [13, theorem V5.17 and lemma X6.12], we are done.

When $N \geqslant 2$ and $2 \leqslant p \leqslant N$ a (much) weaker condition is available to ensure this compact embedding, essentially due to Molchanov (see [21]), and Maz'ya and Shubin [27]. In fact, just to consider a simpler case, the condition

$$\inf_{\boldsymbol{F} \in \mathscr{N}_\rho} \{\mu(\boldsymbol{B}_d(x) \setminus \boldsymbol{F})\} \to \infty \quad \text{when } |x| \to \infty \tag{6.3}$$

suffices, where μ is a positive regular Borel measure on $\mathbb{R}^N$ vanishing on all sets of p-capacity zero, and $\mathscr{N}_\rho$ is the family of $\boldsymbol{F} \Subset \boldsymbol{B}_d(x)$ with $\operatorname{Cap}_p(\boldsymbol{F}) \leqslant \rho \operatorname{Cap}_p(\boldsymbol{B}_d)$ for some $\rho \in (0, 1)$.

When $p = N$ we will use the *relative N-capacity* of A, as given in [13, § VIII1],

$$\operatorname{Cap}_N(A) := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^N \, \mathrm{d}x \colon u \in W_0^{1,N}(\boldsymbol{B}_{2d}(x)) \text{ and } A \subseteq \{u \geqslant 1\}^o \right\}. \tag{6.4}$$

Below we prove an estimate using [27, lemma 4.2] (see also [13, lemma VIII2.5]).

Proposition 6.1. *Let $N \geqslant 2$ and let $2 \leqslant p \leqslant N$. Then there exists a constant $C_5 > 0$ depending on p, N such that, for each $\rho \in (0, 1)$ and $u \in W^{1,p}(\boldsymbol{B}_d(x))$, we have*

$$\int_{\boldsymbol{B}_d(x)} |u|^p \, \mathrm{d}y \leqslant \frac{C_5 d^p}{\rho} \int_{\boldsymbol{B}_d(x)} |\nabla u|^p \, \mathrm{d}y + \frac{C_5 d^N}{\inf_{\boldsymbol{F} \in \mathscr{N}_\rho} \{\mu(\boldsymbol{B}_d(x) \setminus \boldsymbol{F})\}} \int_{\boldsymbol{B}_d(x)} |u|^p \, \mathrm{d}\mu. \tag{6.5}$$

It is worth remarking here, assuming (6.3) and (6.5), that one can follow exactly [27, propositions 4.3 and 4.4] to conclude that the embedding from $W_b^{1,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ is compact.

Before proceeding to the proof of proposition 6.1, we show the following estimate.

Proposition 6.2. *Let $N \geqslant 2$ and let $2 \leqslant p \leqslant N$. Then there is an absolute constant $C_6 > 0$ such that, for all $u \in W^{1,p}(\boldsymbol{B}_d(x))$ with $u \not\equiv 0$ on $\boldsymbol{B}_d(x)$ yet $u \equiv 0$ on $K \Subset \boldsymbol{B}_d(x)$, we have*

$$\operatorname{Cap}_p(K) \leqslant C_6 \mathfrak{L}(\boldsymbol{B}_d) \frac{\int_{\boldsymbol{B}_d(x)} |\nabla u|^p \, \mathrm{d}y}{\int_{\boldsymbol{B}_d(x)} |u|^p \, \mathrm{d}y}. \tag{6.6}$$

Proof. First, recall that *Poincaré's inequality* says that, for every $N \geqslant 1$ and $p \geqslant 2$,

$$\int_{\boldsymbol{B}_d(x)} |u - \bar{u}|^p \, \mathrm{d}y \leqslant \left(\frac{d}{\pi_p}\right)^p \int_{\boldsymbol{B}_d(x)} |\nabla u|^p \, \mathrm{d}y \quad \forall u \in W^{1,p}(\boldsymbol{B}_d(x)). \tag{6.7}$$

Here,

$$\bar{u} := \frac{1}{\mathfrak{L}(\boldsymbol{B}_d)} \int_{\boldsymbol{B}_d(x)} u(y) \, \mathrm{d}y,$$

and $\pi_p = 2\pi((p-1)^{1/p}/p \sin(\pi/p))$ is given by Esposito *et al.* [14]. When $p = 2$, this best constant is given by Payne and Weinberger, as well as Bebendorf [6].

