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A COMBINATORIAL SOLUTION TO MŒGLIN'S PARAMETRIZATION OF ARTHUR PACKETS FOR p-ADIC QUASISPLIT $Sp(N)$ AND $O(N)$

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Abstract We develop a general procedure to study the combinatorial structure of Arthur packets for p-adic quasisplit $Sp(N)$ and $O(N)$ following the works of Mœglin. This will allow us to answer many delicate questions concerning the Arthur packets of these groups, for example the size of the packets.

Keywords: symplectic and orthogonal group; Arthur packet; Jacquet functor

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1. Introduction

Let F be a p-adic field and G be a quasisplit symplectic or special orthogonal group, i.e., $G = Sp(2n)$, $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2n, \eta)$. Here η is a quadratic character associated with a quadratic extension E/F by the local class field theory, and $\mathrm{SO}(2n, \eta)$ is the outer form of the split $\mathrm{SO}(2n)$ with respect to E/F and an outer automorphism θ_0 induced from the conjugate action of $O(2n)$. We let $\theta_0 = \mathrm{id}$ in other cases, and write $\Sigma_0 = \langle\theta_0\rangle$, $G^{\Sigma_0} = G \rtimes \Sigma_0$. So for $G = \mathrm{SO}(2n, \eta)$, $G^{\Sigma_0} \cong O(2n, \eta)$. For simplicity, we denote $G(F)$ by G, which should not cause any confusion in the context. Let $\widehat{G}$ be the complex dual group of G, and ${}^L G$ be the Langlands dual group of G. Here we can simplify the Langlands dual groups as in the following table:

G	${}^L G$
$Sp(2n)$	$\mathrm{SO}(2n+1, \mathbb{C})$
$\mathrm{SO}(2n+1)$	$Sp(2n, \mathbb{C})$
$\mathrm{SO}(2n, \eta)$	$\mathrm{SO}(2n, \mathbb{C}) \rtimes \Gamma_{E/F}$

In the last case, we fix an isomorphism $\mathrm{SO}(2n, \mathbb{C}) \rtimes \Gamma_{E/F} \cong O(2n, \mathbb{C})$. So in either of these cases, there is a natural embedding ξ_N of ${}^L G$ into $\mathrm{GL}(N, \mathbb{C})$ up to $\mathrm{GL}(N, \mathbb{C})$-conjugacy, where $N = 2n + 1$ if $G = Sp(2n)$ or $N = 2n$ otherwise. Let W_F be the Weil group, the local Langlands group can be defined to be

$$L_F := W_F \times \mathrm{SL}(2, \mathbb{C}).$$

An Arthur parameter of G is a $\widehat{G}$-conjugacy class of admissible homomorphisms

$$\underline{\psi} : L_F \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow {}^L G,$$

such that $\underline{\psi}|_{W_F}$ is bounded. We denote the set of Arthur parameters of G by $\Psi(G)$. Let $\widehat{\theta}_0$ be the dual automorphism of θ_0, then Σ_0 acts on $\Psi(G)$ through $\widehat{\theta}_0$, and we denote the corresponding set of Σ_0-orbits by $\bar{\Psi}(G)$. Let $\Pi(G)$ be the set of equivalence classes of irreducible admissible representations of G, and we denote by $\bar{\Pi}(G)$ the set of Σ_0-orbits in $\Pi(G)$. For $\psi \in \bar{\Psi}(G)$, Arthur [1] shows there exists a finite 'multi-set' $\bar{\Pi}_\psi$ of elements in $\bar{\Pi}(G)$, which is related to a certain twisted character on $\mathrm{GL}(N)$ through the twisted endoscopic character identity (cf. [10, § 4]). We call $\bar{\Pi}_\psi$ an Arthur packet of G. Mœglin [7] constructs the elements in $\bar{\Pi}_\psi$, and shows it is in fact *multiplicity free*. As a result, we can also define $\Pi_\psi^{\Sigma_0}$ to be the set of irreducible representations of G^{Σ_0}, whose restriction to G have irreducible constituents in $\bar{\Pi}_\psi$. To understand the structure of $\Pi_\psi^{\Sigma_0}$, we need to introduce the set $\mathrm{Jord}(\psi)$ of Jordan blocks associated with ψ.

For $\psi \in \bar{\Psi}(G)$, by composing with ξ_N we get an equivalence class of N-dimensional self-dual representation of $L_F \times \mathrm{SL}(2, \mathbb{C})$. So we can decompose ψ as follows

$$\psi = \bigoplus_{i=1}^{r} l_i \psi_i = \bigoplus_{i=1}^{r} l_i (\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}). \tag{1.1}$$

Here ρ_i are equivalence classes of irreducible unitary representations of W_F, which can be identified with irreducible unitary supercuspidal representations of $\mathrm{GL}(d_{\rho_i})$ under the local Langlands correspondence (cf. [2, 3], and [9]). And ν_{a_i} (respectively ν_{b_i}) are the $(a_i - 1)$th (respectively $(b_i - 1)$th) symmetric power representations of $\mathrm{SL}(2, \mathbb{C})$. The irreducible constituent $\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}$ has dimension $n_i = n_{(\rho_i, a_i, b_i)}$ and multiplicity l_i. We define the multi-set of Jordan blocks for ψ as follows,

$$\mathrm{Jord}(\psi) := \{(\rho_i, a_i, b_i) \text{ with multiplicity } l_i : 1 \leqslant i \leqslant r\}.$$

Moreover, for any ρ let us define

$$\mathrm{Jord}_\rho(\psi) := \{(\rho', a', b') \in \mathrm{Jord}(\psi) : \rho' = \rho\}.$$

One can define the parity for self-dual irreducible unitary representations ρ of W_F as in [11, § 3]. Then we say (ρ_i, a_i, b_i) is of *orthogonal type* if $a_i + b_i$ is even when ρ_i is of orthogonal type, and $a_i + b_i$ is odd when ρ_i is of symplectic type. Similarly we say (ρ_i, a_i, b_i) is of *symplectic type* if $a_i + b_i$ is odd when ρ_i is of orthogonal type, and $a_i + b_i$ is even when ρ_i is of symplectic type. Let ψ_p be the parameter whose Jordan blocks consist of those in $\mathrm{Jord}(\psi)$ with the same parity as $\widehat{G}$, and let ψ_{np} be any parameter such that

$$\psi = \psi_{np} \oplus \psi_p \oplus \psi_{np}^\vee,$$

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where $\psi_{np}^{\vee}$ is the dual of ψ_{np}. We also denote by $\mathrm{Jord}(\psi)_p$ the set of Jordan blocks in $\mathrm{Jord}(\psi_p)$ without multiplicity. Then let us define

$$\widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}} = \left\{ \varepsilon(\cdot) \in (\mathbb{Z}/2\mathbb{Z})^{\mathrm{Jord}(\psi_p)} : \prod_{(\rho,a,b)\in \mathrm{Jord}(\psi_p)} \varepsilon(\rho, a, b) = 1 \right\}.$$

and

$$\widehat{\mathcal{S}_{\psi}^{\Sigma_0}} = \{\varepsilon \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}} : \varepsilon(\rho, a, b) = \varepsilon(\rho', a', b') \text{ if } (\rho, a, b) = (\rho', a', b') \text{ in } \mathrm{Jord}(\psi)_p\}.$$

If we choose a representative $\underline{\psi} : L_F \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G$, then one can show $\widehat{\mathcal{S}_{\psi}^{\Sigma_0}}$ is canonically isomorphic to the group of characters of the component group of

$$\mathrm{Cent}(\mathrm{Im}\, \underline{\psi}, \widehat{G} \rtimes \langle\widehat{\theta}_0\rangle)/Z(\widehat{G})^{\Gamma_F}.$$

So we also call elements in $\widehat{\mathcal{S}_{\psi}^{\Sigma_0}}$ (and also $\widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$) characters. It follows from Arthur's theory that there is a canonical way to associate any irreducible representation in $\Pi_{\psi}^{\Sigma_0}$ with an element $\varepsilon \in \widehat{\mathcal{S}_{\psi}^{\Sigma_0}}$ (cf. [10, § 8]). Let us denote the direct sum of all irreducible representations associated with $\varepsilon \in \widehat{\mathcal{S}_{\psi}^{\Sigma_0}}$ by $\pi_W^{\Sigma_0}(\psi, \varepsilon)$, then

$$\Pi_{\psi}^{\Sigma_0} = \bigoplus_{\varepsilon \in \widehat{\mathcal{S}_{\psi}^{\Sigma_0}}} \pi_W^{\Sigma_0}(\psi, \varepsilon), \tag{1.2}$$

where we identify $\Pi_{\psi}^{\Sigma_0}$ with the direct sum of all its elements.

Mœglin's construction of $\Pi_{\psi}^{\Sigma_0}$ comes with a parametrization by $\widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$. It also depends on some total order $>_{\psi}$ on $\mathrm{Jord}_{\rho}(\psi_p)$ for each ρ. To describe the condition on $>_{\psi}$, we need to write the Jordan blocks differently. For $(\rho, a, b) \in \mathrm{Jord}(\psi_p)$, let us write $A = (a+b)/2 - 1$, $B = |a-b|/2$, and set $\zeta = \zeta_{a,b} = \mathrm{Sign}(a-b)$ if $a \neq b$ and arbitrary otherwise. Then we can denote (ρ, a, b) also by (ρ, A, B, ζ). We say $>_{\psi}$ is 'admissible' if it satisfies

$$(\mathcal{P}): \quad \forall (\rho, A, B, \zeta), (\rho, A', B', \zeta') \in \mathrm{Jord}(\psi_p) \quad \text{with } A > A', B > B' \text{ and } \zeta = \zeta',$$
$$\text{then } (\rho, A, B, \zeta) >_{\psi} (\rho, A', B', \zeta').$$

Since the sign ζ is relevant in this condition, Mœglin's parametrization will also depend on the choice of $\zeta_{a,b}$, when $a = b$. First, we have

$$\Pi_{\psi}^{\Sigma_0} = \bigoplus_{\varepsilon \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}} \pi_{M,>_{\psi}}^{\Sigma_0}(\psi, \varepsilon). \tag{1.3}$$

The following theorem gives the connection between (1.3) and (1.2).

Theorem 1.1 [10, Theorem 8.9]. *There exists a character of* $\varepsilon_{\psi}^{M/W} \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$ *such that*

$$\pi_{M,>_{\psi}}^{\Sigma_0}(\psi, \varepsilon) = \begin{cases} \pi_W^{\Sigma_0}(\psi, \varepsilon\varepsilon_{\psi}^{M/W}), & \textit{if } \varepsilon\varepsilon_{\psi}^{M/W} \in \widehat{\mathcal{S}_{\psi}^{\Sigma_0}}, \\ 0, & \textit{otherwise.} \end{cases}$$

In [10], we define

$$\varepsilon_\psi^{M/W} := \varepsilon_\psi^{MW/W} \varepsilon_\psi^{M/MW}, \tag{1.4}$$

for $\varepsilon_\psi^{MW/W}$ and $\varepsilon_\psi^{M/MW}$ in $\widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$. Here we recall the definition of these characters.

To define $\varepsilon_\psi^{MW/W}$, we need to first define a set $\mathcal{Z}_{MW/W}(\psi)$ of *unordered pairs* of Jordan blocks from $\mathrm{Jord}(\psi_p)$ as follows. We call a pair $\{(\rho, a, b), (\rho', a', b') \in \mathrm{Jord}(\psi_p)\}$ is contained in $\mathcal{Z}_{MW/W}(\psi)$ if and only if $\rho = \rho'$, and it is in one of the following situations.

(1) Case: a, b are even and a', b' are odd.

(a) If $\zeta_{a,b} = -1$ and $\begin{cases} \zeta_{a',b'} = -1 \Rightarrow (\rho, a, b) >_\psi (\rho, a', b'), a > a' \\ \zeta_{a',b'} = +1 \Rightarrow a > a'. \end{cases}$

(b) If $\zeta_{a,b} = \zeta_{a',b'} = +1$ and $\begin{cases} (\rho, a, b) >_\psi (\rho, a', b') \Rightarrow a' > a, b > b' \\ (\rho, a, b) <_\psi (\rho, a', b') \Rightarrow a > a', b > b'. \end{cases}$

(2) Case: a is odd, b is even and a' is even, b' is odd.

(a) If $\zeta_{a,b} = -1$ and $\begin{cases} \zeta_{a',b'} = -1 \Rightarrow (\rho, a, b) >_\psi (\rho, a', b'), a < a' \\ \zeta_{a',b'} = +1 \text{ and } \begin{cases} (\rho, a, b) >_\psi (\rho, a', b') \Rightarrow a < a' \\ (\rho, a, b) <_\psi (\rho, a', b') \Rightarrow a > a'. \end{cases} \end{cases}$

(b) If $\zeta_{a,b} = \zeta_{a',b'} = +1$ and $\begin{cases} (\rho, a, b) >_\psi (\rho, a', b') \Rightarrow a < a', b > b' \\ (\rho, a, b) <_\psi (\rho, a', b') \Rightarrow a > a', b > b'. \end{cases}$

For $(\rho, a, b) \in \mathrm{Jord}(\psi_p)$, let

$$\mathcal{Z}_{MW/W}(\psi)_{(\rho,a,b)} := \{(\rho', a', b') \in \mathrm{Jord}(\psi_p) : \text{the pair of } (\rho, a, b) \text{ and } (\rho', a', b') \text{ lies in } \mathcal{Z}_{MW/W}(\psi)\}.$$

Then we can define

$$\varepsilon_\psi^{MW/W}(\rho, a, b) := (-1)^{|\mathcal{Z}_{MW/W}(\psi)_{(\rho,a,b)}|}.$$

Next, we define $\varepsilon_\psi^{M/MW}$ according to the following rule. Let $(\rho, a, b) \in \mathrm{Jord}(\psi_p)$.

(1) If $a + b$ is odd, $\varepsilon_\psi^{M/MW}(\rho, a, b) = 1$.

(2) If $a + b$ is even, let

$$m = \sharp\{(\rho, a', b') \in \mathrm{Jord}(\psi) : a', b' \text{ odd}, \zeta_{a',b'} = -1, (\rho, a', b') >_\psi (\rho, a, b)\},$$

and

$$n = \sharp\{(\rho, a', b') \in \mathrm{Jord}(\psi) : a', b' \text{ odd}, (\rho, a', b') <_\psi (\rho, a, b)\}.$$

Then

$$\varepsilon_\psi^{M/MW}(\rho, a, b) = \begin{cases} 1 & \text{if } a, b \text{ even}, \\ (-1)^m & \text{if } a, b \text{ odd}, \zeta_{a,b} = +1, \\ (-1)^{m+n} & \text{if } a, b \text{ odd}, \zeta_{a,b} = -1. \end{cases}$$

Mœglin further parametrizes the irreducible constituents in $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \varepsilon)$. To describe that, we need to briefly go through all the stages of Mœglin's construction of $\Pi^{\Sigma_0}_\psi$. Let us denote by ψ_d the composition of ψ with

$$\Delta : W_F \times \mathrm{SL}(2, \mathbb{C}) \to W_F \times \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}),$$

which is the diagonal embedding of $\mathrm{SL}(2, \mathbb{C})$ into $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ when restricted to $\mathrm{SL}(2, \mathbb{C})$, and is identity on W_F. It is easy to see

$$\mathrm{Jord}(\psi_d) = \bigcup_{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)} \bigcup_{C\in[B,A]} \{(\rho, C, C, +1)\}.$$

We call ψ has *discrete diagonal restriction* if $\psi = \psi_p$ and $\mathrm{Jord}(\psi_d)$ is multiplicity free. Note the second condition is equivalent to saying the intervals $[B, A], [B', A']$ do not intersect for any $(\rho, A, B, \zeta), (\rho, A', B', \zeta')$ in $\mathrm{Jord}_\rho(\psi)$. In this case, $\mathrm{Jord}_\rho(\psi)$ has a natural order $>_\psi$, namely

$$(\rho, A, B, \zeta) >_\psi (\rho, A', B', \zeta') \quad \text{if and only if } A > A'.$$

Among the parameters with discrete diagonal restriction, we call ψ is *elementary* if $A = B$ for all $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$. For the elementary parameters, Mœglin [5] shows $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \varepsilon)$ is irreducible.

Suppose ψ has discrete diagonal restriction, Mœglin shows the irreducible constitutes of $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \varepsilon)$ can be parametrized by pairs of integer-valued functions $(\underline{l}, \underline{\eta})$ over $\mathrm{Jord}(\psi)$, such that

$$\underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]] \quad \text{and} \quad \underline{\eta}(\rho, A, B, \zeta) \in \{\pm 1\}, \tag{1.5}$$

and

$$\varepsilon(\rho, A, B, \zeta) = \varepsilon_{\underline{l},\underline{\eta}}(\rho, A, B, \zeta) := \underline{\eta}(\rho, A, B, \zeta)^{A-B+1}(-1)^{[(A-B+1)/2]+\underline{l}(\rho,A,B,\zeta)}. \tag{1.6}$$

Moreover,

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow\ & \times_{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)} \\ & \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B + \underline{l}(\rho, A, B, \zeta) - 1) & \cdots & -\zeta(A - \underline{l}(\rho, A, B, \zeta) + 1) \end{pmatrix} \\ & \times \pi^{\Sigma_0}_M \Bigg(\bigcup_{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)} \bigcup_{C\in[B+\underline{l}(\rho,A,B,\zeta), A-\underline{l}(\rho,A,B,\zeta)]} \\ & \times (\rho, C, C, \underline{\eta}(\rho, A, B, \zeta)(-1)^{C-B-\underline{l}(\rho,A,B,\zeta)}, \zeta) \Bigg)\end{aligned}$$

as the unique irreducible subrepresentation (see (3.2) and (3.5)). There is an obvious equivalence relation to be made here on pairs $(\underline{l}, \underline{\eta})$, namely

$$(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$$

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if and only if $\underline{l} = \underline{l}'$ and $(\underline{\eta}/\underline{\eta}')(\rho, A, B, \zeta) = 1$ unless $\underline{l}(\rho, A, B, \zeta) = (A - B + 1)/2$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \varepsilon) = \bigoplus_{\{(\underline{l},\underline{\eta}):\, \varepsilon=\varepsilon_{\underline{l},\underline{\eta}}\}/\sim_{\Sigma_0}} \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}). \tag{1.7}$$

To get to the more general case $\psi = \psi_p$, we need to choose an admissible order $>_\psi$ on $\mathrm{Jord}(\psi)$. We can index $\mathrm{Jord}_\rho(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

We say $\psi_\gg$ dominates ψ with respect to $>_\psi$ if $\mathrm{Jord}_\rho(\psi_\gg)$ consists of $(\rho, A_{\gg,i}, B_i + T_{\gg,i}, \zeta_{\gg,i}) := (\rho, A_i + T_i, B_i + T_i, \zeta_i)$ for $T_i \geqslant 0$, and inherits the same admissible order $>_\psi$. We can further choose $\psi_\gg$ to have discrete diagonal restriction with the natural order $>_\psi$. After identifying $\mathrm{Jord}(\psi)$ with $\mathrm{Jord}(\psi_\gg)$ in the natural way, we can define for any pair of functions $(\underline{l}, \underline{\eta})$ satisfying (1.5) and (1.6),

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) := {}& \circ_{\{\rho:\mathrm{Jord}_\rho(\psi)\neq\emptyset\}} \circ_{(\rho,A_i,B_i,\zeta_i)\in\mathrm{Jord}_\rho(\psi)} \\ & \times \mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}),\end{aligned} \tag{1.8}$$

where i is decreasing in the composition of Jacquet functors (see (3.7)). (This definition is different from that in [10, § 8], for there we take a total order $>_\psi$ on $\mathrm{Jord}(\psi)$. But it follows from Lemma 3.2 that only the restriction of $>_\psi$ to $\mathrm{Jord}_\rho(\psi)$ for each ρ matters, and the two definitions will give the same result.) Then Mœglin shows the following facts (cf. [10, Proposition 8.5 and Corollary 8.7]):

(1) $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ only depends on the choice of order $>_\psi$, and it is either irreducible or zero.

(2) If $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \cong \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0$, then $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$.

(3) The decomposition (1.7) still holds.

Finally, for general $\psi \in \bar{\Psi}(G)$, we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \varepsilon) = (\times_{(\rho,a,b)\in\mathrm{Jord}(\psi_{np})} Sp(St(\rho, a), b)) \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_p, \varepsilon)$$

(see (3.3)). Moreover, Mœglin shows

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) := (\times_{(\rho,a,b)\in\mathrm{Jord}(\psi_{np})} Sp(St(\rho, a), b)) \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_p, \underline{l}, \underline{\eta})$$

is irreducible (cf. [4, Theorem 6]), when $\pi^{\Sigma_0}_{M,>_\psi}(\psi_p, \underline{l}, \underline{\eta}) \neq 0$.

To summarize, for $\psi \in \bar{\Psi}(G)$ we can refine the decomposition (1.3) as follows

$$\Pi^{\Sigma_0}_\psi = \bigoplus_{\{(\underline{l},\underline{\eta}):\, \prod_{(\rho,a,b)\in\mathrm{Jord}(\psi_p)} \varepsilon_{\underline{l},\underline{\eta}}(\rho,a,b)=1\}/\sim_{\Sigma_0}} \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}), \tag{1.9}$$

where $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ is either irreducible or zero. So it is natural to ask the following question:

Question 1.2. When $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$?

The main goal of this paper is to answer this question. An explicit answer to this question will certainly allow us to count the size of the Arthur packets (see Example B.1). In fact, it contains more information than that. For example, one can use it to determine the zeros of local normalized intertwining operators, which is important for describing the residue spectrum of automorphic forms (see [7]). Another example is to use it to characterize the image of the local theta correspondence of type I in many cases (see [6]). We hope the results and techniques of this paper will open the door for investigating many other delicate questions concerning the Arthur packets.

Now we describe our results. In the simplest case, one can consider $\mathrm{Jord}(\psi_p)$ consisting of two Jordan blocks $(\rho, A_2, B_2, \zeta), (\rho, A_1, B_1, \zeta)$ satisfying $A_2 \geqslant A_1, B_2 \geqslant B_1$. Let the order be

$$(\rho, A_2, B_2, \zeta) >_\psi (\rho, A_1, B_1, \zeta).$$

In Proposition 5.2, we prove $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if

$$\begin{cases} \eta_2 = (-1)^{A_1-B_1}\eta_1 & \Rightarrow A_2 - l_2 \geqslant A_1 - l_1, \quad B_2 + l_2 \geqslant B_1 + l_1, \\ \eta_2 \neq (-1)^{A_1-B_1}\eta_1 & \Rightarrow B_2 + l_2 > A_1 - l_1. \end{cases}$$

Suppose we change the assumption such that $B_2 \leqslant B_1$. Then in Lemma 6.2, we prove $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if

$$\begin{cases} \eta_2 = (-1)^{A_1-B_1}\eta_1 & \Rightarrow 0 \leqslant l_2 - l_1 \leqslant (A_2 - B_2) - (A_1 - B_1), \\ \eta_2 \neq (-1)^{A_1-B_1}\eta_1 & \Rightarrow l_2 + l_1 > A_1 - B_1. \end{cases}$$

The general case is much more complicated, because we also need to take account of pairs of Jordan blocks, which are not necessarily in adjacent positions under the chosen order. To do so, we need to change the order, and the point is the parametrization will change as well. So in the end, we develop a procedure (cf. § 8), and it will give rise to some explicit combinatorial conditions on $(\underline{l}, \underline{\eta})$ like those in the case above. In Appendix B, we give an example to demonstrate how it works.

