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SPECTRAL INVARIANCE FOR CERTAIN ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS

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Abstract We construct algebras of pseudodifferential operators on a continuous family groupoid $\mathcal{G}$ that are closed under holomorphic functional calculus, contain the algebra of all pseudodifferential operators of order 0 on $\mathcal{G}$ as a dense subalgebra and reflect the smooth structure of the groupoid $\mathcal{G}$, when $\mathcal{G}$ is smooth. As an application, we get a better understanding on the structure of inverses of elliptic pseudodifferential operators on classes of non-compact manifolds. For the construction of these algebras closed under holomorphic functional calculus, we develop three methods: one using semi-ideals, one using commutators and one based on Schwartz spaces on the groupoid.

One of our main results is to reduce the construction of spectrally invariant algebras of order 0 pseudodifferential operators to the analogous problem for regularizing operators. We then show that, in the case of the *generalized 'cusp'-calculi* c_n, $n \geqslant 2$, it is possible to construct algebras of regularizing operators that are closed under holomorphic functional calculus and consist of smooth kernels. For $n = 1$, this was shown not to be possible by the first author in an earlier paper.

Keywords: pseudodifferential operator; operator algebra; spectral invariance; groupoid

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Contents

1. Introduction

Let M be a compact manifold and P an elliptic pseudodifferential operator of order $m \geqslant 0$ on M. Assume that P is invertible as an unbounded operator on $L^2(M)$ (the space of square-integrable $\frac{1}{2}$-densities on M). A classical and often useful result states that then P^{-1} is also a pseudodifferential operator. For non-compact manifolds, the situation is more complicated, essentially because we also want to control the behaviour at infinity of the inverse.

For example, when M has cylindrical ends, a convenient class of pseudodifferential operators is that of b-pseudodifferential operators introduced by Melrose [**31**,**32**] (see also [**23**] and [**42**]). Then it is known that the inverse P^{-1} of an elliptic b-pseudodifferential operator (defined in the L^2 sense) is not necessarily also a b-pseudodifferential operator in the so-called *small b-calculus*. We say that the b-calculus is not *spectrally invariant*. There exist, however, different classes of pseudodifferential operators associated to a manifold with cylindrical ends that are spectrally invariant. In this paper we discuss this property for the c_n-calculi, which are spectrally invariant for $n \geqslant 2$.*

Closely related to spectral invariance is the question of whether a given algebra of pseudodifferential operators, say, of order 0, is closed under holomorphic functional calculus. Let us explain the relevance of this property. Let H be a Hilbert space and $a \in \mathcal{L}(H)$ *relatively invertible* in $\mathcal{L}(H)$, i.e. there exists $\tilde{a} \in \mathcal{L}(H)$ with $a\tilde{a}a = a$ and $\tilde{a}a\tilde{a} = \tilde{a}$. By a characterization of Atkinson [**2**], we know that this is the case if and only if the range $R(a)$ of a is closed. By a classical result of Rickart [**43**] (see also [**9**] and [**11**, Bemerkung 5.7]), 0 is an isolated point of the spectrum $\sigma(a^*a)$ of a^*a, and the orthogonal projection p

* When $n = 1$, the c_n-calculus is nothing but the b-calculus and, when $n = 2$, it is usually called the 'cusp'-calculus (see, for instance, [**34**], which is based on earlier work of Melrose). Here, n should not be confused with the dimension of the manifold but determines the degree of degeneracy in direction to the boundary, more precisely, the c_n-calculus is modelled on the differential operators with degeneracies of the form $x^n\partial_x$ at the boundary, where x stands for the direction normal to the boundary.

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onto the kernel $N(a^*a) = N(a)$ is given by the integral

$$p := \frac{1}{2\pi \mathrm{i}} \int_\gamma (z\,\mathrm{id}_H - a^*a)^{-1}\,\mathrm{d}z, \tag{1.1}$$

where γ is a small circle around the origin in $\mathbb{C}$ that does not intersect $\sigma(a^*a)$. In that case, the operator

$$\tilde{a} = (p + a^*a)^{-1}a^* \tag{1.2}$$

is a relative inverse of a. Because of $(\tilde{a}a)^* = \tilde{a}a$ and $(a\tilde{a})^* = a\tilde{a}$, the operator $\tilde{a}$ is also called an *orthogonal generalized inverse* or a *Moore–Penrose inverse of* a [**39**]; it is easily seen to be uniquely determined.

Thus a spectrally invariant algebra will contain the Moore–Penrose inverses of its elements (when they exist). For more about generalized inverses, we refer to [**39**].

It is natural to ask what properties of a are inherited by the Moore–Penrose inverse $\tilde{a}$. It is immediate from (1.1) and (1.2) that $\tilde{a}$ belongs to the intersection of all subalgebras $A \subseteq \mathcal{L}(H)$ that are symmetric with respect to the $*$-operation from $\mathcal{L}(H)$ and closed under holomorphic functional calculus in $\mathcal{L}(H)$; in particular, any property of a that can be covered by a symmetric subalgebra A of $\mathcal{L}(H)$ that is closed under holomorphic functional calculus is true for the Moore–Penrose inverse as well. Thus it is interesting to find algebras that are closed under holomorphic functional calculus. Without loss of generality, we can always assume that an algebra that is closed under holomorphic functional calculus is also symmetric. Of particular importance for pseudodifferential and micro-local analysis are symmetric, continuously embedded *Fréchet* subalgebras of C^*-algebras that are closed under holomorphic functional calculus (Ψ^*-algebras [**11**]). Indeed, in contrast to the rather rigid C^*-topology, the Fréchet-topology allows a flexible treatment of $\mathcal{C}^\infty$-phenomena within a functional analytic setting [**12**, **14**]. On the other hand, stability under holomorphic functional calculus and symmetry still establishes a strong relation between the structure of a Ψ^*-algebra and that of its C^*-closures, leading to sometimes unexpected insights into the internal structure of a Ψ^*-algebra; for instance, the set of relatively invertible elements in Ψ^*-algebras has been shown to be a locally rational Fréchet manifold [**11**]. Starting from the seminal work [**11**] of Gramsch, the world of Ψ^*-algebras has been explored by many authors, and we refer the reader to [**11**–**13**, **15**, **17**, **21**, **28**] and the references given there for more details. Besides, the K-theory groups, $K_*(A)$, of a symmetric subalgebra of a C^*-algebra $\mathcal{B}$ that is closed under holomorphic functional calculus coincide with those of its closure in $\mathcal{B}$. Our basic example for an algebra that is closed under holomorphic functional calculus is the algebra of classical pseudodifferential operators of order 0 on a closed manifold.

In [**25**], the authors considered a pseudodifferential calculus on continuous family groupoids; this calculus generalizes the pseudodifferential calculus on $\mathcal{C}^{\infty,0}$-foliations used by Connes to prove the index theorem for foliated spaces [**7**] (see [**46**, **47**] for an introduction to the theory of pseudodifferential operators). In a slightly different context, Nistor *et al.* [**40**] and Monthubert and Pierrot [**38**] have studied a pseudodifferential calculus on differentiable groupoids. As demonstrated by the examples in [**37**, **40**] and the survey [**24**], the groupoid approach yields a pseudodifferential calculus for many interesting situations in analysis and geometry, especially on open manifolds and manifolds with singularities, in a unified way. Up to some support condition, this pseudodifferential calculus recovers the existing calculi, and in many cases goes beyond the results that are known in the literature. Let us only mention that the class of algebras that can be defined using groupoids include the ordinary pseudodifferential calculus, a G-equivariant pseudodifferential calculus on bundles of Lie groups, the b-calculus of Melrose and many of its cousins on manifolds with corners, the edge calculus on manifolds with fibred boundaries, the calculus of adiabatic pseudodifferential operators and many others. On the other hand, we know for many of the different pseudodifferential calculi mentioned above that the algebra of operators of order 0 is not closed under holomorphic functional calculus. In fact, due to the support condition in the case of a general continuous family groupoid (which is a quite convenient condition that will insure that the composition is defined), the algebra of operators of order 0 is almost never closed under holomorphic functional calculus.

In the present paper, we develop a general strategy to embed the algebra $\Psi^{0,0}(\mathcal{G})$ of pseudodifferential operators of order 0 on a continuous family groupoid $\mathcal{G}$ into larger algebras $\mathcal{A}$ that are closed under holomorphic functional calculus and still share some of the interesting properties with the algebra $\Psi^{0,0}(\mathcal{G})$. (We shall denote by $\Psi^{m,0}(\mathcal{G})$ the space of order-m pseudodifferential operators on a continuous family groupoid $\mathcal{G}$.) In fact, it is one of the results of this paper that it usually suffices to embed the algebra $\Psi^{-\infty,0}(\mathcal{G})$ of operators of order $-\infty$ in an algebra $\mathcal{J}$ that is closed under holomorphic functional calculus. Up to some technical conditions, $\mathcal{A} := \Psi^{0,0}(\mathcal{G}) + \mathcal{J}$ is then an algebra that is closed under holomorphic functional calculus. For the construction of the algebra $\mathcal{J}$, we suggest three alternatives. The first one relates properties of an algebra to those of a two-sided ideal and its corresponding quotient, the second one is based on commutator methods from operator theory, whereas the third one depends more on the geometry of the groupoid and requires the existence of a length function ϕ with polynomial growth on the groupoid. The role of ϕ is to define a Schwartz space $\mathcal{S}(\mathcal{G})$ on $\mathcal{G}$ that replaces $\mathcal{C}_c^\infty(\mathcal{G}) = \Psi^{-\infty,0}(\mathcal{G})$. In this way, we control the behaviour at infinity of the kernels of our pseudodifferential operators on a typical leaf $d^{-1}(x)$ of the groupoid.

The paper is organized as follows. In § 2 we recall the notion of algebras closed under functional calculus, and consider and answer the question of whether an algebra is closed under functional calculus provided an ideal and the corresponding quotients are. In § 3 we develop the operator theoretical methods (based on commutators) that are used in § 4 to construct algebras $\mathcal{A}$ containing $\Psi^{0,0}(\mathcal{G})$ and closed under functional calculus. In § 5 we introduce the c_n-calculi on manifolds with boundary and with corners. One can embed the c_n-calculi, $n \geqslant 2$, in Ψ^*-algebras that consist of smooth kernels, a result that is proved in § 6. Section 7 is devoted to the study of the Schwartz space $\mathcal{S}(\mathcal{G})$ of a continuous family groupoid and the proof that it is closed under functional calculus. We also define length functions for the groupoids associated to various pseudodifferential calculi on manifolds with corners (the b-calculus, the 'cusp'-calculus, or, more generally, the c_n-calculi).

2. Algebras closed under holomorphic functional calculus

In this section we recall some basic facts about algebras that are closed under holomorphic functional calculus and describe a method to generate algebras closed under holomorphic functional calculus. This method is based on permanence properties of the closure under holomorphic functional calculus when passing to two-sided ideals, to quotients, or back from ideals and quotients to the algebra.

2.1. Definitions

We begin by recalling the definition of an algebra closed under holomorphic functional calculus.

Definition 2.1. Let $\mathcal{B}$ be a Banach algebra with unit e. A not necessarily unital subalgebra $A \subseteq \mathcal{B}$ is said to be *closed under the holomorphic functional calculus in* $\mathcal{B}$ provided that, for every $a = \lambda e + x \in \mathbb{C}e + A$ and all $f \in \mathcal{O}(\sigma_{\mathcal{B}}(a))$, we have $f(a) \in \mathbb{C}e + A$.

Here, $\mathcal{O}(\sigma_{\mathcal{B}}(a))$ denotes the algebra of germs of holomorphic functions defined on a neighbourhood of $\sigma_{\mathcal{B}}(a)$, the spectrum of a in the Banach algebra $\mathcal{B}$, and

$$f(a) := \frac{1}{2\pi \mathrm{i}} \int_{\gamma} f(z)(ze - a)^{-1} \,\mathrm{d}z \in \mathcal{B} \tag{2.1}$$

is the operator given by the usual holomorphic functional calculus within the Banach algebra $\mathcal{B}$. For any algebra A with unit, we shall denote by A^{-1} the set of invertible elements of A.

Remark 2.2. The following observations are immediate.

(a) Let $\mathcal{B}$ be a unital C^*-algebra and $A \subseteq \mathcal{B}$ a symmetric subalgebra (i.e. closed under taking adjoints). Then we have $\sigma_{\mathcal{B}}(a) = \sigma_{\mathcal{A}_e}(a)$, where $\mathcal{A}_e$ is the completion of $\mathbb{C}e + A$ with respect to the norm induced by $\mathcal{B}$, so Definition 2.1 recovers [**8**, Definition 1, p. 285].

(b) An arbitrary intersection of algebras closed under holomorphic functional calculus in $\mathcal{B}$ is again closed under holomorphic functional calculus in $\mathcal{B}$.

(c) If $A \subseteq \mathcal{B}$ is closed under the holomorphic functional calculus in $\mathcal{B}$, then A is *spectrally invariant* in $\mathcal{B}$, i.e. for the groups of invertible elements, we have

$$(\mathbb{C}e + A) \cap \mathcal{B}^{-1} = (\mathbb{C}e + A)^{-1},$$

or, equivalently, $j^{-1}(\mathcal{B}^{-1}) = (\mathbb{C}e + A)^{-1}$ if $j : \mathbb{C}e + A \hookrightarrow \mathcal{B}$ denotes the natural inclusion.

It is often useful to consider algebras closed under holomorphic functional calculus that are in addition complete with respect to a finer topology. For example, the class of Ψ^*-algebras introduced by Gramsch [**11**] in connection with a perturbation theory for singular integral and pseudodifferential operators is, in fact, the appropriate setting to describe $\mathcal{C}^\infty$-phenomena of micro-local analysis within a functional analytic framework [**12**].

Definition 2.3. Let $\mathcal{B}$ be a unital C^*-algebra. A symmetric, spectrally invariant subalgebra $\mathcal{A} \subseteq \mathcal{B}$, $e \in \mathcal{A}$, is called a *Ψ^*-algebra* in $\mathcal{B}$ if and only if there exists a Fréchet topology $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{A}$ making the embedding $\iota : (\mathcal{A}, \mathcal{T}_{\mathcal{A}}) \hookrightarrow (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ continuous.

In case $e \notin \mathcal{A}$, the algebra $\mathcal{A}$ is said to be a *non-unital* Ψ^*-algebra, provided that $\mathbb{C}e \oplus \mathcal{A}$ is a Ψ^*-algebra.

Following the usual convention, in the sequel a Ψ^*-algebra is always unital. A unital Fréchet algebra $\mathcal{A}$ is said to be *submultiplicative* if the topology $\mathcal{T}_{\mathcal{A}}$ can be generated by a countable system $(q_j)_{j\in\mathbb{N}}$ of submultiplicative semi-norms, i.e. semi-norms satisfying $q_j(xy) \leqslant q_j(x)q_j(y)$ and $q_j(e) = 1$.

We recall a few basic facts about (non-unital) Ψ^*-algebras; most of them are obvious consequences of the definition.

Proposition 2.4. *Let $\mathcal{B}$ be a unital C^*-algebra and $\mathcal{A} \subseteq \mathcal{B}$ a subalgebra.*

(a) *If $\mathcal{A}$ is a non-unital Ψ^*-algebra, then the Fréchet-topology on $\mathbb{C}e \oplus \mathcal{A}$ induces a Fréchet-topology $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{A}$; in particular, $(\mathcal{A}, \mathcal{T}_{\mathcal{A}}) \hookrightarrow (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is continuous.*

(b) *If $\mathcal{A}$ is a Ψ^*-algebra, then the group of invertible elements $\mathcal{A}^{-1}$ is open and the inversion*

$$\mathcal{A}^{-1} \ni x \mapsto x^{-1} \in \mathcal{A} \tag{2.2}$$

is continuous.