Next, let $u \in W^{1,p}(\boldsymbol{B}_d(x))$ satisfy our hypotheses. Suppose that $u \geqslant 0$ and

$$\frac{1}{\mathfrak{L}(\boldsymbol{B}_d)} \int_{\boldsymbol{B}_d(x)} u^p \, \mathrm{d}y = 1.$$

Then we easily see that $0 < \bar{u} \leqslant 1$ by *Hölder's inequality*, and $\phi := 1 - u \equiv 1$ on K. Note that $\bar{\phi} \geqslant 0$. As $\|u\|^p_{p,\boldsymbol{B}_d(x)} = \mathfrak{L}(\boldsymbol{B}_d)$ and $\|\bar{u}\|^p_{p,\boldsymbol{B}_d(x)} = \bar{u}^p \mathfrak{L}(\boldsymbol{B}_d)$, it follows that

$$0 \leqslant \bar{\phi} = 1 - \bar{u} = [\mathfrak{L}(\boldsymbol{B}_d)]^{-1/p}\{\|u\|_{p,\boldsymbol{B}_d(x)} - \|\bar{u}\|_{p,\boldsymbol{B}_d(x)}\}.$$

As $0 \leqslant \|u\|_{p,\boldsymbol{B}_d(x)} - \|\bar{u}\|_{p,\boldsymbol{B}_d(x)} \leqslant \|u - \bar{u}\|_{p,\boldsymbol{B}_d(x)}$, by (6.7), one has

$$\|\bar{\phi}\|^p_{p,\boldsymbol{B}_d(x)} \leqslant \|u - \bar{u}\|^p_{p,\boldsymbol{B}_d(x)} \leqslant \left(\frac{d}{\pi_p}\right)^p \|\nabla u\|^p_{p,\boldsymbol{B}_d(x)},$$

from which, and again with (6.7), we observe that

$$\begin{aligned} \|\phi\|^p_{p,\boldsymbol{B}_d(x)} &\leqslant 2^{p-1}\{\|\phi - \bar{\phi}\|^p_{p,\boldsymbol{B}_d(x)} + \|\bar{\phi}\|^p_{p,\boldsymbol{B}_d(x)}\} \\ &\leqslant \left(\frac{2d}{\pi_p}\right)^p \|\nabla u\|^p_{p,\boldsymbol{B}_d(x)}. \end{aligned} \tag{6.8}$$

Finally, we can use the symmetry of $\boldsymbol{B}_d(x)$ to extend ϕ to a new function ψ in $\mathbb{R}^N$ such that $\psi(y) := \phi(y)$ when $y \in \bar{\boldsymbol{B}}_d(x)$, and

$$\psi(y) := \phi\left(x + (y - x)\frac{d^2}{|y - x|^2}\right)$$

when $y \in \boldsymbol{B}^c_d(x)$ in every ray emanating from x. Thus,

$$\|\psi\|^p_{p,\boldsymbol{B}_{3d}(x)} \leqslant \mathfrak{c}_5 \|\phi\|^p_{p,\boldsymbol{B}_d(x)} \quad \text{and} \quad \||\nabla\psi|\|^p_{p,\boldsymbol{B}_{3d}(x)} \leqslant \mathfrak{c}_5 \||\nabla\phi|\|^p_{p,\boldsymbol{B}_d(x)}$$

for a constant $\mathfrak{c}_5 > 0$ independent of d (by the definition of a definite integral). Let $\eta \in C^\infty_c(\boldsymbol{B}_{3d})$ satisfy $\eta = 1$ when $|x| \leqslant d$ and $\eta = 0$ when $|x| \geqslant 2d$, with $\||\nabla\eta|\|_{\infty,\mathbb{R}^N} \leqslant d^{-1}$. Then we have

$$\begin{aligned} \operatorname{Cap}_p(K) &\leqslant \||\nabla(\psi\eta)|\|^p_{p,\mathbb{R}^N} \\ &\leqslant 2^{p-1}\mathfrak{c}_5\{\||\nabla u|\|^p_{p,\boldsymbol{B}_d(x)} + \||\nabla\eta|\|^p_{\infty,\mathbb{R}^N}\|\phi\|^p_{p,\boldsymbol{B}_d(x)}\}, \end{aligned}$$

which together with (6.8) yields estimate (6.6) for $C_6 = 2^{p-1}\mathfrak{c}_5[1 + (2/\pi_p)^p]$. □