The key input in our procedure is an explicit formula describing how the parametrization changes with respect to the change of order $>_\psi$ when $\psi = \psi_p$ (cf. § 6). Since this result could have interests by itself, we would like to state it here. Let us consider any two adjacent Jordan blocks $(\rho, A_i, B_i, \zeta_i)$ $(i = 1, 2)$ under an admissible order $>_\psi$ with

$$(\rho, A_2, B_2, \zeta_2) >_\psi (\rho, A_1, B_1, \zeta_1).$$

Suppose the new order $>'_\psi$ obtained by switching the two is still admissible. Then by definition, either $\zeta_1 \neq \zeta_2$ or one of $\{[B_i, A_i]\}_{i=1,2}$ is included in the other. Let us define ψ_- by

$$\mathrm{Jord}(\psi_-) = \mathrm{Jord}(\psi) \backslash \{(\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\}.$$

Theorem 1.3 (cf. Theorems 6.1 and 6.3). *Suppose*

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0,$$

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then the restrictions of $(\underline{l}, \underline{\eta})$ *and* $(\underline{l}', \underline{\eta}')$ *to* Jord(ψ_-) *are equivalent* $(\sim_{\Sigma_0})$ *and the following conditions are satisfied.*

(1) *If* $\zeta_1 = \zeta_2$*, it suffices to consider the case* $[B_2, A_2] \supseteq [B_1, A_1]$*. Then we are in one of the following situations.*

(a) *If* $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$ *and* $\eta_1' = (-1)^{A_2-B_2}\eta_2'$*, then*

$$\begin{cases} l_1 = l_1' \\ l_2 - l_2' = (A_1 - B_1 - 2l_1) + 1 \\ \eta_1' = (-1)^{A_2-B_2}\eta_1. \end{cases}$$

(b) *If* $\eta_2 = (-1)^{A_1-B_1}\eta_1$ *and* $\eta_1' \neq (-1)^{A_2-B_2}\eta_2'$*, then*

$$\begin{cases} l_1 = l_1' \\ l_2' - l_2 = (A_1 - B_1 - 2l_1) + 1 \\ \eta_1' = (-1)^{A_2-B_2}\eta_1. \end{cases}$$

(c) *If* $\eta_2 = (-1)^{A_1-B_1}\eta_1$ *and* $\eta_1' = (-1)^{A_2-B_2}\eta_2'$*, then*

$$\begin{cases} l_1 = l_1' \\ (l_2' - l_1') + (l_2 - l_1) = (A_2 - B_2) - (A_1 - B_1) \\ \eta_1' = (-1)^{A_2-B_2}\eta_1. \end{cases}$$

(2) *If* $\zeta_1 \neq \zeta_2$*, then*

$$\begin{cases} l_2' = l_2 \\ l_1' = l_1 \\ \eta_2 = (-1)^{A_1-B_1+1}\eta_2' \\ \eta_1 = (-1)^{A_2-B_2+1}\eta_1'. \end{cases}$$

In both cases, we have denoted

$$l_i = \underline{l}(\rho, A_i, B_i, \zeta_i), \quad l_i' = \underline{l}'(\rho, A_i, B_i, \zeta_i),$$

and

$$\eta_i = \underline{\eta}(\rho, A_i, B_i, \zeta_i), \quad \eta_i' = \underline{\eta}'(\rho, A_i, B_i, \zeta_i),$$

for $i = 1, 2$.

2. Conventions

Now we want to set up some conventions for this paper. We follow the notations in the introduction. Since only ψ_p is relevant in answering Question 1.2, we assume $\psi = \psi_p$ in the rest of the paper. We also require $(\underline{l}, \underline{\eta})$ to always satisfy

$$\prod_{(\rho,a,b)\in \mathrm{Jord}(\psi)} \varepsilon_{\underline{l},\underline{\eta}}(\rho, a, b) = 1. \tag{2.1}$$

So we will not write down this condition later in the paper.

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In many arguments of the paper, we need to fix a self-dual irreducible unitary supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$. So if $\psi_\gg$ is a dominating parameter of ψ, we would like to define

$$\mathrm{Jac}_{X^c} := \circ_{\rho' \neq \rho} \circ_{(\rho', A', B', \zeta') \in \mathrm{Jord}_{\rho'}(\psi)} \mathrm{Jac}_{(\rho', A'_\gg, B'_\gg, \zeta') \mapsto (\rho', A', B', \zeta')}$$

and

$$\mathcal{C}_{X^c} := \times_{\rho' \neq \rho} \times_{(\rho', A', B', \zeta') \in \mathrm{Jord}_{\rho'}(\psi)} \begin{pmatrix} \zeta' B'_\gg & \cdots & \zeta'(B'+1) \\ \vdots & & \vdots \\ \zeta' A'_\gg & \cdots & \zeta'(A'+1) \end{pmatrix}.$$

Since we are taking $\rho' \neq \rho$ in Jac_{X^c} (respectively $\mathcal{C}_{X^c}$), it will 'commute' with all kinds of Jacquet functors (respectively induced modules) defined with respect to ρ in our arguments (see Lemma 3.2, Corollary 4.3). Later we use this property freely without mentioning it.

Finally, for a fixed ρ, we often need to put apart some subset of $\mathrm{Jord}_\rho(\psi)$ in different ways. Here we want to quantify the corresponding notions.

(1) Suppose $(\rho, A, B, \zeta) \in \mathrm{Jord}_\rho(\psi)$ and r is a positive integer, we say (ρ, A, B, ζ) (or $[A, B]$) is in level r 'far away', if

$$B > 2^r \cdot \sum_{(\rho, A', B', \zeta') \in \mathrm{Jord}_\rho(\psi)} (A' - B' + 1),$$

and we write

$$(\rho, A, B, \zeta) \gg_r 0 \quad \text{or } (\rho, A, B, \zeta) \gg 0 \text{ when } r = 1.$$

(2) Suppose $(\rho, A, B, \zeta) \in \mathrm{Jord}_\rho(\psi)$ and r is a positive integer, we say (ρ, A, B, ζ) (or $[A, B]$) is in level r 'far away' from a subset J of $\mathrm{Jord}_\rho(\psi)$, if

$$B > 2^{r|J|} \cdot \left(\sum_{(\rho, A', B', \zeta') \in J} A' + |J| \sum_{(\rho, A', B', \zeta') \in \mathrm{Jord}_\rho(\psi)} (A' - B' + 1) \right),$$

and we write

$$(\rho, A, B, \zeta) \gg_r J \quad \text{or} \quad (\rho, A, B, \zeta) \gg J \text{ when } r = 1.$$

(3) For a subset J of $\mathrm{Jord}_\rho(\psi)$, we denote its complement in $\mathrm{Jord}_\rho(\psi)$ by J^c. We say J is 'separated' from J^c, if the following conditions are satisfied.

(a) For any $(\rho, A, B, \zeta) \in J$ and $(\rho, A', B', \zeta') \in J^c$,

$$\text{either } B' > A \text{ or } B > A'.$$

(b) For any admissible order $>_J$ on J, there exists a dominating set of Jordan blocks $J_\gg$ of J with discrete diagonal restriction, such that for any $(\rho, A, B, \zeta) \in J$ and $(\rho, A', B', \zeta') \in J^c$,

$$\text{if } B' > A \text{ then } B' > A_\gg.$$

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(c) There exists an admissible order $>_{J^c}$ on J^c, under which one can find a dominating set of Jordan blocks $J^c_{\gg}$ of J^c with discrete diagonal restriction, such that for any $(\rho, A, B, \zeta) \in J$ and $(\rho, A', B', \zeta') \in J^c$,

$$\text{if } B > A' \text{ then } B > A'_{\gg}.$$

In application, what is important is only the fact that these notions ('far away', 'separated') can be quantified, but not the specific way that we quantify them. For example, once we can measure what it means for some Jordan blocks to be 'far away' from all the others, we can just take them as far as we want in practice.

3. Parabolic induction and Jacquet module

We review the notations in [10, 11]. For $\mathrm{GL}(n)$, let us take B to be the group of upper-triangular matrices and T to be the group of diagonal matrices, then the standard Levi subgroup M can be identified with

$$\mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_r)$$

for any partition of $n = n_1 + \cdots + n_r$ as follows

$$\begin{pmatrix} \mathrm{GL}(n_1) & & \\ & \ddots & \\ & & \mathrm{GL}(n_r) \end{pmatrix}$$
$$(g_1, \ldots, g_r) \longrightarrow \mathrm{diag}\{g_1, \ldots, g_r\}.$$

For $\pi = \pi_1 \otimes \cdots \otimes \pi_r$, where π_i is a finite-length admissible representation of $\mathrm{GL}(n_i)$ for $1 \leqslant i \leqslant r$, we denote the normalized parabolic induction $\mathrm{Ind}^G_P(\pi)$ by

$$\pi_1 \times \cdots \times \pi_r.$$

An irreducible supercuspidal representation of a general linear group can always be written in a unique way as $\rho||^x := \rho \otimes |\det(\cdot)|^x$ for an irreducible unitary supercuspidal representation ρ and a real number x. For a finite-length arithmetic progression of real numbers of common length 1 or -1

$$x, \ldots, y$$

and an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$, it is a general fact that

$$\rho||^x \times \cdots \times \rho||^y$$

has a unique irreducible subrepresentation, denoted by $\langle \rho; x, \ldots, y \rangle$ or $\langle x, \ldots, y \rangle$. If $x \geqslant y$, it is called a Steinberg representation; if $x < y$, it is called a Speh representation. Such sequence of ordered numbers is called a *segment*, and we denote it by $[x, y]$ or $\{x, \ldots, y\}$. In particular, when $x = -y > 0$, we can let $a = 2x + 1 \in \mathbb{Z}$ and write

$$St(\rho, a) := \left\langle \frac{a-1}{2}, \ldots, -\frac{a-1}{2} \right\rangle.$$

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We also define a *generalized segment* to be a matrix

$$\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} \tag{3.1}$$

such that each row is a decreasing (respectively increasing) segment and each column is an increasing (respectively decreasing) segment. The normalized induction

$$\times_{i\in[1,m]}\langle\rho; x_{i1}, \ldots, x_{in}\rangle$$

has a unique irreducible subrepresentation, and we denote it by $\langle\rho; \{x_{ij}\}_{m\times n}\rangle$. If there is no ambiguity with ρ, we also write it as $\langle\{x_{ij}\}_{m\times n}\rangle$ or

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}. \tag{3.2}$$

Moreover,

$$\langle\rho; \{x_{ij}\}_{m\times n}\rangle \cong \langle\rho; \{x_{ij}\}^T_{m\times n}\rangle$$

where $\{x_{ij}\}^T_{m\times n}$ is the transpose of $\{x_{ij}\}_{m\times n}$. The dual of $\langle\rho; \{x_{ij}\}_{m\times n}\rangle$ is

$$\langle\rho; \{x_{ij}\}_{m\times n}\rangle^\vee \cong \begin{pmatrix} -x_{mn} & \cdots & -x_{m1} \\ \vdots & & \vdots \\ -x_{1n} & \cdots & -x_{11} \end{pmatrix}.$$

Let a, b be positive integers, we define $Sp(St(\rho, a), b)$ to be the unique irreducible subrepresentation of

$$St(\rho, a)|\,|^{-(b-1)/2} \times St(\rho, a)|\,|^{-(b-3)/2} \times \cdots \times St(\rho, a)|\,|^{(b-1)/2}. \tag{3.3}$$

Then one can see $Sp(St(\rho, a), b)$ is given by the following generalized segment

$$\begin{bmatrix} (a-b)/2 & \cdots & 1-(a+b)/2 \\ \vdots & & \vdots \\ (a+b)/2-1 & \cdots & -(a-b)/2 \end{bmatrix}.$$

If $G = Sp(2n)$, let us define it with respect to

$$\begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix},$$

where

$$J_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

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Let us take B to be subgroup of upper-triangular matrices in G and T to be subgroup of diagonal matrices in G, then the standard Levi subgroup M can be identified with

$$\mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_r) \times G_-$$

for any partition $n = n_1 + \cdots + n_r + n_-$ and $G_- = Sp(2n_-)$ as follows

$$\begin{pmatrix} \mathrm{GL}(n_1) & & & & & & 0 \\ & \ddots & & & & & \\ & & \mathrm{GL}(n_r) & & & & \\ & & & G_- & & & \\ & & & & \mathrm{GL}(n_r) & & \\ & & & & & \ddots & \\ 0 & & & & & & \mathrm{GL}(n_1) \end{pmatrix}$$

$$(g_1, \ldots g_r, g) \longrightarrow \mathrm{diag}\{g_1, \ldots, g_r, g, {}_tg_r^{-1}, \ldots, {}_tg_1^{-1}\}, \tag{3.4}$$

where ${}_tg_i = J_{n_i} {}^tg_i J_{n_i}^{-1}$ for $1 \leqslant i \leqslant r$. Note n_- can be 0, in which case we simply write $Sp(0) = 1$. For $\pi = \pi_1 \otimes \cdots \otimes \pi_r \otimes \sigma$, where π_i is a finite-length admissible representation of $\mathrm{GL}(n_i)$ for $1 \leqslant i \leqslant r$ and σ is a finite-length admissible representation of G_-, we denote the normalized parabolic induction $\mathrm{Ind}_P^G(\pi)$ by

$$\pi_1 \times \cdots \times \pi_r \rtimes \sigma. \tag{3.5}$$

These notations can be easily extended to special orthogonal groups. If $G = \mathrm{SO}(N)$ split, we define it with respect to J_N. When N is odd, the situation is exactly the same as the symplectic case. When $N = 2n$, there are two distinctions. First, the standard Levi subgroups given through the embedding (3.4) do not exhaust all standard Levi subgroups of $\mathrm{SO}(2n)$. To get all of them, we need to take the θ_0-conjugate of M given in (3.4), where

$$\theta_0 = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Note $M^{\theta_0} \neq M$ only when $n_- = 0$ and $n_r > 1$. In this paper, we only take those Levi subgroups M given in (3.4). Second, if the partition $n = n_1 + \cdots + n_r + n_-$ satisfies $n_r = 1$ and $n_- = 0$, then we can rewrite it as $n = n_1 + \cdots + n_{r-1} + n'_-$ with $n'_- = 1$, and the corresponding Levi subgroup is the same. This is because $\mathrm{GL}(1) \cong \mathrm{SO}(2)$. If $G = \mathrm{SO}(2n, \eta)$, the standard Levi subgroups of $\mathrm{SO}(2n, \eta)$ will be the outer form of those θ_0-stable standard Levi subgroups of $\mathrm{SO}(2n)$. In particular, they can be identified with $\mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_r) \times \mathrm{SO}(n_-, \eta)$ and $n_- \neq 0$.

Next we want to define parabolic induction and Jacquet module for the category $\mathrm{Rep}(G^{\Sigma_0})$ of finite-length representations of G^{Σ_0}. Let $P = MN$ be a standard parabolic subgroup of G. If M is θ_0-stable, we write $M^{\Sigma_0} := M \rtimes \Sigma_0$. Otherwise, we let $M^{\Sigma_0} = M$. Suppose $\sigma^{\Sigma_0} \in \mathrm{Rep}(M^{\Sigma_0})$, $\pi^{\Sigma_0} \in \mathrm{Rep}(G^{\Sigma_0})$.

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(1) If $M^{\theta_0} = M$, we define the normalized parabolic induction $\mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0}$ to be the extension of the representation $\mathrm{Ind}_P^G(\sigma^{\Sigma_0}|_M)$ by an induced action of Σ_0, and we define the normalized Jacquet module $\mathrm{Jac}_{P^{\Sigma_0}} \pi^{\Sigma_0}$ to be the extension of the representation $\mathrm{Jac}_P(\pi^{\Sigma_0}|_G)$ by an induced action of Σ_0.

(2) If $M^{\theta_0} \neq M$, we define the normalized parabolic induction $\mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0}$ to be $\mathrm{Ind}_G^{G^{\Sigma_0}} \mathrm{Ind}_P^G(\sigma^{\Sigma_0}|_M)$, and we define the normalized Jacquet module $\mathrm{Jac}_{P^{\Sigma_0}} \pi^{\Sigma_0}$ to be $\mathrm{Jac}_P(\pi^{\Sigma_0}|_G)$.

Let ρ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(d_\rho)$, and $M = \mathrm{GL}(d_\rho) \times G_-$ be the Levi component of a standard maximal parabolic subgroup P of G. For $\pi^{\Sigma_0} \in \mathrm{Rep}(G^{\Sigma_0})$, we can decompose the semisimplification of the Jacquet module

$$s.s.\mathrm{Jac}_{P^{\Sigma_0}}(\pi^{\Sigma_0}) = \bigoplus_i \tau_i \otimes \sigma_i,$$

where $\tau_i \in \mathrm{Rep}(\mathrm{GL}(d_\rho))$ and $\sigma_i \in \mathrm{Rep}(G_-^{\Sigma_0})$, both of which are irreducible. We define $\mathrm{Jac}_x \pi^{\Sigma_0}$ for any real number x to be

$$\mathrm{Jac}_x(\pi) = \bigoplus_{\tau_i = \rho|\,|^x} \sigma_i. \tag{3.6}$$

If we have an ordered sequence of real numbers $\{x_1, \ldots, x_s\}$, we can define

$$\mathrm{Jac}_{x_1,\ldots,x_s} \pi^{\Sigma_0} = \mathrm{Jac}_{x_s} \circ \cdots \circ \mathrm{Jac}_{x_1} \pi^{\Sigma_0}.$$

For a generalized segment X (cf. (3.1)), we define $\mathrm{Jac}_X := \circ_{x \in X} \mathrm{Jac}_x$, where x ranges over X from top to bottom and left to right. Similarly, we can define Jac_x^{op} analogous to Jac_x, but with respect to $\rho^\vee$ and the standard Levi subgroup $\mathrm{GL}(n_-) \times \mathrm{GL}(d_{\rho^\vee})$.

For $\psi \in \bar{\Psi}(G)$, let $\psi_\gg$ be a dominating parameter of ψ with respect to certain admissible order $>_\psi$. For $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$, we define

$$\mathrm{Jac}_{(\rho, A_\gg, B_\gg, \zeta) \mapsto (\rho, A, B, \zeta)} := \mathrm{Jac}_{X^\gg_{(\rho, A, B, \zeta)}} \tag{3.7}$$

where

$$X^\gg_{(\rho,A,B,\zeta)} = \begin{bmatrix} \zeta B_\gg & \cdots & \zeta(B+1) \\ \vdots & & \vdots \\ \zeta A_\gg & \cdots & \zeta(A+1) \end{bmatrix}.$$

The following lemmas are very useful when we want to permute the Jacquet functors defined in (3.6).

Lemma 3.1 [11, Lemma 5.6]. *If $\pi^{\Sigma_0} \in Rep(G^{\Sigma_0})$ and $|x - y| \neq 1$, then*

$$Jac_{x,y} \pi^{\Sigma_0} = Jac_{y,x} \pi^{\Sigma_0}.$$

Lemma 3.2. *Let ρ, ρ' be two distinct unitary irreducible supercuspidal representations of general linear groups, and x, y be any two real numbers. For $\pi^{\Sigma_0} \in Rep(G^{\Sigma_0})$,*

$$Jac'_y \circ Jac_x \pi^{\Sigma_0} = Jac_x \circ Jac'_y \pi^{\Sigma_0},$$

where Jac_x (respectively Jac'_y) is defined with respect to ρ (respectively ρ').

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Proof. The proof is the same as Lemma 3.1. □

There are some explicit formulas for computing the Jacquet modules in the case of classical groups and general linear groups (cf. [11, § 5]). Since we use them quite often, let us recall them here. We fix a unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$, and take '$\overset{s.s.}{=}$' for equality after semisimplification.

For any decreasing segment $\{a, \ldots, b\}$ and $\zeta = \pm 1$,

$$\mathrm{Jac}_x \langle \rho'; \zeta a, \ldots, \zeta b \rangle = \begin{cases} \langle \rho'; \zeta(a-1), \ldots, \zeta b \rangle, & \text{if } x = \zeta a \text{ and } \rho' \cong \rho, \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\mathrm{Jac}^{op}_x \langle \rho'; \zeta a, \ldots, \zeta b \rangle = \begin{cases} \langle \rho'; \zeta a, \ldots, \zeta(b+1) \rangle, & \text{if } x = \zeta b \text{ and } \rho' \cong \rho^\vee, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\pi_i \in \mathrm{Rep}(\mathrm{GL}(n_i))$ for $i = 1$ or 2, we have

$$\mathrm{Jac}_x(\pi_1 \times \pi_2) \overset{s.s.}{=} (\mathrm{Jac}_x \pi_1) \times \pi_2 \oplus \pi_1 \times (\mathrm{Jac}_x \pi_2),$$

and

$$\mathrm{Jac}^{op}_x(\pi_1 \times \pi_2) \overset{s.s.}{=} (\mathrm{Jac}^{op}_x \pi_1) \times \pi_2 \oplus \pi_1 \times (\mathrm{Jac}^{op}_x \pi_2).$$

Suppose $\pi^{\Sigma_0} \in \mathrm{Rep}(G)$ and $\tau \in \mathrm{Rep}(\mathrm{GL}(d))$, we have

$$\mathrm{Jac}_x(\tau \rtimes \pi^{\Sigma_0}) \overset{s.s.}{=} (\mathrm{Jac}_x \tau) \rtimes \pi^{\Sigma_0} \oplus (\mathrm{Jac}^{op}_{-x} \tau) \rtimes \pi^{\Sigma_0} \oplus \tau \rtimes \mathrm{Jac}_x \pi^{\Sigma_0}.$$

Finally, we want to recall the following vanishing result for Jacquet modules of elements in the Arthur packets. This will become very useful when we want to simplify the results of Jacquet modules after applying the above formulas.

Proposition 3.3 [10, Proposition 8.3]. *Suppose $\psi \in \bar{\Psi}(G)$ and $\pi^{\Sigma_0} \in \Pi^{\Sigma_0}_\psi$. Let ρ be a unitary irreducible supercuspidal representation of $\mathrm{GL}(d_\rho)$.*

(1) *For $\zeta \in \{\pm 1\}$ and segment $[x, y]$ with $0 \leqslant x \leqslant y$,*

$$Jac_{\zeta x, \ldots, \zeta y}\, \pi^{\Sigma_0} = 0,$$

unless there exists a sequence of Jordan blocks $\{(\rho, A_i, B_i, \zeta)\}_{i=1}^n \subseteq \mathrm{Jord}_\rho(\psi)$ such that $B_1 = x$, $A_n > y$, and $B_i \leqslant B_{i+1} \leqslant A_i + 1$.

(2) *For $x \in \mathbb{R}$, let $m = \sharp\{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi) : \zeta B = x\}$. If $n > m$, then*

$$Jac_{\underbrace{x, \ldots, x}_{n}}\, \pi^{\Sigma_0} = 0.$$

4. Some irreducibility results

In this section, we want to recall some irreducibility results. We start with general linear groups. For any two segments $[x, y]$ and $[x', y']$ such that $(x - y)(x' - y') \geqslant 0$, we say they are *linked* if as sets $[x, y] \not\subseteq [x', y']$, $[x', y'] \not\subseteq [x, y]$, and $[x, y] \cup [x', y']$ can form a segment after imposing the same order. The following theorem is fundamental in determining the reducibility of an induced representation of $\mathrm{GL}(n)$.

Theorem 4.1 (Zelevinsky [12]). *For unitary irreducible supercuspidal representations ρ, ρ' of general linear groups, and segments $[x, y]$, $[x', y']$ such that $(x-y)(x'-y') \geqslant 0$,*

$$\langle \rho; x, \ldots, y\rangle \times \langle \rho'; x', \ldots, y'\rangle$$

is reducible if and only if $\rho \cong \rho'$ and $[x, y], [x', y']$ are linked. In case it is reducible, it consists of the unique irreducible subrepresentations of

$$\langle \rho; x, \ldots, y\rangle \times \langle \rho; x', \ldots, y'\rangle \quad \text{and} \quad \langle \rho; x', \ldots, y'\rangle \times \langle \rho; x, \ldots, y\rangle.$$

To extend this theorem to generalized segments, we have to extend the notion of 'link' first. For any two generalized segments $\{x_{ij}\}_{m\times n}$ and $\{y_{ij}\}_{m'\times n'}$ with the same monotone properties for the rows and columns, we say they are *linked* if $[x_{m1}, x_{1n}]$, $[y_{m'1}, y_{1n'}]$ are linked, and the four sides of the rectangle formed by $\{x_{ij}\}_{m\times n}$ do not have inclusive relations with the corresponding four sides of the rectangle formed by $\{y_{ij}\}_{m'\times n'}$ (e.g., $[x_{11}, x_{1n}] \nsubseteq [y_{11}, y_{1n'}]$ and $[x_{11}, x_{1n}] \nsupseteq [y_{11}, y_{1n'}]$). It is easy to check that if $\{x_{ij}\}_{m\times n}$ and $\{y_{ij}\}_{m'\times n'}$ are linked, then $\{x_{ij}\}^T_{m\times n}$ and $\{y_{ij}\}^T_{m'\times n'}$ are also linked. So for generalized segments $\{x_{ij}\}_{m\times n}$ and $\{y_{ij}\}_{m'\times n'}$ with different monotone properties for the rows and columns, we say they are *linked* if $\{x_{ij}\}^T_{m\times n}$ and $\{y_{ij}\}_{m'\times n'}$ are linked, or equivalently $\{x_{ij}\}_{m\times n}$ and $\{y_{ij}\}^T_{m'\times n'}$ are linked. One can check that this notion of 'link' is equivalent to the one in [8].