(c) *If $\mathcal{A}$ is a non-unital Ψ^*-algebra, then, for any $a \in \mathcal{A}$, there exists an analytic map $h : \varrho_{\mathcal{B}}(a) \to \mathcal{A}$ where $\varrho_{\mathcal{B}}(a) \subseteq \mathbb{C}$ is the resolvent set, such that*

$$(\lambda e - a)^{-1} = \frac{1}{\lambda}e + h(\lambda).$$

(d) *Any Ψ^*-algebra, unital or not, is closed under functional calculus. If $\mathcal{A}$ is non-unital and $a = \lambda e + x \in \mathbb{C}e \oplus \mathcal{A}$ is arbitrary, then $f(a) - f(\lambda)e \in \mathcal{A}$ for all $f \in \mathcal{O}(\sigma_{\mathcal{B}}(a))$.*

Proof. For (a), it suffices to show that $\mathcal{A}$ is closed in $\mathbb{C}e \oplus \mathcal{A}$. Let $a_j \in \mathcal{A}$ be with $\lim_{j\to\infty a_j} = \lambda e + a \in \mathbb{C}e \oplus \mathcal{A}$. Without loss of generality, we can assume $\lambda = 1$ and $a = 0$. By the spectral invariance of $\mathbb{C}e \oplus \mathcal{A}$, we find $j \in \mathbb{N}$, $\mu \in \mathbb{C}$ and $b \in \mathcal{A}$ with

$$e = a_j(\mu e + b) = \mu a_j + a_j b \in \mathcal{A},$$

which contradicts our assumption $\varepsilon \notin \mathcal{A}$.

For (b), we first note that $\mathcal{A}^{-1}$ is open because of $\mathcal{A}^{-1} = \iota^{-1}(\mathcal{B}^{-1})$ and then use an old result of Banach [**3**] which says that the inversion in a Fréchet algebra is continuous if and only if the group of invertible elements is a G_δ-set.

To prove (c), we first note that the resolvent $\varrho_{\mathcal{B}}(a) \ni \lambda \mapsto (\lambda e - a)^{-1} \in \mathbb{C}e \oplus \mathcal{A}$ is analytic by the continuity of the inversion. Moreover, the resolvent identity together with the spectral invariance of $\mathbb{C}e \oplus \mathcal{A}$ in $\mathcal{B}$ yields

$$(\lambda e - a)^{-1} = \frac{1}{\lambda}e + \underbrace{\frac{a}{\lambda}(\lambda e - a)^{-1}}_{=:h(\lambda)} \in \mathbb{C}e \oplus \mathcal{A},$$

showing that $h : \varrho_{\mathcal{B}}(a) \to \mathcal{A}$ has the desired properties.

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Finally, part (d) is an immediate consequence of part (c) and the defining formula

$$f(a) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} f(\lambda)(\lambda e - a)^{-1} \, \mathrm{d}\lambda$$

for the holomorphic functional calculus. □

Definition 2.5. Let $\mathcal{B}$ be a Banach algebra with unit e, and $\varphi : A \to \mathcal{B}$ be a morphism of algebras, which we assume to preserve the unit if A has one. Then A is called *locally spectral invariant with respect to* φ if there exists $\varepsilon > 0$ such that we have

$$(e + \varphi(x))^{-1} \in \mathbb{C} + \varphi(A)$$

for all $x \in A$ with $\|\varphi(x)\|_{\mathcal{B}} < \varepsilon$. In that case, we say that A has *property* (P_A) in $\mathcal{B}$. The morphism φ is to be understood from the context. Moreover, A is said to have *property* $(\tilde{P}_A)$ if A is unital and $\varphi^{-1}(\mathcal{B}^{-1}) = A^{-1}$.

By [**11**, Lemma 5.3], properties (P_A) and $(\tilde{P}_A)$ are closely related in many interesting cases.

Lemma 2.6. *Let $\varphi : A \to \mathcal{B}$ be an injective morphism of unital algebras. Then the following hold.*

(a) *If $\mathcal{B}$ is a unital Banach algebra and $\varphi(A)$ is dense in $\mathcal{B}$, then we have $(P_A) \Leftrightarrow (\tilde{P}_A)$.*

(b) *If $\mathcal{B}$ is a unital C^*-algebra and $\varphi(A)$ is symmetric in $\mathcal{B}$, then we have $(P_A) \Leftrightarrow (\tilde{P}_A)$.*

In other words, property $(\tilde{P}_A)$ is not stronger than the (apparently weaker) condition (P_A), provided that either A is dense in B or A is symmetric.

We close this subsection with a lemma that helps to characterize spectrally invariant subalgebras. It will be used in § 7 to prove the spectral invariance of the Schwartz convolution algebra on a continuous family groupoid. We include here a proof that is closely related to the one of the previous lemma, for the sake of completeness.

Lemma 2.7. *Let $\mathcal{B}$ be a Banach algebra with unit e and $\mathcal{A}$ a Banach algebra such that $e \in \mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A}$ is dense in $\mathcal{B}$ and $r_{\mathcal{A}}(a) = r_{\mathcal{B}}(a)$ holds for all $a \in \mathcal{A}$, where $r_D(a)$ denotes the spectral radius of a in a Banach algebra D. Then we have*

$$\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1},$$

i.e. $\mathcal{A}$ is spectrally invariant in $\mathcal{B}$. Moreover, $\mathcal{A}$ is closed under holomorphic functional calculus in $\mathcal{B}$.

Proof. Let $a \in \mathcal{A} \cap \mathcal{B}^{-1}$ be arbitrary. By the density of $\mathcal{A}$ in $\mathcal{B}$, there exists $y \in \mathcal{A}$ with $\|ay - e\|_{\mathcal{B}} \leqslant \frac{1}{2}$, hence $\varrho_{\mathcal{A}}(ay - e) \leqslant \frac{1}{2}$. In particular, ay is invertible in $\mathcal{A}$, and we get $a^{-1} = a^{-1}(ay)(ay)^{-1} = y(ay)^{-1} \in \mathcal{A}$, which completes the proof. □

The above statement generalizes right away to non-unital algebras.

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2.2. Spectral invariance and ideals

We are now going to look more closely at the question of how spectral invariance of an algebra is related to that of its quotients, bearing in mind Proposition 2.4 (that a compatible Fréchet topology on a spectrally invariant subalgebra implies stability under holomorphic functional calculus). We start by describing the general setting.

From now on and throughout this section, $\mathcal{B}$ will be a Banach algebra with unit e, $A \subseteq \mathcal{B}$ will be a subalgebra with $e \in A$, $\mathcal{J} \subseteq \mathcal{B}$ will be a proper, closed two-sided ideal in $\mathcal{B}$ and $I \subseteq A$ will be a two-sided ideal in A with $I \subseteq \mathcal{J}$. Then the map

$$\varphi : A/I \to \mathcal{B}/\mathcal{J} : a + I \mapsto a + \mathcal{J} \tag{2.3}$$

is a well-defined homomorphism of unital algebras. Also, note that φ is one-to-one if and only if $\mathcal{J} \cap A = I$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{B}/\mathcal{J} & \longrightarrow & 0 \end{array} \tag{2.4}$$

The next theorem relates the different properties (P_I), (P_A), $(P_{A/I})$ and $(\tilde{P}_{A/I})$ to one another. A special case of part (a) can be found in [**5**, Appendix]; (b) is from [**13**, § 5].

Theorem 2.8. *Let A, I, $\mathcal{B}$, $\mathcal{J}$ and φ be as in (2.3) above. We shall write $(\tilde{P})$ instead of $(\tilde{P}_{A/I})$ for simplicity. Property (P_I) is considered with respect to the obvious morphism $I \to \mathbb{C}e \oplus \mathcal{J}$. Then we have the following.*

(a) *If $I \subseteq \mathcal{J}$ is dense, then (P_I) together with $(\tilde{P})$ imply (P_A).*

(b) *If $I \subseteq \mathcal{J}$ is dense, then (P_A) implies $(P_{A/I})$.*

(c) *If $I \subseteq \mathcal{J}$ and $A \subseteq \mathcal{B}$ are dense, then (P_A) implies $(\tilde{P})$.*

(d) *(P_A) implies (P_I).*

(e) *$(\tilde{P})$ implies $(P_{A/I})$. Moreover, if $\varphi : A/I \to \mathcal{B}/\mathcal{J}$ is one-to-one and $\varphi(A/I)$ is dense in $\mathcal{B}/\mathcal{J}$, then we have $(P_{A/I})$ if and only if we have $(\tilde{P})$.*

Proof. Let $\varepsilon > 0$ always be the constant from Definition 2.5.

(a) By the continuity of the inversion in the unital Banach algebra $\mathcal{B}/\mathcal{J}$, we can find $0 < \delta < \frac{1}{2}\varepsilon < 1$ such that $\|(b + \mathcal{J})^{-1} - (e + \mathcal{J})\|_{\mathcal{B}/\mathcal{J}} < \frac{1}{8}\varepsilon$, for all $b \in \mathcal{B}$ satisfying $\|(b + \mathcal{J}) - (e + \mathcal{J})\|_{\mathcal{B}/\mathcal{J}} < \delta$.

Consider now $a \in A$ with $\|a - e\|_{\mathcal{B}} < \delta$. From $\|(a - e) + \mathcal{J}\|_{\mathcal{B}/\mathcal{J}} < \delta$ and $(\tilde{P})$, we obtain that there exists $a_1 \in A$ with $aa_1 - e =: x_1 \in I$ and $\|a_1 - e + \mathcal{J}\|_{\mathcal{B}/\mathcal{J}} < \frac{1}{8}\varepsilon$. Using the density of I in $\mathcal{J}$, we find $x_2 \in I$ with $\|a_1 - e + x_2\|_{\mathcal{B}} < \frac{1}{4}\varepsilon$, and hence

$$\|x_1 + ax_2\|_{\mathcal{B}} \leqslant \|a\|_{\mathcal{B}}\|a_1 - e + x_2\|_{\mathcal{B}} + \|a - e\|_{\mathcal{B}} < \varepsilon,$$

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and by (P_I) we get $z \in I$ with $(e + x_1 + ax_2)^{-1} = e + z$, i.e.

$$a(a_1 + x_2)(e + z) = e,$$

which gives $a \in A^{-1}$, and completes the proof of (a).

(b) Let $0 < \delta < \frac{1}{2}\varepsilon$ and $a + I \in A/I$ be with $\|\varphi(a + I) - (e + \mathcal{J})\|_{\mathcal{B}/\mathcal{J}} < \delta$. By the density of I in $\mathcal{J}$, there exists $x \in I$ with $\|a + x - e\|_{\mathcal{B}} < \varepsilon$, and hence (P_A) gives $a_1 \in A$ with $(a + x)a_1 = e = a_1(a + x)$. We thus obtain

$$\varphi(a + I)^{-1} = \varphi(a_1 + I) \in \varphi(A/I),$$

and hence (b) is proved.

(c) Let $a + I \in A/I$ be with $\varphi(a + I) \in (\mathcal{B}/\mathcal{J})^{-1}$. Thus there exists $b \in \mathcal{B}$ and $y \in \mathcal{J}$ with $ab - y - e = 0$. By the density assumption, we obtain $x \in I$ and $a_1 \in A$ with $\|aa_1 - x - e\|_{\mathcal{B}} < \varepsilon$, and hence $aa_1 - x \in A^{-1}$ by (P_A). Let $a_2 \in A$ be such that $e = (aa_1 - x)a_2 = aa_1a_2 - xa_2$. Consequently, $a_1a_2 + I \in A/I$ is a right-inverse of $a + I$ in A/I. Similarly, we also obtain a left-inverse of $a + I$, which gives $a + I \in (A/I)^{-1}$, and completes the proof of (c).

To prove (d), let $\varepsilon > 0$ and $x \in I$ be with $\|x\|_{\mathcal{B}} < \varepsilon$. Then $e + x$ is invertible in A because of (P_A). Then (P_I) is a consequence of (P_A) and the identity

$$(e + x)^{-1} - e = -x + \underbrace{(e + x)^{-1}}_{\in A} x^2 \in I.$$

Finally, part (e) is a simple, straightforward computation using Lemma 2.6. □

Remark 2.9. It follows that if $I \subseteq \mathcal{J}$ and ϕ is injective, then we can replace $(\tilde{P})$ with $(P_{A/I})$ in Theorem 2.8 (a).

We are mostly interested in deciding when A is spectrally invariant in $\mathcal{B}$. The following special case of Theorem 2.8 will be used in the sequel.

Corollary 2.10. *Let $\mathcal{B}$ be a unital C^*-algebra, and $\mathcal{J}$, A and I as above, but additionally symmetric with respect to the $*$-operation in $\mathcal{B}$. Assume that I is dense in $\mathcal{J}$. Then A is spectrally invariant in $\mathcal{B}$ provided that:*

(a) *$\mathbb{C}e \oplus I$ is spectrally invariant in $\mathbb{C}e \oplus \mathcal{J}$; and*

(b) *$\varphi^{-1}((B/\mathcal{J})^{-1}) = (A/I)^{-1}$.*

3. Semi-ideals

We now turn to a method of constructing algebras closed under holomorphic functional calculus, or, more generally, Ψ^*-algebras. More precisely, in order to be able to deal with non-unital algebras, we need to study composition with possibly unbounded operators and the semi-ideals generated in this process.

3.1. Definitions

We now introduce semi-ideals.

Definition 3.1. A subspace $J \subseteq B$ of a unital algebra B is said to be a *semi-ideal* in B provided that we have $xby \in J$ for all $x, y \in J$ and all $b \in B$.

Remark 3.2.

(a) A left, right or two-sided ideal is obviously a semi-ideal.

(b) Let B be a unital algebra and $J \subsetneq B$ be a proper semi-ideal. Then we have $e \notin J$ and $J \cap B^{-1} = \emptyset$, because otherwise we would contradict the identities $b = ebe$ and $e = x(x^{-1})^2 x$.

Proposition 3.3. *Let $\mathcal{B}$ be a unital Banach algebra, let $\mathcal{A} \subseteq \mathcal{B}$ be closed under holomorphic functional calculus and let $J \subsetneq \mathcal{A}$ be a proper semi-ideal.*

(a) $0 \in \sigma_{\mathcal{B}}(x)$ *for all* $x \in J$.

(b) *J is closed under the holomorphic functional calculus in $\mathcal{B}$.*

Proof. The first statement is an immediate consequence of Remark 3.2 (b) and the spectral invariance of $\mathcal{A}$ in $\mathcal{B}$ by Remark 2.2 (c). Now, if $a = \lambda e + x \in \mathbb{C}e + J$ and $f \in \mathcal{O}(\sigma_{\mathcal{B}}(a))$ are arbitrary, then there exists $g \in \mathcal{O}(\sigma_{\mathcal{B}}(a))$ such that $f(z) = f(\lambda) + f'(\lambda)(z-\lambda) + (z-\lambda)^2 g(z)$. This gives $f(a) - f(\lambda)e = f'(\lambda)x + xg(a)x \in J$ and completes the proof. □

If $\mathcal{B}$ is, in addition, a C^*-algebra, we can assume, without loss of generality, that algebras closed under holomorphic functional calculus are symmetric. More precisely, we have the following result.

Lemma 3.4. *If $A \subseteq \mathcal{B}$ is closed under the holomorphic functional calculus in the C^*-algebra $\mathcal{B}$, then $A_* := \{a \in A : a^* \in A\}$ is symmetric and closed under the holomorphic functional calculus in $\mathcal{B}$.*

Proof. It is sufficient to note that we have $f(a)^* = f^*(a^*)$ if $f \in \mathcal{O}(\sigma_{\mathcal{B}}(a))$ and $[f^* : z \mapsto \overline{f(\bar{z})}] \in \mathcal{O}(\sigma_{\mathcal{B}}(a^*))$. □

3.2. A commutator method

We shall use several procedures to construct subalgebras closed under holomorphic functional calculus. The first one leads to Ψ^*-algebras using commutator methods, whereas the second one produces a semi-ideal, hence also an algebra closed under functional calculus. We begin by recalling the construction of submultiplicative Ψ^*-algebras using commutators with closed, symmetric operators. These techniques were first used in [**4**] for the characterization of pseudodifferential operators on $\mathbb{R}^n$, and later on investigated systematically in [**15**], for instance. Here we follow the presentations in [**15**, **21**]. Let us start with a description of the general setting.

Let $\mathcal{K}$ be a Hilbert space, $(\mathcal{A}, (\|\cdot\|_j)_{j\in\mathbb{N}})$ be a submultiplicative Ψ^*-algebra in $\mathcal{L}(\mathcal{K})$ with $\|\cdot\|_1 = \|\cdot\|_{\mathcal{L}(\mathcal{K})}$. Also, let $\mathfrak{T}$ be a finite set of densely defined, closed, symmetric operators $T : \mathcal{K} \supseteq \mathcal{D}(T) \to \mathcal{K}$. By [**21**, Lemma 2.17], each $T \in \mathfrak{T}$ induces a closed *-derivation

$$\delta_T : \mathcal{A} \supseteq \mathcal{D}(\delta_T) \to \mathcal{A}.$$

Here, we have $a \in \mathcal{D}(\delta_T)$ if and only if $a(\mathcal{D}(T)) \subseteq \mathcal{D}(T)$, there exists $\delta_T(a) \in \mathcal{A}$ with $\delta_T(a)\varphi = \mathrm{i}(Ta\varphi - aT\varphi)$ for all $\varphi \in \mathcal{D}(T)$, and the same is also true for a^*.