Proof of proposition 6.1. Let $u \in W^{1,p}(\boldsymbol{B}_d(x))$ and let $\tau > 0$. Set

$$E_\tau := \{y \in \boldsymbol{B}_d(x) \colon |u(y)| \leqslant \tau\}.$$

As $0 \leqslant |u| \leqslant [|u| - \tau]^+ + \tau$, we have

$$\|u\|_{p,\boldsymbol{B}_d(x)} \leqslant \tau[\mathfrak{L}(\boldsymbol{B}_d)]^{1/p} + \||u| - \tau\|_{p,\boldsymbol{B}_d(x)\setminus E_\tau},$$

so that

$$\|u\|_{p,\boldsymbol{B}_d(x)} \leqslant 2\||u| - \tau_u\|_{p,\boldsymbol{B}_d(x)\setminus E_{\tau_u}} \tag{6.9}$$

for $\tau_u := \|u\|_{p,\boldsymbol{B}_d(x)}[\mathfrak{L}(\boldsymbol{B}_d)]^{-1/p}/2$. When $\operatorname{Cap}_p(E_{\tau_u}) \geqslant \rho \operatorname{Cap}_p(\boldsymbol{B}_d)$ we can apply (6.6) and (6.9) to the function $[|u| - \tau_u]^+ \geqslant 0$, which vanishes on E_{τ_u}, to observe that

$$\rho \operatorname{Cap}_p(\boldsymbol{B}_d) \leqslant \operatorname{Cap}_p(E_{\tau_u}) \leqslant 2^p C_6 \mathfrak{L}(\boldsymbol{B}_d) \frac{\int_{\boldsymbol{B}_d(x)} |\nabla u|^p \, \mathrm{d}y}{\int_{\boldsymbol{B}_d(x)} |u|^p \, \mathrm{d}y}.$$

So, noting that $\operatorname{Cap}_p(\boldsymbol{B}_d) = d^{N-p}\operatorname{Cap}_p(\boldsymbol{B}_1)$ and $\mathfrak{L}(\boldsymbol{B}_d) = d^N\mathfrak{L}(\boldsymbol{B}_1)$, we derive that

$$\int_{\boldsymbol{B}_d(x)} |u|^p \,\mathrm{d}y \leqslant \frac{C_6\mathfrak{L}(\boldsymbol{B}_1)}{\operatorname{Cap}_p(\boldsymbol{B}_1)}\frac{(2d)^p}{\rho}\int_{\boldsymbol{B}_d(x)} |\nabla u|^p \,\mathrm{d}y.$$

When $\operatorname{Cap}_p(E_{\tau_u}) \leqslant \rho\operatorname{Cap}_p(\boldsymbol{B}_d)$ we can follow [27, p. 936] to observe that

$$\int_{\boldsymbol{B}_d(x)} |u|^p \,\mathrm{d}\mu \geqslant \tau_u^p\{\mu(\boldsymbol{B}_d(x)\setminus E_{\tau_u})\} \geqslant \frac{\inf_{\boldsymbol{F}\in\mathscr{N}_\rho}\{\mu(\boldsymbol{B}_d(x)\setminus\boldsymbol{F})\}}{d^N2^p\mathfrak{L}(\boldsymbol{B}_1)}\int_{\boldsymbol{B}_d(x)} |u|^p \,\mathrm{d}y.$$

As a result, we are done with $C_5 = \max\{2^pC_6\mathfrak{L}(\boldsymbol{B}_1)\operatorname{Cap}_p^{-1}(\boldsymbol{B}_1), 2^p\mathfrak{L}(\boldsymbol{B}_1)\} > 0$. □

Now, we can follow [21, theorems 6.1 and 6.2] to show that condition (6.2) will work.