Theorem 4.2 (Mœglin–Waldspurger [8]). *For unitary irreducible supercuspidal representations ρ, ρ' of general linear groups, and generalized segments $\{x_{ij}\}_{m\times n}$, $\{y_{ij}\}_{m'\times n'}$,*

$$\langle \rho; \{x_{ij}\}_{m\times n}\rangle \times \langle \rho'; \{y_{ij}\}_{m'\times n'}\rangle$$

is irreducible unless $\rho \cong \rho'$ and $\{x_{ij}\}_{m\times n}, \{y_{ij}\}_{m'\times n'}$ are linked.

We will be mostly using the following corollary of this theorem.

Corollary 4.3. *Let ρ, ρ' be unitary irreducible supercuspidal representations of general linear groups, and $\{x_{ij}\}_{m\times n}$, $\{y_{ij}\}_{m'\times n'}$ be generalized segments. Suppose $\rho \ncong \rho'$, or $\{x_{ij}\}_{m\times n}, \{y_{ij}\}_{m'\times n'}$ are not linked, then*

$$\langle \rho; \{x_{ij}\}_{m\times n}\rangle \times \langle \rho'; \{y_{ij}\}_{m'\times n'}\rangle \quad \cong \quad \langle \rho'; \{y_{ij}\}_{m'\times n'}\rangle \times \langle \rho; \{x_{ij}\}_{m\times n}\rangle.$$

Proof. One just needs to notice there a Weyl group action transform the inducing representation $\langle \rho; \{x_{ij}\}_{m\times n}\rangle \otimes \langle \rho'; \{y_{ij}\}_{m'\times n'}\rangle$ to $\langle \rho'; \{y_{ij}\}_{m'\times n'}\rangle \otimes \langle \rho; \{x_{ij}\}_{m\times n}\rangle$. Then the corollary follows from the fact that both induced representations are irreducible. □

Next, let us consider G^{Σ_0}.

Lemma 4.4 [7, Lemma 8.2]. *Let $\psi \in \bar{\Psi}(G)$ and $\pi^{\Sigma_0} \in \Pi_\psi^{\Sigma_0}$. For any self-dual irreducible unitary supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$ and real number x,*

$$\rho||^x \rtimes \pi^{\Sigma_0}$$

is irreducible, provided for all $(\rho, A, B, \zeta) \in \mathrm{Jord}_\rho(\psi)$, we have either

$$B > |x| \quad \text{or} \quad |x| > A+1.$$

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We will not give the proof of this lemma here, but we would like to discuss the idea behind the proof. Let τ be an irreducible representation of $\mathrm{GL}(d)$, and π^{Σ_0} be an irreducible representation of G^{Σ_0}. To show $\tau \rtimes \pi^{\Sigma_0}$ is irreducible, there is the following criterion.

Lemma 4.5. *Suppose there exits a unique irreducible subrepresentation*

$$\sigma \hookrightarrow \tau \rtimes \pi^{\Sigma_0}$$

such that σ is multiplicity free in $s.s.(\tau \rtimes \pi^{\Sigma_0})$, and

$$\sigma \hookrightarrow \tau^\vee \rtimes \pi^{\Sigma_0}.$$

Then $\tau \rtimes \pi^{\Sigma_0}$ is irreducible.

Proof. Since $\sigma \hookrightarrow \tau^\vee \rtimes \pi^{\Sigma_0}$, we know $\tau \rtimes \pi^{\Sigma_0}$ has a quotient isomorphic to σ. Then by the fact that $\sigma \hookrightarrow \tau \rtimes \pi^{\Sigma_0}$ and σ is multiplicity free in $s.s.(\tau \rtimes \pi^{\Sigma_0})$, we see σ is a direct summand of $\tau \rtimes \pi^{\Sigma_0}$. This means $\tau \rtimes \pi^{\Sigma_0}$ necessarily has another irreducible subrepresentation. But this contradicts to the uniqueness of σ. □

By the same idea, we can generalize Lemma 4.4 to the following proposition.

Proposition 4.6. *Let $\psi \in \bar{\Psi}(G)$ and $\pi^{\Sigma_0} \in \Pi_\psi^{\Sigma_0}$. For any self-dual irreducible unitary supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$, and*

$$\tau = \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta y & \cdots & \zeta y' \end{pmatrix}$$

such that $y \geqslant x \geqslant x' > 0$ and $\zeta = \pm 1$, if for all $(\rho, A, B, \zeta) \in \mathrm{Jord}_\rho(\psi)$, we have either

$$B > y \quad \textit{or} \quad x' > A + 1,$$

then $\tau \rtimes \pi^{\Sigma_0}$ is irreducible. Moreover,

$$\tau \rtimes \pi^{\Sigma_0} \cong \tau^\vee \rtimes \pi^{\Sigma_0} \tag{4.1}$$

in this case.

Proof. Taking conjugation by elements in G^{Σ_0}, one can transform the inducing representation $\tau \otimes \pi^{\Sigma_0}$ to $\tau^\vee \otimes \pi^{\Sigma_0}$. So $\tau \rtimes \pi^{\Sigma_0} \cong \tau^\vee \rtimes \pi^{\Sigma_0}$ if $\tau \rtimes \pi^{\Sigma_0}$ is irreducible. To apply Lemma 4.5, let us choose an irreducible subrepresentation $\sigma \hookrightarrow \tau \rtimes \pi^{\Sigma_0}$. Let

$$X = \begin{bmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta y & \cdots & \zeta y' \end{bmatrix},$$

then by our assumption, $\mathrm{Jac}_z \pi^{\Sigma_0} = 0$ for any $z \in X$. Also because $y \geqslant x \geqslant x' > 0$, we have

$$\mathrm{Jac}_X(\tau \rtimes \pi^{\Sigma_0}) \overset{s.s.}{=} (\mathrm{Jac}_X \tau) \rtimes \pi^{\Sigma_0} = \pi^{\Sigma_0}.$$

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This means σ is the unique irreducible subrepresentation of $\tau \rtimes \pi^{\Sigma_0}$, and it is multiplicity free in $s.s.(\tau \rtimes \pi^{\Sigma_0})$. Then it suffices for us to show $\sigma \hookrightarrow \tau^\vee \rtimes \pi^{\Sigma_0}$. By Lemma 4.4, we have

$$\begin{aligned}
\sigma &\hookrightarrow \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \times \rho||^{\zeta y} \times \cdots \times \rho||^{\zeta y'} \rtimes \pi^{\Sigma_0} \\
&\cong \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \times \rho||^{\zeta y} \times \cdots \times \rho||^{\zeta(y'+1)} \times \rho||^{-\zeta y'} \rtimes \pi^{\Sigma_0} \\
&\cong \rho||^{-\zeta y'} \times \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \times \rho||^{\zeta y} \times \cdots \times \rho||^{\zeta(y'+1)} \rtimes \pi^{\Sigma_0} \\
&\cdots \quad \cdots \\
&\cong \rho||^{-\zeta y'} \times \cdots \times \rho||^{-\zeta y} \times \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \rtimes \pi^{\Sigma_0}.
\end{aligned}$$

By induction on $y - x$, we can assume

$$\sigma' := \begin{pmatrix} \zeta x & \cdots & \zeta x' \\ \vdots & & \vdots \\ \zeta(y-1) & \cdots & \zeta(y'-1) \end{pmatrix} \rtimes \pi^{\Sigma_0}$$

is irreducible. Then

$$\sigma' \cong \begin{pmatrix} -\zeta(y'-1) & \cdots & -\zeta(y-1) \\ \vdots & & \vdots \\ -\zeta x' & \cdots & -\zeta x \end{pmatrix} \rtimes \pi^{\Sigma_0}$$

as we have seen in the beginning. Since $\mathrm{Jac}_z \sigma' = 0$ for $z \in [-\zeta y', -\zeta y]$, then

$$\mathrm{Jac}_{-\zeta y', \ldots, -\zeta y}(\rho||^{-\zeta y'} \times \cdots \times \rho||^{-\zeta y} \times \sigma') = \sigma'.$$

Therefore,

$$\sigma \hookrightarrow \rho||^{-\zeta y'} \times \cdots \times \rho||^{-\zeta y} \times \sigma'$$

as the unique irreducible subrepresentation. It follows

$$\sigma \hookrightarrow \langle -\zeta y', \ldots, -\zeta y \rangle \rtimes \sigma' \cong \langle -\zeta y', \ldots, -\zeta y \rangle \rtimes \begin{pmatrix} -\zeta(y'-1) & \cdots & -\zeta(y-1) \\ \vdots & & \vdots \\ -\zeta x' & \cdots & -\zeta x \end{pmatrix} \rtimes \pi^{\Sigma_0}$$

as the unique irreducible subrepresentation. Hence

$$\sigma \hookrightarrow \begin{pmatrix} -\zeta y' & \cdots & -\zeta y \\ \vdots & & \vdots \\ -\zeta x' & \cdots & -\zeta x \end{pmatrix} \rtimes \pi^{\Sigma_0}.$$

This finishes the proof. □

5. Basic case and generalization

We describe the *basic case* as follows. Let us fix a self-dual unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$. There exists

$$\{(\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\} \subseteq \mathrm{Jord}(\psi)$$

such that $A_2 \geqslant A_1$, $B_2 \geqslant B_1$, and $\zeta_1 = \zeta_2 = \zeta$. These two Jordan blocks are 'separated' from the other blocks in $\mathrm{Jord}_\rho(\psi)$. Moreover, let

$$\mathrm{Jord}(\psi_-) = \mathrm{Jord}(\psi) \backslash \{(\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\},$$

we require ψ_- has discrete diagonal restriction. We can extend the natural order on $\mathrm{Jord}(\psi_-)$ to $\mathrm{Jord}(\psi)$ as follows

$$(\rho, A, B, \zeta) >_\psi (\rho, A', B', \zeta') \quad \text{if and only if } A \geqslant A'.$$

In particular,

$$(\rho, A_2, B_2, \zeta_2) >_\psi (\rho, A_1, B_1, \zeta_1).$$

For functions $\underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]]$ and $\underline{\eta}(\rho, A, B, \zeta) \in \mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Jord}(\psi)$, we denote

$$l_1 = \underline{l}(\rho, A_1, B_1, \zeta_1), \quad l_2 = \underline{l}(\rho, A_2, B_2, \zeta_2),$$

and

$$\eta_1 = \underline{\eta}(\rho, A_1, B_1, \zeta_1), \quad \eta_2 = \underline{\eta}(\rho, A_2, B_2, \zeta_2).$$

Lemma 5.1 (Mœglin). *In the basic case, suppose*

$$[A_2, B_2] = [A_1, B_1],$$

then $\pi^{\Sigma_0}_{M, >_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ *if and only if*

$$\begin{cases} \eta_2 = (-1)^{A_1 - B_1}\eta_1 & \Rightarrow l_1 = l_2, \\ \eta_2 \neq (-1)^{A_1 - B_1}\eta_1 & \Rightarrow l_1 = l_2 = (A_1 - B_1 + 1)/2. \end{cases}$$

This lemma is in [4, Lemma 3.4] and it is fundamental for all the results that we are going to derive in this paper. The lemma can also be generalized as follows.

Proposition 5.2. *In the basic case, if* $\pi^{\Sigma_0}_{M, >_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, *then*

$$\begin{cases} \eta_2 = (-1)^{A_1 - B_1}\eta_1 & \Rightarrow A_2 - l_2 \geqslant A_1 - l_1, \quad B_2 + l_2 \geqslant B_1 + l_1, \\ \eta_2 \neq (-1)^{A_1 - B_1}\eta_1 & \Rightarrow B_2 + l_2 > A_1 - l_1. \end{cases} \tag{5.1}$$

Conversely, if (5.1) is satisfied, then $\pi^{\Sigma_0}_{M, >_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, *moreover*

$$\pi^{\Sigma_0}_{M, >_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta B_2 & \cdots & -\zeta A_2 \\ \vdots & & \vdots \\ \zeta(B_2 + l_2 - 1) & \cdots & -\zeta(A_2 - l_2 + 1) \end{pmatrix}$$

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$$\times \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1) \end{pmatrix}$$
$$\times\ \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2-l_2, B_2+l_2, 0, \eta_2, \zeta),$$
$$(\rho, A_1-l_1, B_1+l_1, 0, \eta_1, \zeta)).$$

We give the proof of Proposition 5.2 in Appendix A. Next we would like to generalize the basic case to the following situation. Suppose we can index $\mathrm{Jord}_\rho(\psi)$ for each ρ such that $A_i \geqslant A_{i-1}$ and $B_i \geqslant B_{i-1}$. Moreover, we can divide $\mathrm{Jord}_\rho(\psi)$ into chunks of

$$\{(\rho, A_i, B_i, \zeta_i), (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1})\} \quad \text{with } \zeta_i = \zeta_{i-1}, \text{ or } \{(\rho, A_j, B_j, \zeta_j)\}, \tag{5.2}$$

such that each of them is 'separated' from the others in $\mathrm{Jord}_\rho(\psi)$. We call this the *generalized basic case*. There is a natural order $>_\psi$ on $\mathrm{Jord}_\rho(\psi)$, i.e.,

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

Proposition 5.3. *In the generalized basic case, $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if the condition (5.1) is satisfied for each chunk of pair $\{(\rho, A_i, B_i, \zeta_i), (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1})\}$ in (5.2) for all ρ.*

Proof. We first prove the sufficiency of the nonvanishing condition by induction on the number of intersected pairs in $\mathrm{Jord}(\psi)$. Let ρ be fixed. For $\mathrm{Jord}_\rho(\psi)$, suppose n is the biggest integer such that $[A_n, B_n]$ and $[A_{n-1}, B_{n-1}]$ intersects. Let

$$\mathrm{Jord}(\psi_-) = \mathrm{Jord}(\psi) \backslash \{(\rho, A_n, B_n, \zeta_n)\}.$$

By induction we can assume

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T_n, B_n+T_n, l_n, \eta_n, \zeta_n)) \neq 0$$

for the smallest T_n such that $[A_n+T_n, B_n+T_n]$ does not intersect with $[A_{n-1}, B_{n-1}]$. For those intersected pairs $\{(\rho, A_i, B_i, \zeta_i), (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1})\}$, we can put them apart by shifting $(\rho, A_i, B_i, \zeta_i)$ to $(\rho, A_i+T_i, B_i+T_i, \zeta_i)$ again for the smallest T_i. Let us write $T_j = 0$ for those $(\rho, A_j, B_j, \zeta_j)$ remained in $\mathrm{Jord}_\rho(\psi)$. As a result we can get a parameter $\psi_\gg$ dominating ψ with discrete diagonal restriction such that

$$(\rho, A_{\gg,i}, B_{\gg,i}, \zeta_i) = (\rho, A_i+T_i, B_i+T_i, \zeta_i).$$

Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i\neq n} \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \times \mathcal{C}_{X^c}$$
$$\rtimes\ \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T_n, B_n+T_n, l_n, \eta_n, \zeta_n)),$$

where i is increasing. We would like to show

$$\mathrm{Jac}_{(\rho, A_n+T_n, B_n+T_n, \zeta_n) \mapsto (\rho, A_n, B_n, \zeta_n)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T_n, B_n+T_n, l_n, \eta_n, \zeta_n)) \neq 0.$$

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Note by our assumption,

$$\mathrm{Jac}_{(\rho,A_n+T_n,B_n+T_n,\zeta_n)\mapsto(\rho,A_n,B_n,\zeta_n)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta})\neq 0.$$

So after we apply the same Jacquet functor to the full-induced representation above, we should get something nonzero. To compute this Jacquet module, one notes $B_n+1>A_i+T_i$ for $T_i\neq 0$, so it can only be

$$\times_{i\neq n}\begin{pmatrix}\zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1)\\ \vdots & & \vdots\\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1)\end{pmatrix}\times\mathcal{C}_{X^c}\rtimes\mathrm{Jac}_{(\rho,A_n+T_n,B_n+T_n,\zeta_n)\mapsto(\rho,A_n,B_n,\zeta_n)}$$
$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_n+T_n,B_n+T_n,l_n,\eta_n,\zeta_n))\neq 0.$$

This gives what we want.

Next for the necessity of the nonvanishing condition, we can assume $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})\neq 0$. We still fix ρ and choose a dominating parameter $\psi_{\gg}$ with discrete diagonal restriction in the way as above. Then by definition

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta})\hookrightarrow\times_i\begin{pmatrix}\zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1)\\ \vdots & & \vdots\\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1)\end{pmatrix}\times\mathcal{C}_{X^c}\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}).$$

It is easy to see that those generalized segments in the induced representations are not linked. So we can change their orders in the induction. In particular, we can take any generalized segment to the front. As a result

$$\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta})\neq 0,$$

for any i. This gives the condition that we want with respect to ρ. By varying ρ, we prove the necessity of the condition. □

Remark 5.4. Suppose a subset of Jordan blocks of $\mathrm{Jord}_\rho(\psi)$ satisfies the condition in the generalized basic case, then we say the Jordan blocks in this set have 'good shape'.

5.1. Some necessary conditions on nonvanishing

In this section, we want to use Proposition 5.2 to give some necessary conditions on the nonvanishing of $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})$ in general. Let us fix ρ and index the Jordan blocks in $\mathrm{Jord}_\rho(\psi)$ such that

$$(\rho,A_i,B_i,\zeta_i)>_\psi(\rho,A_{i-1},B_{i-1},\zeta_{i-1}).$$

Let $(\rho,A_k,B_k,\zeta_k)>_\psi(\rho,A_{k-1},B_{k-1},\zeta_{k-1})$ be two adjacent blocks under the order $>_\psi$ and $\zeta_k=\zeta_{k-1}$.

Lemma 5.5. *Suppose $A_k\geqslant A_{k-1}$ and $B_k\geqslant B_{k-1}$. If $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})\neq 0$, then l_k, η_k, l_{k-1}, η_{k-1} satisfy the condition (5.1).*

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Proof. Let $\psi_{\gg}$ be a dominating parameter with discrete diagonal restriction. We also define $\psi^{(k)}$ from $\psi_{\gg}$ by shifting $(\rho, A_i + T_i, B_i + T_i, \zeta_i)$ back to $(\rho, A_i, B_i, \zeta_i)$ for $i \leqslant k$. Then

$$
\begin{aligned}
\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i<k-1} & \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \\
& \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{I} \\
& \times \underbrace{\begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}, \underline{l}, \underline{\eta}) \\
\hookrightarrow \times_{i<k-1} & \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \\
& \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{I} \\
& \times \underbrace{\begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+T_{k-1}+1) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots & & \vdots \\ \zeta_k(A_{k-1}+T_k) & \cdots & \zeta_k(A_{k-1}+T_{k-1}+1) & \cdots & \zeta_k(A_{k-1}+1) \end{pmatrix}}_{II_1} \\
& \times \underbrace{\begin{pmatrix} \zeta_k(A_{k-1}+T_k+1) & \cdots & \zeta_k(A_{k-1}+T_{k-1}+2) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+T_{k-1}+1) \end{pmatrix}}_{II_2} \\
& \times \underbrace{\begin{pmatrix} \zeta_k(A_{k-1}+T_{k-1}+1) & \cdots & \zeta_k(A_{k-1}+2) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_{k-1}) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II_3} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}, \underline{l}, \underline{\eta}),
\end{aligned}
$$

where i increases. Note (I) is interchangeable with (II_1) and (II_2), and $B_k + T_{k-1} + 1 > A_i + T_i + 1$ for $i < k-1$. As a result,

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$$\mathrm{Jac}_{(\rho,A_k+T_k,B_k+T_k,\zeta_k)\mapsto(\rho,A_k+T_{k-1},B_k+T_{k-1},\zeta_k)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})\neq 0.$$

Then by Proposition 5.2, $l_k,\eta_k,l_{k-1},\eta_{k-1}$ satisfy the condition (5.1). □

Lemma 5.6. *Suppose* $[A_k,B_k]\supseteq[A_{k-1},B_{k-1}]$. *If* $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})\neq 0$, *then* $l_k,\eta_k,l_{k-1},\eta_{k-1}$ *satisfy the following condition:*

$$\begin{cases}\eta_k=(-1)^{A_{k-1}-B_{k-1}}\eta_{k-1} & \Rightarrow 0\leqslant l_k-l_{k-1}\leqslant(A_k-B_k)-(A_{k-1}-B_{k-1}),\\ \eta_k\neq(-1)^{A_{k-1}-B_{k-1}}\eta_{k-1} & \Rightarrow l_k+l_{k-1}>A_{k-1}-B_{k-1}.\end{cases}\tag{5.3}$$

Proof. Let $\psi_\gg$ be a dominating parameter with discrete diagonal restriction. We also define $\psi^{(k)}$ from $\psi_\gg$ by shifting $(\rho,A_i+T_i,B_i+T_i,\zeta_i)$ back to (ρ,A_i,B_i,ζ_i) for $i\leqslant k$. Then

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})\hookrightarrow\times_{i<k-1}&\begin{pmatrix}\zeta_i(B_i+T_i)&\cdots&\zeta_i(B_i+1)\\ \vdots&&\vdots\\ \zeta_i(A_i+T_i)&\cdots&\zeta_i(A_i+1)\end{pmatrix}\\ &\times\underbrace{\begin{pmatrix}\zeta_{k-1}(B_{k-1}+T_{k-1})&\cdots&\zeta_{k-1}(B_{k-1}+1)\\ \vdots&&\vdots\\ \zeta_{k-1}(A_{k-1}+T_{k-1})&\cdots&\zeta_{k-1}(A_{k-1}+1)\end{pmatrix}}_{I}\\ &\times\underbrace{\begin{pmatrix}\zeta_k(B_k+T_k)&\cdots&\zeta_k(B_k+1)\\ \vdots&&\vdots\\ \zeta_k(A_k+T_k)&\cdots&\zeta_k(A_k+1)\end{pmatrix}}_{II}\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)},\underline{l},\underline{\eta}),\end{aligned}$$

where i increases. Note (I) and (II) are interchangeable due to $[A_k+1,B_k+1]\supseteq[A_{k-1}+1,B_{k-1}+1]$. Since $B_k+T_{k-1}+1>A_i+T_i+1$ for $i<k-1$, we have

$$\mathrm{Jac}_{(\rho,A_k+T_k,B_k+T_k,\zeta_k)\mapsto(\rho,A_k+T_{k-1},B_k+T_{k-1},\zeta_k)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})\neq 0.$$

In particular,

$$\mathrm{Jac}_{(\rho,A_k+T_k,B_k+T_k,\zeta_k)\mapsto(\rho,A_k+T_{k-1}+B_{k-1}-B_k,B_{k-1}+T_{k-1},\zeta_k)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})\neq 0.$$

By Proposition 5.2, the condition (5.1) is satisfied for $(\rho,A_k+T_{k-1}+B_{k-1}-B_k,B_{k-1}+T_{k-1},l_k,\eta_k,\zeta_k)$ and $(\rho,A_{k-1}+T_{k-1},B_{k-1}+T_{k-1},l_{k-1},\eta_{k-1},\zeta_{k-1})$, i.e.,

- If $\eta_k=(-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$, then

$$\begin{cases}(A_k+T_{k-1}+B_{k-1}-B_k)-l_k\geqslant(A_{k-1}+T_{k-1})-l_{k-1}\Rightarrow l_k-l_{k-1}\\ \quad\leqslant(A_k-B_k)-(A_{k-1}-B_{k-1}),\\ (B_{k-1}+T_{k-1})+l_k\geqslant(B_{k-1}+T_{k-1})+l_{k-1}\Rightarrow l_k-l_{k-1}\geqslant 0.\end{cases}$$

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- If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$, then

$$(B_{k-1}+T_{k-1})+l_k > (A_{k-1}+T_{k-1})-l_{k-1} \Rightarrow l_k+l_{k-1} > A_{k-1}-B_{k-1}.$$

This finishes the proof. $\square$

Lemma 5.7. *Suppose* $[A_k, B_k] \subseteq [A_{k-1}, B_{k-1}]$. *If* $\pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, *then* $l_k, l_{k-1}, \eta_k, \eta_{k-1}$ *satisfy the following condition:*

$$\begin{cases} \eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1} & \Rightarrow 0 \leqslant l_{k-1}-l_k \leqslant (A_{k-1}-B_{k-1})-(A_k-B_k), \\ \eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1} & \Rightarrow l_k+l_{k-1} > A_k-B_k. \end{cases} \tag{5.4}$$

Proof. Let $\psi_\gg$ be a dominating parameter with discrete diagonal restriction. We also define $\psi^{(k)}$ from $\psi_\gg$ by shifting $(\rho, A_i+T_i, B_i+T_i, \zeta_i)$ back to $(\rho, A_i, B_i, \zeta_i)$ for $i \leqslant k$. Then

$$\begin{aligned} \pi_{M,>_\psi}^{\Sigma_0}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i<k-1} & \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \\ & \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{I} \\ & \times \underbrace{\begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II} \rtimes \pi_{M,>_\psi}^{\Sigma_0}(\psi^{(k)}, \underline{l}, \underline{\eta}), \end{aligned}$$

where i increases. Note (I) and (II) are interchangeable due to $[A_k+1, B_k+1] \subseteq [A_{k-1}+1, B_{k-1}+1]$. Since $B_k+T_{k-1}+1 > A_i+T_i+1$ for $i < k-1$, we have

$$\mathrm{Jac}_{(\rho, A_k+T_k, B_k+T_k, \zeta_k) \mapsto (\rho, A_k+T_{k-1}, B_k+T_{k-1}, \zeta_k)} \pi_{M,>_\psi}^{\Sigma_0}(\psi_\gg, \underline{l}, \underline{\eta}) \neq 0.$$

In particular,

$$\mathrm{Jac}_{(\rho, A_k+T_k, B_k+T_k, \zeta_k) \mapsto (\rho, A_{k-1}+T_{k-1}, B_k+T_{k-1}+A_{k-1}-A_k, \zeta_k)} \pi_{M,>_\psi}^{\Sigma_0}(\psi_\gg, \underline{l}, \underline{\eta}) \neq 0.$$

By Proposition 5.2, the condition (5.1) is satisfied for $(\rho, A_{k-1}+T_{k-1}, B_k+T_{k-1}+A_{k-1}-A_k, l_k, \eta_k, \zeta_k)$ and $(\rho, A_{k-1}+T_{k-1}, B_{k-1}+T_{k-1}, l_{k-1}, \eta_{k-1}, \zeta_{k-1})$, i.e.,

- If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$, then

$$\begin{cases} (A_{k-1}+T_{k-1})-l_k \geqslant (A_{k-1}+T_{k-1})-l_{k-1} \Rightarrow l_{k-1}-l_k \geqslant 0, \\ (B_k+T_{k-1}+A_{k-1}-A_k)+l_k \geqslant (B_{k-1}+T_{k-1})+l_{k-1} \Rightarrow l_{k-1}-l_k \\ \qquad \leqslant (A_{k-1}-B_{k-1})-(A_k-B_k). \end{cases}$$

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- If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$, then

$$(B_k + T_{k-1} + A_{k-1} - A_k) + l_k > (A_{k-1} + T_{k-1}) - l_{k-1} \Rightarrow l_k + l_{k-1} > A_k - B_k.$$

This finishes the proof. □

6. Change of order formulas

For $\psi = \psi_p \in \bar{\Psi}(G)$, we want to show how Mœglin's parametrization of elements in $\Pi_{\psi}^{\Sigma_0}$ changes as we change the order $>_{\psi}$. So we fix an admissible order $>_{\psi}$ and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_{\rho})$. We index the Jordan blocks in $\mathrm{Jord}_{\rho}(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) >_{\psi} (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

Let $(\rho, A_k, B_k, \zeta_k) >_{\psi} (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ be two adjacent blocks under the order $>_{\psi}$. We denote by $>'_{\psi}$ the order obtained from $>_{\psi}$ by switching $(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ and $(\rho, A_k, B_k, \zeta_k)$. And we assume $>'_{\psi}$ is also admissible. Then we are in the following two cases.