Furthermore, as shown in [**21**, § 2.2], the finite set $\{\delta_T : T \in \mathfrak{T}\}$ leads to the following scale of symmetric subalgebras of $\mathcal{A}$:

$$\begin{aligned}
&\Psi^0(\mathfrak{T}) := \mathcal{A},\\
&\Psi^1(\mathfrak{T}) := \bigcap_{T\in\mathfrak{T}} \mathcal{D}(\delta_T),\\
&\Psi^r(\mathfrak{T}) := \{a \in \Psi^{r-1}(\mathfrak{T}),\ \delta_T(a) \in \Psi^{r-1}(\mathfrak{T}) \text{ for all } T \in \mathfrak{T}\}, \quad r \geqslant 2.
\end{aligned}$$

Each of the algebras $\Psi^r(\mathfrak{T})$ is endowed with a system of semi-norms, namely,

$$q_{0,j}(a) := \|a\|_j \quad \text{for } a \in \Psi^0(\mathfrak{T}),\ j \in \mathbb{N},$$

and

$$q_{r,j}(a) := q_{r-1,j}(a) + \sum_{T\in\mathfrak{T}} q_{r-1,j}(\delta_T(a)) \quad \text{for } a \in \Psi^r(\mathfrak{T}),\ r, j \in \mathbb{N}.$$

Finally, we endow the algebra

$$\Psi^\infty(\mathfrak{T}) := \bigcap_{r=0}^{\infty} \Psi^r(\mathfrak{T})$$

with the system of semi-norms $(q_{r,j})_{r,j\in\mathbb{N}}$.

Similarly, the set $\mathfrak{T}$ induces the scale of so-called $\mathfrak{T}$-*Sobolev spaces* by

$$\begin{aligned}
&\mathcal{H}^0(\mathfrak{T}) := \mathcal{K},\\
&\mathcal{H}^1(\mathfrak{T}) := \bigcap_{T\in\mathfrak{T}} \mathcal{D}(T),\\
&\mathcal{H}^r(\mathfrak{T}) := \{x \in \mathcal{H}^{r-1}(\mathfrak{T}),\ Tx \in \mathcal{H}^{r-1}(\mathfrak{T}) \text{ for all } T \in \mathfrak{T}\}, \quad r \geqslant 2.
\end{aligned}$$

As above, the spaces $\mathcal{H}^r(\mathfrak{T})$ are endowed with the iterated graph norms with respect to the system $\mathfrak{T}$, i.e. $p_0(x) := \|x\|_{\mathcal{K}}$ and

$$p_r(x) := p_{r-1}(x) + \sum_{T\in\mathfrak{T}} p_{r-1}(Tx), \quad x \in \mathcal{H}^r(\mathfrak{T}), \quad r \geqslant 1.$$

The intersection

$$\mathcal{H}^\infty(\mathfrak{T}) := \bigcap_{r=0}^{\infty} \mathcal{H}^r(\mathfrak{T})$$

is endowed with the system of norms $(p_r)_{r\in\mathbb{N}}$.

The main properties of this construction are summarized in the next theorem. For a proof, see [**21**, Theorem 2.24], or [**15**, § 2] for the special case $\mathcal{A} = \mathcal{L}(\mathcal{K})$.

Theorem 3.5. *The algebra* $(\Psi^\infty(\mathfrak{T}), (q_{r,j}))$ *is a submultiplicative* Ψ^**-algebra in* $\mathcal{L}(\mathcal{K})$ *with* $\Psi^\infty(\mathfrak{T}) \subseteq \mathcal{A}$. *The* $\mathfrak{T}$*-Sobolev-spaces* $\mathcal{H}^r(\mathfrak{T})$ *are Hilbert spaces,* $\mathcal{H}^\infty(\mathfrak{T})$ *is a projective limit of a sequence of Hilbert spaces and, for each* $r \in \mathbb{N} \cup \{\infty\}$, *the natural map*

$$\Psi^r(\mathfrak{T}) \times \mathcal{H}^r(\mathfrak{T}) \to \mathcal{H}^r(\mathfrak{T}) : (a, \varphi) \mapsto a(\varphi)$$

is bilinear and continuous.

3.3. Commutators and semi-ideals

The second construction that we shall need associates to the algebra $\mathcal{A}$ and the system $\mathfrak{T}$ a semi-ideal in $\mathcal{L}(\mathcal{K})$. Indeed, let $\mathcal{J}_0(\mathfrak{T}) := \mathcal{A}$, and denote by $\mathcal{J}_1(\mathfrak{T})$ the space of all $x \in \mathcal{J}_0(\mathfrak{T})$ such that, for all $T, T_1, T_2 \in \mathfrak{T}$, we have the following.

(a) $x(\mathcal{K}) \subseteq \mathcal{D}(T)$ and $\omega_T^\ell(x) := Tx \in \mathcal{J}_0(\mathfrak{T})$.

(b) There exists $\omega_T^r(x) \in \mathcal{J}_0(\mathfrak{T})$ with $\omega_T^r(x)f = xTf$ for all $f \in \mathcal{D}(T)$.

(c) $x(\mathcal{K}) \subseteq \mathcal{D}(T_1)$, and there exists $\omega_{T_1,T_2}^{\ell,r}(x) \in \mathcal{J}_0(\mathfrak{T})$ with $\omega_{T_1,T_2}^{\ell,r}(x)f = T_1 x T_2 f$ for all $f \in \mathcal{D}(T_2)$.

Moreover, let $\mathcal{J}_{k+1}(\mathfrak{T})$ be the space of all $x \in \mathcal{J}_k(\mathfrak{T})$ such that we have $\omega_T^\ell(x)$, $\omega_T^r(x)$, $\omega_{T_1,T_2}^{\ell,r}(x) \in \mathcal{J}_k(\mathfrak{T})$ for all $T, T_1, T_2 \in \mathfrak{T}$.

We endow the spaces $\mathcal{J}_k(\mathfrak{T})$ with the following systems $(p_{j,k})_{j\in\mathbb{N}_0}$ of norms: let $p_{j,0}(x) := \|x\|_j$, for $x \in \mathcal{J}_0(\mathfrak{T})$, and let

$$p_{j,k+1}(x) := p_{j,k}(x) + \sum_{T\in\mathfrak{T}} (p_{j,k}(\omega_T^\ell(x)) + p_{j,k}(\omega_T^r(x))) + \sum_{T_1,T_2\in\mathfrak{T}} p_{j,k}(\omega_{T_1,T_2}^{\ell,r}(x)),$$

for $x \in \mathcal{J}_{k+1}(\mathfrak{T})$. Moreover, the projective limit

$$\mathcal{J}_\infty(\mathfrak{T}) := \bigcap_{k=0}^{\infty} \mathcal{J}_k(\mathfrak{T})$$

is endowed with the projective topology given by the system of norms $(p_{j,k})_{j,k\in\mathbb{N}_0}$.

Let us collect the main properties of this construction in the following theorem.

Theorem 3.6. *Let* $(\mathcal{A}, (\|\cdot\|_j)_{j\in\mathbb{N}})$ *be a submultiplicative* Ψ^**-algebra. Then, for* $k \in \mathbb{N}_0 \cup \{\infty\}$, *we have the following.*

(a) $(\mathcal{J}_k(\mathfrak{T}), (p_{j,k}))$ *is a submultiplicative Fréchet algebra. The canonical embedding* $\mathcal{J}_k(\mathfrak{T}) \hookrightarrow \mathcal{J}_0(\mathfrak{T})$ *is continuous.*

(b) $\Psi^\infty(\mathfrak{T})\mathcal{J}_k(\mathfrak{T}) \subseteq \mathcal{J}_k(\mathfrak{T})$ *and* $\mathcal{J}_k(\mathfrak{T})\Psi^\infty(\mathfrak{T}) \subseteq \mathcal{J}_k(\mathfrak{T})$; *the two canonical bilinear maps*

$$\Psi^\infty(\mathfrak{T}) \times \mathcal{J}_k(\mathfrak{T}) \to \mathcal{J}_k(\mathfrak{T}) \quad \text{and} \quad \mathcal{J}_k(\mathfrak{T}) \times \Psi^\infty(\mathfrak{T}) \to \mathcal{J}_k(\mathfrak{T})$$

are jointly continuous.

(c) *$\mathcal{J}_k(\mathfrak{T})$ is a semi-ideal in the Ψ^*-algebra $\mathcal{A}$; in particular, $\mathcal{J}_k(\mathfrak{T})$ is closed under the holomorphic functional calculus in $\mathcal{L}(H)$. Moreover, the canonical map*

$$\mathcal{J}_k(\mathfrak{T}) \times \mathcal{A} \times \mathcal{J}_k(\mathfrak{T}) \to \mathcal{J}_k(\mathfrak{T}) \tag{3.1}$$

is jointly continuous.

Proof. The proofs are by induction with respect to k. Since the arguments for the steps from k to $k+1$ are the same as for $k=1$, the proofs of the steps are omitted.

For (a), let us first assume that we have $x, y \in \mathcal{J}_1(\mathfrak{T})$. Then we have $xy \in \mathcal{J}_1(\mathfrak{T})$, with

$$\omega_T^\ell(xy) = \omega_T^\ell(x)y, \omega_T^r(xy) = x\omega_T^r(y) \quad \text{and} \quad \omega_{T_1,T_2}^{\ell,r}(xy) = \omega_{T_1}^\ell(x)\omega_{T_2}^r(y)$$

for all $T, T_1, T_2 \in \mathfrak{T}$; the submultiplicativity is now immediate whereas for the completeness of $\mathcal{J}_1(\mathfrak{T})$ we have to use the closedness of the operators $T \in \mathfrak{T}$. For (b), note that, for $a \in \Psi^\infty(\mathfrak{T})$ and $x \in \mathcal{J}_1(\mathfrak{T})$, a straightforward computation gives $ax \in \mathcal{J}_1(\mathfrak{T})$ and $xa \in \mathcal{J}_1(\mathfrak{T})$, with

$$\begin{aligned}
\omega_T^\ell(ax) &= a\omega_T^\ell(x) - \mathrm{i}\delta_T(a)x,\\
\omega_T^r(ax) &= a\omega_T^r(x),\\
\omega_{T_1,T_2}^{\ell,r}(ax) &= a\omega_{T_1,T_2}^{\ell,r}(x) - \mathrm{i}\delta_{T_1}(a)\omega_{T_2}^r(x),\\
\omega_T^\ell(xa) &= \omega_T^\ell(x)a,\\
\omega_T^r(xa) &= \omega_T^r(x)a - \mathrm{i}x\delta_T(a),\\
\omega_{T_1,T_2}^{\ell,r}(xa) &= \omega_{T_1,T_2}^{\ell,r}(x)a + \mathrm{i}\omega_{T_1}^\ell(x)\delta_{T_2}(a)
\end{aligned}$$

for all $T, T_1, T_2 \in \mathfrak{T}$.

For the semi-ideal property (c), it suffices to note that, for $a \in \mathcal{A}$ and $x, y \in \mathcal{J}_k(\mathfrak{T})$, we have

$$\omega_T^\ell(xay) = \omega_T^\ell(x)ay, \quad \omega_T^r(xay) = xa\omega_T^r(y) \quad \text{and} \quad \omega_{T_1,T_2}^{\ell,r}(xay) = \omega_{T_1}^\ell(x)a\omega_{T_2}^r(xy)$$

for all $T, T_1, T_2 \in \mathfrak{T}$, which gives the joint continuity of (3.1) as well. □

Remark 3.7. Note that it is not clear, and in general not true, that the spaces $\mathcal{J}_k(\mathfrak{T})$ are symmetric subspaces of $\mathcal{L}(\mathcal{K})$. However, we easily obtain this property by considering the spaces

$$\mathcal{J}_k(\mathfrak{T})_* := \{x \in \mathcal{J}_k(\mathfrak{T}) : x^* \in \mathcal{J}_k(\mathfrak{T})\}. \tag{3.2}$$

It is straightforward to check that Theorem 3.6 also remains true for the smaller spaces $\mathcal{J}_k(\mathfrak{T})_*$. By a slight abuse of notation, we will sometimes write $\mathcal{J}_k(\mathfrak{T})$ for the spaces $\mathcal{J}_k(\mathfrak{T})_*$.

Corollary 3.8. *We have that $\mathcal{J}_m(\mathfrak{T})$ is a non-unital Ψ^*-algebra, for any $m \in \mathbb{N} \cup \{\infty\}$.*

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4. An extended pseudodifferential calculus

Recall that the notion of a continuous family groupoid defined in [**41**] generalizes that of $\mathcal{C}^{\infty,0}$-foliations as considered in [**7**]. More precisely, a *continuous family groupoid* is a locally compact topological groupoid such that $\mathcal{G}$ is covered by some open subsets Ω such that the following conditions are satisfied.

(1) Each chart Ω is homeomorphic to two open subsets of $\mathbb{R}^k \times \mathcal{G}^{(0)}$, $T_d \times U_d$ and $T_r \times U_r$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} & T_r \times U_r & \xleftarrow{\simeq} & \Omega & \xrightarrow{\simeq} & T_d \times U_d & \\ \swarrow & & \swarrow{\scriptstyle r} & & \searrow{\scriptstyle d} & & \searrow \\ U_r & \xleftarrow{=} & r(\Omega) & & d(\Omega) & \xrightarrow{=} & U_d \end{array} \tag{4.1}$$

(2) Each coordinate change (for r, respectively d) is given by $(t, u) \mapsto (\phi(t, u), u)$ where ϕ is of class $\mathcal{C}^{\infty,0}$, i.e. $u \mapsto \phi(\cdot, u)$ is a continuous map from U_* to $\mathcal{C}^\infty(T_*, T'_*)$, $* = d, r$.

In addition, one requires that the composition and the inversion be $\mathcal{C}^{\infty,0}$ morphisms [**41**]. For the sake of simplicity, we will always assume that the space $M := \mathcal{G}^{(0)}$ of units is compact and our groupoid $\mathcal{G}$ is Hausdorff. (Recall that M is always Hausdorff.)

4.1. Groupoid algebras

To any continuous family groupoid $\mathcal{G}$, there is associated an algebra of pseudodifferential operators: namely, let $\Psi^{m,0}(\mathcal{G})$ be the space of continuous, uniformly supported, invariant families of pseudodifferential operators $(P_x)_{x\in M}$ on the fibres of the groupoid (see [**25**]). For later purposes, note that this implies in particular the following: let $\Omega \cong T \times U$ be a chart as in (4.1) and $\varphi \in \mathcal{C}^{\infty,0}_{\mathrm{c}}(\Omega)$. Then there exists $p \in \mathcal{C}_{\mathrm{c}}(U, S^m(T; \mathbb{R}^k_\eta))$ such that for each $x \in U$, $\varphi P_x \varphi$ corresponds to the pseudodifferential operator $p(x, y, D_y)$ on T.

It is convenient to know that as in [**16**] we obtain a natural Fréchet topology on each of the spaces $\Psi^{m,0}(\mathcal{G})$ such that $\Psi^{0,0}(\mathcal{G})$ is a Fréchet algebra and $\Psi^{-\infty,0}(\mathcal{G})$ is a closed ideal (see also [**20**, **29**]).

In this section, we are going to show how the methods of the previous section can be used to construct algebras closed under holomorphic functional calculus that contain the algebras $\Psi^{-\infty,0}(\mathcal{G})$ and $\Psi^{0,0}(\mathcal{G})$ of pseudodifferential operators on the groupoid and share some of their algebraic and analytic properties. As it might be expected, it is difficult to construct, in general, algebras closed under holomorphic functional calculus that retain all geometric properties of the given groupoid. Nevertheless, we do construct algebras closed under holomorphic functional calculus that retain at least some of the geometric properties of the groupoid.

First, let us fix some notation. Throughout this section, $\mathcal{K}_x := L^2(\mathcal{G}_x; r^*\mathcal{D}^{1/2})$ stands for the completion of the space $\mathcal{C}_c^\infty(\mathcal{G}_x; r^*\mathcal{D}^{1/2})$ with respect to the sesquilinear pairing

$$(f, g)_x := \int_{\mathcal{G}_x} f(\gamma)\overline{g(\gamma)}.$$

Moreover, the Hilbert spaces $\mathcal{K}_x$, $x \in M$, can be glued together to a new, big Hilbert space $\mathcal{H}$ containing all necessary information for us.