First, when $N \geqslant 3$ and $2 \leqslant p < N$, we use $\mathfrak{L}(\boldsymbol{F}) \leqslant \mathfrak{c}_1[\operatorname{Cap}_p(\boldsymbol{F})]^{N/(N-p)}$ to see that, for $d > 0$, $\mathfrak{L}(\boldsymbol{F}) \leqslant \varrho\mathfrak{L}(\boldsymbol{B}_d)$ implies that $\boldsymbol{F} \in \mathscr{N}_\rho$, provided that $\varrho > 0$ is sufficiently small. In fact, we have

$$\mathfrak{L}(\boldsymbol{F}) \leqslant \mathfrak{c}_1[\operatorname{Cap}_p(\boldsymbol{F})]^{N/(N-p)} \ll \sigma_d\mathfrak{L}(\boldsymbol{B}_d) \leqslant \sigma_d\mathfrak{c}_1[\operatorname{Cap}_p(\boldsymbol{B}_d)]^{N/(N-p)}, \tag{6.10}$$

so that $\operatorname{Cap}_p(\boldsymbol{F}) \leqslant \sigma_d^{(N-p)/N}\operatorname{Cap}_p(\boldsymbol{B}_d)$, where $\varrho \ll \sigma_d < 1$ are constants. When $N \geqslant 2$ and $p = N$, one instead applies $\mathfrak{L}(\boldsymbol{F})/\mathfrak{L}(\boldsymbol{B}_{2d}) \leqslant \mathfrak{c}_6\operatorname{Cap}_N(\boldsymbol{F})$ for an absolute constant $\mathfrak{c}_6 > 0$ (see [13, lemma VIII1.4]) to observe that $\mathfrak{L}(\boldsymbol{F}) \leqslant \varrho\mathfrak{L}(\boldsymbol{B}_d)$ implies that $\boldsymbol{F} \in \mathscr{N}_\rho$ again for sufficiently small $\varrho > 0$, as

$$\frac{\mathfrak{L}(\boldsymbol{F})}{\mathfrak{L}(\boldsymbol{B}_{2d})} \leqslant \mathfrak{c}_6\operatorname{Cap}_N(\boldsymbol{F}) \ll \sigma_d\frac{\mathfrak{L}(\boldsymbol{B}_d)}{\mathfrak{L}(\boldsymbol{B}_{2d})} \leqslant \sigma_d\mathfrak{c}_6\operatorname{Cap}_N(\boldsymbol{B}_d). \tag{6.11}$$

All of these discussions indicate that (6.3) can be replaced by a stronger condition

$$\inf_{\boldsymbol{F}\in\mathscr{M}_\varrho}\{\mu(\boldsymbol{B}_d(x)\setminus\boldsymbol{F})\} \to \infty \quad \text{when } |x| \to \infty. \tag{6.12}$$

Here, μ is a measure on $\mathbb{R}^N$ that is absolutely continuous with respect to the Lebesgue measure, and $\mathscr{M}_\varrho$ is the family of $\boldsymbol{F} \Subset \boldsymbol{B}_d(x)$ such that $\mathfrak{L}(\boldsymbol{F}) \leqslant \varrho\mathfrak{L}(\boldsymbol{B}_d)$ for very small $\varrho \in (0,1)$.

Finally, let $\mathrm{d}\mu := b\,\mathrm{d}x$ with b satisfying (6.2), and let $\boldsymbol{F} \in \mathscr{M}_\varrho$ with $\mathfrak{L}(\boldsymbol{B}_d(x)\setminus\boldsymbol{F}) \geqslant \frac{3}{4}\mathfrak{L}(\boldsymbol{B}_d)$. Then, for all $M > 0$, condition (6.2) implies that

$$\mathfrak{L}(\{y \in \boldsymbol{B}_d(x) \colon b(y) \geqslant M\} \cap \{\boldsymbol{B}_d(x)\setminus\boldsymbol{F}\}) \geqslant \tfrac{1}{2}\mathfrak{L}(\boldsymbol{B}_d),$$

from which one deduces that

$$\inf_{\boldsymbol{F}\in\mathscr{M}_\varrho}\int_{\boldsymbol{B}_d(x)\setminus\boldsymbol{F}} b(x)\,\mathrm{d}x \geqslant \tfrac{1}{2}M\mathfrak{L}(\boldsymbol{B}_d).$$

That is, (6.12) is satisfied.

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