6.1. Case $\zeta_k = \zeta_{k-1}$

In this case, we can assume without loss of generality that $[A_k, B_k] \supseteq [A_{k-1}, B_{k-1}]$. For functions $\underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]]$ and $\underline{\eta}(\rho, A, B, \zeta) \in \mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Jord}(\psi)$, we denote

$$l_k = \underline{l}(\rho, A_k, B_k, \zeta_k), \quad l_{k-1} = \underline{l}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}),$$

and

$$\eta_k = \underline{\eta}(\rho, A_k, B_k, \zeta_k), \quad \eta_{k-1} = \underline{\eta}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}).$$

From $(\underline{l}, \underline{\eta})$ satisfying (5.3), we want to construct another pair $(\underline{l}', \underline{\eta}')$ such that

$$\underline{l}'(\cdot) = \underline{l}(\cdot) \quad \text{and} \quad \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $\mathrm{Jord}(\psi)\backslash\{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}$. Let us denote

$$l'_k = \underline{l}'(\rho, A_k, B_k, \zeta_k), \quad l'_{k-1} = \underline{l}'(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}),$$

and

$$\eta'_k = \underline{\eta}'(\rho, A_k, B_k, \zeta_k), \quad \eta'_{k-1} = \underline{\eta}'(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}).$$

Then we define $l'_k, l'_{k-1}, \eta'_k, \eta'_{k-1}$ according to the following formulas.

- If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$, then $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ l_k - l'_k = (A_{k-1} - B_{k-1} - 2l_{k-1}) + 1 \\ \eta'_{k-1} = (-1)^{A_k-B_k}\eta_{k-1}. \end{cases}$$

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- If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and

$$l_k - l_{k-1} < (A_k - B_k)/2 - (A_{k-1} - B_{k-1}) + l_{k-1},$$

then $\eta'_{k-1} \neq (-1)^{A_k-B_k}\eta'_k$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ l'_k - l_k = (A_{k-1} - B_{k-1} - 2l_{k-1}) + 1 \\ \eta'_{k-1} = (-1)^{A_k-B_k}\eta_{k-1}. \end{cases}$$

- If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and

$$l_k - l_{k-1} \geqslant (A_k - B_k)/2 - (A_{k-1} - B_{k-1}) + l_{k-1},$$

then $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ (l'_k - l'_{k-1}) + (l_k - l_{k-1}) = (A_k - B_k) - (A_{k-1} - B_{k-1}) \\ \eta'_{k-1} = (-1)^{A_k-B_k}\eta_{k-1}. \end{cases}$$

One can check $(\underline{l}', \underline{\eta}')$ satisfies (5.4). We denote this transformation by S^+. We can also define its 'inverse' S^-, namely we start with any $(\underline{l}', \underline{\eta}')$ satisfying (5.4), and we define $l_k, l_{k-1}, \eta_k, \eta_{k-1}$ according to the following formulas.

- If $\eta'_{k-1} \neq (-1)^{A_k-B_k}\eta'_k$, then $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ l'_k - l_k = (A_{k-1} - B_{k-1} - 2l_{k-1}) + 1 \\ \eta'_{k-1} = (-1)^{A_k-B_k}\eta_{k-1}. \end{cases}$$

- If $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$ and

$$l'_k - l'_{k-1} < (A_k - B_k)/2 - (A_{k-1} - B_{k-1}) + l'_{k-1},$$

then $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ l_k - l'_k = (A_{k-1} - B_{k-1} - 2l_{k-1}) + 1 \\ \eta'_{k-1} = (-1)^{A_k-B_k}\eta_{k-1}. \end{cases}$$

- If $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$ and

$$l'_k - l'_{k-1} \geqslant (A_k - B_k)/2 - (A_{k-1} - B_{k-1}) + l'_{k-1},$$

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then $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and

$$\begin{cases} l_{k-1} = l'_{k-1} \\ (l'_k - l'_{k-1}) + (l_k - l_{k-1}) = (A_k - B_k) - (A_{k-1} - B_{k-1}) \\ \eta'_{k-1} = (-1)^{A_k - B_k}\eta_{k-1}. \end{cases}$$

One can also check $(\underline{l}, \underline{\eta})$ satisfies (5.3). Moreover, we have

$$S^- \circ S^+(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}, \underline{\eta}),$$

and

$$S^+ \circ S^-(\underline{l}', \underline{\eta}') \sim_{\Sigma_0} (\underline{l}', \underline{\eta}').$$

So S^+ (respectively S^-) induces a bijection between $(\underline{l}, \eta)$ satisfying (5.3) and $(\underline{l}', \underline{\eta}')$ satisfying (5.4) modulo the equivalence relation $\sim_{\Sigma_0}$ on both sides.

Theorem 6.1. *Suppose* $(\underline{l}', \underline{\eta}') = S^+(\underline{l}, \underline{\eta})$, *then*

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}').$$

Let $\psi_\gg$ be a dominating parameter of ψ such that $\mathrm{Jord}_\rho(\psi_\gg) = \mathrm{Jord}_\rho(\psi)$, and $\mathrm{Jord}_{\rho'}(\psi_\gg)$ has discrete diagonal restriction for $\rho' \neq \rho$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \mathrm{Jac}_{X^c}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}),$$

and

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') = \mathrm{Jac}_{X^c}\pi^{\Sigma_0}_{M,>'_\psi}(\psi_\gg, \underline{l}', \underline{\eta}').$$

So it suffices to prove the proposition for such $\psi_\gg$. Therefore, in the following discussions of the proof of this proposition, we always assume $\mathrm{Jord}_{\rho'}(\psi)$ has discrete diagonal restriction for $\rho' \neq \rho$, and if we choose some dominating $\psi_\gg$ of ψ, we always assume $\mathrm{Jord}_{\rho'}(\psi_\gg) = \mathrm{Jord}_{\rho'}(\psi)$ for $\rho' \neq \rho$.

6.1.1. Reduction. Let $(\underline{l}', \underline{\eta}') = S^+(\underline{l}, \underline{\eta})$. We want to reduce the proposition to the following case:

$$\begin{aligned} &(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) \quad \text{for } i \neq k, \\ &(\rho, A_k, B_k, \zeta_k) \gg (\rho, A_{k-2}, B_{k-2}, \zeta_{k-2}) \quad \text{and } 0. \end{aligned} \tag{6.1}$$

We do this in two steps. First we reduce it to the case:

$$\begin{aligned} &(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) \quad \text{for } i > k, \\ &(\rho, A_k, B_k, \zeta_k) \gg \bigcup_{j=1}^{k-2}\{(\rho, A_j, B_j, \zeta_j)\} \quad \text{and } 0. \end{aligned} \tag{6.2}$$

Let us choose a dominating parameter $\psi_\gg$ with respect to $>_\psi$ such that $T_i = 0$ for $i < k-1$,

$$(\rho, A_i + T_i, B_i + T_i, \zeta_i) \gg (\rho, A_{i-1} + T_{i-1}, B_{i-1} + T_{i-1}, \zeta_{i-1}) \quad \text{for } i \geqslant k$$

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and

$$(\rho, A_k + T_{k-1}, B_k + T_{k-1}, \zeta_k) \gg \bigcup_{j=1}^{k-2}\{(\rho, A_j, B_j, \zeta_j)\} \quad \text{and} \quad 0.$$

From $\psi_{\gg}$, we can obtain a dominating parameter $\psi'_{\gg}$ with respect to $>'_{\psi}$ such that $T'_i = T_i$ for $i \neq k, k-1$, and $T'_k = T_{k-1}, T'_{k-1} = T_k$. Let us also denote T_{k-1} by T, and construct $\psi^T_{\gg}$ from $\psi_{\gg}$ by changing T_k to T. Let $\psi^{(k)}_{\gg}$ be obtained from $\psi_{\gg}$ by changing T_k, T_{k-1} to zero.

Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow &\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix} \\ &\times \begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+1) \end{pmatrix} \\ &\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}),\end{aligned}$$

where the two generalized segments are interchangeable. So $\pi^{\Sigma_0}_{M,>_\psi}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) \neq 0$, and

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow &\begin{pmatrix} \zeta_k(B_k+T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T) & \cdots & \zeta_k(A_k+1) \end{pmatrix} \\ &\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).\end{aligned}$$

By (6.2),

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^T_{\gg}, \underline{l}', \underline{\eta}') = \pi^{\Sigma_0}_{M,>_\psi}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) \neq 0.$$

Then

$$\begin{aligned}\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow &\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi^T_{\gg}, \underline{l}', \underline{\eta}') \\ \hookrightarrow &\underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_{I}\end{aligned}$$

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$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II}$$

$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{III}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).$$

We can interchange (*II*) and (*III*). If $B_{k-1} \neq B_k$, then $\mathrm{Jac}_{\zeta_{k-1}(B_{k-1}+T)}\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') = 0$. So we can 'combine' (*I*) and (*III*), i.e.,

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{I+III}$$

$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).$$

If $B_{k-1} = B_k$, let us write

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_{I}$$

$$\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T-1) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T-1) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{III_-}$$

$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).$$

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There exists an irreducible constituent σ of

$$\underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_{I} \times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix}$$

such that

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_\gg, \underline{l}', \underline{\eta}') \hookrightarrow \sigma \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T-1) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T-1) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{III_-}$$

$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_\gg, \underline{l}, \underline{\eta}).$$

Suppose $\mathrm{Jac}_{\zeta_{k-1}(B_{k-1}+T)}\sigma \neq 0$, then $\mathrm{Jac}_{\zeta_{k-1}(B_{k-1}+T)}\sigma$ is contained in

$$\underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_{I} \times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix}$$

which is irreducible. So

$$\sigma \hookrightarrow \rho||^{\zeta_{k-1}(B_{k-1}+T)} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_{I}$$

$$\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix}.$$

Hence

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_\gg, \underline{l}', \underline{\eta}') \hookrightarrow \rho||^{\zeta_{k-1}(B_{k-1}+T)} \times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_{I}$$

$$\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_k(B_k+T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II}$$

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$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T-1) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T-1) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{III_-} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).$$

If $A_k = A_{k-1}$, then $[A_k, B_k] = [A_{k-1}, B_{k-1}]$, and there is nothing to prove. So we can assume $A_k > A_{k-1}$. Then

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \rho||^{\zeta_{k-1}(B_{k-1}+T)} \times \begin{pmatrix} \zeta_k(B_k+T) \\ \vdots \\ \zeta_k(A_k+T) \end{pmatrix}$$

$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T+1) \end{pmatrix}}_{I}$$

$$\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T+1) \\ \vdots \\ \zeta_{k-1}(A_{k-1}+T) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta_k(B_k+T-1) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T-1) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II_-}$$

$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T-1) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T-1) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{III_-} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).$$

As a result, we have

$$\mathrm{Jac}_{\zeta_{k-1}(B_{k-1}+T),\zeta_k(B_k+T)} \pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \neq 0,$$

which is impossible. Therefore, we must have $\mathrm{Jac}_{\zeta_{k-1}(B_{k-1}+T)}\sigma = 0$, and hence

$$\sigma = \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+T) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+T) \end{pmatrix}}_{I_+}.$$

In this case,

$$\mathrm{Jac}_{\zeta_{k-1}(B_{k-1}+T-1)} \pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') = 0.$$

So we again have

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T'_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{(I+III)}$$

$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).$$

Since $[\zeta_{k-1}(A_{k-1}+T'_{k-1}), \zeta_{k-1}(A_{k-1}+1)] \supseteq [\zeta_k(A_k+T), \zeta_k(A_k+1)]$, we can interchange $(I+III)$ and (II). Therefore,

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(k)}_{\gg}, \underline{l}', \underline{\eta}') = \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).$$

After applying $\circ_{i>k}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)}$ to both sides, we get

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}').$$

Secondly, we want to further reduce it to (6.1). So let us assume we are in case (6.2). We can choose a dominating parameter $\psi_{\gg}$ with discrete diagonal restriction so that $T_i = 0$ for $i > k$ and $i = k-1$. We also require

$$(\rho, A_i+T_i, B_i+T_i, \zeta_i) \gg (\rho, A_{i-1}+T_{i-1}, B_{i-1}+T_{i-1}, \zeta_{i-1}) \quad \text{for } i < k,$$

and

$$(\rho, A_k, B_k, \zeta_k) \gg (\rho, A_{k-2}+T_{k-2}, B_{k-2}+T_{k-2}, \zeta_{k-2}) \quad \text{and} \quad 0.$$

Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i<k-1} \underbrace{\begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix}}_{I_i}$$

$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+1) \end{pmatrix}}_{II}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}),$$

where i increases. Since $B_k+1 > A_i+T_i+1$ for $i < k-1$, we can interchange (II) with (I_i). Let $\psi^{(k)}_{\gg}$ be obtained from $\psi_{\gg}$ by changing T_k to zero. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \times_{i<k-1} \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).$$

By (6.1),

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(k)}_{\gg}, \underline{l}', \underline{\eta}') \neq 0.$$

Since

$$\mathrm{Jac}_{(\rho,A_k+T_k,B_k+T_k,\zeta_k)\mapsto(\rho,A_k,B_k,\zeta_k)}$$

commutes with

$$\circ_{i<k-1}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_k)\mapsto(\rho,A_i,B_i,\zeta_i)},$$

we have

$$\circ_{i<k-1}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_k)\mapsto(\rho,A_i,B_i,\zeta_i)}\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg},\underline{l},\underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}).$$

Similarly,

$$\circ_{i<k-1}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_k)\mapsto(\rho,A_i,B_i,\zeta_i)}\pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(k)}_{\gg},\underline{l}',\underline{\eta}') = \pi^{\Sigma_0}_{M,>'_\psi}(\psi,\underline{l}',\underline{\eta}').$$

So

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi,\underline{l}',\underline{\eta}').$$

This finishes our reduction step.

6.1.2. Critical case. From the previous reduction, we can now assume (6.1):

$$\begin{aligned}(\rho,A_i,B_i,\zeta_i) &\gg (\rho,A_{i-1},B_{i-1},\zeta_{i-1}) \quad \text{for } i\neq k,\\ (\rho,A_k,B_k,\zeta_k) &\gg (\rho,A_{k-2},B_{k-2},\zeta_{k-2}) \quad \text{and } 0.\end{aligned}$$

In this critical case, we can actually get the nonvanishing condition.

Lemma 6.2. *Suppose we are in case (6.1).*

(1) $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})\neq 0$ *if and only if*

$$\begin{cases}\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1} & \Rightarrow 0\leqslant l_k - l_{k-1}\leqslant (A_k-B_k)-(A_{k-1}-B_{k-1}),\\ \eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1} & \Rightarrow l_k + l_{k-1} > A_{k-1}-B_{k-1}.\end{cases}$$

(2) $\pi^{\Sigma_0}_{M,>'_\psi}(\psi,\underline{l}',\underline{\eta}')\neq 0$ *if and only if*

$$\begin{cases}\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k & \Rightarrow 0\leqslant l'_k - l'_{k-1}\leqslant (A_k-B_k)-(A_{k-1}-B_{k-1}),\\ \eta'_{k-1} \neq (-1)^{A_k-B_k}\eta'_k & \Rightarrow l'_k + l'_{k-1} > A_{k-1}-B_{k-1}.\end{cases}$$

Proof. We only show (1), and (2) is similar. One first notes the necessity of the nonvanishing condition has been shown in Lemma 5.6, so we get an upper bound for the size of the packet $|\Pi^{\Sigma_0}_\psi|$. In fact we can also get a lower bound for it. Let us define ψ^* by changing (ρ,A_k,B_k,ζ) to $(\rho,A_{k-1},B_k-A_k+A_{k-1},\zeta)$. Then the functor $\mathrm{Jac}_{(\rho,A_k,B_k,\zeta)\mapsto(\rho,A_{k-1},B_k-A_k+A_{k-1},\zeta)}$ induces a surjection from $\Pi^{\Sigma_0}_\psi$ to $\Pi^{\Sigma_0}_{\psi^*}$:

$$\pi^{\Sigma_0}_{M,>_{\psi^*}}(\psi,\underline{l},\underline{\eta}) = \mathrm{Jac}_{(\rho,A_k,B_k,\zeta)\mapsto(\rho,A_{k-1},B_k-A_k+A_{k-1},\zeta)}\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}).$$

So $|\Pi^{\Sigma_0}_{\psi^*}| < |\Pi^{\Sigma_0}_{\psi}|$. By Proposition 5.2, we have $\pi^{\Sigma_0}_{M,>'_\psi}(\psi^*, \underline{l}', \underline{\eta}') \neq 0$ if and only if

$$\begin{cases} \eta'_{k-1} = (-1)^{A_k - B_k}\eta'_k & \Rightarrow 0 \leqslant l'_k - l'_{k-1} \leqslant (A_{k-1} - (B_k - A_k + A_{k-1})) - (A_{k-1} - B_{k-1}), \\ \eta'_{k-1} \neq (-1)^{A_k - B_k}\eta'_k & \Rightarrow l'_k + l'_{k-1} > A_{k-1} - B_{k-1}. \end{cases}$$

Comparing this condition with the necessary condition for $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, one can easily see that $|\Pi^{\Sigma_0}_{\psi^*}|$ is equal to the upper bound for $|\Pi^{\Sigma_0}_{\psi}|$. Therefore, $|\Pi^{\Sigma_0}_{\psi}|$ must be equal to its upper bound, i.e., the necessary condition for $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ is also sufficient. □

Now we begin to prove the change of order formula in this case. Let us define ψ_- by

$$\mathrm{Jord}(\psi_-) = \mathrm{Jord}(\psi) \backslash \{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\},$$

then ψ_- has discrete diagonal restriction. Let $\zeta = \zeta_k = \zeta_{k-1}$. We are going to break the proof into four steps.

Step One: We want to show if $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0$, then we can choose $(\underline{l}', \underline{\eta}')$ within its $(\sim_{\Sigma_0})$ equivalence class such that

$$\underline{l}'(\cdot) = \underline{l}(\cdot) \text{ and } \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $\mathrm{Jord}(\psi_-)$.

- Suppose $\underline{l}(\cdot) = 0$ over $\mathrm{Jord}(\psi_-)$. We can define ψ_{e_-} by

$$\mathrm{Jord}(\psi_{e_-}) := \bigcup_{(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp) \in \mathrm{Jord}(\psi_-)} \; \bigcup_{C^\sharp \in [A^\sharp, B^\sharp]} \{(\rho^\sharp, C^\sharp, C^\sharp, \zeta^\sharp)\}.$$

And we define ψ_e by adding $(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$. From $(\underline{l}, \underline{\eta})$, we obtain $(\underline{l}_e, \underline{\eta}_e)$ such that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_e, \underline{l}_e, \underline{\eta}_e).$$

Suppose

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi_e, \underline{l}'_e, \underline{\eta}'_e) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_e, \underline{l}_e, \underline{\eta}_e).$$

By computing $\varepsilon^{M/W}_{\psi_e}$ with respect to $>_\psi$ and $>'_\psi$ (cf. (1.4)), one finds

$$\underline{\eta}'_e(\cdot) = \underline{\eta}_e(\cdot)$$

over $\mathrm{Jord}(\psi_{e_-})$. Therefore, if we let

$$\underline{l}'(\cdot) = \underline{l}(\cdot) = 0 \quad \text{and} \quad \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $\mathrm{Jord}(\psi_-)$, and

$$\underline{l}'(\cdot) = \underline{l}'_e(\cdot) \quad \text{and} \quad \underline{\eta}'(\cdot) = \underline{\eta}'_e(\cdot)$$

over $(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$, then

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') = \pi^{\Sigma_0}_{M,>'_\psi}(\psi_e, \underline{l}'_e, \underline{\eta}'_e) = \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).$$

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- Let $(\psi_0, \underline{l}_0)$ be obtained from $(\psi, \underline{l})$ by changing $(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp)$ to $(\rho^\sharp, A^\sharp - l^\sharp, B^\sharp + l^\sharp, \zeta^\sharp)$ and letting $\underline{l}_0(\rho^\sharp, A^\sharp - l^\sharp, B^\sharp + l^\sharp, \zeta^\sharp) = 0$ for all $(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp) \in \mathrm{Jord}(\psi_-)$, where $l^\sharp = \underline{l}(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp)$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_{(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp) \in \mathrm{Jord}(\psi_-)} \tau(\rho^\sharp, A^\sharp - l^\sharp, B^\sharp + l^\sharp, \zeta^\sharp) \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_0, \underline{l}_0, \underline{\eta}),$$

as the unique irreducible subrepresentation, where

$$\tau(\rho^\sharp, A^\sharp - l^\sharp, B^\sharp + l^\sharp, \zeta^\sharp) := \begin{pmatrix} \zeta^\sharp B^\sharp & \cdots & -\zeta^\sharp A^\sharp \\ \vdots & & \vdots \\ \zeta^\sharp(B^\sharp + l^\sharp - 1) & \cdots & -\zeta^\sharp(A^\sharp - l^\sharp + 1) \end{pmatrix}.$$

Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi_0, \underline{l}_0, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi_0, \underline{l}'_0, \underline{\eta}')$. We know from the previous discussion that

$$\underline{l}'_0(\cdot) = \underline{l}_0(\cdot) = 0 \quad \text{and} \quad \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $\mathrm{Jord}(\psi_0) \backslash \{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}$. From $\underline{l}'_0$ we can obtain $\underline{l}'$ such that

$$\underline{l}'(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp) = l^\sharp = \underline{l}(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp)$$

for $(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp) \in \mathrm{Jord}(\psi_-)$. Then

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \hookrightarrow \times_{(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp) \in \mathrm{Jord}(\psi_-)} \tau(\rho^\sharp, A^\sharp - l^\sharp, B^\sharp + l^\sharp, \zeta^\sharp) \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_0, \underline{l}'_0, \underline{\eta}'),$$

as the unique irreducible subrepresentation. Therefore, $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}')$. This finishes the first step.