To be more precise, fix a positive, faithful measure ν on M. Then note that each $f \in \mathcal{C}_c^{\infty,0}(\mathcal{G}; r^*\mathcal{D}^{1/2})$ induces a section

$$\tilde{f} : M \to \prod_{x \in M} \mathcal{K}_x : x \mapsto f|_{\mathcal{G}_x} \in \mathcal{C}_c^\infty(\mathcal{G}_x; r^*\mathcal{D}^{1/2}) \subseteq \mathcal{K}_x,$$

where, as usual, 'section' simply means $\tilde{f}(x) \in \mathcal{K}_x$ for all $x \in M$. Now, for any $f, g \in \mathcal{C}_c^{\infty,0}(\mathcal{G}; r^*\mathcal{D}^{1/2})$, the function $M \ni x \mapsto (\tilde{f}(x), \tilde{g}(x))_x$ is continuous, hence ν-measurable. Consequently, the set $\mathfrak{M}$ of all sections

$$h : M \to \prod_{x \in M} \mathcal{K}_x$$

such that the map $M \ni x \mapsto (\tilde{f}(x), h(x))_x$ is ν-measurable for all $f \in \mathcal{C}_c^{\infty,0}(\mathcal{G}, r^*\mathcal{D}^{1/2})$ induces the structure of a ν-measurable field of Hilbert spaces on the family $(\mathcal{K}_x)_{x \in M}$ [**10**, Definition 2.1.3.1, Proposition 2.1.4.4]. The set $\mathcal{H}$ of all $h \in \mathfrak{M}$ satisfying

$$\int_M \|h(x)\|^2_{\mathcal{K}_x} \, \mathrm{d}\nu(x) < \infty$$

is in fact a Hilbert space and we write $\mathcal{H} =: \int_M^\oplus \mathcal{K}_x \, \mathrm{d}\nu(x)$. The space $\mathcal{C}_c^{\infty,0}(\mathcal{G}; r^*\mathcal{D}^{1/2})$ is then a dense subspace of $\mathcal{H}$.

Recall that an operator $P \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* provided there exists a family $(\hat{P}(x))_{x \in M}$ of operators $\hat{P}(x) \in \mathcal{L}(H_x)$ such that, for any $h \in \mathfrak{M}$,

$$(Ph)(x) = \hat{P}(x)h(x)$$

almost everywhere and $[x \mapsto \|\hat{P}(x)\|_{\mathcal{L}(\mathcal{K}x)}] \in L^\infty(M; \nu)$ [**10**, Definition 2.2.3.2]. As usual, we write in that case $P = \int_M^\oplus \hat{P}(x) \, \mathrm{d}\nu(x)$. A straightforward computation gives [**10**, Proposition 2.2.3.2]

$$\|P\|_{\mathcal{L}(\mathcal{H})} = \operatorname{ess\,sup}_x \|\hat{P}(x)\|_{\mathcal{L}(\mathcal{K}x)}.$$

The set of all decomposable operators is in fact a C^*-subalgebra, which we denote by $\mathcal{L}_D(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$.

Finally, for any $\gamma \in \mathcal{G}$, the operators

$$U_\gamma : \mathcal{C}_c^\infty(\mathcal{G}_{d(\gamma)}; r^*\mathcal{D}^{1/2}) \to \mathcal{C}_c^\infty(\mathcal{G}_{r(\gamma)}; r^*\mathcal{D}^{1/2}), \quad U_\gamma(f)(\gamma') = f(\gamma'\gamma),$$

extend by continuity to isometric isomorphisms $U_\gamma : \mathcal{K}_{d(\gamma)} \to \mathcal{K}_{r(\gamma)}$, and hence induce an action of the groupoid $\mathcal{G}$ on $\mathcal{H}$. Decomposable operators $P \in \mathcal{L}_D(\mathcal{H})$ with $\hat{P}(r(\gamma))U_\gamma =$

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$U_\gamma \hat{P}(d(\gamma))$ for all $\gamma \in \mathcal{G}$ are called *invariant* with respect to the action of the groupoid, and we denote the C^*-subalgebra of all invariant operators by $\mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$.

By the results of [**25**, § 3], the regular representations π_x, $x \in M$, of the algebra $\Psi^{0,0}(\mathcal{G})$ fit together to a faithful *-representation

$$\pi_r : \Psi^{0,0}(\mathcal{G}) \to \mathcal{L}_D^{\mathcal{G}}(\mathcal{H}) : \pi_r(P) = \int_M^{\oplus} \pi_x(P)\, \mathrm{d}\nu(x)$$

of unital algebras. In the sequel, we will identify $\Psi^{0,0}(\mathcal{G})$ with its image under π_r and construct subalgebras $A \subseteq \mathcal{L}(\mathcal{H})$ closed under holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$ and containing $\Psi^{0,0}(\mathcal{G})$ as a subalgebra. Because of Lemma 3.4 and the following lemma, we can always assume that the algebras A are symmetric subalgebras of $\mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$.

Lemma 4.1. *Let $A \subseteq \mathcal{L}_D(\mathcal{H})$ be closed under holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$. Then $A^{\mathcal{G}} := A \cap \mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$ is also $\mathcal{G}$-invariant and closed under holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$.*

Proof. This follows because $\mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$. □

As a first step towards constructing algebras closed under holomorphic functional calculus, we are going to show that we can reduce the problem of finding such algebras A essentially to the construction of algebras that contain $\Psi^{-\infty,0}(\mathcal{G})$ and are closed under the holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$. Indeed, let

$$\mathcal{J} := C_r^*(\mathcal{G}) = \overline{\Psi^{-\infty,0}(\mathcal{G})}^{\mathcal{L}(\mathcal{H})},$$
$$\mathcal{B} := \mathfrak{A}_r(\mathcal{G}) = \overline{\Psi^{0,0}(\mathcal{G})}^{\mathcal{L}(\mathcal{H})},$$

and suppose that we have a subspace $I = I^* \subseteq \mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$ with the following properties:

$$\Psi^{-\infty,0}(\mathcal{G}) \subseteq I \subseteq \mathcal{J}, \tag{4.2}$$
$$I \text{ is a } \Psi^{0,0}(\mathcal{G})\text{-left and -right module} \tag{4.3}$$

and

$$I \text{ has property } (P_I) \text{ in } \mathcal{B}. \tag{4.4}$$

Theorem 4.2. *Let I be as in Equations (4.2), (4.3) above and $A := \Psi^{0,0}(\mathcal{G}) + I \subseteq \mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$. Then we have*

$$(\mathbb{C}\,\mathrm{id}_{\mathcal{H}} + A) \cap \mathcal{L}(\mathcal{H})^{-1} = (\mathbb{C}\,\mathrm{id}_{\mathcal{H}} + A)^{-1}. \tag{4.5}$$

In particular, if there exists a Fréchet topology on I making the $\Psi^{0,0}(\mathcal{G})$-module action as well as the embedding $I \hookrightarrow \mathcal{L}(\mathcal{H})$ continuous, then A is closed under the holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$ and is a Ψ^*-algebra containing $\Psi^{0,0}(\mathcal{G})$.

Proof. Because of (4.3), the space $I \subseteq A$ is an ideal in the algebra A.

By Lemma 2.6 and Theorem 2.8 (a), it suffices to prove that A/I has property $(\tilde{P}_{A/I})$ in $\mathcal{B}/\mathcal{J}$. So, let $\bar{a} = a + I \in A/I$ be such that $\varphi(\bar{a}) \in (\mathcal{B}/\mathcal{J})^{-1}$. Without loss of generality, we may assume $a \in \Psi^{0,0}(\mathcal{G})$. On the other hand, the homogeneous principal symbol map σ_0 induces an isomorphism $\hat{\sigma}_0 : \mathcal{B}/\mathcal{J} \to \mathcal{C}(S^*(\mathcal{G}))$, with $\sigma_0(a) \in \mathcal{C}^{\infty,0}(S^*(\mathcal{G}))^{-1}$ (recall that M is assumed to be compact, and hence $S^*(\mathcal{G})$ is also compact). Using the exactness of the sequence

$$0 \to \Psi^{m-1,0}(\mathcal{G}; E) \to \Psi^{m,0}(\mathcal{G}; E) \xrightarrow{\sigma_m} \mathcal{C}_{\mathrm{c}}^{\infty,0}(S^*(\mathcal{G}), \mathrm{End}(E) \otimes \mathcal{P}_m) \to 0 \tag{4.6}$$

and the asymptotic completeness of $\Psi^{0,0}(\mathcal{G})$ together with a formal Neumann series, we obtain $b \in \Psi^{0,0}(\mathcal{G})$ with

$$\mathrm{id}_{\mathcal{H}} - ab \in \Psi^{-\infty,0}(\mathcal{G}), \qquad \mathrm{id}_{\mathcal{H}} - ba \in \Psi^{-\infty,0}(\mathcal{G}).$$

Thus, $\bar{a}$ is invertible in $\Psi^{0,0}(\mathcal{G})/\Psi^{-\infty,0}(\mathcal{G})$, hence also in $\Psi^{0,0}(\mathcal{G})/(I \cap \Psi^{0,0}(\mathcal{G}))$ because of $\Psi^{-\infty,0}(\mathcal{G}) \subseteq I \cap \Psi^{0,0}(\mathcal{G})$. This gives the property $(\tilde{P}_{A/I})$, and completes the proof of (4.5).

As for the supplement it suffices to endow the algebra A with the quotient topology induced from $\Psi^{0,0}(\mathcal{G}) \oplus I$. The rest is clear. □

The following lemma enables us to apply the methods developed in the previous section to pseudodifferential operators on the groupoid $\mathcal{G}$.

Lemma 4.3. *Let $T \in \Psi^{m,0}(\mathcal{G}; \mathcal{D}^{1/2})$ be arbitrary. Then the unbounded operator*

$$T : \mathcal{H} \supseteq \mathcal{C}_{\mathrm{c}}^{\infty}(\mathcal{G}; r^*\mathcal{D}^{1/2}) \to \mathcal{H}$$

is closable.

Moreover, if T equals its formal adjoint $T^* \in \Psi^{m,0}(\mathcal{G}; \mathcal{D}^{1/2})$, then the closure of T is symmetric.

For notational simplicity, we shall not distinguish between $T \in \Psi^{m,0}(\mathcal{G}; \mathcal{D}^{1/2})$ and its minimal closed extension $\bar{T} : \mathcal{H} \supseteq \mathcal{D}(\bar{T}) \to \mathcal{H}$ in the sequel.

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{C}_{\mathrm{c}}^{\infty}(\mathcal{G}; r^*\mathcal{D}^{1/2})$ with $f_j \to 0$ in $\mathcal{H}$ and $Tf_j \to f$ in $\mathcal{H}$ for some $f \in \mathcal{H}$. We have to show $f = 0$. Let $T^* \in \Psi^{m,0}(\mathcal{G}; \mathcal{D}^{1/2})$ be the formal adjoint of T. Then we obtain, for all $h \in \mathcal{C}_{\mathrm{c}}^{\infty,0}(\mathcal{G}, r^*\mathcal{D}^{1/2})$

$$\begin{aligned}
\langle f, h\rangle_{\mathcal{H}} &= \lim \langle Tf_j, h\rangle_{\mathcal{H}} \\
&= \lim \int_M \langle T_x f_j|_{\mathcal{G}_x}, h|_{\mathcal{G}_x}\rangle_{\mathcal{K}_x}\, \mathrm{d}\nu(x) \\
&= \lim \int_M \langle f_j|_{\mathcal{G}_x}, T_x^* h|_{\mathcal{G}_x}\rangle_{\mathcal{K}_x}\, \mathrm{d}\nu(x) \\
&= \lim \langle f_j, T^* h\rangle_{\mathcal{H}} = 0.
\end{aligned}$$

Since $\mathcal{C}_{\mathrm{c}}^{\infty,0}(\mathcal{G}, r^*\mathcal{D}^{1/2})$ is dense in $\mathcal{H}$, this gives $f = 0$. The rest is clear. □

Let us briefly outline how the operator theoretic methods from the previous sections are used in the following two subsections to embed the algebra $\Psi^{0,0}(\mathcal{G})$ of pseudodifferential operators of order 0 in an algebra that is closed under holomorphic functional calculus.

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In § 4.2 we construct a submultiplicative Ψ^*-algebra $\mathcal{A}_0 \subset \mathcal{L}(\mathcal{H})$ with $\Psi^{0,0}(\mathcal{G}) \subseteq \mathcal{A}_0$ by using a variant of the commutator methods described in Theorem 3.5. Our construction will be such that $\mathcal{A}_0 \subseteq \mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$ and $\Psi^{0,0}(\mathcal{G})$ are dense in $\mathcal{A}_0$ [**21**, Corollary 2.5]. Moreover, elements in $\mathcal{A}_0$ are equivariant families of operators that admit locally a pseudodifferential presentation with a continuous family of symbols. In a second step, we take in § 4.3 a finite set $\mathfrak{T} \subseteq \Psi^{1,0}(\mathcal{G}; \mathcal{D}^{1/2})$ of formally self-adjoint first order pseudodifferential operators and construct the semi-ideal $\mathcal{J}_\infty(\mathfrak{T})$ and the Ψ^*-algebra $\Psi^\infty(\mathfrak{T})$ corresponding to $\mathcal{A}_0$ and $\mathfrak{T}$ as in Theorem 3.6. By Theorem 1 of [**25**] we then have $\Psi^{-\infty,0}(\mathcal{G}) \subseteq \mathcal{J}_\infty(\mathfrak{T})$ and $\Psi^{0,0}(\mathcal{G}) \subseteq \Psi^\infty(\mathfrak{T})$, hence (4.2), (4.3) and (4.4) hold for the symmetrized semi-ideal $I := \mathcal{J}_\infty(\mathfrak{T})_*$ by Theorem 3.6, thus $\mathcal{A}_1 := \Psi^{0,0}(\mathcal{G}) + \mathcal{J}_\infty(\mathfrak{T})_*$ is a (submultiplicative) Ψ^*-algebra containing $\Psi^{0,0}(\mathcal{G})$ by Theorem 4.2. Again by [**21**, Corollary 2.5], we can even assume that $\Psi^{0,0}(\mathcal{G})$ is dense in $\mathcal{A}_1$. More properties of the algebra $\mathcal{A}_1$ are listed in Proposition 4.8

The algebra $\mathcal{A}_1$ clearly depends on several choices. To finally complete the construction we take the intersection over all possible choices involved in the construction of the algebra $\mathcal{A}_1$, and obtain an algebra $\mathcal{A}_\infty$ that is independent of any choices, is closed under holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$, and contains $\Psi^{0,0}(\mathcal{G})$.

It is worth stressing that the construction is not limited to finite sets $\mathfrak{T} \subset \Psi^{1,0}(\mathcal{G}; \mathcal{D}^{1/2})$. Indeed, since the arbitrary intersection of algebras closed under functional calculus in $\mathcal{L}(\mathcal{H})$ is still closed under holomorphic functional calculus in $\mathcal{L}(\mathcal{H})$, we can admit also countable subsets $\mathfrak{T} \subseteq \Psi^{1,0}(\mathcal{G}; \mathcal{D}^{1/2})$ for our construction. This can for instance be used for non-compact situations.

It remains to construct the algebras $\mathcal{A}_0$ and $\mathcal{J}_\infty(\mathfrak{T})_*$.

4.2. Construction of the algebra $\mathcal{A}_0$

Let $\chi : \Omega \xrightarrow{\cong} U \times T \subseteq M \times \mathbb{R}^k_y$ be a chart of $\mathcal{G}$ as in (4.1), and $\varphi_0, \psi_0 \in \mathcal{C}_c^{\infty,0}(\Omega)$ be arbitrary. We define the *local symbol* of $a \in \mathcal{L}_D(\mathcal{H})$ on Ω with respect to φ_0 and ψ_0 by

$$\sigma_\Omega(a; \varphi_0, \psi_0)(x, y, \eta) := \mathrm{e}^{-\mathrm{i}y\eta} \psi_0(x, y) \hat{a}(x)[y' \mapsto \varphi_0(x, y')\mathrm{e}^{\mathrm{i}y'\eta}](y)$$

for $x \in U$, $y \in T$, and $\eta \in \mathbb{R}^k_\eta$. Thus, we get

$$\begin{aligned} \sup_{x,\eta} \int_T |\sigma_\Omega(a; \varphi_0, \psi_0)(x, y, \eta)|^2 \, \mathrm{d}y &\leqslant c(\varphi_0, \psi_0) \sup_x \|\hat{a}(x)\|^2_{\mathcal{L}(\mathcal{K}x)} \\ &= c(\varphi_0, \psi_0) \|a\|^2_{\mathcal{L}_D(\mathcal{H})}. \end{aligned} \tag{4.7}$$

Since the operators $\mathrm{i}\partial_{y_s}$, $\mathrm{i}y_r\partial_{y_s}$, $r \neq s$, $\mathrm{i}y_s\partial y_s + \frac{1}{2}\mathrm{i}$ and M_{y_r} can be realized as closed symmetric operators on $\mathcal{H}$, we can consider the corresponding submultiplicative Ψ^*-algebra $\tilde{\mathcal{A}}_\Omega$ with respect to the C^*-algebra $\mathcal{L}_D(\mathcal{H})$ as in Theorem 3.5. A straightforward computation then yields $\Psi^{0,0}(\mathcal{G}) \subseteq \tilde{\mathcal{A}}_\Omega$. Observe that for $a \in \tilde{\mathcal{A}}_\Omega$, the derivatives $\eta_r\partial_{\eta_s}\sigma_\Omega(a; \varphi_0, \psi_0)$ and $\partial_{y_s}\sigma_\Omega(a; \varphi_0, \psi_0)$ of the local symbol of a can be realized as the local symbol of the commutator of a with one of the operators mentioned above, hence they satisfy an L^2 estimate similar to (4.7). As in [**6**] or [**22**], an application of Sobolev's embedding theorem leads to the following estimate.

Lemma 4.4. *Let $\alpha, \beta \in \mathbb{N}^n$ be arbitrary. Then there exists a continuous semi-norm $q_{\alpha,\beta}$ on $\tilde{\mathcal{A}}_\Omega$ such that*

$$\sup_{x,y,\eta} |\langle\eta\rangle^{|\beta|}\partial_\eta^\beta \partial_y^\alpha \sigma_\Omega(a;\varphi_0,\psi_0)(x,y,\eta)| \leqslant q_{\alpha,\beta}(a).$$

Let $\mathcal{A}_\Omega$ be the closure of $\Psi^{0,0}(\mathcal{G})$ in the submultiplicative Ψ^*-algebra $\tilde{\mathcal{A}}_\Omega \cap \mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$. By [**21**, Corollary 2.5], $\mathcal{A}_\Omega$ is a Ψ^*-algebra in $\mathcal{L}_D^{\mathcal{G}}(\mathcal{H})$, and $\Psi^{0,0}(\mathcal{G})$ is dense in $\mathcal{A}_\Omega$. Note that the local symbols of elements in $\mathcal{A}_\Omega$ are in the variable x locally uniform limits of symbols in $\mathcal{C}(U, S^0(T;\mathbb{R}^k_\eta))$, hence they remain continuous in x, which gives the following proposition.

Proposition 4.5. *Let $a \in \mathcal{A}_\Omega$ be arbitrary and $\chi : \Omega \xrightarrow{\cong} U \times T$ be as above. Then for all $\varphi, \psi \in \mathcal{C}_c^\infty(\Omega)$ there exists a symbol $\sigma_\Omega(a;\varphi,\psi) \in \mathcal{C}(U, S^0(T;\mathbb{R}^k_\eta))$ such that we have*

$$\chi_*(\psi a \varphi) = \sigma_\Omega(a;\varphi,\psi)(x,y,D_y). \tag{4.8}$$

Choosing a (countable) cover $\mathcal{G} = \bigcup_{\Omega\in\mathcal{V}} \Omega$ of $\mathcal{G}$ by open charts Ω as above, we can define $\mathcal{A}_0$ to be the closure of $\Psi^{0,0}(\mathcal{G})$ in the submultiplicative Ψ^*-algebra $\bigcap_{\Omega\in\mathcal{V}} \mathcal{A}_\Omega$. Then $\Psi^{0,0}(\mathcal{G})$ is dense in $\mathcal{A}_0$, and each $a \in \mathcal{A}_0$ has a representation (4.8) with respect to a symbol $\sigma_\Omega(a;\varphi,\psi) \in \mathcal{C}(U, S^0(T;\mathbb{R}^n_\eta))$.

As explained above, in a next step we consider now the semi-ideal $\mathcal{J}_\infty(\mathfrak{T})$.

4.3. Construction of the semi-ideal $\mathcal{J}_\infty(\mathfrak{T})$

The question of which properties can actually be obtained by choosing the set $\mathfrak{T}$ of closed, symmetric operators appropriately is more complicated because it includes in particular the analysis of pseudodifferential operators on non-compact manifolds, hence, we will be rather short at this point, and sketch only what is within reach.

Choose an at-most-countable set of sections $S \in \mathcal{C}^\infty(M, A(\mathcal{G}))$, $S \in \mathfrak{S}$, such that for each $x \in M$ the set $\{S(x) \in T_x\mathcal{G}_x : S \in \mathfrak{S}_x\}$ generates $T_x\mathcal{G}_x$ as a real vector space for some finite subset $\mathfrak{S}_x \subseteq \mathfrak{S}$, and let $\mathfrak{T} \subseteq \Psi^{1,0}(\mathcal{G};\mathcal{D}^{1/2})$ be the corresponding set of right-invariant, d-vertical vector fields. After multiplying them by i, the elements in $\mathfrak{T}$ have symmetric, minimal closed extensions by Lemma 4.3. For simplicity, let us assume that $\mathfrak{T}$ is finite; otherwise, we have to consider the projective limit of the corresponding semi-ideals with respect to an increasing sequence $\mathfrak{T}_j \subseteq \mathfrak{T}_{j+1} \subseteq \mathfrak{T}$. Let $\mathcal{H}^m(\mathfrak{T})$, $m \in \mathbb{N}_0$, be the corresponding scale of $\mathfrak{T}$-Sobolev spaces. Note that the spaces $\mathcal{H}^m(\mathfrak{T})$ have a decomposition as a direct integral of the form

$$\mathcal{H}^m(\mathfrak{T}) = \int_M^\oplus H^m_{\mathfrak{T}_x}(\mathcal{G}_x, r^*\mathcal{D}^{1/2})\,\mathrm{d}\nu(x),$$

where $H^m_{\mathfrak{T}_x}(\mathcal{G}_x, r^*\mathcal{D}^{1/2})$ is the Sobolev space of order $m \in \mathbb{N}_0$ associated to the vector fields $\mathfrak{T}_x := \mathfrak{T}|_{\mathcal{G}_x} \subseteq \Psi^{1,0}(\mathcal{G};\mathcal{D}^{1/2})$. We extend these scales of Sobolev spaces by duality to $m \in \mathbb{Z}$, then the following result follows immediately from the definition of the ideal $\mathcal{J}_\infty(\mathfrak{T})_*$.

Lemma 4.6. *Let $a \in \mathcal{J}_\infty(\mathfrak{T})_*$ be arbitrary. Then a induces for each $m \in \mathbb{N}$ and each $x \in M$ bounded operators $a : \mathcal{H}^{-m}(\mathfrak{T}) \to \mathcal{H}^m(\mathfrak{T})$ and*

$$\hat{a}(x) : H^{-m}_{\mathfrak{T}_x}(\mathcal{G}_x, r^*\mathcal{D}^{1/2}) \to H^m_{\mathfrak{T}_x}(\mathcal{G}_x, r^*\mathcal{D}^{1/2})$$

with locally uniform estimates in the transverse parameter x.

Consequently, $\hat{a}(x)$ has a smooth kernel provided Sobolev's embedding

$$H^m_{\mathfrak{T}_x}(\mathcal{G}_x, r^*\mathcal{D}^{1/2}) \hookrightarrow \mathcal{C}_b(\mathcal{G}_x, r^*\mathcal{D}^{1/2}) \tag{4.9}$$

holds for $2m > \dim \mathcal{G}_x$. Since everything takes place within the algebra $\mathcal{A}_0$ with locally uniform estimates in the transverse parameter, the smooth kernel depends in addition continuously on the transverse parameter x.

Note that (4.9) holds for instance if the manifolds $\mathcal{G}_x$ are of bounded geometry [**45**, Appendix]. For the question, when the groupoid is of bounded geometry we refer to [**1**]. Let us denote by $\mathcal{C}^\infty_b(\mathcal{G}_x \times \mathcal{G}_x, \mathrm{END}(\mathcal{D}^{1/2}))$ the space of all smooth sections $\mathcal{G}_x \times \mathcal{G}_x \to \mathrm{END}(\mathcal{D}^{1/2})$ that are uniformly bounded, as are all their covariant derivatives. A combination of Lemma 4.6 and (4.9) then yields a characterization of the Schwartz kernels of the operators $\hat{a}(x)$.

Proposition 4.7. *Suppose that the manifolds $\mathcal{G}_x$ are of bounded geometry for all $x \in M$, and let $a \in \mathcal{J}_\infty(\mathfrak{T})_*$ be arbitrary. Then $\hat{a}(x) \in \mathcal{L}(\mathcal{K}x)$ is an operator with $\mathcal{C}^\infty$-kernel $k_{\hat{a}}(x, \cdot, \cdot) \in \mathcal{C}^\infty_b(\mathcal{G}_x \times \mathcal{G}_x, \mathrm{END}(\mathcal{D}^{1/2}))$.*

We summarize the results of the above discussion in the following proposition.

Proposition 4.8. *Suppose that for any $x \in M$, the fibre $\mathcal{G}_x$ is a manifold of bounded geometry. Then there exists a Ψ^*-algebra $\mathcal{A}_1$ containing $\Psi^{0,0}(\mathcal{G})$ as a dense subalgebra such that each $P \in \mathcal{A}_1$ is given by a $\mathcal{G}$-invariant family $(P_x)_{x \in M}$ of pseudodifferential operators P_x on $\mathcal{G}_x$.*

Note that the definition of the Ψ^*-algebra $\mathcal{A}_1$ depends on many choices. As explained before in § 4.2 we obtain an algebra $\mathcal{A}_\infty$ independent of choices but still closed under functional calculus by considering intersections over all possible choices.

Proof. This follows directly from the plan for the construction of $\mathcal{A}_1$ after Lemma 4.3, Proposition 4.5 and Proposition 4.8. □

See also [**44**] for some related results.

Remark 4.9. The 'commutator method' presented here may be replaced by an equivalent method using an elliptic, positive-order operator in place of a family of vector fields (see [**48**, § 4.2]).

The above results provide us with Ψ^*-algebras that are useful in practice, because they consist of pseudodifferential operators. These algebras will necessarily contain operators that are not properly supported (unless our manifold is compact without corners). Nevertheless, these algebras consist of bounded operators, so their Schwartz kernels must satisfy some decay conditions far from the diagonal. It is difficult in general to quantize these decay conditions. One possibility is to consider commutators with functions that approximate the distance function as in [**18**] or [**45**]. If the groupoid allows a length function with polynomial growth as in § 7, we can improve this by introducing Schwartz spaces. Also, for certain explicitly given groupoids much more is possible; we have elaborated this is § 6 for the case of generalized cusp-calculi on compact manifolds with corners.

5. Algebras on manifolds with corners

In this section we recall the constructions of various groupoids associated to manifolds with corners (see for instance [**25**, **37**, **40**]). We shall use these results to define length functions on some of these groupoids, which in turn is useful when defining Schwartz spaces associated to manifolds with corners, in § 7. Note that we do not require that the manifolds used in this section have embedded hyperfaces. Also, we shall use these constructions in the particular case of manifolds with boundary in the next section in order to construct algebras with smooth kernels. The reader interested only in the next section can skip this section, and only refer back to it when necessary.

Let X be a manifold with corners, and x a point of X; we denote by $F(x)$ the connected component of the set of points having the same codimension as x which contains x, and by $N_xF(x) = T_xX/T_xF(x)$ the normal space to the boundary at x. One can define several groupoids associated to X, giving various pseudodifferential calculi, such as the b-calculus, the cusp-calculus, and its generalizations (the c_n-calculi).

Let

$$\mathcal{G}(X) = \{(x, y, \alpha) \mid x, y \in X,\ \operatorname{codim}(x) = \operatorname{codim}(y),\ \alpha : N_yF(y) \xrightarrow{\sim} N_xF(x)\},$$

where α is given, through trivializations $N_yF(y) \simeq \mathbb{R}_+^k$ based on inward-pointing normals and $N_xF(x) \simeq \mathbb{R}_+^k$, by a matrix which has one and only one non-zero element on each line and each column, and this element is positive. It is precisely the product of a diagonal matrix with all terms strictly positive by a permutation matrix.

The groupoid structure of $\mathcal{G}(X)$ is given by $d, r : \mathcal{G}(X) \to X$ with $r(x, y, \alpha) = x$, and $d(x, y, \alpha) = y$; the composition law is induced by the composition of the isomorphisms, in the sense that $(x, y, \alpha)(y, z, \beta) = (x, z, \alpha\beta)$.

One can endow $\mathcal{G}(X)$ with several different differential structures such that the resulting groupoids are homeomorphic but not diffeomorphic. Let Ω and Ω' be two charts of X of same codimension. Thus

$$\Omega \overset{\pi}{\simeq} U \times \mathbb{R}_+^A \quad \text{and} \quad \Omega' \overset{\pi'}{\simeq} U' \times \mathbb{R}_+^{A'},$$

where U and U' are open subsets of some $\mathbb{R}^n$ and A and A' are the sets of local hyperfaces contained in Ω and Ω'. The sets A and A' have the same cardinal, namely the codimension of Ω. Fix a bijection $\sigma : A' \to A$.

If $t \in \mathbb{R}_+^{A'}$ and $\lambda \in (\mathbb{R}_+^*)^{A'}$, then let $B'_t = \{i \in A',\ t_i = 0\}$. The product of the matrix of $\sigma_{|B'_t}$ by the diagonal matrix consisting of the λ_i, for $i \in B'_t$, is denoted by $m_{\sigma,t,\lambda}$; it defines an isomorphism $\alpha_{\sigma,t,\lambda}$ through the trivializations induced by π and π':

$$\begin{array}{ccc} \alpha_{\sigma,t,\lambda} : N_{\pi'^{-1}(u',t)}F(\pi'^{-1}(u',t)) & \xrightarrow{\simeq} & N_{\pi^{-1}(u,\sigma(\lambda t))}F(\pi^{-1}(u,\sigma(\lambda t))) \\ \simeq\downarrow & & \simeq\downarrow \\ \mathbb{R}_+^{B'_t} & \xrightarrow{m_{\sigma,t,\lambda}} & \mathbb{R}_+^{B} \end{array}$$

5.1. The b-calculus differential structure

Using the notation above, consider the map

$$\begin{aligned} \psi_\sigma : U \times U' \times \mathbb{R}_+^{A'} \times (\mathbb{R}_+^*)^{A'} &\to \mathcal{G}(X) \\ (u,u',t,\lambda) &\mapsto (\pi^{-1}(u,\sigma(\lambda t)), \pi'^{-1}(u',t), \alpha_{\sigma,t,\lambda}). \end{aligned}$$

This map is injective, and one thus obtains a C^∞ structure on $\mathcal{G}(X)$, which turns it into a Lie groupoid, whose fibres are submanifolds without boundary; it is amenable.

Definition 5.1. The groupoid of the b-calculus, $\Gamma_1(X)$ is the union of the connected components containing the unit of each d-fibre of $\mathcal{G}(X)$.

Remark 5.2. It is instructive to consider the special case when X is a manifold with connected boundary ∂X and defining function ρ. Then one can prove that

$$\Gamma_1(X) \simeq \{(x,y,\lambda) \in X \times X \times (\mathbb{R}_+^*),\ \rho(x) = \lambda\rho(y)\}.$$

The identification with the b-calculus is obtained by observing that $\partial M \times \partial M \times (\mathbb{R}_+^*) \subset \Gamma_1(X)$ and that $\log\lambda = \log\rho(x) - \log\rho(y)$ if $(x,y,\lambda) \in \Gamma_1(X)$ and $x \notin \partial M$.

5.2. The cusp-calculus and c_n-calculi differential structures

Consider a continuous, strictly increasing map, smooth for $t \neq 0$, such that

$$\begin{aligned} \tau_n : \mathbb{R}_+ &\to \mathbb{R}_+ \\ t &\mapsto \begin{cases} \frac{1}{e}(-\log(t))^{-1/n} & \text{if } t \in \left(0, \frac{1}{e}\right), \\ 0 & \text{if } t = 0, \\ t & \text{if } t \geqslant 1. \end{cases} \end{aligned}$$

If U is an open subset of a Euclidean space, we will also denote by $\tau_n : U \times \mathbb{R}_+^{A'} \to U \times \mathbb{R}_+^{A'}$ the map obtained by applying τ_n to each coordinate of $\mathbb{R}_+^{A'}$.