Step Two: We want to give some restrictions on $(\underline{l}', \underline{\eta}')$ in terms of $(\underline{l}, \underline{\eta})$, when $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0$. From the previous step, we can assume

$$\underline{l}'(\cdot) = \underline{l}(\cdot) \quad \text{and} \quad \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $\mathrm{Jord}(\psi_-)$. Next we consider the partial cuspidal support of $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$, which is defined as follows. Recall the cuspidal support of an irreducible admissible representation π^{Σ_0} of G^{Σ_0} is a G^{Σ_0}-conjugacy class of pairs $(M^{\Sigma_0}, \sigma^{\Sigma_0})$, where

$$M^{\Sigma_0} \cong \prod_i \mathrm{GL}(n_i) \times G_-^{\Sigma_0}$$

is a Levi subgroup of G^{Σ_0}, and

$$\sigma^{\Sigma_0} \cong (\otimes_i \tau_i) \otimes \pi_-^{\Sigma_0}$$

is an irreducible supercuspidal representation of M^{Σ_0}. We call $(G_-^{\Sigma_0}, \pi_-^{\Sigma_0})$ the partial cuspidal support of π^{Σ_0}. The partial cuspidal support of $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ can be computed as follows. Let $\psi_\gg$ be a dominating parameter of ψ with discrete diagonal restriction, obtained by shifting $(\rho, A_k, B_k, \zeta_k)$ to $(\rho, A_k + T_k, B_k + T_k, \zeta_k)$. Then $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ has

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the same partial cuspidal support as $\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta})$. Combined with the generalized Aubert involution (cf. [10, Definition 6.15]), it is also equal to that of

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^0_{\gg}, 0, \underline{\eta}), \tag{6.3}$$

where $\mathrm{Jord}(\psi^0_{\gg})$ is obtained from $\mathrm{Jord}(\psi_{\gg})$ by changing $(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp)$ to $(\rho^\sharp, A^\sharp - l^\sharp, B^\sharp + l^\sharp, +)$ for $l^\sharp = \underline{l}(\rho^\sharp, A^\sharp, B^\sharp, \zeta^\sharp)$. This is a discrete series representation and its cuspidal support can be computed as in [11]. To handle the combinatorics involved, we define an operation $(*)$ on the equivalence classes of $(\rho, A, B, 0, \eta, +) \sim (\rho, A+T, B+T, 0, \eta, +)$ as follows:

$$(\rho, A, B, 0, \eta, +) * (\rho, A', B', 0, \eta', +) \sim (\rho, A^*, B^*, 0, \eta^*, +).$$

(1) If $\eta \neq (-1)^{A'-B'}\eta'$, then

$$\begin{cases} A^* - B^* = (A-B) + (A'-B') + 1 \\ \eta^* = \eta'. \end{cases}$$

(2) If $\eta = (-1)^{A'-B'}\eta'$ and

(a) $A - B > A' - B' \Rightarrow \begin{cases} A^* - B^* = (A-B) - (A'-B') - 1 \\ \eta^* \neq \eta'. \end{cases}$

(b) $A - B < A' - B' \Rightarrow \begin{cases} A^* - B^* = (A'-B') - (A-B) - 1 \\ \eta^* = \eta'. \end{cases}$

If $\eta = (-1)^{A'-B'}\eta'$ and $A - B = A' - B'$, we define the product to be $\emptyset$. We can force $\emptyset$ to be the identity element under this operation, then it is easy to check that $\{\emptyset\} \sqcup \{(\rho, A, B, 0, \eta, +)\}/\sim$ forms a group. The partial cuspidal support of (6.3) is equal to that of

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^\sharp, 0, \underline{\eta}^\sharp;\ *_i(\rho, A_i - l_i, B_i + l_i, 0, \eta_i, +)),$$

where $\mathrm{Jord}(\psi^\sharp) := \mathrm{Jord}(\psi^0_{\gg}) \backslash \mathrm{Jord}_\rho(\psi^0_{\gg})$, $\underline{\eta}^\sharp$ is obtained by restriction, and the product $*$ is taken in the decreasing order with respect to $>_\psi$. In the same way, one can show the partial cuspidal support of $\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}')$ is equal to that of

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^\sharp, 0, \underline{\eta}^\sharp;\ *_i(\rho, A_i - l'_i, B_i + l'_i, 0, \eta'_i, +)),$$

where the product $*$ is taken in the decreasing order with respect to $>'_\psi$. As a result, we must have

$$*_i(\rho, A_i - l_i, B_i + l_i, 0, \eta_i, +) \sim *_i(\rho, A_i - l'_i, B_i + l'_i, 0, \eta'_i, +).$$

It follows

$$(\rho, A_k - l_k, B_k + l_k, 0, \eta_k, +) * (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta_{k-1}, +)$$

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is equivalent to

$$(\rho, A_{k-1} - l'_{k-1}, B_{k-1} + l'_{k-1}, 0, \eta'_{k-1}, +) * (\rho, A_k - l'_k, B_k + l'_k, 0, \eta'_k, +).$$

So we are necessarily in one of the following situations.

(1) If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$, then one of the following cases is satisfied.

(a) $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and

$$\begin{aligned}(A_{k-1} - B_{k-1} - 2l_{k-1}) + (A_k - B_k - 2l_k) + 2 \\ = (A_k - B_k - 2l'_k) - (A_{k-1} - B_{k-1} - 2l'_{k-1})\end{aligned}$$

i.e.,

$$(l_k + l_{k-1}) - (l'_k - l'_{k-1}) = A_{k-1} - B_{k-1} + 1.$$

(b) $\eta_{k-1} \neq (-1)^{A_k-B_k}\eta'_{k-1}$ and

$$\begin{aligned}(A_{k-1} - B_{k-1} - 2l_{k-1}) + (A_k - B_k - 2l_k) + 2 \\ = (A_{k-1} - B_{k-1} - 2l'_{k-1}) - (A_k - B_k - 2l'_k)\end{aligned}$$

i.e.,

$$(l_k + l_{k-1}) + (l'_k - l'_{k-1}) = A_k - B_k + 1.$$

(2) If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} \neq (-1)^{A_k-B_k}\eta'_k$, then one of the following cases is satisfied.

(a) $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and

$$\begin{aligned}(A_k - B_k - 2l_k) - (A_{k-1} - B_{k-1} - 2l_{k-1}) \\ = (A_{k-1} - B_{k-1} - 2l'_{k-1}) + (A_k - B_k - 2l'_k) + 2\end{aligned}$$

i.e.,

$$(l'_k + l'_{k-1}) - (l_k - l_{k-1}) = A_{k-1} - B_{k-1} + 1.$$

(b) $\eta_{k-1} \neq (-1)^{A_k-B_k}\eta'_{k-1}$ and

$$\begin{aligned}(A_{k-1} - B_{k-1} - 2l_{k-1}) - (A_k - B_k - 2l_k) \\ = (A_{k-1} - B_{k-1} - 2l'_{k-1}) + (A_k - B_k - 2l'_k) + 2\end{aligned}$$

i.e.,

$$(l'_k + l'_{k-1}) + (l_k - l_{k-1}) = A_k - B_k + 1.$$

(3) If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$, then one of the following cases is satisfied.

(a) $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and

$$\begin{aligned}(A_{k-1} - B_{k-1} - 2l_{k-1}) - (A_k - B_k - 2l_k) \\ = (A_k - B_k - 2l'_k) - (A_{k-1} - B_{k-1} - 2l'_{k-1})\end{aligned}$$

i.e.,

$$(l_k - l_{k-1}) + (l'_k - l'_{k-1}) = (A_k - B_k) - (A_{k-1} - B_{k-1}).$$

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(b) $\eta_{k-1} \neq (-1)^{A_k - B_k} \eta'_{k-1}$ and

$$\begin{aligned}(A_{k-1} - B_{k-1} - 2l_{k-1}) - (A_k - B_k - 2l_k) \\ = (A_{k-1} - B_{k-1} - 2l'_{k-1}) - (A_k - B_k - 2l'_k)\end{aligned}$$

i.e.,

$$l_k - l_{k-1} = l'_k - l'_{k-1}.$$

(4) If $\eta_k \neq (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1}$ and $\eta'_{k-1} \neq (-1)^{A_k - B_k} \eta'_k$, then the following case is satisfied.

(a) $\eta_{k-1} \neq (-1)^{A_k - B_k} \eta'_{k-1}$ and

$$\begin{aligned}(A_{k-1} - B_{k-1} - 2l_{k-1}) + (A_k - B_k - 2l_k) \\ = (A_{k-1} - B_{k-1} - 2l'_{k-1}) + (A_k - B_k - 2l'_k)\end{aligned}$$

i.e.,

$$l_k + l_{k-1} = l'_k + l'_{k-1}.$$

Since in our change of order formulas, we always have

$$\eta_{k-1} = (-1)^{A_k - B_k} \eta'_{k-1},$$

it is enough to eliminate those cases in which this is not satisfied. This is not easy in general, but at least we can do this when $l_{k-1} = 0$.

Step Three: We would like to derive the change of order formula when $l_{k-1} = 0$. Let us define ψ_e by

$$\mathrm{Jord}(\psi_e) := \bigcup_{C_{k-1} \in [A_{k-1}, B_{k-1}]} \{(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})\} \cup \mathrm{Jord}(\psi) \backslash \{(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}.$$

Then we can assume $\pi^{\Sigma_0}_{M, >_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M, >_\psi}(\psi_e, \underline{l}_e, \underline{\eta}_e) \neq 0$. Suppose

$$\pi^{\Sigma_0}_{M, >_\psi}(\psi_e, \underline{l}_e, \underline{\eta}_e) = \pi^{\Sigma_0}_{M, >'_\psi}(\psi_e, \underline{l}'_e, \underline{\eta}'_e).$$

One can show as in *Step one* that

$$\underline{l}'_e(\cdot) = \underline{l}_e(\cdot) \quad \text{and} \quad \underline{\eta}'_e(\cdot) = \underline{\eta}_e(\cdot)$$

over $\mathrm{Jord}(\psi_-)$. Moreover, by computing $\varepsilon^{M/W}_{\psi_e}$ with respect to $>_\psi$ and $>'_\psi$, one finds $\underline{\eta}'_e$ is alternating over $\{\bigcup_{C_{k-1} \in [A_{k-1}, B_{k-1}]} (\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})\}$. So from $(\underline{l}'_e, \underline{\eta}'_e)$, we can obtain $(\underline{l}', \underline{\eta}')$ by letting $l'_{k-1} = 0$ and $\eta'_{k-1} = \underline{\eta}'_e(\rho, B_{k-1}, B_{k-1}, \zeta)$. Then

$$\pi^{\Sigma_0}_{M, >'_\psi}(\psi, \underline{l}', \underline{\eta}') = \pi^{\Sigma_0}_{M, >'_\psi}(\psi_e, \underline{l}'_e, \underline{\eta}'_e).$$

It follows from *Step two* that we have several restrictions on $(\underline{l}', \underline{\eta}')$. To eliminate the case that $\eta_{k-1} \neq (-1)^{A_k - B_k} \eta'_{k-1}$, we would like to compute the difference between η_{k-1} and η'_{k-1} explicitly. The idea is again to compute $\varepsilon^{M/W}_{\psi_e}$ with respect to $>_\psi$ and $>'_\psi$ (cf. (1.4)). To distinguish these two orders, we write $\varepsilon^{M/W}_{\psi_e}$ for $>_\psi$ and $\varepsilon'^{M/W}_{\psi_e}$ for $>'_\psi$. Then

$$\eta_{k-1} \varepsilon^{M/W}_{\psi_e}(\rho, B_{k-1}, B_{k-1}, \zeta) = \eta'_{k-1} \varepsilon'^{M/W}_{\psi_e}(\rho, B_{k-1}, B_{k-1}, \zeta).$$

To apply the formula for $\varepsilon^{M/W}_{\psi_e}$ (respectively $\varepsilon'^{M/W}_{\psi_e}$), we need to write $(\rho, A_k, B_k, \zeta) = (\rho, a_k, b_k)$.

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- Suppose $\zeta = +1$.

(1) $A_k \in \mathbb{Z}$, then $\begin{cases} a_k, b_k \text{ even} & \Rightarrow \eta_{k-1} = -\eta'_{k-1} \\ a_k, b_k \text{ odd} & \Rightarrow \eta_{k-1} = \eta'_{k-1}. \end{cases}$

(2) $A_k \notin \mathbb{Z}$, then $\begin{cases} a_k \text{ odd}, b_k \text{ even} & \Rightarrow \eta_{k-1} = -\eta'_{k-1} \\ a_k \text{ even}, b_k \text{ odd} & \Rightarrow \eta_{k-1} = \eta'_{k-1}. \end{cases}$

- Suppose $\zeta = -1$.

(1) $A_k \in \mathbb{Z}$, then $\begin{cases} a_k, b_k \text{ even} & \Rightarrow \eta_{k-1} = -\eta'_{k-1} \\ a_k, b_k \text{ odd} & \Rightarrow \eta_{k-1} = \eta'_{k-1}. \end{cases}$

(2) $A_k \notin \mathbb{Z}$, then $\begin{cases} a_k \text{ even}, b_k \text{ odd} & \Rightarrow \eta_{k-1} = -\eta'_{k-1} \\ a_k \text{ odd}, b_k \text{ even} & \Rightarrow \eta_{k-1} = \eta'_{k-1}. \end{cases}$

It follows from the computations here that

$$\eta_{k-1} = (-1)^{inf(a_k,b_k)-1}\eta'_{k-1}.$$

Recall $A_k - B_k + 1 = inf(a_k, b_k)$, so this is exactly what we want. Adding this condition, the remaining cases in *Step two* are as follows.

(1) If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$, then $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and

$$l_k - l'_k = A_{k-1} - B_{k-1} + 1.$$

(2) If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} \neq (-1)^{A_k-B_k}\eta'_k$, then $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and

$$l'_k - l_k = A_{k-1} - B_{k-1} + 1.$$

(3) If $\eta_k = (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$ and $\eta'_{k-1} = (-1)^{A_k-B_k}\eta'_k$, then $\eta_{k-1} = (-1)^{A_k-B_k}\eta'_{k-1}$ and

$$l_k + l'_k = (A_k - B_k) - (A_{k-1} - B_{k-1}).$$

So this finishes the proof of the change of order formula in the case $l_{k-1} = 0$.

Step Four: In this last step, we want to show that if $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0$ with $l_{k-1} \neq 0$, then $(\underline{l}', \underline{\eta}') \sim_{\Sigma_0} S^+(\underline{l}, \underline{\eta})$. Note when $[A_k, B_k] = [A_{k-1}, B_{k-1}]$, this is obvious. So from now on, we assume

$$[A_k, B_k] \neq [A_{k-1}, B_{k-1}].$$

By the nonvanishing condition in Lemma 6.2, we have $l_k \geqslant l_{k-1}$. Since we have assumed $l_{k-1} \neq 0$, then $l_k \neq 0$.

First we would like to reduce it to the case $B_k = B_{k-1}$. Suppose $B_{k-1} > B_k$, let us define ψ^* from ψ by shifting $(\rho, A_{k-1}, B_{k-1}, \zeta)$ to $(\rho, A_{k-1} - B_{k-1} + B_k, B_k, \zeta)$. Then we have

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^*, \underline{l}', \underline{\eta}') = \mathrm{Jac}_{(\rho,A_{k-1},B_{k-1},\zeta)\mapsto(\rho,A_{k-1}-B_{k-1}+B_k,B_k,\zeta)}\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}').$$

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So $\mathrm{Jac}_{(\rho,A_{k-1},B_{k-1},\zeta)\mapsto(\rho,A_{k-1}-B_{k-1}+B_k,B_k,\zeta)}$ induces a bijection from $\Pi_{\psi}^{\Sigma_0}$ to $\Pi_{\psi^*}^{\Sigma_0}$ by Lemma 6.2. On the other side, we claim

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi^*,\underline{l},\underline{\eta})=\mathrm{Jac}_{(\rho,A_{k-1},B_{k-1},\zeta)\mapsto(\rho,A_{k-1}-B_{k-1}+B_k,B_k,\zeta)}\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta}).$$

To see this, we let $\psi_{\gg}$ be a dominating parameter with respect to $>_\psi$, obtained from ψ by shifting (ρ,A_k,B_k,ζ) to (ρ,A_k+T,B_k+T,ζ), and $\psi_{\gg}$ has discrete diagonal restriction. Then

$$\begin{aligned}\pi_{M,>_\psi}^{\Sigma_0}(\psi_{\gg},\underline{l},\underline{\eta})&\hookrightarrow\begin{pmatrix}\zeta(B_k+T)&\cdots&\zeta(B_k+1)\\ \vdots&&\vdots\\ \zeta(A_k+T)&\cdots&\zeta(A_k+1)\end{pmatrix}\rtimes\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta})\\ &\hookrightarrow\underbrace{\begin{pmatrix}\zeta(B_k+T)&\cdots&\zeta(B_k+1)\\ \vdots&&\vdots\\ \zeta(A_k+T)&\cdots&\zeta(A_k+1)\end{pmatrix}}_{*-1}\\ &\quad\times\underbrace{\begin{pmatrix}\zeta B_{k-1}&\cdots&\zeta(B_k+1)\\ \vdots&&\vdots\\ \zeta A_{k-1}&\cdots&\zeta(A_{k-1}-B_{k-1}+B_k+1)\end{pmatrix}}_{*-2}\rtimes\sigma\end{aligned}$$

where

$$\sigma:=\mathrm{Jac}_{(\rho,A_{k-1},B_{k-1},\zeta)\mapsto(\rho,A_{k-1}-B_{k-1}+B_k,B_k,\zeta)}\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta}).$$

Since we can interchange $(*-1)$ and $(*-2)$, it is easy to see $\pi_{M,>_\psi}^{\Sigma_0}(\psi^*,\underline{l},\underline{\eta})=\sigma$. This shows our claim. So if $\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta})=\pi_{M,>'_\psi}^{\Sigma_0}(\psi,\underline{l}',\underline{\eta}')\neq 0$, then $\pi_{M,>_\psi}^{\Sigma_0}(\psi^*,\underline{l},\underline{\eta})=\pi_{M,>'_\psi}^{\Sigma_0}(\psi^*,\underline{l}',\underline{\eta}')$. And suppose we know $(\underline{l},\underline{\eta})$ is related to $(\underline{l}',\underline{\eta}')$ according to our formula with respect to ψ^* modulo the equivalence relation $\sim_{\Sigma_0}$, then it is easy to see they are related in the same way with respect to ψ. Hence $(\underline{l}',\underline{\eta}')\sim_{\Sigma_0}S^+(\underline{l},\underline{\eta})$.

Now we only consider the case $B_{k-1}=B_k$, and by our previous assumption we have $A_k>A_{k-1}$. Let ψ^{**} be defined from ψ by changing (ρ,A_k,B_k,ζ) and $(\rho,A_{k-1},B_{k-1},\zeta)$ to (ρ,A_k-1,B_k+1,ζ) and $(\rho,A_{k-1}-1,B_{k-1}+1,\zeta)$, respectively. Then we claim for $l_{k-1}\neq 0$,

$$\begin{aligned}\mathrm{Jac}_{\zeta B_{k-1},\ldots,-\zeta A_{k-1}}\circ\mathrm{Jac}_{\zeta B_k,\ldots,-\zeta A_k}\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta})&=\pi_{M,>_\psi}^{\Sigma_0}(\psi_-,\underline{l}_-,\underline{\eta}_-;\\ &(\rho,A_k-1,B_k+1,l_k-1,\eta_k,\zeta),(\rho,A_{k-1}-1,B_{k-1}+1,l_{k-1}-1,\eta_{k-1},\zeta)).\end{aligned}$$

In particular, this means we get a bijection from $\Pi_{\psi}^{\Sigma_0}\backslash\{\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta}):l_{k-1}=0\}$ to $\Pi_{\psi^{**}}^{\Sigma_0}$. To prove the claim, we first show for $l_{k-1}\neq 0$,

$$\begin{aligned}\mathrm{Jac}_{\zeta B_k,\ldots,-\zeta A_k}\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta})&=\pi_{M,>_\psi}^{\Sigma_0}(\psi_-,\underline{l}_-,\underline{\eta}_-;\\ &(\rho,A_k-1,B_k+1,l_k-1,\eta_k,\zeta),(\rho,A_{k-1},B_{k-1},l_{k-1},\eta_{k-1},\zeta)).\end{aligned}$$

Again let $\psi_{\gg}$ be a dominating parameter with respect to $>_\psi$, obtained from ψ by shifting (ρ,A_k,B_k,ζ) to (ρ,A_k+T,B_k+T,ζ), and $\psi_{\gg}$ has discrete diagonal restriction. Then

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$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \langle \zeta(B_k+T), \ldots, -\zeta(A_k+T)\rangle \times \begin{pmatrix} \zeta(B_k+1+T) & \cdots & \zeta(B_k+2) \\ \vdots & & \vdots \\ \zeta(A_k-1+T) & \cdots & \zeta A_k \end{pmatrix}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1, B_k+1, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta))$$

$$\hookrightarrow \langle \zeta(B_k+T), \ldots, -\zeta A_k\rangle \times \begin{pmatrix} \zeta(B_k+1+T) & \cdots & \zeta(B_k+2) \\ \vdots & & \vdots \\ \zeta(A_k-1+T) & \cdots & \zeta A_k \end{pmatrix}$$

$$\times \underbrace{\langle -\zeta(A_k+1), \ldots, -\zeta(A_k+T)\rangle}_{**-1} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1, B_k+1, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)).$$

Since $A_k > A_{k-1}$, we can take the dual of $(**-1)$ by (4.1). Therefore,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \langle \zeta B_k, \ldots, -\zeta A_k\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1, B_k+1, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)).$$

By applying $\mathrm{Jac}_{\zeta B_k,\ldots,-\zeta A_k}$ to the full- induced representation above, we get $\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1, B_k+1, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta))$. So

$$\mathrm{Jac}_{\zeta B_k,\ldots,-\zeta A_k}\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1, B_k+1, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta)).$$

Next for the same T,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1+T, B_k+1+T, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta))$$

$$\hookrightarrow \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta A_{k-1}\rangle}_{**-2} \times \underbrace{\begin{pmatrix} \zeta(B_k+1+T) & \cdots & \zeta(B_k+2) \\ \vdots & & \vdots \\ \zeta(A_k-1+T) & \cdots & \zeta A_k \end{pmatrix}}_{**-3}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1, B_k+1, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}-1, B_{k-1}+1, l_{k-1}-1, \eta_{k-1}, \zeta)).$$

Since $B_k = B_{k-1}$, we can interchange $(**-2)$ and $(**-3)$. So

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1, B_k+1, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}, B_{k-1}, l_{k-1}, \eta_{k-1}, \zeta))$$

$$\hookrightarrow \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta A_{k-1}\rangle}_{**-2} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k-1, B_k+1, l_k-1, \eta_k, \zeta), (\rho, A_{k-1}-1, B_{k-1}+1, l_{k-1}-1, \eta_{k-1}, \zeta)).$$

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After applying $\mathrm{Jac}_{\zeta B_{k-1},\ldots,-\zeta A_{k-1}}$ to the full-induced representation above, we get $\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_k-1,B_k+1,l_k-1,\eta_k,\zeta),(\rho,A_{k-1}-1,B_{k-1}+1,l_{k-1}-1,\eta_{k-1},\zeta))$. So

$$\begin{aligned}&\mathrm{Jac}_{\zeta B_{k-1},\ldots,-\zeta A_{k-1}}\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_k-1,B_k+1,l_k-1,\eta_k,\zeta),\\&\qquad(\rho,A_{k-1},B_{k-1},l_{k-1},\eta_{k-1},\zeta))\\&\quad=\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_k-1,B_k+1,l_k-1,\eta_k,\zeta),\\&\qquad(\rho,A_{k-1}-1,B_{k-1}+1,l_{k-1}-1,\eta_{k-1},\zeta)).\end{aligned}$$