As above, one can define

$$\begin{aligned} \psi_{\sigma,n} : U \times U' \times \mathbb{R}_+^{A'} \times (\mathbb{R}_+^*)^{A'} &\to \mathcal{G}(X) \\ (u,u',t,\lambda) &\mapsto (\pi^{-1}(u,\sigma(\tau_n(\lambda t))), \pi'^{-1}(u',\tau_n(t)), \alpha_{\sigma,t,\lambda}), \end{aligned}$$

which endows $\mathcal{G}(X)$ with a new differential structure.

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Definition 5.3. Let $n \geqslant 2$. The groupoid of the c_n-calculus, $\Gamma_n(X)$, is the union of the connected components containing the unit of each d-fibre of the groupoid $\mathcal{G}(X)$ endowed with the structure defined by $\psi_{\sigma,n-1}$ (using τ_{n-1}).

Recall that the c_2-calculus is also known as the *cusp-calculus* (see, for instance, [**30**, **33**, **34**]).

Remark 5.4. When X is a manifold with connected boundary, endowed with a defining function of the boundary, ρ, then

$$\Gamma_{n+1}(X) \simeq \{(u,v,\mu) \in X \times X \times \mathbb{R} \mid \mu\rho(u)^n\rho(v)^n = \rho(u)^n - \rho(v)^n\}$$

as smooth manifolds, which can be seen directly from the definition. Moreover, the structural morphisms of the groupoid Γ_{n+1} become $d(u,v,\mu) = v$, $r(u,v,\mu) = u$ and $(u,v,\mu)(v,w,\lambda) = (u,w,\mu+\lambda)$. The Lie algebroid of this groupoid is seen to consist of the vector fields $X \in \Gamma(TM)$ such that $X(\rho(x)^{-n})$ is a smooth function on M.

5.3. Comparison of the c_n-pseudodifferential calculi

The groupoids defined above only differ by their differential structures; in fact, they are all homeomorphic. This is intuitively clear since τ_n is a homeomorphism (but it is *not* a diffeomorphism) which induces an algebraic isomorphism of groupoids. An immediate application is the fact that the b-calculus and c_n-calculi have the same norm closure.

To keep notation simple, we will only consider here the case of a manifold with boundary. Then by considering a collar neighbourhood of the boundary, one gets a partition

$$X = X_1 \cup X_2$$

with $\pi : X_1 \simeq \partial X \times [0,1)$ and $X_2 = X \setminus \pi^{-1}(\partial X \times [0,1/e))$; the boundary defining function used here is

$$\rho(x) = \begin{cases} et & \text{if } x \in X \setminus X_2 \text{ with } \pi(x) = (u,t), \\ 1 & \text{if } x \in X_2 \end{cases}$$

(ρ is not a smooth function on X, but it is smooth on ∂X).

This allows us to define a homeomorphism

$$\left.\begin{aligned} \Theta_{n+1} : \Gamma(X) &\to \Gamma_{n+1}(X) \\ (x,y,\lambda) &\mapsto (u,v,\mu) \end{aligned}\right\} \tag{5.1}$$

with

$$u = \begin{cases} x & \text{if } x \in X_2, \\ \pi^{-1} \circ \tau_n \circ \pi(x) & \text{if } x \in X \setminus X_2, \end{cases}$$
$$v = \begin{cases} y & \text{if } y \in X_2, \\ \pi^{-1} \circ \tau_n \circ \pi(y) & \text{if } y \in X \setminus X_2, \end{cases}$$
$$\mu = \log(\lambda).$$

One can check then that $\mu\rho(u)^n\rho(v)^n = \rho(u)^n - \rho(v)^n$ by considering separately the following cases.

(1) If $x \in X \setminus X_2$ and $y \in X \setminus X_2$ then $u \in X \setminus X_2$ and $v \in X \setminus X_2$, since $\tau_n(t) \leqslant 1/e$ if $t \leqslant 1/e$. Thus if $\rho(u) \neq 0$, $\rho(u)^{-n} = -\log(\lambda t)$ and $\rho(v)^{-n} = -\log(t)$, which implies that $\rho(v)^{-n} - \rho(u)^{-n} = \log(\lambda)$. If $\rho(u) = 0$, then $\rho(v) = 0$ and the equality is trivial.

(2) If $x \in X \setminus X_2$ and $y \in X_2$, then $u \in X \setminus X_2$ and $v = y \in X_2$. Thus $\rho(u)^{-n} = -\log(\rho(x)/e)$, and $\rho(v)^{-n} = 1$. But $\rho(x) = \lambda\rho(y) = \lambda$, so that $\rho(v)^{-n} - \rho(u)^{-n} = \log(\lambda)$.

(3) If $x \in X_2$ and $y \in X_2$, then $u = x \in X_2$ and $v = y \in X_2$. Thus $\rho(u) = \rho(v) = 1$ and the equality is trivial.

6. Algebras of smooth kernels

In this section we shall study regularizing operators on certain differentiable groupoids on a manifold with boundary.

Recall that a *differentiable groupoid* is a continuous family groupoid $\mathcal{G}$ such that the space of units M, as well as the space of arrows $\mathcal{G}^{(1)} = \mathcal{G}$, are differentiable manifolds (possibly with corners), all structural maps are differentiable, and the domain map $d : \mathcal{G} \to M$ is a submersion of manifolds with corners. Note that the latter in particular implies that the fibres $\mathcal{G}_x := d^{-1}(x)$ are smooth manifolds without corners that are in general non-compact. Pseudodifferential operators on differentiable groupoids have been considered in [**24**, **38**, **40**] in more detail.

We shall now use the results of the previous sections to construct an algebra of regularizing operators that is closed under holomorphic functional calculus and whose kernels are smooth *including on the boundary*, for suitable $\mathcal{G}$. This is non-trivial, in view of the results of [**21**, **32**], where it is proved that this is not possible for the b-calculus. We begin by formulating the problem more precisely.

Let $\mathcal{G} \to M$ be a Hausdorff differentiable groupoid on a manifold with corners M. We want to construct algebras A with the following properties:

(1) $\Psi^{-\infty}(\mathcal{G}) = \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{G}) \subset A \subset \mathcal{C}^\infty(\mathcal{G}) \cap C^*(\mathcal{G})$; and

(2) A is a (possibly non-unital) Ψ^*-algebra.

Definition 6.1. An algebra A satisfying properties (a) and (b) right above is called *a Ψ^*-algebra of smooth kernels on $\mathcal{G}$.*

In [**21**] it is proved that there is no Ψ^*-algebra of smooth kernels on $\mathcal{G} = \Gamma_1(M)$, where $\Gamma_1(M)$ is the smooth groupoid associated to the b-calculus (see § 5). However, we shall now show how to construct algebras of smooth kernels on $\mathcal{G}$, if $\mathcal{G} = \Gamma_n(M)$ are the groupoids defining the c_n-calculi on a manifold with boundary M, provided that $n \geqslant 2$ (see § 4 for the definition of $\Gamma_n(M)$). Some of us have learned that it is possible to construct algebras of smooth kernels on $\Gamma_n(M)$ from Richard Melrose.

Let $I := \dot{\mathcal{C}}^\infty(M \times M)$ be the space of smooth functions on $M \times M$ that vanish to infinite order on the boundary (the boundary here is the union of hyperfaces of $M \times M$). Then $\dot{\mathcal{C}}^\infty(M \times M)$ is an algebra of smooth kernels, and hence an algebra of order $-\infty$ pseudodifferential operators on $M_0 := M \smallsetminus \partial M$. Moreover, these operators are bounded on $L^2(M_0)$. We have to note here that the correct density on M_0 for the c_n-calculus is, on a tubular neighbourhood of ∂M, of the form $hx^{-n}|\mathrm{d}x|\,|\mathrm{d}y|$, where $x \in [0,\infty)$ is a boundary defining function on M and $|\mathrm{d}y|$ is a density on ∂M, and $h \in \mathcal{C}^\infty(M)$.

Lemma 6.2. *The space $I \subset C^*(\mathcal{G})$ is a non-unital Ψ^*-algebra.*

Proof. Let Δ be the Laplace operator for some compatible metric (i.e. such that $x^n\partial_x$ has length one). Take $\mathfrak{T} = \{\Delta, x^{-1}\}$, regarded as unbounded operators on $L^2(M)$, and apply the semi-ideal construction to $C^*(\Gamma_n(M))$ and $\mathfrak{T}$. This yields an algebra $J := \mathcal{J}_{-\infty}(\mathfrak{T})_*$ satisfying

$$J \subset \{T \in C^*(\Gamma_n(M)) \mid x^{-i}\Delta^j T \Delta^k x^{-l} \text{ is bounded } \forall i,j,k,l\}.$$

Clearly, the opposite inclusion is also true by the definition of $\mathcal{J}_{-\infty}(\mathfrak{T})$.

Let $H^m(M)$ be the domain of $\Delta^{m/2}$, if $m \geqslant 0$, or let $H^m(M)$ be the dual of $H^{-m}(M)$, if m is negative. Also, let $H^{-\infty}(M) = \cup H^m(M)$ and $H^\infty(M) = \cap H^m(M)$. Since every $T \in J$ maps $H^{-\infty}(M) \to H^\infty(M)$ and J is symmetric, we obtain that T is a smoothing operator. Thus, T is an operator with integral kernel given by a smooth function $K(x,y)$. Since $x^{-i}K(x,y,x',y')x'^{-j}$ also must define a bounded operator, we see that K vanishes to infinite order at the boundary. Consequently, $J = I$, and Corollary 3.8 completes the proof. □

Remark 6.3. In the proof above, we obtain the same conclusion by considering $\mathfrak{T} = \{x^{-1}, X_1, \dots, X_m\}$, where $X_1, \dots, X_m \in \Gamma(TM)$ is a system of generators for the vector fields corresponding to the c_n-calculus, that is, b-vector fields satisfying $X_j(x^{-n+1}) \in \mathcal{C}^\infty(M)$.

We proceed now to describe the regularizing operators in the c_n-calculi on a manifold with boundary M [**24**] in a way that is most convenient for our purposes.

Let $(\mathcal{A}, (\|\cdot\|_j)_{j\in\mathbb{N}})$ be a submultiplicative Fréchet algebra. Assume there is given an action $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ of $\mathbb{R}$ by automorphisms on $\mathcal{A}$ (so $\alpha_t \circ \alpha_s = \alpha_{t+s}$). If, for any $a \in \mathcal{A}$, the map $\mathbb{R} \ni t \mapsto \alpha_t(a)$ is smooth and there exist polynomials P_n, *independent of* n, such that

$$\|\alpha_t(a)\|_n \leqslant P_n(|t|)\|a\|_n, \tag{6.1}$$

then we say that the action of $\mathbb{R}$ on $\mathcal{A}$ is *with polynomial growth*. In that case, we can introduce on the Schwartz space $\mathcal{S}(\mathbb{R}, \mathcal{A})$ an algebra structure by

$$f * g(t) = \int_{\mathbb{R}} f(s)\alpha_s(g(t-s))\,\mathrm{d}s.$$

Moreover, $\mathcal{S}(\mathbb{R}, \mathcal{A})$ acquires a family of seminorms $\|\cdot\|_{n,i,j}$,

$$\|f\|_{n,i,j} = \int_{t\in\mathbb{R}} \|t^i \partial_t^j f(t)\|_n \,\mathrm{d}t,$$

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making it a Fréchet algebra, that is submultiplicative with respect to an *equivalent* system of seminorms that we now define.

First, we can assume that $P_n(t) = C_n(1 + t^{M_n})$, for some $C_n > 0$ and $M_n \in \mathbb{N}$. Then, a direct verification using the binomial expansion of $s^i = (s - t + t)^i$ and the submultiplicativity of the seminorm $\|\cdot\|_n$ of $\mathcal{A}$ shows that

$$\|f * g\|_{n,i,j} = \int \left\| t^i \partial_t^j \int f(s)\alpha_s(g(t-s))\,\mathrm{d}s \right\|_n \mathrm{d}t$$
$$\leqslant C_n \sum_{\beta+\gamma=i} \binom{i}{\beta} (\|f\|_{n,\beta,0} + \|f\|_{n,\beta+M_n,0}) \|g\|_{n,\gamma,j}. \tag{6.2}$$

(Here $\binom{a}{b} = a!b!^{-1}(a-b)!^{-1}$ are the binomial coefficients.)

Let us adjoin a unit denoted e to $\mathcal{S}(\mathbb{R}, \mathcal{A})$ and extend the norms $\|\cdot\|_{n,i,j}$ to $\mathcal{S}(\mathbb{R}, \mathcal{A})^+ := \mathcal{S}(\mathbb{R}, \mathcal{A}) \oplus \mathbb{C}e$ by $\|(f, \lambda e)\|_{n,i,j} := \|f\|_{n,i,j} + |\lambda|$. Equation (6.2) implies that left multiplication by f on $\mathcal{S}(\mathbb{R}, \mathcal{A})^+$ is continuous in the norm $\sum_{\alpha \leqslant i} \|\cdot\|_{n,\alpha,j}$. The corresponding operator norm, $|||f|||_{n,i,j}$ is then submultiplicative, by definition and satisfies

$$|||f|||_{n,i,j} \leqslant \|f\|_{n,i,j} + K_{n,i,j} \sum_{l \leqslant i+M_n} \|f\|_{n,l,0}.$$

(The first term appears due to the fact that we have adjoined a unit to $\mathcal{S}(\mathbb{R}, \mathcal{A})$. Also, $K_{n,i,j}$ is a constant that is independent of f.) On the other hand,

$$\|f\|_{n,i,j} = \|f * e\|_{n,i,j} \leqslant |||f|||_{n,i,j} \|e\|_{n,i,j} = |||f|||_{n,i,j},$$

which proves that the families of seminorms $|||\cdot|||_{n,i,j}$ and $\|\cdot\|_{n,i,j}$ define the same topology on $\mathcal{S}(\mathbb{R}, \mathcal{A})$.

Let M be a compact manifold with boundary ∂M. On $\partial M \times [0, \infty)$ we consider the vector field $X_n = (1 + x^n)^{-1} x^n \partial_x$, with $x \in [0, \infty)$. Our choice of this vector field is justified by the fact that

$$X_n(x^{-n+1}) = (-n+1)(1 + x^n)^{-1} \in \mathcal{C}^\infty([0, \infty)),$$

and hence it defines a vector field corresponding to the c_n-calculus, $n \geqslant 2$. Moreover, this vector field defines, by integration, an action of $\mathbb{R}$ on $\partial M \times \partial M \times [0, \infty)$, which is trivial on ∂M.

Let $\mathcal{A} := \mathcal{S}(\partial M \times \partial M \times [0, \infty))$, with product

$$(fg)(m_1, m_2, t) = \int_{\partial M} f(m_1, m, t) g(m, m_2, t)\,\mathrm{d}m.$$

Then $\mathcal{A}$ is isomorphic, as an algebra, with the complete projective tensor product $\Psi^{-\infty}(\partial M) \otimes_\pi \mathcal{S}([0, \infty))$.

Lemma 6.4. *Assume $n \geqslant 2$. Then the action of $\mathbb{R}$ on $\mathcal{A} := \mathcal{S}(\partial M \times \partial M \times [0, \infty))$ integrating the action of the vector field $X_n = (1+x^n)^{-1} x^n \partial_x$ is with polynomial growth.*

Proof. Let $S^0(\mathbb{R})$ be the space of classical symbols of order 0 on $\mathbb{R}$, with its natural Fréchet space structure. Consider the function

$$f_n(x) = (-n+1)^{-1}x^{-n+1} + x : (0,\infty) \to \mathbb{R}. \tag{6.3}$$

Then f_n is a bijection such that the induced map

$$f_n^* : \mathcal{S}([0,\infty)) \to \sum_{k=0}^{n-2} S^{-k/(n-1)}(\mathbb{R})$$

is an equivariant map, i.e. the action of $\mathbb{R}$ on $\mathcal{S}(\mathbb{R})$ being by translation. Moreover, f_n^* is an isomorphism onto its image, which consists of the linear combinations of symbols that are of order $-\infty$ on the positive semi-axis $[0,\infty) \subset \mathbb{R}$. In particular, the image of f_n^* is closed.

Since the action of $\mathbb{R}$ by translation on $S^k(\mathbb{R})$ is with polynomial growth, the given action of $\mathbb{R}$ on $\mathcal{S}([0,\infty))$ is also with polynomial growth. □

Probably the above lemma is the only reason why we have to restrict to $n \geqslant 2$ in order to construct algebras with smooth kernels on the c_n-calculus groupoid $\Gamma_n(M)$.

Fix a smooth function $\phi \in \mathcal{C}^\infty([0,\infty))$, $\phi(x) = 1$ if x is in a certain neighbourhood of 0, $\phi(x) = 0$ if $x \geqslant 1$, and let

$$A_1 = \phi\mathcal{S}(\mathbb{R},\mathcal{A})\phi := \phi\mathcal{S}(\mathbb{R},\mathcal{S}(\partial M \times \partial M \times [0,\infty)))\phi,$$

where $\mathcal{A} := \mathcal{S}(\partial M \times \partial M \times [0,\infty))$, as in the above lemma. The algebra A_1 acts on $L^2(\partial M \times [0,\infty))$.

Let $\Gamma_n(M)$ be the groupoid defining the c_n-calculus. Then $\Psi^{-\infty}(\Gamma_n(M))$ identifies with an algebra of bounded operators on $L^2(M)$ (with the canonical induced measure).

We are ready to prove the following theorem.

Theorem 6.5. *Let M be a compact, smooth manifold with connected boundary. Then $\mathfrak{A} = A_1 + I \subset C^*(\Gamma_n(M))$ and is a non-unital Ψ^*-algebra with smooth kernels.*

Proof. We shall use the results of the previous sections. First, however, we have to prove that $\mathfrak{A} := A_1 + I$ is an algebra.

Indeed, by Theorem 3.6, it is enough to check that $A_1 \subset \Psi^\infty(\mathfrak{X})$, where $\mathfrak{X} = \{x^{-1}, X_0, \ldots, X_m\}$ is chosen as in Remark 6.3. We identify a tubular neighbourhood of ∂M with a subset of $\partial M \times [0,\infty)$. To prove this, we first notice that $X_0, \ldots, X_m \in \Psi^1(\Gamma_n(M))$, $A_1 \subset \Psi^{-\infty}(\Gamma_n(M))$, and hence any product of the form

$$X_{i_1} \cdots X_{i_j} a X_{i_{j+1}} \cdots X_{i_r}$$

consists of bounded operators. Then, let us write by a_f the operator of convolution on $\partial M \times [0,\infty)$ with the function $f \in \mathcal{S}(\mathbb{R})$ (the action of $\mathbb{R}$ is the one integrating the vector field $X_n = (1+x^n)^{-1}x^n\partial_x$). Also, let us observe that $\mathcal{A} = \mathcal{S}(\partial M \times \partial M \times [0,\infty))$ identifies with an algebra of operators on $L^2(\partial M \times [0,\infty))$. Then the map

$$\mathcal{S}(\mathbb{R}) \otimes_\pi \mathcal{A} \ni f \otimes b \to a_f b \in \mathcal{S}(\mathbb{R},\mathcal{A})$$

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is an isomorphism. We need to check that δ, $\delta(a) := [x^{-1}, a]$ maps $\mathcal{S}(\mathbb{R}, \mathcal{A})$ to itself. Clearly, $\delta(b) = 0$, for any $b \in \mathcal{A}$. If $f \in \mathcal{S}(\mathbb{R})$, then $\delta(f) \in \mathcal{S}(\mathbb{R}, \mathcal{A})$. This proves our claim that $\mathcal{S}(\mathbb{R}, \mathcal{A})$ is stable with respect to δ. In conclusion, $A_1 \in \Psi^\infty(\mathfrak{T})$ and hence $A_1 I + I A_1 \subset I$ (see Theorem 3.6).

The algebra $\mathfrak{A}$ has a Fréchet topology induced from the Fréchet topologies of A_1 and I. To prove that it is a Ψ^*-algebra, we thus only need to prove that it is spectrally invariant. To this end, we shall use the results of Theorem 2.8. Let $\mathcal{B} = C^*(\Gamma_n(M))$, regarded as a subalgebra of the algebra of bounded operators on $L^2(M)$. Also, let $\mathcal{J}$ be the norm closure of $x\mathcal{B}$. Let $\mathcal{K}$ be the algebra of compact operators on $L^2(\partial M)$. Then $\mathcal{B}/\mathcal{J} \simeq C_0(\mathbb{R}, \mathcal{K})$ by standard results on groupoid C^*-algebras. (This statement can also be extracted from either of [**25**] or [**35**] by using, for example, the exact sequence associated to the invariant subset $\partial M \times \{0\} \subset \partial M \times [0, \infty)$.) Let $\varphi : \mathfrak{A}/I \to \mathcal{B}/\mathcal{J}$ be the induced morphism. Then the range of this morphism is $\mathcal{S}(\mathbb{R}, \mathcal{C}^\infty(\partial M \times \partial M)) \subset \mathcal{B}/\mathcal{J} \simeq C_0(\mathbb{R}, \mathcal{K})$ and hence φ is locally spectral invariant.

Theorem 2.8 or, more precisely, Corollary 2.10 then shows that $\mathfrak{A}$ is spectrally invariant. This completes the proof. □

We are planning to clarify the relation between our algebra $\mathfrak{A}$ for the c_2-calculus and the cusp-calculus as defined by Melrose (an exposition can be found in [**34**]) in another paper.

7. The Schwartz space of a continuous family groupoid

In this section we define a notion of Schwartz space on a continuous family groupoid $\mathcal{G}$, i.e. a space of rapidly decreasing functions as well as their derivatives. This was introduced in [**36**] in the case of differentiable groupoids. We prove, using also some methods introduced in [**19**], that this is a subalgebra of $C^*(\mathcal{G})$, stable under holomorphic functional calculus.

7.1. The Schwartz convolution algebra

Let $\mathcal{G}$ be a Hausdorff, continuous family groupoid. Denote its Lie algebroid by A. Then A is, by definition, the restriction to M of the vertical tangent spaces along the fibres of $d : \mathcal{G} \to M$. Fix a 1-density on A. This will then give rise to a 1-density on each of the manifolds $\mathcal{G}_x := d^{-1}(x)$, and hence to a smooth measure μ_x on each of $\mathcal{G}_x$. Moreover, the measures μ_x are invariant with respect to right translations, and hence they form a Haar system.

Definition 7.1. Let μ be the Haar system on $\mathcal{G}$ introduced above. A length function with polynomial growth on $\mathcal{G}$ is a continuous function $\phi : \mathcal{G} \to \mathbb{R}_+$ such that

(1) $\phi(g_1 g_2) \leqslant \phi(g_1) + \phi(g_2)$,

(2) $\forall g \in \mathcal{G}$, $\phi(g^{-1}) = \phi(g)$,

(3) ϕ is proper,

(4) $\exists c, N$, $\forall x \in \mathcal{G}^{(0)}$, $\forall r \in \mathbb{R}_+$, $\mu_x(\phi^{-1}([0, r])) \leqslant c(r^N + 1)$.

The polynomial growth condition ensures that there exists $k_0 \in \mathbb{N}$ and $C \in \mathbb{R}$ such that for any $k \geqslant k_0$, and for any $x \in G^{(0)}$,

$$\int_{G_x} \frac{1}{(1+\phi(g))^k} \, d\mu_x \leqslant C.$$

Let v be a section of the Lie algebroid of $\mathcal{G}$, $A(\mathcal{G})$; such a section defines a differential operator of order 1 on $\mathcal{G}$. Thus if $v_1, \dots, v_l$ are sections of $A(\mathcal{G})$, and if $f \in \mathcal{C}_0(\mathcal{G}, \Omega^{1/2})$, then $v_1 \cdots v_k \cdot f \cdot v_{k+1} \cdots v_l$ is a distribution on $\mathcal{G}$. It belongs to $\mathcal{C}_0(\mathcal{G}, \Omega^{1/2})$ provided that there exists $g \in \mathcal{C}_0(\mathcal{G}, \Omega^{1/2})$ such that, for any $a \in \mathcal{C}_c^{\infty,0}(G)$, $g \cdot a = (v_1 \cdots v_k \cdot f \cdot v_{k+1} \cdots v_l) \cdot a$.

Definition 7.2. Let $\mathcal{G}$ be a continuous family groupoid and ϕ be a length function with polynomial growth on $\mathcal{G}$. Define

$$\begin{aligned}\mathcal{S}^{k,d}(\mathcal{G}, \phi) = \Big\{ f \in \mathcal{C}_0(G, \Omega^{1/2}),\ \forall v_1, \dots, v_d \in \mathcal{C}(A(\mathcal{G})),\ \forall i \leqslant d,& \\ v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_d \in \mathcal{C}_0(\mathcal{G}, \Omega^{1/2})& \\ \text{and } \sup_{g \in \mathcal{G}} |v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_d(g)| (1+\phi(g))^k < \infty \Big\}.&\end{aligned}$$

The Schwartz space of $\mathcal{G}$ with respect to ϕ is

$$\mathcal{S}(\mathcal{G}, \phi) = \bigcap_{k,d \in \mathbb{N}} \mathcal{S}^{k,d}(\mathcal{G}, \phi).$$

The space $\mathcal{S}^{k,d}(\mathcal{G}, \phi)$ can be endowed with the norm

$$\|f\|_{k,d} = \sup_{i \leqslant l \leqslant d} \sup_{\substack{v_1, \dots, v_l \in \mathcal{C}(A(\mathcal{G})) \\ \|v_j\| \leqslant 1}} \sup_{g \in \mathcal{G}} |v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_l(g)| (1+\phi(g))^k.$$

Remark 7.3.

(1) If $l \geqslant k$ and $f \in \mathcal{S}^{l,d}(\mathcal{G}, \phi)$ then $f \in \mathcal{S}^{k,d}(\mathcal{G}, \phi)$ and $\|f\|_{k,d} \leqslant \|f\|_{l,d}$.

(2) If $f \in \mathcal{S}^{k,d}(\mathcal{G}, \phi)$ and $k \geqslant k_0$ then for any $x \in G^{(0)}$ and any $v_1, \dots, v_l \in \mathcal{C}(A(\mathcal{G}))$ (with $l \leqslant d$), one has

$$\begin{aligned}\int_{\mathcal{G}_x} |v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_l(g)|^2 &\leqslant \|f\|_{k,d}^2 \int_{\mathcal{G}_x} \frac{1}{(1+\phi(g))^{2k}} \, d\mu_x \\ &\leqslant C \|f\|_{k,d}^2\end{aligned}$$

so that $v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_l \in L^2(\mathcal{G}_x)$, and

$$\|v_1 \cdots v_i \cdot f \cdot v_{i+1} \cdots v_l\|_{L^2(\mathcal{G}_x)} \leqslant \sqrt{C} \|f\|_{k,d}.$$

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Proposition 7.4. *For any $k, d \in \mathbb{N}$, with k such that*

$$\int_{\mathcal{G}_x} \frac{1}{(1+\phi(g))^k} \, \mathrm{d}\mu_x$$

converges for any $x \in G^{(0)}$, $\mathcal{S}^{k,d}(\mathcal{G}, \phi)$ is a dense subalgebra of $C^*_{\mathrm{r}}(\mathcal{G})$, and there exists a constant λ_k such that $\| \cdot \|_{C^*_{\mathrm{r}}(\mathcal{G})} \leqslant \| \cdot \|_{k,d}$.

Proof. If $f_1, f_2 \in \mathcal{S}^{k,d}(\mathcal{G}, \phi)$. Let $x = d(g)$, we then have

$$\begin{aligned}
&|v_1 \cdots v_i \cdot f_1 * f_2 \cdot v_{i+1} \cdots v_l(g)| \\
&\quad = \left| \int_{\mathcal{G}_x} v_1 \cdots v_i \cdot f_1(gg'^{-1}) f_2 \cdot v_{i+1} \cdots v_l(g') \right| \mathrm{d}\mu_x \\
&\quad \leqslant \int_{\substack{\mathcal{G}_x \\ \phi(g') \geqslant \phi(g)/2}} |v_1 \cdots v_i \cdot f_1(gg'^{-1}) f_2 \cdot v_{i+1} \cdots v_l(g')| \, \mathrm{d}\mu_x \\
&\qquad + \int_{\substack{\mathcal{G}_x \\ \phi(gg'^{-1}) \geqslant \phi(g)/2}} |v_1 \cdots v_i \cdot f_1(gg'^{-1}) f_2 \cdot v_{i+1} \cdots v_l(g')| \, \mathrm{d}\mu_x \\
&\quad \leqslant \int_{\substack{\mathcal{G}_x \\ \phi(g') \geqslant \phi(g)/2}} \|f_1\|_{k,d} \frac{1}{(1+\phi(gg'^{-1}))^k} \|f_2\|_{k,d} \frac{1}{(1+\phi(g'))^k} \, \mathrm{d}\mu_x \\
&\qquad + \int_{\substack{\mathcal{G}_x \\ \phi(gg'^{-1}) \geqslant \phi(g)/2}} \|f_1\|_{k,d} \frac{1}{(1+\phi(gg'^{-1}))^k} \|f_2\|_{k,d} \frac{1}{(1+\phi(g'))^k} \, \mathrm{d}\mu_x \\
&\quad \leqslant \frac{2^k}{(1+\phi(g))^k} \|f_1\|_{k,d} \|f_2\|_{k,d} \left(\int_{\mathcal{G}_x} \frac{1}{(1+\phi(gg'^{-1}))^k} + \int_{\mathcal{G}_x} \frac{1}{(1+\phi(g'))^k} \right) \mathrm{d}\mu_x \\
&\quad \leqslant \frac{2^{k+1} C}{(1+\phi(g))^k} \|f_1\|_{k,d} \|f_2\|_{k,d}
\end{aligned}$$

so that

$$|v_1 \cdots v_i \cdot f_1 * f_2 \cdot v_{i+1} \cdots v_l(g)| (1+\phi(g))^k \leqslant 2^{k+1} C \|f_1\|_{k,d} \|f_2\|_{k,d}, \tag{7.1}$$

which implies that $f_1 * f_2 \in \mathcal{S}^{k,d}(\mathcal{G}, \phi)$. Consequently, $\mathcal{S}^{k,d}(\mathcal{G}, \phi)$ is an algebra.

To prove that $\mathcal{S}^{k,d}(\mathcal{G}, \phi)$ is a subspace of $C^*_{\mathrm{r}}(\mathcal{G})$, we need to show that if $f \in \mathcal{S}^{k,d}(\mathcal{G}, \phi)$, and for any $x \in G^{(0)}$ and $\xi \in C_{\mathrm{c}}(\mathcal{G}_x)$, one has

$$\|f * \xi\|_{L^2(\mathcal{G}_x)} \leqslant \|f\|_{k,d} \|\xi\|_{L^2(\mathcal{G}_x)}$$

up to a constant.

Denote by λ_k the constant $2^{k+1} C$. Then the Cauchy–Schwarz inequality implies that

$$\|f\xi\|^2_{L^2(\mathcal{G}_x)} \leqslant \|\xi\|_{L^2(\mathcal{G}_x)} \|(f^* f)\xi\|_{L^2(\mathcal{G}_x)}.$$

By induction,

$$\|f\xi\|^{2^{n+1}}_{L^2(\mathcal{G}_x)} \leqslant \|\xi\|^{2^{n+1}-1}_{L^2(\mathcal{G}_x)} \|(f^* f)^{2^n} \xi\|_{L^2(\mathcal{G}_x)}.$$

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But if $h \in \mathcal{S}^{k,d}(\mathcal{G}, \phi)$,

$$\|h\xi\|^2_{L^2(\mathcal{G}_x)} = \int_{\mathcal{G}_x} (h(gg'^{-1})\xi g')^2 \, \mathrm{d}\mu_x \leqslant \|h\|^2_{k,d} \int_{\mathcal{G}_x} \left(\frac{\xi g'}{(1+\phi(gg'^{-1}))^k} \right)^2 \mathrm{d}\mu_x.$$

The inequality (7.1) also gives

$$\|(f^*f)^{2^n}\|_{k,d} \leqslant \lambda_k^{2^{n+1}} \|f^{2^{n+1}}\|_{k,d},$$

so that

$$\|f\xi\|^{2^{n+1}}_{L^2(\mathcal{G}_x)} \leqslant \|\xi\|^{2^{n+1}-1}_{L^2(\mathcal{G}_x)} \lambda_k^{2^{n+1}} \|f\|^{2^{n+1}}_{k,d} \left(\int_{\mathcal{G}_x} \left(\frac{\xi g'}{(1+\phi(gg'^{-1}))^k} \right)^2 \right)^{1/2} \mathrm{d}\mu_x.$$

Taking $n \to \infty$,

$$\|f\xi\|_{L^2(\mathcal{G}_x)} \leqslant \|\xi\|_{L^2(\mathcal{G}_x)} \lambda_k \|f\|_{k,d},$$

which implies that $f \in C^*_{\mathrm{r}}(\mathcal{G})$, and $\|f\|_{C^*_{\mathrm{r}}(\mathcal{G})} \leqslant \lambda_k \|f\|_{k,d}$. □

7.2. The theorem

We are now ready to prove the main result of this section, the fact that the algebras $\mathcal{S}(\mathcal{G}, \pi)$ are closed under holomorphic functional calculus.