This finishes the proof of our claim. At last, we want to compute

$$\sigma^{**}:=\mathrm{Jac}_{\zeta B_{k-1},\ldots,-\zeta A_{k-1}}\circ\mathrm{Jac}_{\zeta B_k,\ldots,-\zeta A_k}\pi^{\Sigma_0}_{M,>'_\psi}(\psi,\underline{l}',\underline{\eta}')$$

for $l'_{k-1}\neq 0$. Let $\psi'_\gg$ be a dominating parameter with respect to $>'_\psi$, obtained from ψ by shifting $(\rho,A_{k-1},B_{k-1},\zeta)$ to $(\rho,A_{k-1}+T',B_{k-1}+T',\zeta)$, and $\psi'_\gg$ has discrete diagonal restriction. Then

$$\begin{aligned}\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_\gg,\underline{l}',\underline{\eta}')&\hookrightarrow\langle\zeta(B_{k-1}+T'),\ldots,-\zeta(A_{k-1}+T')\rangle\times\langle\zeta B_k,\ldots,-\zeta A_k\rangle\\&\qquad\times\begin{pmatrix}\zeta(B_{k-1}+1+T')&\cdots&\zeta(B_{k-1}+2)\\\vdots&&\vdots\\\zeta(A_{k-1}-1+T')&\cdots&\zeta A_{k-1}\end{pmatrix}\rtimes\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;\\&\qquad(\rho,A_k-1,B_k+1,l'_k-1,\eta'_k,\zeta),\\&\qquad(\rho,A_{k-1}-1,B_{k-1}+1,l'_{k-1}-1,\eta'_{k-1},\zeta))\\&\hookrightarrow\underbrace{\langle\zeta(B_{k-1}+T'),\ldots,\zeta(B_{k-1}+1)\rangle}_{I}\times\underbrace{\langle\zeta B_{k-1},\ldots,-\zeta(A_{k-1}+T')\rangle}_{II}\\&\qquad\times\underbrace{\langle\zeta B_k,\ldots,-\zeta A_k\rangle}_{III}\times\underbrace{\begin{pmatrix}\zeta(B_{k-1}+1+T')&\cdots&\zeta(B_{k-1}+2)\\\vdots&&\vdots\\\zeta(A_{k-1}-1+T')&\cdots&\zeta A_{k-1}\end{pmatrix}}_{IV}\\&\qquad\rtimes\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_k-1,B_k+1,l'_k-1,\eta'_k,\zeta),\\&\qquad(\rho,A_{k-1}-1,B_{k-1}+1,l'_{k-1}-1,\eta'_{k-1},\zeta)).\end{aligned}$$

We can interchange (*IV*) with (*III*) and (*II*). Also (*II*) and (*III*) are interchangeable. So

$$\begin{aligned}\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_\gg,\underline{l}',\underline{\eta}')&\hookrightarrow\underbrace{\langle\zeta(B_{k-1}+T'),\ldots,\zeta(B_{k-1}+1)\rangle}_{I}\\&\qquad\times\underbrace{\begin{pmatrix}\zeta(B_{k-1}+1+T')&\cdots&\zeta(B_{k-1}+2)\\\vdots&&\vdots\\\zeta(A_{k-1}-1+T')&\cdots&\zeta A_{k-1}\end{pmatrix}}_{IV}\end{aligned}$$

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$$\times \underbrace{\langle \zeta B_k, \ldots, -\zeta A_k \rangle}_{III} \times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta (A_{k-1} + T') \rangle}_{II}$$

$$\rtimes \pi^{\Sigma_0}_{M, >'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l'_k - 1, \eta'_k, \zeta),$$

$$(\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta))$$

$$\hookrightarrow \underbrace{\langle \zeta (B_{k-1} + T'), \ldots, \zeta (B_{k-1} + 1) \rangle}_{I}$$

$$\times \underbrace{\begin{pmatrix} \zeta (B_{k-1} + 1 + T') & \cdots & \zeta (B_{k-1} + 2) \\ \vdots & & \vdots \\ \zeta (A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV}$$

$$\times \underbrace{\langle \zeta B_k, \ldots, -\zeta A_{k-1} \rangle}_{III_1} \times \underbrace{\langle -\zeta (A_{k-1} + 1), \ldots - \zeta A_k \rangle}_{III_2}$$

$$\times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta (A_{k-1} + T') \rangle}_{II}$$

$$\times \underbrace{\begin{pmatrix} \zeta (B_k + 1) & \cdots & \zeta (B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta (A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_{V} \rtimes \pi^{\Sigma_0}_{M, >'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-;$$

$$(\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta),$$

$$(\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)).$$

Since $\mathrm{Jac}_{\zeta (B_{k-1} + 1 + T')} \pi^{\Sigma_0}_{M, >'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') = 0$, we can 'combine' (I) and (IV). We can also interchange (III_2) with (II) and (V), and then take dual of (III_2). As a result,

$$\pi^{\Sigma_0}_{M, >'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\begin{pmatrix} \zeta (B_{k-1} + T') & \cdots & \zeta (B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta (A_{k-1} - 1 + T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV_+} \times \underbrace{\langle \zeta B_k, \ldots, -\zeta A_{k-1} \rangle}_{III_1}$$

$$\times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta (A_{k-1} + T') \rangle}_{II}$$

$$\times \underbrace{\begin{pmatrix} \zeta (B_k + 1) & \cdots & \zeta (B_k + A_{k-1} - A_k + 2) \\ \vdots & & \vdots \\ \zeta (A_k - 1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_{V}$$

$$\times \underbrace{\langle \zeta A_k, \ldots, \zeta (A_{k-1} + 1) \rangle}_{(III_2)^\vee} \rtimes \pi^{\Sigma_0}_{M, >'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-;$$

$$(\rho, A_{k-1} - 1, B_k + A_{k-1} - A_k + 1, l'_k - 1, \eta'_k, \zeta),$$

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$$(\rho, A_{k-1}-1, B_{k-1}+1, l'_{k-1}-1, \eta'_{k-1}, \zeta)).$$

Since $A_k > A_{k-1} > B_{k-1}$ and $\mathrm{Jac}_{\zeta A_k} \pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') = 0$, we can further 'combine' $(III_2)^\vee$ and (V).

$$\begin{aligned}
\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow\ & \underbrace{\begin{pmatrix} \zeta(B_{k-1}+T') & \cdots & \zeta(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta(A_{k-1}-1+T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV_+} \times \underbrace{\langle \zeta B_k, \ldots, -\zeta A_{k-1} \rangle}_{III_1} \\
& \times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta(A_{k-1}+T') \rangle}_{II} \\
& \times \underbrace{\begin{pmatrix} \zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2) \\ \vdots & & \vdots \\ \zeta A_k & \cdots & \zeta(A_{k-1}+1) \end{pmatrix}}_{V_+} \\
& \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{k-1}-1, B_k+A_{k-1}-A_k+1, l'_k-1, \eta'_k, \zeta), \\
& \quad (\rho, A_{k-1}-1, B_{k-1}+1, l'_{k-1}-1, \eta'_{k-1}, \zeta)) \\
\hookrightarrow\ & \underbrace{\begin{pmatrix} \zeta(B_{k-1}+T') & \cdots & \zeta(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta(A_{k-1}-1+T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV_+} \times \underbrace{\langle \zeta B_k, \ldots, -\zeta A_{k-1} \rangle}_{III_1} \\
& \times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta A_{k-1} \rangle}_{II_1} \times \underbrace{\langle -\zeta(A_{k-1}+1), \ldots - \zeta(A_{k-1}+T') \rangle}_{II_2} \\
& \times \underbrace{\begin{pmatrix} \zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2) \\ \vdots & & \vdots \\ \zeta A_k & \cdots & \zeta(A_{k-1}+1) \end{pmatrix}}_{V_+} \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\
& \quad (\rho, A_{k-1}-1, B_k+A_{k-1}-A_k+1, l'_k-1, \eta'_k, \zeta), \\
& \quad (\rho, A_{k-1}-1, B_{k-1}+1, l'_{k-1}-1, \eta'_{k-1}, \zeta)).
\end{aligned}$$

So we can interchange (II_2) with (V_+), and take dual of (II_2). Note $(II_2)^\vee$ is interchangeable with (V_+). Since $A_{k-1} > B_{k-1}$, $(II_2)^\vee$ is also interchangeable with (II_1) and (III_1). Therefore,

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi'_{\gg}, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B_{k-1}+T') & \cdots & \zeta(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta(A_{k-1}-1+T') & \cdots & \zeta A_{k-1} \end{pmatrix}}_{IV_+}$$

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$$\times \underbrace{\langle \zeta(A_{k-1}+T'), \ldots \zeta(A_{k-1}+1)\rangle}_{(II_2)^\vee}$$

$$\times \underbrace{\langle \zeta B_k, \ldots, -\zeta A_{k-1}\rangle}_{III_1} \times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta A_{k-1}\rangle}_{II_1}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2) \\ \vdots & & \vdots \\ \zeta A_k & \cdots & \zeta(A_{k-1}+1) \end{pmatrix}}_{V_+} \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-;$$

$$(\rho, A_{k-1}-1, B_k+A_{k-1}-A_k+1, l'_k-1, \eta'_k, \zeta),$$
$$(\rho, A_{k-1}-1, B_{k-1}+1, l'_{k-1}-1, \eta'_{k-1}, \zeta)).$$

Consequently,

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\langle \zeta B_k, \ldots, -\zeta A_{k-1}\rangle}_{III_1} \times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta A_{k-1}\rangle}_{II_1}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2) \\ \vdots & & \vdots \\ \zeta A_k & \cdots & \zeta(A_{k-1}+1) \end{pmatrix}}_{V_+} \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-;$$

$$(\rho, A_{k-1}-1, B_k+A_{k-1}-A_k+1, l'_k-1, \eta'_k, \zeta),$$
$$(\rho, A_{k-1}-1, B_{k-1}+1, l'_{k-1}-1, \eta'_{k-1}, \zeta))$$

$$\hookrightarrow \underbrace{\langle \zeta B_k, \ldots, -\zeta A_{k-1}\rangle}_{III_1} \times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta A_{k-1}\rangle}_{II_1}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2) \\ \vdots & & \vdots \\ \zeta(A_k-1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_{V} \times \underbrace{\langle \zeta A_k, \ldots, \zeta(A_{k-1}+1)\rangle}_{(III_2)^\vee}$$

$$\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_{k-1}-1, B_k+A_{k-1}-A_k+1, l'_k-1, \eta'_k, \zeta),$$
$$(\rho, A_{k-1}-1, B_{k-1}+1, l'_{k-1}-1, \eta'_{k-1}, \zeta)).$$

Then we take dual of $(III_2)^\vee$, and interchange (III_2) with (V).

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \hookrightarrow \underbrace{\langle \zeta B_k, \ldots, -\zeta A_{k-1}\rangle}_{III_1} \times \underbrace{\langle \zeta B_{k-1}, \ldots, -\zeta A_{k-1}\rangle}_{II_1}$$

$$\underbrace{\langle -\zeta(A_{k-1}+1), \ldots, -\zeta A_k\rangle}_{III_2} \times \underbrace{\begin{pmatrix} \zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2) \\ \vdots & & \vdots \\ \zeta(A_k-1) & \cdots & \zeta A_{k-1} \end{pmatrix}}_{V}$$

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$$\rtimes\ \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_{k-1}-1,B_k+A_{k-1}-A_k+1,l'_k-1,\eta'_k,\zeta),$$
$$(\rho,A_{k-1}-1,B_{k-1}+1,l'_{k-1}-1,\eta'_{k-1},\zeta)).$$

Suppose

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi,\underline{l}',\underline{\eta}')\hookrightarrow \underbrace{\langle\zeta B_k,\ldots,-\zeta A_{k-1}\rangle}_{III_1}\times\underbrace{\langle\zeta B_{k-1},\ldots,-\zeta A_k\rangle}_{II_1+III_2}$$
$$\times\underbrace{\begin{pmatrix}\zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2)\\ \vdots & & \vdots\\ \zeta(A_k-1) & \cdots & \zeta A_{k-1}\end{pmatrix}}_{V}\rtimes\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;$$
$$(\rho,A_{k-1}-1,B_k+A_{k-1}-A_k+1,l'_k-1,\eta'_k,\zeta),$$
$$(\rho,A_{k-1}-1,B_{k-1}+1,l'_{k-1}-1,\eta'_{k-1},\zeta)).$$

Since we can interchange (III_1) with (II_1+III_2), and $B_k=B_{k-1}$, we have

$$\sigma^{**}\hookrightarrow\underbrace{\begin{pmatrix}\zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2)\\ \vdots & & \vdots\\ \zeta(A_k-1) & \cdots & \zeta A_{k-1}\end{pmatrix}}_{V}\rtimes\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;$$
$$(\rho,A_{k-1}-1,B_k+A_{k-1}-A_k+1,l'_k-1,\eta'_k,\zeta),$$
$$(\rho,A_{k-1}-1,B_{k-1}+1,l'_{k-1}-1,\eta'_{k-1},\zeta)).$$

Otherwise, we would have

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi,\underline{l}',\underline{\eta}')\hookrightarrow \underbrace{\langle\zeta B_k,\ldots,-\zeta A_{k-1}\rangle}_{III_1}\times\underbrace{\langle-\zeta(A_{k-1}+1),\ldots,-\zeta A_k\rangle}_{III_2}$$
$$\underbrace{\langle\zeta B_{k-1},\ldots,-\zeta A_{k-1}\rangle}_{II_1}\times\underbrace{\begin{pmatrix}\zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2)\\ \vdots & & \vdots\\ \zeta(A_k-1) & \cdots & \zeta A_{k-1}\end{pmatrix}}_{V}$$
$$\rtimes\ \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_{k-1}-1,B_k+A_{k-1}-A_k+1,l'_k-1,\eta'_k,\zeta),$$
$$(\rho,A_{k-1}-1,B_{k-1}+1,l'_{k-1}-1,\eta'_{k-1},\zeta)).$$

Then we again have

$$\sigma^{**}\hookrightarrow\underbrace{\begin{pmatrix}\zeta(B_k+1) & \cdots & \zeta(B_k+A_{k-1}-A_k+2)\\ \vdots & & \vdots\\ \zeta(A_k-1) & \cdots & \zeta A_{k-1}\end{pmatrix}}_{V}\rtimes\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;$$
$$(\rho,A_{k-1}-1,B_k+A_{k-1}-A_k+1,l'_k-1,\eta'_k,\zeta),$$
$$(\rho,A_{k-1}-1,B_{k-1}+1,l'_{k-1}-1,\eta'_{k-1},\zeta)).$$

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Note that the full-induced representation above has a unique irreducible subrepresentation:

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l'_k - 1, \eta'_k, \zeta), (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)).$$

So it must be equal to σ^{**}. To summarize, if $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0$ for $l_{k-1} \neq 0$, then we have $l'_{k-1} \neq 0$ by the previous step. After applying

$$\mathrm{Jac}_{\zeta B_{k-1},\ldots,-\zeta A_{k-1}} \circ \mathrm{Jac}_{\zeta B_k,\ldots,-\zeta A_k}$$

to both sides we get

$$\begin{aligned}
&\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l_k - 1, \eta_k, \zeta), \\
&\qquad (\rho, A_{k-1} - 1, B_{k-1} + 1, l_{k-1} - 1, \eta_{k-1}, \zeta)) \\
&= \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_k - 1, B_k + 1, l'_k - 1, \eta'_k, \zeta), \\
&\qquad (\rho, A_{k-1} - 1, B_{k-1} + 1, l'_{k-1} - 1, \eta'_{k-1}, \zeta)).
\end{aligned}$$

By induction on l_{k-1}, we can assume $(l_k - 1, \eta_k; l_{k-1} - 1, \eta_{k-1})$ is related to $(l'_k - 1, \eta'_k; l'_{k-1} - 1, \eta'_{k-1})$ according to our formula with respect to ψ^{**}. Then it is easy to deduce that $(l_k, \eta_k; l_{k-1}, \eta_{k-1})$ and $(l'_k, \eta'_k; l'_{k-1}, \eta'_{k-1})$ are also related according to our formula with respect to ψ. Hence $(\underline{l}', \underline{\eta}') \sim_{\Sigma_0} S^+(\underline{l}, \underline{\eta})$.

6.2. Case $\zeta_k \neq \zeta_{k-1}$

In this case, there is no extra conditions on $[A_k, B_k], [A_{k-1}, B_{k-1}]$. For functions $\underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]]$ and $\underline{\eta}(\rho, A, B, \zeta) \in \mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Jord}(\psi)$, we denote

$$l_k = \underline{l}(\rho, A_k, B_k, \zeta_k), \quad l_{k-1} = \underline{l}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}),$$

and

$$\eta_k = \underline{\eta}(\rho, A_k, B_k, \zeta_k), \quad \eta_{k-1} = \underline{\eta}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}).$$

From $(\underline{l}, \underline{\eta})$, we want to construct another pair $(\underline{l}', \underline{\eta}')$ such that

$$\underline{l}'(\cdot) = \underline{l}(\cdot) \quad \text{and} \quad \underline{\eta}'(\cdot) = \underline{\eta}(\cdot)$$

over $\mathrm{Jord}(\psi)\backslash\{(\rho, A_k, B_k, \zeta_k), (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})\}$. Let us denote

$$l'_k = \underline{l}'(\rho, A_k, B_k, \zeta_k), \quad l'_{k-1} = \underline{l}(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}),$$

and

$$\eta'_k = \underline{\eta}'(\rho, A_k, B_k, \zeta_k), \quad \eta'_{k-1} = \underline{\eta}'(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}).$$

Then we define $l'_k, l'_{k-1}, \eta'_k, \eta'_{k-1}$ according to the following formulas.

$$\begin{cases}
l'_k = l_k \\
l'_{k-1} = l_{k-1} \\
\eta_k = (-1)^{A_{k-1} - B_{k-1} + 1} \eta'_k \\
\eta_{k-1} = (-1)^{A_k - B_k + 1} \eta'_{k-1}.
\end{cases}$$

We denote this transformation by U. Since the situation is symmetric here, we have $U \circ U = \mathrm{id}$.

Theorem 6.3. *Suppose* $(\underline{l}', \underline{\eta}') = U(\underline{l}, \underline{\eta})$*, then*

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}').$$

Let $\psi_\gg$ be a dominating parameter of ψ such that $\mathrm{Jord}_\rho(\psi_\gg) = \mathrm{Jord}_\rho(\psi)$, and $\mathrm{Jord}_{\rho'}(\psi_\gg)$ has discrete diagonal restriction for $\rho' \neq \rho$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \mathrm{Jac}_{X^c} \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}),$$

and

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') = \mathrm{Jac}_{X^c} \pi^{\Sigma_0}_{M,>'_\psi}(\psi_\gg, \underline{l}', \underline{\eta}').$$

So it suffices to prove the proposition for such $\psi_\gg$. Therefore, in the following discussions of the proof of this proposition, we always assume $\mathrm{Jord}_{\rho'}(\psi)$ has discrete diagonal restriction for $\rho' \neq \rho$, and if we choose some dominating $\psi_\gg$ of ψ, we always assume $\mathrm{Jord}_{\rho'}(\psi_\gg) = \mathrm{Jord}_{\rho'}(\psi)$ for $\rho' \neq \rho$.

6.2.1. First reduction. Let $(\underline{l}', \underline{\eta}') = U(\underline{l}, \underline{\eta})$. We want to reduce the proposition to the following cases:

$$(\rho, A_i, B_i, \zeta_i) \gg_r (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) \quad \text{for all } i, \text{ and } (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}) \gg_r 0. \tag{6.4}$$

We denote the case with respect to r by $(6.4)_r$. We do this in two steps. First we reduce it to the cases:

$$\begin{cases} (\rho, A_i, B_i, \zeta_i) \gg_r (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}) & \text{for } i > k-1, \\ (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1}) \gg_r \bigcup_{j=1}^{k-2} \{(\rho, A_j, B_j, \zeta_j)\} & \text{and} \quad 0. \end{cases} \tag{6.5}$$

We denote the case with respect to r by $(6.5)_r$. Let us choose a dominating parameter $\psi_\gg$ with respect to $>_\psi$ such that $T_i = 0$ for $i < k-1$,

$$(\rho, A_i + T_i, B_i + T_i, \zeta_i) \gg_r (\rho, A_{i-1} + T_{i-1}, B_{i-1} + T_{i-1}, \zeta_{i-1}) \quad \text{for } i \geqslant k.$$

We further require the existence of T such that $0 \leqslant T < T_k$,

$$\begin{aligned} (\rho, A_{k-1} + T_{k-1}, B_{k-1} + T_{k-1}, \zeta_{k-1}) &\gg_r (\rho, A_k + T_k - T, B_k + T_k - T, \zeta_k) \\ &\gg_r \bigcup_{j=1}^{k-2} \{(\rho, A_j, B_j, \zeta_j)\} \quad \text{and} \quad 0. \end{aligned}$$

Let $\psi^{(k)}_\gg$ be obtained from $\psi_\gg$ by changing T_k, T_{k-1} to zero. Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1} + 1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1} + T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1} + 1) \end{pmatrix}$$

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$$\times \begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+1) \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}),$$

where the two generalized segments are interchangeable for $\zeta_k \neq \zeta_{k-1}$. Let $\psi^T_{\gg}$ be obtained from $\psi_{\gg}$ by changing $(\rho, A_k+T_k, B_k+T_k, \zeta_k)$ to $(\rho, A_k+T_k-T, B_k+T_k-T, \zeta_k)$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta_k(B_k+T_k-T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k-T) & \cdots & \zeta_k(A_k+1) \end{pmatrix}$$
$$\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}).$$

By $(6.5)_r$, we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^T_{\gg}, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi^T_{\gg}, \underline{l}', \underline{\eta}').$$

Since

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(k)}_{\gg}, \underline{l}', \underline{\eta}') = \mathrm{Jac}_{(\rho,A_{k-1}+T_{k-1},B_{k-1}+T_{k-1},\zeta_{k-1})\mapsto(\rho,A_{k-1},B_{k-1},\zeta_{k-1})} \circ$$
$$\mathrm{Jac}_{(\rho,A_k+T_k-T,B_k+T_k-T,\zeta_k)\mapsto(\rho,A_k,B_k,\zeta_k)} \pi^{\Sigma_0}_{M,>'_\psi}(\psi^T_{\gg}, \underline{l}', \underline{\eta}')$$

and $\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta})$ is contained in

$$\mathrm{Jac}_{(\rho,A_{k-1}+T_{k-1},B_{k-1}+T_{k-1},\zeta_{k-1})\mapsto(\rho,A_{k-1},B_{k-1},\zeta_{k-1})} \circ$$
$$\mathrm{Jac}_{(\rho,A_k+T_k-T,B_k+T_k-T,\zeta_k)\mapsto(\rho,A_k,B_k,\zeta_k)} \pi^{\Sigma_0}_{M,>_\psi}(\psi^T_{\gg}, \underline{l}, \underline{\eta}),$$

then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^{(k)}_{\gg}, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi^{(k)}_{\gg}, \underline{l}', \underline{\eta}').$$

After applying $\circ_{i>k}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)}$ to both sides, we get $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}')$.

Secondly we want to further reduce it to $(6.4)_r$. Let us assume we are in case $(6.5)_{r'}$ for r' sufficiently large with respect to r. We can choose a dominating parameter $\psi_{\gg}$ with respect to $>_\psi$ such that $T_i = 0$ for $i > k$, and

$$(\rho, A_{k+1}, B_{k+1}, \zeta_{k+1}) \gg_r (\rho, A_i+T_i, B_i+T_i, \zeta_i)$$
$$\gg_r (\rho, A_{i-1}+T_{i-1}, B_{i-1}+T_{i-1}, \zeta_{i-1}) \quad \text{for } i \leqslant k.$$

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Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) \neq 0$, then

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) \hookrightarrow \times_{i<k-1} &\begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1)\end{pmatrix} \\ &\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1)\end{pmatrix} \\ &\times \begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+1)\end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}),\end{aligned}$$

where i increases. We can also assume $B_{k-1}+1 > A_{k-2}+T_{k-2}+1$. Then we can change the order of the generalized segments as follows,

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) \hookrightarrow &\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1)\end{pmatrix} \\ &\times \begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+1)\end{pmatrix} \\ &\times_{i<k-1} \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1)\end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) \\ \cong &\begin{pmatrix} \zeta_k(B_k+T_k) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k) & \cdots & \zeta_k(A_k+1)\end{pmatrix} \\ &\times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1)\end{pmatrix} \\ &\times_{i<k-1} \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1)\end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}).\end{aligned}$$

We can choose $0 \leqslant T < T_k$ such that

$$\begin{aligned}(\rho, A_{k-1}+T_{k-1}, B_{k-1}+T_{k-1}, \zeta_{k-1}) &\gg_r (\rho, A_k+T_k-T, B_k+T_k-T, \zeta_k) \\ &\gg_r (\rho, A_{k-2}+T_{k-2}, B_{k-2}+T_{k-2}, \zeta_{k-2}) \quad \text{and } 0.\end{aligned}$$

Let $\psi^T_{\gg}$ be obtained from $\psi_{\gg}$ by changing $(\rho, A_k+T_k, B_k+T_k, \zeta_k)$ to $(\rho, A_k+T_k-$

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$T, B_k + T_k - T, \zeta_k)$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^T_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta_k(B_k+T_k-T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k-T) & \cdots & \zeta_k(A_k+1) \end{pmatrix} \times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix} \times_{i<k-1} \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$$

$$\cong \times_{i<k-1} \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1) \end{pmatrix} \times \begin{pmatrix} \zeta_k(B_k+T_k-T) & \cdots & \zeta_k(B_k+1) \\ \vdots & & \vdots \\ \zeta_k(A_k+T_k-T) & \cdots & \zeta_k(A_k+1) \end{pmatrix} \times \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(B_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T_{k-1}) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).$$

By $(6.4)_r$, we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^T_\gg, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi^T_\gg, \underline{l}', \underline{\eta}').$$

Since

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') = \mathrm{Jac}_{(\rho, A_{k-1}+T_{k-1}, B_{k-1}+T_{k-1}, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} \circ \mathrm{Jac}_{(\rho, A_k+T_k-T, B_k+T_k-T, \zeta_k) \mapsto (\rho, A_k, B_k, \zeta_k)} \circ \circ_{i<k-1} \mathrm{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \pi^{\Sigma_0}_{M,>'_\psi}(\psi^T_\gg, \underline{l}', \underline{\eta}')$$

and $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ is contained in

$$\mathrm{Jac}_{(\rho, A_{k-1}+T_{k-1}, B_{k-1}+T_{k-1}, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} \circ \mathrm{Jac}_{(\rho, A_k+T_k-T, B_k+T_k-T, \zeta_k) \mapsto (\rho, A_k, B_k, \zeta_k)} \circ \circ_{i<k-1} \mathrm{Jac}_{(\rho, A_i+T_i, B_i+T_i, \zeta_i) \mapsto (\rho, A_i, B_i, \zeta_i)} \pi^{\Sigma_0}_{M,>_\psi}(\psi^T_\gg, \underline{l}, \underline{\eta}),$$

then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}').$$

This finishes the first reduction.

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6.2.2. Second reduction. We want to reduce the proposition further to the cases:

$$(\rho, A_i, B_i, \zeta_i) \gg_r (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}), \quad \text{and} \quad l_i = 0 \text{ for all } i. \tag{6.6}$$

Let us denote the case with respect to r by $(6.6)_r$. Suppose we are in case $(6.4)_{r'}$ for r' sufficiently large with respect to r. Let ψ^T be obtained by changing $(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ to $(\rho, A_{k-1}+T, B_{k-1}+T, \zeta_{k-1})$ such that

$$B_{k-1}+T > A_k \quad \text{and} \quad B_{k+1} > A_{k-1}+T.$$

Let

$$\mathrm{Jord}(\psi_-) = \{(\rho, A_i - l_i, B_i + l_i, \zeta_i) : i \neq k, k-1\}.$$

We define $(\underline{l}_-, \underline{\eta}_-)$ such that $\underline{l}_-(\rho, A_i - l_i, B_i + l_i, \zeta_i) = 0$ and $\underline{\eta}_-(\rho, A_i - l_i, B_i + l_i, \zeta_i) = \eta_i$. Then

$$\begin{aligned}
\pi^{\Sigma_0}_{M,>'_\psi}(\psi^T, \underline{l}', \underline{\eta}') \hookrightarrow & \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}(A_{k-1}+T) \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) \end{pmatrix} \\
& \times_{i \neq k-1} \begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i+l_i-1) & \cdots & -\zeta_i(A_i-l_i+1) \end{pmatrix} \\
& \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \eta_-; \\
& \quad (\rho, A_{k-1}-l_{k-1}+T, B_{k-1}+l_{k-1}+T, 0, \eta'_{k-1}, \zeta_{k-1}), \\
& \quad (\rho, A_k - l_k, B_k + l_k, 0, \eta'_k, \zeta_k)).
\end{aligned}$$

We choose t such that

$$\begin{aligned}
(\rho, A_{k-1}-l_{k-1}, B_{k-1}+l_{k-1}, \zeta_{k-1}) &\gg_r (\rho, A_k - l_k - t, B_k + l_k - t, \zeta_k) \\
&\gg_r (\rho, A_{k-2}-l_{k-2}, B_{k-2}+l_{k-2}, \zeta_{k-2}).
\end{aligned}$$

Then

$$\begin{aligned}
\pi^{\Sigma_0}_{M,>'_\psi}(\psi^T, \underline{l}', \underline{\eta}') \hookrightarrow & \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}(A_{k-1}+T) \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) \end{pmatrix} \\
& \times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i+l_i-1) & \cdots & -\zeta_i(A_i-l_i+1) \end{pmatrix}}_{I_i} \\
& \times \underbrace{\begin{pmatrix} \zeta_k(B_k+l_k) & \cdots & \zeta_k(B_k+l_k-t+1) \\ \vdots & & \vdots \\ \zeta_k(A_k-l_k) & \cdots & \zeta_k(A_k-l_k-t+1) \end{pmatrix}}_{II}
\end{aligned}$$

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$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+l_{k-1}+T) & \cdots & \zeta_{k-1}(B_{k-1}+l_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}-l_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}}_{III}$$
$$\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1}-l_{k-1}, B_{k-1}+l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}),$$
$$(\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k)).$$

It is clear that the generalized segments (*III*) and (*II*) are interchangeable. We would like to show (*III*) and (I_i) are also interchangeable for $i \neq k-1$. It suffices to make the following observations:

(1) If $\zeta_i = \zeta_{k-1}$
  (a) $i > k$, one observes $B_i > B_{k-1}+l_{k-1}+T$
  (b) $i < k-1$, one observes $B_{k-1}+l_{k-1} > B_i + l_i$.
(2) If $\zeta_i \neq \zeta_{k-1}$
  (a) $i > k$, one observes $A_i > A_{k-1}-l_{k-1}+T$
  (b) $i < k-1$, one observes $B_{k-1}+l_{k-1} > A_i$
  (c) $i = k$, one observes $[A_k, A_k - l_k + 1] \subseteq [A_{k-1}+T-l_{k-1}, A_{k-1}-l_{k-1}+1]$.

Therefore,

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^T, \underline{l}', \underline{\eta}') \hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}(A_{k-1}+T) \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) \end{pmatrix}$$
$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+l_{k-1}+T) & \cdots & \zeta_{k-1}(B_{k-1}+l_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}-l_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}}_{III}$$
$$\times_{i\neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i+l_i-1) & \cdots & -\zeta_i(A_i-l_i+1) \end{pmatrix}}_{I_i}$$
$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+l_k) & \cdots & \zeta_k(B_k+l_k-t+1) \\ \vdots & & \vdots \\ \zeta_k(A_k-l_k) & \cdots & \zeta_k(A_k-l_k-t+1) \end{pmatrix}}_{II}$$
$$\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1}-l_{k-1}, B_{k-1}+l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}),$$
$$(\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k)).$$

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Next we want to take dual of

$$\underbrace{\begin{pmatrix} -\zeta_{k-1}(A_{k-1}+1) & \cdots & -\zeta_{k-1}(A_{k-1}+T) \\ \vdots & & \vdots \\ -\zeta_{k-1}(A_{k-1}-l_{k-1}+2) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) \end{pmatrix}}_{IV}$$

from

$$\begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}(A_{k-1}+T) \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) \end{pmatrix}.$$

It is clear that (*IV*) and (*III*) are interchangeable. To see (*IV*) and (*II*) are interchangeable, one notes $\zeta_k \neq \zeta_{k-1}$ and $[A_{k-1}+T, A_{k-1}+1] \supseteq [A_k - l_k, B_k + l_k]$. To see (*IV*) and ($I_i$) are also interchangeable, it suffices to make the following observations:

(1) If $\zeta_i = \zeta_{k-1}$
   (a) $i > k$, one observes $A_i > A_{k-1}+T$
   (b) $i < k-1$, one observes $A_{k-1}+l_{k-1} > A_i$.
(2) If $\zeta_i \neq \zeta_{k-1}$
   (a) $i \geqslant k$, one observes $B_i > A_{k-1}+1$
   (b) $i < k-1$, one observes $A_{k-1}-l_{k-1} > B_i + l_i$.

As a result,

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^T, \underline{l}', \underline{\eta}') \hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}A_{k-1} \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}$$
$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+l_{k-1}+T) & \cdots & \zeta_{k-1}(B_{k-1}+l_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}-l_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}}_{III}$$
$$\times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i+l_i-1) & \cdots & -\zeta_i(A_i-l_i+1) \end{pmatrix}}_{I_i}$$
$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+l_k) & \cdots & \zeta_k(B_k+l_k-t+1) \\ \vdots & & \vdots \\ \zeta_k(A_k-l_k) & \cdots & \zeta_k(A_k-l_k-t+1) \end{pmatrix}}_{II}$$

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$$\times \underbrace{\begin{pmatrix} -\zeta_{k-1}(A_{k-1}+1) & \cdots & -\zeta_{k-1}(A_{k-1}+T) \\ \vdots & & \vdots \\ -\zeta_{k-1}(A_{k-1}-l_{k-1}+2) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) \end{pmatrix}}_{IV}$$
$$\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1}-l_{k-1}, B_{k-1}+l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), (\rho, A_k-l_k-t, B_k+l_k-t, 0, \eta'_k, \zeta_k)).$$

By (4.1), we can take the dual of (*IV*). Therefore,

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^T, \underline{l}', \underline{\eta}') \hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}A_{k-1} \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}$$
$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+l_{k-1}+T) & \cdots & \zeta_{k-1}(B_{k-1}+l_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}-l_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}}_{III}$$
$$\times_{i\neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i+l_i-1) & \cdots & -\zeta_i(A_i-l_i+1) \end{pmatrix}}_{I_i}$$
$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+l_k) & \cdots & \zeta_k(B_k+l_k-t+1) \\ \vdots & & \vdots \\ \zeta_k(A_k-l_k) & \cdots & \zeta_k(A_k-l_k-t+1) \end{pmatrix}}_{II}$$
$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) & \cdots & \zeta_{k-1}(A_{k-1}-l_{k-1}+2) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{(IV)^\vee}$$
$$\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1}-l_{k-1}, B_{k-1}+l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), (\rho, A_k-l_k-t, B_k+l_k-t, 0, \eta'_k, \zeta_k)).$$

As before, one can show $(IV)^\vee$ are interchangeable with (*II*) and (I_i). Then

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi^T, \underline{l}', \underline{\eta}') \hookrightarrow \begin{pmatrix} \zeta_{k-1}(B_{k-1}+T) & \cdots & -\zeta_{k-1}A_{k-1} \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1+T) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}$$

$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(B_{k-1}+l_{k-1}+T) & \cdots & \zeta_{k-1}(B_{k-1}+l_{k-1}+1) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}-l_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}}_{III}$$

$$\times \underbrace{\begin{pmatrix} \zeta_{k-1}(A_{k-1}-l_{k-1}+1+T) & \cdots & \zeta_{k-1}(A_{k-1}-l_{k-1}+2) \\ \vdots & & \vdots \\ \zeta_{k-1}(A_{k-1}+T) & \cdots & \zeta_{k-1}(A_{k-1}+1) \end{pmatrix}}_{(IV)^\vee}$$

$$\times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i+l_i-1) & \cdots & -\zeta_i(A_i-l_i+1) \end{pmatrix}}_{I_i}$$

$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+l_k) & \cdots & \zeta_k(B_k+l_k-t+1) \\ \vdots & & \vdots \\ \zeta_k(A_k-l_k) & \cdots & \zeta_k(A_k-l_k-t+1) \end{pmatrix}}_{II}$$

$$\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1}-l_{k-1}, B_{k-1}+l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), (\rho, A_k-l_k-t, B_k+l_k-t, 0, \eta'_k, \zeta_k)).$$

This implies

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \hookrightarrow \begin{pmatrix} \zeta_{k-1}B_{k-1} & \cdots & -\zeta_{k-1}A_{k-1} \\ \vdots & & \vdots \\ \zeta_{k-1}(B_{k-1}+l_{k-1}-1) & \cdots & -\zeta_{k-1}(A_{k-1}-l_{k-1}+1) \end{pmatrix}$$

$$\times_{i \neq k-1} \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i+l_i-1) & \cdots & -\zeta_i(A_i-l_i+1) \end{pmatrix}}_{I_i}$$

$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k+l_k) & \cdots & \zeta_k(B_k+l_k-t+1) \\ \vdots & & \vdots \\ \zeta_k(A_k-l_k) & \cdots & \zeta_k(A_k-l_k-t+1) \end{pmatrix}}_{II}$$

$$\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1}-l_{k-1}, B_{k-1}+l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), (\rho, A_k-l_k-t, B_k+l_k-t, 0, \eta'_k, \zeta_k)).$$

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One can further show $\pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}')$ is the unique irreducible subrepresentation. On the other hand,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \times_i \underbrace{\begin{pmatrix} \zeta_i B_i & \cdots & -\zeta_i A_i \\ \vdots & & \vdots \\ \zeta_i(B_i + l_i - 1) & \cdots & -\zeta_i(A_i - l_i + 1) \end{pmatrix}}_{I_i}$$

$$\times \underbrace{\begin{pmatrix} \zeta_k(B_k + l_k) & \cdots & \zeta_k(B_k + l_k - t + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k - l_k) & \cdots & \zeta_k(A_k - l_k - t + 1) \end{pmatrix}}_{II}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta_{k-1}, \zeta_{k-1}), (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta_k, \zeta_k)).$$

By $(6.6)_r$,

$$\begin{aligned} &\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta'_{k-1}, \zeta_{k-1}), \\ &\quad (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta'_k, \zeta_k)) \\ &= \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \eta_-; (\rho, A_{k-1} - l_{k-1}, B_{k-1} + l_{k-1}, 0, \eta_{k-1}, \zeta_{k-1}), \\ &\quad (\rho, A_k - l_k - t, B_k + l_k - t, 0, \eta_k, \zeta_k)). \end{aligned}$$

Hence $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}')$. This finishes the second reduction.

6.2.3. Final resolution. Now we want to resolve the case $(6.6)_r$. Since $l_i = 0$, we can also view ψ as an elementary parameter, denoted by ψ_e. The function $\underline{\eta}$ over $\mathrm{Jord}(\psi)$ determines a function ε_e over $\mathrm{Jord}(\psi_e)$, i.e., for $C_i \in [A_i, B_i]$,

$$\varepsilon_e(\rho, C_i, C_i, \zeta_i) = \eta_i(-1)^{C_i - B_i}.$$

Similarly, we can define ε'_e. It is obvious that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, 0, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_{\psi_e}}(\psi_e, \varepsilon_e).$$

Let ψ^T be obtained by changing $(\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})$ to $(\rho, A_{k-1} + T, B_{k-1} + T, \zeta_{k-1})$ such that

$$B_{k-1} + T > A_k \quad \text{and} \quad B_{k+1} > A_{k-1} + T.$$

Then

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi, 0, \underline{\eta}') = \mathrm{Jac}_{(\rho, A_{k-1}+T, B_{k-1}+T, \zeta_{k-1}) \mapsto (\rho, A_{k-1}, B_{k-1}, \zeta_{k-1})} \pi^{\Sigma_0}_{M,>'_\psi}(\psi^T, 0, \underline{\eta}').$$

The order $>'_\psi$ induces an order $>'_{\psi_e}$ on $\mathrm{Jord}(\psi_e)$, and we define

$$\pi^{\Sigma_0}_{M,>'_{\psi_e}}(\psi_e, \varepsilon'_e) := \circ_{C_{k-1} \in [B_{k-1}, A_{k-1}]} \mathrm{Jac}_{(\rho, C_{k-1}+T, C_{k-1}+T, \zeta_{k-1}) \mapsto (\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})} \pi^{\Sigma_0}_{M,>'_{\psi_e}}(\psi^T_e, \varepsilon'_e).$$

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Since

$$\pi^{\Sigma_0}_{M,>'_{\psi_e}}(\psi^T_e, \varepsilon'_e) = \pi^{\Sigma_0}_{M,>'_{\psi}}(\psi^T, 0, \underline{\eta}'),$$

and

$$\mathrm{Jac}_{(\rho,A_{k-1}+T,B_{k-1}+T,\zeta_{k-1})\mapsto(\rho,A_{k-1},B_{k-1},\zeta_{k-1})} = \circ_{C_{k-1}\in[B_{k-1},A_{k-1}]} \mathrm{Jac}_{(\rho,C_{k-1}+T,C_{k-1}+T,\zeta_{k-1})\mapsto(\rho,C_{k-1},C_{k-1},\zeta_{k-1})}$$

we get

$$\pi^{\Sigma_0}_{M,>'_{\psi}}(\psi, 0, \underline{\eta}') = \pi^{\Sigma_0}_{M,>'_{\psi_e}}(\psi_e, \varepsilon'_e).$$

So it is enough to show $\pi^{\Sigma_0}_{M,>_{\psi_e}}(\psi_e, \varepsilon_e) = \pi^{\Sigma_0}_{M,>'_{\psi_e}}(\psi_e, \varepsilon'_e)$. Note

$$\pi^{\Sigma_0}_{M,>_{\psi_e}}(\psi_e, \varepsilon_e) = \pi^{\Sigma_0}_{W}(\psi_e, \varepsilon_e\varepsilon^{M/W}_{\psi_e})$$
$$\pi^{\Sigma_0}_{M,>'_{\psi_e}}(\psi_e, \varepsilon'_e) = \pi^{\Sigma_0}_{W}(\psi_e, \varepsilon'_e\varepsilon'^{M/W}_{\psi_e}),$$

where $\varepsilon^{M/W}_{\psi_e}$ (respectively $\varepsilon'^{M/W}_{\psi_e}$) is defined with respect to the order $>_{\psi_e}$ (respectively $>'_{\psi_e}$) (cf. (1.4)). Then we just need to verify

$$\varepsilon_e\varepsilon^{M/W}_{\psi_e} = \varepsilon'_e\varepsilon'^{M/W}_{\psi_e},$$

or equivalently,

$$\frac{\varepsilon_e}{\varepsilon'_e} = \frac{\varepsilon^{M/W}_{\psi_e}}{\varepsilon'^{M/W}_{\psi_e}} = \frac{\varepsilon^{M/MW}_{\psi_e}}{\varepsilon'^{M/MW}_{\psi_e}} \cdot \frac{\varepsilon^{MW/W}_{\psi_e}}{\varepsilon'^{MW/W}_{\psi_e}}. \tag{6.7}$$

We divide it into two cases:

(1) If $A_i \in \mathbb{Z}$,

$$\varepsilon^{MW/W}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = \varepsilon'^{MW/W}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = 1.$$

And

$$\varepsilon^{M/MW}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = \begin{cases} (-1)^m & \text{if } \zeta_i = +1, \\ (-1)^{m+n} & \text{if } \zeta_i = -1, \end{cases}$$

where

$$m = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = -1, (\rho, C_j, C_j, \zeta_j) >_{\psi_e} (\rho, C_i, C_i, \zeta_i)\},$$
$$n = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : (\rho, C_i, C_i, \zeta_i) >_{\psi_e} (\rho, C_j, C_j, \zeta_j)\}.$$

And

$$\varepsilon'^{M/MW}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = \begin{cases} (-1)^{m'} & \text{if } \zeta_i = +1, \\ (-1)^{m'+n'} & \text{if } \zeta_i = -1, \end{cases}$$

where

$$m' = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = -1, (\rho, C_j, C_j, \zeta_j) >'_{\psi_e} (\rho, C_i, C_i, \zeta_i)\},$$
$$n' = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : (\rho, C_i, C_i, \zeta_i) >'_{\psi_e} (\rho, C_j, C_j, \zeta_j)\}.$$

(a) $i \neq k, k-1$

- $\varepsilon_e(\rho, C_i, C_i, \zeta_i)/\varepsilon'_e(\rho, C_i, C_i, \zeta_i) = 1$
- $\varepsilon^{M/MW}_{\psi_e}(\rho, C_i, C_i, \zeta_i)/\varepsilon'^{M/MW}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = 1$

(b) $i = k$

- $\varepsilon_e(\rho, C_k, C_k, \zeta_k)/\varepsilon'_e(\rho, C_k, C_k, \zeta_k) = (-1)^{A_{k-1}-B_{k-1}+1}$
- $\varepsilon^{M/MW}_{\psi_e}(\rho, C_k, C_k, \zeta_k)/\varepsilon'^{M/MW}_{\psi_e}(\rho, C_k, C_k, \zeta_k) = (-1)^{A_{k-1}-B_{k-1}+1}$

(c) $i = k-1$

- $\varepsilon_e(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})/\varepsilon'_e(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) = (-1)^{A_k-B_k+1}$
- $\varepsilon^{M/MW}_{\psi_e}(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})/\varepsilon'^{M/MW}_{\psi_e}(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) = (-1)^{A_k-B_k+1}$.

(2) $A_i \notin \mathbb{Z}$

$$\varepsilon^{M/MW}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = \varepsilon'^{M/MW}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = 1.$$

And

$$\varepsilon^{MW/W}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = \begin{cases} (-1)^m & \text{if } \zeta_i = +1, \\ (-1)^n & \text{if } \zeta_i = -1, \end{cases}$$

where

$$m = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = -1, (\rho, C_j, C_j, \zeta_j) >_{\psi_e} (\rho, C_i, C_i, \zeta_i)\},$$
$$n = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = +1, (\rho, C_i, C_i, \zeta_i) >_{\psi_e} (\rho, C_j, C_j, \zeta_j)\}.$$

And

$$\varepsilon'^{MW/W}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = \begin{cases} (-1)^{m'} & \text{if } \zeta_i = +1, \\ (-1)^{n'} & \text{if } \zeta_i = -1, \end{cases}$$

where

$$m' = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = -1, (\rho, C_j, C_j, \zeta_j) >'_{\psi_e} (\rho, C_i, C_i, \zeta_i)\},$$
$$n' = \sharp\{C_j \in [A_j, B_j] \text{ for all } j : \zeta_j = +1, (\rho, C_i, C_i, \zeta_i) >'_{\psi_e} (\rho, C_j, C_j, \zeta_j)\}.$$

(a) $i \neq k, k-1$

- $\varepsilon_e(\rho, C_i, C_i, \zeta_i)/\varepsilon'_e(\rho, C_i, C_i, \zeta_i) = 1$
- $\varepsilon^{MW/W}_{\psi_e}(\rho, C_i, C_i, \zeta_i)/\varepsilon'^{MW/W}_{\psi_e}(\rho, C_i, C_i, \zeta_i) = 1$

(b) $i = k$

- $\varepsilon_e(\rho, C_k, C_k, \zeta_k)/\varepsilon'_e(\rho, C_k, C_k, \zeta_k) = (-1)^{A_{k-1}-B_{k-1}+1}$
- $\varepsilon^{MW/W}_{\psi_e}(\rho, C_k, C_k, \zeta_k)/\varepsilon'^{MW/W}_{\psi_e}(\rho, C_k, C_k, \zeta_k) = (-1)^{A_{k-1}-B_{k-1}+1}$

(c) $i = k-1$

- $\varepsilon_e(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})/\varepsilon'_e(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) = (-1)^{A_k-B_k+1}$
- $\varepsilon^{MW/W}_{\psi_e}(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1})/\varepsilon'^{MW/W}_{\psi_e}(\rho, C_{k-1}, C_{k-1}, \zeta_{k-1}) = (-1)^{A_k-B_k+1}$.

It follows from the calculations above that (6.7) holds, and this ends the proof of Theorem 6.3.

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7. Reduction operations

In this section, we want to give three operations, which will be used in our general procedure to reduce the problem of finding nonvanishing conditions for $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$.