Theorem 7.5. *The Schwartz space of $\mathcal{G}$ with respect to ϕ, $\mathcal{S}(\mathcal{G}, \phi)$, is closed under holomorphic functional calculus in $C^*_{\mathrm{r}}(\mathcal{G})$.*

Proof. The methods of [**19**] extend without difficulty to our case.

As above, let k_0 be such that

$$\int_{\mathcal{G}_x} \frac{1}{(1+\phi(g))^{k_0}} \, \mathrm{d}\mu_x$$

converges for any $x \in G^{(0)}$.

Lemma 7.6. *If $l \geqslant k \geqslant k_0$ then $\mathcal{S}^{l,d}(\mathcal{G}, \phi)$ is stable under holomorphic calculus in $\mathcal{S}^{k,d}(\mathcal{G}, \phi)$.*

Proof. To prove this, by Lemma 2.7, we will show that

$$\lim_{n\to\infty} \|f^n\|^{1/n}_{k,d} = \lim_{n\to\infty} \|f^n\|^{1/n}_{l,d}.$$

Indeed, this implies that $\mathcal{S}^{l,d}(\mathcal{G}, \phi)$, which is a dense subalgebra of $\mathcal{S}^{k,d}(\mathcal{G}, \phi)$, is also full.

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Now, as above, we have

$$
\begin{aligned}
&|v_1 \cdots v_i \cdot f^n \cdot v_{i+1} \cdots v_l(g)| \\
&\quad = \left| \int_{g_1 \cdots g_n = g} (v_1 \cdots v_i \cdot f)(g_1) f(g_2) \cdots f(g_{n-1})(f \cdot v_{i+1} \cdots v_l)(g_n) \right| \\
&\quad \leqslant \sum_{i=1}^{n} \int_{\substack{g_1 \cdots g_n = g \\ \phi(g_i) \geqslant \phi(g)/n}} |v_1 \cdots v_i \cdot f(g_1)|\,|f(g_2)| \cdots |f(g_{n-1})|\,|f \cdot v_{i+1} \cdots v_l(g_n)| \\
&\quad \leqslant \sum_{i=1}^{n} \|f\|_{k,d}^{n-1} \|f\|_{l,d} \int_{\substack{g_1 \cdots g_n = g \\ \phi(g_i) \geqslant \phi(g)/n}} \frac{1}{(1+\phi(g_i))^l} \prod_{j \neq i} \frac{1}{(1+\phi(g_j))^k} \\
&\quad \leqslant \|f\|_{k,d}^{n-1} \|f\|_{l,d} \frac{1}{(1+(\phi(g)/n))^l} \sum_{i=1}^{n} \int_{\substack{g_1 \cdots g_n = g \\ \phi(g_i) \geqslant \phi(g)/n}} \prod_{j \neq i} \frac{1}{(1+\phi(g_j))^k}.
\end{aligned}
$$

As the latter integrals are lower than C,

$$|f^n(g)|(1+\phi(g))^l \leqslant \|f\|_{k,d}^{n-1} \|f\|_{l,d} n^l \cdot nC,$$

which gives

$$\|f^n\|_{l,d}^{1/n} \leqslant n^{(1+l)/n} C^{1/n} \|f\|_{k,d}^{1-(1/n)} \|f\|_{l,d}^{1/n}$$

thus

$$\lim_{n\to\infty} \|f^n\|_{l,d}^{1/n} \leqslant \|f\|_{k,d}.$$

Let us now apply this inequality to f^m, we get

$$\lim_{n\to\infty} \|f^{mn}\|_{l,d}^{1/n} \leqslant \|f^m\|_{k,d},$$

so that if $m \to \infty$,

$$\lim_{n\to\infty} \|f^n\|_{l,d}^{1/n} = \lim_{m,n\to\infty} \|f^{mn}\|_{l,d}^{1/mn} \leqslant \lim_{m\to\infty} \|f^m\|_{k,d}^{1/m}.$$

Because of $\|f^m\|_{k,d} \leqslant \|f^m\|_{l,d}$ we get $\lim_{n\to\infty} \|f^n\|_{l,d}^{1/n} = \lim_{n\to\infty} \|f^n\|_{k,d}^{1/n}$. □

Lemma 7.7. *If $k \geqslant k_0$, one has*

$$\mathcal{S}^{k,d}(\mathcal{G},\phi) * C^*(\mathcal{G}) * \mathcal{S}^{k,d}(\mathcal{G},\phi) \subset \mathcal{S}^{0,d}(\mathcal{G},\phi).$$

Proof. If $f_1, f_2 \in \mathcal{S}^{k,d}(\mathcal{G},\phi)$ and $f \in C^*(\mathcal{G})$, then

$$|v_1 \cdots v_i \cdot f_1 * f * f_2 \cdot v_{i+1} \cdots v_l(g)| = \left| \int_{\mathcal{G}_{s(g)}} (v_1 \cdots v_i \cdot f_1)(gg'^{-1})(f * f_2 \cdot v_{i+1} \cdots v_l)(g') \right|.$$

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But if we denote by f_3 the function $g' \mapsto (v_1 \cdots v_i \cdot f_1)(gg'^{-1})$, then the Cauchy–Schwarz inequality gives

$$\begin{aligned}|v_1 \cdots v_i \cdot f_1 * f * f_2 \cdot v_{i+1} \cdots v_l(g)| &\leqslant \|f_3\|_{L^2(\mathcal{G}_{s(g)})} \|f * f_2 \cdot v_{i+1} \cdots v_l\|_{L^2(\mathcal{G}_{s(g)})} \\ &\leqslant C\|f_3\|_{k,d} \|f\|_{C_{\mathrm{r}}^*(\mathcal{G})} \|f_2\|_{L^2(G_{s(g)})} \\ &\leqslant C\|f_3\|_{k,d} \|f\|_{C_{\mathrm{r}}^*(\mathcal{G})} C\|f_2\|_{k,d},\end{aligned}$$

and hence $f_1 * f * f_2 \in \mathcal{S}^{0,d}(\mathcal{G}, \phi)$. □

Lemma 7.8. *If $k \geqslant k_0$, then $\mathcal{S}^{k,d}(\mathcal{G}, \phi)$ is stable under holomorphic functional calculus in $C_{\mathrm{r}}^*(\mathcal{G})$.*

Proof. If $f \in \mathcal{S}^{k,d}(\mathcal{G}, \phi)$, then Lemma 7.7 implies that

$$\|f^n\|_{0,d} \leqslant C^2 \|f\|_{k,d} \|f^{n-2}\|_{C_{\mathrm{r}}^*(\mathcal{G})} \|f\|_{k,d}$$

thus

$$\lim_{n\to\infty} \|f^n\|_{0,d}^{1/n} = \lim_{n\to\infty} \|f^n\|_{C_{\mathrm{r}}^*(\mathcal{G})}^{1/n}$$

(the inverse inequality is given by Proposition 7.4). To prove this lemma, it remains to show that

$$\lim_{n\to\infty} \|f^n\|_{0,d}^{1/n} = \lim_{n\to\infty} \|f^n\|_{k,d}^{1/n}.$$

If $f_1, f_2 \in \mathcal{S}^{2k,d}(\mathcal{G}, \phi)$, then

$$\begin{aligned}|v_1 \cdots v_i \cdot f_1 * f_2 \cdot v_{i+1} \cdots v_l(g)| &\leqslant \int_{\mathcal{G}_x} |v_1 \cdots v_i \cdot f_1(gg'^{-1})||f_2 \cdot v_{i+1} \cdots v_l(g')| \\ &\leqslant \int_{\substack{\mathcal{G}_{s(g)} \\ \phi(gg'^{-1}) \geqslant \phi(g)/2}} \frac{\|f_1\|_{2k,d}\|f_2\|_{0,d}}{(1+\phi(gg'^{-1}))^k(1+\phi(g))^k} \\ &\qquad + \int_{\substack{\mathcal{G}_{s(g)} \\ \phi(g') \geqslant \phi(g)/2}} \frac{\|f_1\|_{0,d}\|f_2\|_{2k,d}}{(1+\phi(g'))^k(1+\phi(g))^k} \\ &\leqslant \frac{C}{(1+\phi(g))^k} (\|f_1\|_{0,d}\|f_2\|_{2k,d} + \|f_1\|_{2k,d}\|f_2\|_{0,d})\end{aligned}$$

hence $\|f_1 f_2\|_{k,d} \leqslant C(\|f_1\|_{0,d}\|f_2\|_{2k,d} + \|f_1\|_{2k,d}\|f_2\|_{0,d})$.

Applying this to $f_1 = f_2 = f$, we get

$$\|f^{2n}\|_{k,d} \leqslant 2C\|f\|_{0,d}\|f\|_{2k,d}.$$

But, by Lemma 7.6,

$$\lim_{n\to\infty} \|f^n\|_{2k,d}^{1/n} = \lim_{n\to\infty} \|f^n\|_{k,d}^{1/n},$$

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so that

$$\lim_{n\to\infty} \|f^n\|_{k,d}^{1/n} \leqslant \lim_{n\to\infty} \|f^n\|_{0,d}^{1/n},$$

which gives an equality as the opposite inequality comes from Remark 7.3. □

We are ready now to complete the proof of Theorem 7.5. The Schwartz space of $\mathcal{G}$ with respect to ϕ is thus an intersection of subalgebras of $C^*_{\mathrm{r}}(\mathcal{G})$ which are stable under holomorphic functional calculus, hence $\mathcal{S}(\mathcal{G},\phi)$ is stable under holomorphic functional calculus in $C^*_{\mathrm{r}}(\mathcal{G})$. □

Corollary 7.9. *Let $\mathcal{G}$ be a continuous family groupoid, and let ϕ be a length function with polynomial growth. Denote by $\Psi^0_s(\mathcal{G})$ the sum of $\Psi^0(\mathcal{G})$ and of the Schwartz space of $\mathcal{G}$ with respect to ϕ, $\mathcal{S}(\mathcal{G},\phi)$. Then $\Psi^0_s(\mathcal{G})$ is stable under holomorphic functional calculus.*

Moreover, if $P \in \Psi^0_s(\mathcal{G})$ is Fredholm, then there exists $Q \in \Psi^0_s(\mathcal{G})$ such that $PQ - I$ and $QP - I$ are both compact operators.

Proof. Let us first show that $\Psi^0_s(\mathcal{G})$ is an algebra, which amounts to proving that if $k_1, k_2 \in I^{m,0}_{\mathrm{c}}(\mathcal{G},\mathcal{G}^{(0)})$ and $f \in \mathcal{S}(\mathcal{G},\phi)$ then $k_1 * f * k_2 \in \mathcal{S}(\mathcal{G},\phi)$.

The Lie algebroid of $\mathcal{G}$ being a $\mathcal{C}^{\infty,0}$ fibre bundle, it is possible to choose a set $v_1,\dots,v_N \in \mathcal{C}(A(\mathcal{G}))$ such that for any $x \in \mathcal{G}^{(0)}$, $(v_1(x),\dots,v_n(x))$ generates $T_x\mathcal{G}_x$. Now let

$$\Delta = \sum_{i=1}^{N} v_i^2.$$

This differential operator is of degree 2 and admits a parametrix.

If $k_1, k_2 \in I^{m,0}_{\mathrm{c}}(\mathcal{G},\mathcal{G}^{(0)})$ then for any l there exist $k_1', k_2' \in I^{m-2l,0}_{\mathrm{c}}(\mathcal{G},\mathcal{G}^{(0)})$ and $r_1, r_2 \in \mathcal{C}^{\infty,0}_{\mathrm{c}}(\mathcal{G})$ such that

$$k_1 = k_1' * k_\Delta^l + r_1, \qquad k_2 = k_\Delta^l * k_2' + r_2.$$

But when $k \in I^{-N-1,0}_{\mathrm{c}}(\mathcal{G},\mathcal{G}^{(0)})$, it is given by an absolutely convergent oscillatory integral, thus it belongs to $\mathcal{C}_{\mathrm{c}}(\mathcal{G})$. Hence if l is large enough, k_1' and k_2' belong to $\mathcal{S}(\mathcal{G})$. But

$$k_1 * f * k_2 = k_1' * k_\Delta^l * f * k_\Delta^l * k_2' + k_1' * k_\Delta^l * f * r_2 + r_1 * f * k_\Delta^l * k_2' + r_1 * f * r_2$$

so that $k_1 * f * k_2 \in \mathcal{S}(\mathcal{G})$ since $k_\Delta^l * f * k_\Delta^l \in \mathcal{S}(\mathcal{G})$ by definition and $\mathcal{S}(\mathcal{G})$ is an algebra. The first part of this result is then a direct consequence of Theorem 4.2.

Consider the exact sequence

$$0 \to \mathcal{K} \to \mathfrak{A}(\mathcal{G}) \to \mathfrak{A}(\mathcal{G})/\mathcal{K} \to 0.$$

If $P \in \Psi^0_s(\mathcal{G})$ is Fredholm, then its image in $\mathfrak{A}(\mathcal{G})/\mathcal{K}$ is invertible, thus, Theorem 2.8 implies that its inverse lives in $\Psi^0_s(\mathcal{G})/(\Psi^0_s(\mathcal{G}) \cap \mathcal{K})$. This shows that there exists a parametrix in $\Psi^0_s(\mathcal{G})$. □

As Paolo Piazza has kindly informed us, Schwartz-type spaces seem to be important also for questions related to higher signatures [**26**, **27**].

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7.3. Schwartz spaces associated to the c_n groupoids

The continuous family groupoids defined in § 5 can be endowed with length functions in order to define, for each of them, a Schwartz space, thus to obtain an algebra of pseudodifferential operators closed under holomorphic functional calculus. To keep notations simple, we only consider here the case of a compact manifold with boundary. In the case of the b-calculus, we have

$$\Gamma(X) = \{(x, y, \lambda) \in X \times X \times (\mathbb{R}_+^*),\ \rho(x) = \lambda\rho(y)\}.$$

Define $\phi(x, y, \lambda) = |\log(\lambda)|$. It was shown in [**36**] that this defines a length function.

Before considering the case of the c_n-calculi, we need an easy lemma.

Lemma 7.10. *Let $f : \mathcal{G} \to \mathcal{G}'$ be a homeomorphism of continuous family groupoids that preserves the Haar systems. Then any length function with polynomial growth on $\mathcal{G}$ induces a length function with polynomial growth on $\mathcal{G}'$.*

Proof. Assume that $\mathcal{G}$ has a length function with polynomial growth, ϕ. Let $\phi' = \phi \circ f : \mathcal{G}' \to \mathbb{R}_+$. This function is clearly a length function with polynomial growth, since f is a homeomorphism, and the Haar system on $\mathcal{G}'$ is induced by that on $\mathcal{G}$. □

Recall from § 4 that there is a homeomorphism $\Theta_n : \Gamma(X) \to \Gamma_n(X)$ where $\Gamma(X)$ is the groupoid of the b-calculus, and $\Gamma_n(X)$ is the groupoid of the c_n-calculus.

Proposition 7.11. *Let $\phi_n = \phi \circ \Theta_n^{-1}$. Then ϕ_n is a length function. Under the identification*

$$\Gamma_{n+1}(X) = \{(u, v, \mu) \in X \times X \times \mathbb{R},\ \mu\rho(u)^n\rho(v)^n = \rho(u)^n - \rho(v)^n\},$$

one gets $\phi_n(u, v, \mu) = |\mu|$.

Proof. This is clear since $\Theta_n(x, y, \lambda) = (u, v, \log(\lambda))$. □

Proposition 7.11 and Lemma 7.10 thus provide us with an alternative approach to the construction of algebras of pseudodifferential operators closed under holomorphic functional calculus.

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