7.1. Pull

7.1.1. Case of unequal length. We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$. We index the Jordan blocks in $\mathrm{Jord}_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg \bigcup_{j=1}^{n} \{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover,

$$[A_n, B_n] \supsetneq [A_{n-1}, B_{n-1}] \quad \text{and} \quad \zeta_n = \zeta_{n-1}.$$

We denote by $>'_\psi$ the order obtained from $>_\psi$ by switching $(\rho, A_n, B_n, \zeta_n)$ and $(\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})$. It is still admissible. Let S^+_n be the corresponding transformation on $(\underline{l}, \underline{\eta})$. We define ψ_- by

$$\mathrm{Jord}(\psi_-) = \mathrm{Jord}(\psi) \backslash \{(\rho, A_n, B_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})\}.$$

We denote the restriction of $(\underline{l}, \underline{\eta})$ to $\mathrm{Jord}(\psi_-)$ by $(\underline{l}_-, \underline{\eta}_-)$.

Proposition 7.1. *For any $(\underline{l}, \underline{\eta})$, $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if the following three conditions are satisfied:*

(1)

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n), \\ (\rho, A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

for some T_n, T_{n-1}, *such that*

$$[A_n + T_n, B_n + T_n] \supsetneq [A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}]$$

and $(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_n + T_n, B_n + T_n, \zeta_n)$ *for* $i > n$.

(2)

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), \\ (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

for some T *such that* $B_i > A_n + T$ *for* $i > n$.

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(3)

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n),$$
$$(\rho, A_{n-1}+T, B_{n-1}+T, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0$$

for some T *such that* $B_i > A_{n-1}+T$ *for* $i > n$, *and* $(\underline{l}', \underline{\eta}') = S^+_n(\underline{l}, \underline{\eta})$.

Conversely, if $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, *then (1), (2), (3) still hold after we replace '*some*' by '*all*'.*

Proof. The converse is obvious. So we mainly need to show the sufficiency of the above three conditions. Let $\zeta = \zeta_n = \zeta_{n-1}$, and $(\underline{l}', \underline{\eta}') = S^+_n(\underline{l}, \underline{\eta})$ as in the proposition. Since $[A_n, B_n] \supsetneq [A_{n-1}, B_{n-1}]$, we necessarily have $[A_n+1, B_n+1] \supsetneq [A_{n-1}, B_{n-1}]$ or $[A_n, B_n] \supsetneq [A_{n-1}+1, B_{n-1}+1]$. So we divide it into two cases.

Suppose $[A_n+1, B_n+1] \supsetneq [A_{n-1}, B_{n-1}]$, we claim $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if the following conditions are satisfied:

- $$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+1, B_n+1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0;$$
- $$\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1}+T, B_{n-1}+T, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0$$

  for some T such that $B_i > A_{n-1}+T$ for $i > n$.

It suffices to take T sufficiently large so that $B_i > A_{n-1}+T$ for $i > n$, and

$$(\rho, A_{n-1}+T, B_{n-1}+T, \zeta_{n-1}) \gg \bigcup_{j=1}^{n-2}\{(\rho, A_j, B_j, \zeta_j)\} \cup \{(\rho, A_n+1, B_n+1, \zeta_n)\}.$$

By Theorem 6.1, we have

$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n+1, B_n+1, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) =$$
$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+1, B_n+1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0.$$

So

$$\pi^{\Sigma_0}_{\gg} := \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n+1, B_n+1, l'_n, \eta'_n, \zeta_n),$$
$$(\rho, A_{n-1}+T, B_{n-1}+T, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0$$

and

$$\pi^{\Sigma_0}_{\gg} \hookrightarrow \begin{pmatrix} \zeta(B_{n-1}+T) & \cdots & \zeta(B_{n-1}+1) \\ \vdots & & \vdots \\ \zeta(A_{n-1}+T) & \cdots & \zeta(A_{n-1}+1) \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n+1, B_n+1, l'_n, \eta'_n, \zeta_n),$$

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$$
\begin{aligned}
&(\rho, A_{n-1}, B_{n-1}, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1}))\\
&= \begin{pmatrix} \zeta(B_{n-1}+T) & \cdots & \zeta(B_{n-1}+1) \\ \vdots & & \vdots \\ \zeta(A_{n-1}+T) & \cdots & \zeta(A_{n-1}+1) \end{pmatrix}\\
&\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+1, B_n+1, l_n, \eta_n, \zeta_n),\\
&(\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})).
\end{aligned}
$$

Note

$$
\begin{aligned}
&\mathrm{Jac}_{(\rho,A_n+1,B_n+1,\zeta)\mapsto(\rho,A_n,B_n,\zeta)}\pi^{\Sigma_0}_{\gg} =\\
&\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1}+T, B_{n-1}+T, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0.
\end{aligned}
$$

So after we apply the same Jacquet functor to the full-induced representation above, we should get something nonzero. To compute this Jacquet module, one notes

$$\zeta(B_{n-1}+T), -\zeta(A_{n-1}+1) \notin \{\zeta(B_n+1), \ldots, \zeta(A_n+1)\},$$

so it can only be

$$
\begin{aligned}
&\begin{pmatrix} \zeta(B_{n-1}+T) & \cdots & \zeta(B_{n-1}+1) \\ \vdots & & \vdots \\ \zeta(A_{n-1}+T) & \cdots & \zeta(A_{n-1}+1) \end{pmatrix} \rtimes \mathrm{Jac}_{(\rho,A_n+1,B_n+1,\zeta)\mapsto(\rho,A_n,B_n,\zeta)}\\
&\quad \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+1, B_n+1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1}))\\
&= \begin{pmatrix} \zeta(B_{n-1}+T) & \cdots & \zeta(B_{n-1}+1) \\ \vdots & & \vdots \\ \zeta(A_{n-1}+T) & \cdots & \zeta(A_{n-1}+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0.
\end{aligned}
$$

Hence $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$. This shows our claim in the first case.

Suppose $[A_n, B_n] \supsetneq [A_{n-1}+1, B_{n-1}+1]$, we claim $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if the following conditions are satisfied:

- $$\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1}+1, B_{n-1}+1, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0;$$
- $$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T, B_n+T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

  for *some* T such that $B_i > A_n + T$ for $i > n$.

The argument of this case is essentially the same as the previous one. Again it suffices to take T sufficiently large so that $B_i > A_n + T$ for $i > n$, and

$$(\rho, A_n+T, B_n+T, \zeta_n) \gg \bigcup_{j=1}^{n-2}\{(\rho, A_j, B_j, \zeta_j)\} \cup \{(\rho, A_{n-1}+1, B_{n-1}+1, \zeta_{n-1})\}.$$

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By Theorem 6.1, we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n, B_n, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}+1, B_{n-1}+1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) =$$
$$\pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1}+1, B_{n-1}+1, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \neq 0.$$

So

$$\pi^{\Sigma_0}_{\gg} := \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T, B_n+T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}+1, B_{n-1}+1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

and

$$\begin{aligned}\pi^{\Sigma_0}_{\gg} \hookrightarrow &\begin{pmatrix} \zeta(B_n+T) & \cdots & \zeta(B_n+1) \\ \vdots & & \vdots \\ \zeta(A_n+T) & \cdots & \zeta(A_n+1)\end{pmatrix} \\ &\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n, B_n, l_n, \eta_n, \zeta_n), \\ &\quad (\rho, A_{n-1}+1, B_{n-1}+1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \\ = &\begin{pmatrix} \zeta(B_n+T) & \cdots & \zeta(B_n+1) \\ \vdots & & \vdots \\ \zeta(A_n+T) & \cdots & \zeta(A_n+1)\end{pmatrix} \\ &\rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), \\ &\quad (\rho, A_{n-1}+1, B_{n-1}+1, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})).\end{aligned}$$

Note

$$\mathrm{Jac}_{(\rho, A_{n-1}+1, B_{n-1}+1, \zeta)\mapsto(\rho, A_{n-1}, B_{n-1}, \zeta)}\pi^{\Sigma_0}_{\gg} =$$
$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T, B_n+T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0.$$

So after we apply the same Jacquet functor to the full-induced representation above, we should get something nonzero. To compute this Jacquet module, one notes

$$\zeta(B_n+T), -\zeta(A_n+1) \notin \{\zeta(B_{n-1}+1), \ldots, \zeta(A_{n-1}+1)\},$$

so it can only be

$$\begin{aligned}&\begin{pmatrix} \zeta(B_n+T) & \cdots & \zeta(B_n+1) \\ \vdots & & \vdots \\ \zeta(A_n+T) & \cdots & \zeta(A_n+1)\end{pmatrix} \rtimes \mathrm{Jac}_{(\rho, A_{n-1}+1, B_{n-1}+1, \zeta)\mapsto(\rho, A_{n-1}, B_{n-1}, \zeta)} \\ &\quad \pi^{\Sigma_0}_{M,>'_\psi}(\psi_-, \underline{l}'_-, \underline{\eta}'_-; (\rho, A_n, B_n, l'_n, \eta'_n, \zeta_n), (\rho, A_{n-1}+1, B_{n-1}+1, l'_{n-1}, \eta'_{n-1}, \zeta_{n-1})) \\ &= \begin{pmatrix} \zeta(B_n+T) & \cdots & \zeta(B_n+1) \\ \vdots & & \vdots \\ \zeta(A_n+T) & \cdots & \zeta(A_n+1)\end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0.\end{aligned}$$

Hence $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>'_\psi}(\psi, \underline{l}', \underline{\eta}') \neq 0$. This shows our claim in the second case.

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By combining our claims in both cases in some alternating way, we can shift both $[A_n, B_n], [A_{n-1}, B_{n-1}]$ to $[A_n + T_n, B_n + T_n], [A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}]$ for any T_n, T_{n-1} such that

$$[A_n + T_n, B_n + T_n] \supsetneq [A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}]$$

and $(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_n + T_n, B_n + T_n, \zeta_n)$ for $i > n$. Then the statement of this proposition is clear. □

Remark 7.2. The way we use this proposition is to take all T_n, T_{n-1} and T to be large.

7.1.2. Case of equal length. We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$. We index the Jordan blocks in $\mathrm{Jord}_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg \bigcup_{j=1}^{n} \{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover,

$$[A_n, B_n] = [A_{n-1}, B_{n-1}] \quad \text{and} \quad \zeta_n = \zeta_{n-1}.$$

Note

$$\text{there exists no } i < n-1 \text{ satisfying } \zeta_i = \zeta_n,\ A_i > A_n \text{ and } B_i > B_n. \tag{7.1}$$

We define ψ_- by

$$\mathrm{Jord}(\psi_-) = \mathrm{Jord}(\psi) \backslash \{(\rho, A_n, B_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})\}.$$

We denote the restriction of $(\underline{l}, \underline{\eta})$ to $\mathrm{Jord}(\psi_-)$ by $(\underline{l}_-, \underline{\eta}_-)$.

Proposition 7.3. *For any $(\underline{l}, \underline{\eta})$, $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if the following two conditions are satisfied:*

(1)

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n, B_n + T_n, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + T_{n-1}, B_{n-1} + T_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

for some $T_n = T_{n-1}$ *such that* $(\rho, A_i, B_i, \zeta_i) \gg (\rho, A_n + T_n, B_n + T_n, \zeta_n)$ *for* $i > n$.

(2)

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0$$

for some T *such that* $B_i > A_n + T$ *for* $i > n$.

Conversely, if $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then (1), (2) still hold after we replace 'some' *by* 'all'*.*

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Proof. The converse is obvious. So we mainly need to show the sufficiency of the above two conditions. Let $[A_n, B_n] = [A, B]$ and $\zeta_n = \zeta$. It is enough to prove the proposition by taking $T_n = T_{n-1} = 1$ in the first condition. So let us suppose

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0.$$

We can take T sufficiently large such that $B_i > A_n + T$ for $i > n$, and

$$(\rho, A_n + T, B_n + T, \zeta_n) \gg \bigcup_{j=1}^{n-2} \{(\rho, A_j, B_j, \zeta_j)\} \cup \{(\rho, A_{n-1} + 1, B_{n-1} + 1, \zeta_{n-1})\}.$$

Let

$$\pi^{\Sigma_0}_{\gg} := \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})).$$

Then

$$\pi^{\Sigma_0}_{\gg} \hookrightarrow \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})).$$

Since

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T, B_n + T, l_n, \eta_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0,$$

then

$$\mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)} \pi^{\Sigma_0}_{\gg} \neq 0. \tag{7.2}$$

Note $\zeta(B+T) \notin [\zeta(B+1), \zeta(A+1)]$. So this implies

$$\mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \neq 0.$$

Then there exists $C \in [A + 1, B + 1]$ and an irreducible representation σ such that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + 1, B_n + 1, l_n, \eta_n, \zeta_n), (\rho, A_{n-1} + 1, B_{n-1} + 1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})) \hookrightarrow \langle \zeta C, \ldots, \zeta(A+1) \rangle \rtimes \sigma.$$

By (7.1), we must have $C = B + 1$. Therefore,

$$\pi^{\Sigma_0}_{\gg} \hookrightarrow \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix} \times \begin{pmatrix} \zeta(B+1) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \sigma.$$

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Let us denote the full-induced representation above by $(*-1)$. By Frobenius reciprocity, σ is an irreducible constituent of

$$\begin{aligned}\mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+1, B_n+1, l_n, \eta_n, \zeta_n),\\ (\rho, A_{n-1}+1, B_{n-1}+1, l_{n-1}, \eta_{n-1}, \zeta_{n-1})).\end{aligned}$$

In fact it is not hard to show that the Jacquet module above consists of representations in

$$\begin{aligned}\Pi^{\Sigma_0}(\psi_-, (\rho, A_n+1, B_n+1, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1}))\bigcup\\ \Pi^{\Sigma_0}(\psi_-, (\rho, A_n+1, B_{n-1}, \zeta_n), (\rho, A_{n-1}, B_n+1, \zeta_{n-1})).\end{aligned}$$

So in particular, σ is an element in the above packets. We claim

$$\mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)}\sigma \neq 0. \tag{7.3}$$

Otherwise, one finds

$$\mathrm{Jac}_{\zeta C',\ldots,\zeta C''}\sigma = 0$$

for any $C' \in [B+1, A+T]$ and $C'' \in [A+1, A+T]$. This implies

$$\mathrm{Jac}_{(\rho,A+T,B+T,\zeta)\mapsto(\rho,A,B,\zeta)}(*-1) = \sigma.$$

So $(*-1)$ has a unique irreducible subrepresentation, and hence

$$\pi^{\Sigma_0}_{\gg} \hookrightarrow \begin{pmatrix}\zeta(B+T) & \cdots & \zeta(B+1)\\ \vdots & & \vdots\\ \zeta(A+T) & \cdots & \zeta(A+1)\end{pmatrix} \rtimes \sigma.$$

Then $\mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)}\sigma = 0$ implies $\mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)}\pi^{\Sigma_0}_{\gg} = 0$, but this contradicts to (7.2). As a consequence, σ can only be in

$$\Pi^{\Sigma_0}(\psi_-, (\rho, A_n+1, B_n+1, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})).$$

Now by (7.3), we have

$$\sigma \hookrightarrow \begin{pmatrix}\zeta C\\ \vdots\\ \zeta(A+1)\end{pmatrix} \rtimes \sigma'$$

for some $C \in [B+1, A+1]$ and some irreducible representation σ'. For the same reason as before, we must have $C = B+1$. This also implies $\sigma' \in \Pi^{\Sigma_0}_{\psi}$. Therefore,

$$\pi^{\Sigma_0}_{\gg} \hookrightarrow \underbrace{\begin{pmatrix}\zeta(B+T) & \cdots & \zeta(B+2)\\ \vdots & & \vdots\\ \zeta(A+T) & \cdots & \zeta(A+2)\end{pmatrix} \times \begin{pmatrix}\zeta(B+1)\\ \vdots\\ \zeta(A+1)\end{pmatrix} \times \begin{pmatrix}\zeta(B+1)\\ \vdots\\ \zeta(A+1)\end{pmatrix}}_{(*-2)} \rtimes \sigma'.$$

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There exists an irreducible constituent τ of $(*-2)$ such that

$$\pi_{\gg}^{\Sigma_0} \hookrightarrow \tau \rtimes \sigma'.$$

By (7.1), $\mathrm{Jac}_{\zeta(C),\ldots,\zeta(A+1)}\sigma' = 0$ for all $C \in [B+1, A+1]$. Then we can conclude from (7.2) that

$$\mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)}\tau \neq 0.$$

So there exists $C \in [B+1, A+1]$ and an irreducible representation τ' such that

$$\tau \hookrightarrow \begin{pmatrix} \zeta C \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \tau'.$$

From $(*-2)$, we see C can only be $B+1$. Hence, τ' is an irreducible constituent of

$$\mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)}(*-2) = 2 \cdot \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix} \times \begin{pmatrix} \zeta(B+1) \\ \vdots \\ \zeta(A+1) \end{pmatrix}.$$

If $\mathrm{Jac}_{\zeta(B+1)}\tau' \neq 0$, then it is necessary that

$$\begin{aligned} \mathrm{Jac}_{\zeta(B+1)}\tau' &= \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix} \times \begin{pmatrix} \zeta(B+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \\ &\cong \begin{pmatrix} \zeta(B+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \times \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix}, \end{aligned}$$

which is irreducible. Consequently,

$$\mathrm{Jac}_{\zeta(B+1),\ldots\zeta(A+1)} \circ \mathrm{Jac}_{\zeta(B+1),\ldots\zeta(A+1)}\pi_{\gg}^{\Sigma_0} \neq 0.$$

But this is impossible by (7.1). Therefore, we must have

$$\tau' = \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+1) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}.$$

To summarize, we get

$$\pi_{\gg}^{\Sigma_0} \hookrightarrow \begin{pmatrix} \zeta(B+1) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \times \begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+1) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix} \rtimes \sigma'.$$

Hence,

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \mathrm{Jac}_{(\rho,A+T,B+T,\zeta)\mapsto(\rho,A,B,\zeta)} \circ \mathrm{Jac}_{\zeta(B+1),\ldots,\zeta(A+1)}\pi_{\gg}^{\Sigma_0} \neq 0.$$

This finishes the proof. □

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7.2. Expand

We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$. We index the Jordan blocks in $\mathrm{Jord}_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg_2 \bigcup_{j=1}^{n} \{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover, for $i < n$,

$$A_n \geqslant A_i \text{ and there exists no } [A_i, B_i] \subseteq [A_n, B_n] \text{ with } \zeta_i = \zeta_n.$$

Let t_n be the smallest integer such that $B_n - t_n = B_i$ for some $i < n$ and $\zeta_i = \zeta_n$. If such t_n does not exist, we let $t_n := [B_n]$. We define ψ_- by

$$\mathrm{Jord}(\psi_-) = \mathrm{Jord}(\psi) \backslash \{(\rho, A_n, B_n, \zeta_n)\}.$$

We denote the restriction of $(\underline{l}, \underline{\eta})$ to $\mathrm{Jord}(\psi_-)$ by $(\underline{l}_-, \underline{\eta}_-)$.

Proposition 7.4. *We fix a positive integer $t \leqslant t_n$. Then for any $(\underline{l}, \underline{\eta})$, $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if*

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)) \neq 0.$$

Moreover,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)) \hookrightarrow \begin{pmatrix} \zeta_n(B_n - t) & \cdots & -\zeta_n(A_n + t) \\ \vdots & & \vdots \\ \zeta_n(B_n - 1) & \cdots & -\zeta_n(A_n + 1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}),$$

as the unique irreducible subrepresentation, and

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \circ_{i \in [1,t]} Jac_{\zeta_n(B_n - i), \ldots, -\zeta_n(A_n + i)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + t, B_n - t, l_n + t, \eta_n, \zeta_n)).$$

Proof. We first consider the case $t = 1$. Let $\psi_\gg$ dominates ψ with discrete diagonal restriction such that

$$(\rho, A_{n+1}, B_{n+1}, \zeta_{n+1}) \gg (\rho, A_n + T_n + 1, B_n + T_n - 1, \zeta_n) \gg (\rho, A_{n-1} + T_{n-1}, B_n + T_{n-1}, \zeta_{n-1}).$$

Let $\psi_{\gg,-}$ be obtained from $\psi_\gg$ by removing $(\rho, A_n + T_n, B_n + T_n, \zeta_n)$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n + T_n + 1, B_n + T_n - 1, l_n + 1, \eta_n, \zeta_n))$$

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$$\hookrightarrow \langle \zeta_n(B_n+T_n-1), \ldots, -\zeta_n(A_n+T_n+1)\rangle$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T_n, B_n+T_n, l_n, \eta_n, \zeta_n)).$$

Suppose

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+1, B_n-1, l_n+1, \eta_n, \zeta_n)) \neq 0.$$

Let

$$\mathrm{Jac}_{X_{>n}} := \circ_{i>n}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)}$$
$$\mathrm{Jac}_{X'_n} := \mathrm{Jac}_{(\rho,A_n+T_n+1,B_n+T_n-1,\zeta_n)\mapsto(\rho,A_n+1,B_n-1,\zeta_n)}$$
$$\mathrm{Jac}_{X_{<n}} := \circ_{i<n}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)}$$

where i decreases. Then after we apply

$$\mathrm{Jac}_{X_{>n}} \circ \mathrm{Jac}_{X'_n} \circ \mathrm{Jac}_{X_{<n}}$$

and Jac_{X^c} to the full-induced representation

$$\langle \zeta_n(B_n+T_n-1), \ldots, -\zeta_n(A_n+T_n+1)\rangle$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T_n, B_n+T_n, l_n, \eta_n, \zeta_n)), \tag{7.4}$$

we should get something nonzero.

For $i<n$, one notes

$$\zeta_n(B_n+T_n-1), \zeta_n(A_n+T_n+1) \notin [\zeta_i(A_i+T_i), \zeta_i(B_i+1)].$$

So $\mathrm{Jac}_{X_{<n}}$(7.4) becomes

$$\begin{aligned}&\langle \zeta_n(B_n+T_n-1), \ldots, -\zeta_n(A_n+T_n+1)\rangle\\ &\quad\rtimes \mathrm{Jac}_{X_{<n}}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T, B_n+T, l_n, \eta_n, \zeta_n))\\ &= \langle \zeta_n(B_n+T_n-1), \ldots, -\zeta_n(A_n+T_n+1)\rangle\\ &\quad\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n-1)}_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T, B_n+T, l_n, \eta_n, \zeta_n)),\end{aligned}$$

where $\psi^{(n-1)}_{\gg,-}$ is obtained from $\psi_{\gg,-}$ by changing $(\rho, A_i+T_i, B_i+T_i, \zeta_i)$ to $(\rho, A_i, B_i, \zeta_i)$ for $i<n$.

For $i=n$, we can further write $\mathrm{Jac}_{X'_n}$ as

$$\mathrm{Jac}_{\zeta_n(A_n+T_n+1),\ldots,\zeta_n(A_n+2)} \circ \mathrm{Jac}_{(\rho,A_n+T_n,B_n+T_n,\zeta_n)\mapsto(\rho,A_n,B_n,\zeta_n)} \circ \mathrm{Jac}_{\zeta_n(B_n+T_n-1),\ldots,\zeta_n B_n}.$$

First, we claim $\mathrm{Jac}_{\zeta_n(B_n+T_n-1),\ldots,\zeta_n B_n}$ can only apply to

$$\langle \zeta_n(B_n+T_n-1), \ldots, -\zeta_n(A_n+T_n+1)\rangle.$$

Otherwise, there exists $x \in [\zeta_n(B_n+T_n-1), \zeta_n B_n]$ such that

$$\mathrm{Jac}_x \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n-1)}_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T, B_n+T, l_n, \eta_n, \zeta_n)) \neq 0.$$

This can only happen when there exists $i<n$ such that $B_i \geqslant B_n$ and $\zeta_i = \zeta_n$, but that contradicts to our assumption. As a result,

$$\begin{aligned}\mathrm{Jac}_{\zeta_n(B_n+T_n-1),\ldots,\zeta_n B_n} \circ \mathrm{Jac}_{X_{<n}}(7.4) &= \langle \zeta_n(B_n-1), \ldots, -\zeta_n(A_n+T_n+1)\rangle\\ \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi^{(n-1)}_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T, B_n+T, l_n, \eta_n, \zeta_n)).&\end{aligned}$$

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Secondly, we claim $\mathrm{Jac}_{(\rho,A_n+T_n,B_n+T_n,\zeta_n)\mapsto(\rho,A_n,B_n,\zeta_n)}$ can only apply to

$$\pi_{M,>_\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)},\underline{l}_-,\underline{\eta}_-;(\rho,A_n+T_n,B_n+T_n,l_n,\eta_n,\zeta_n)).$$

This is because

$$\zeta_n(B_n-1),\zeta_n(A_n+T_n+1)\notin[\zeta_n(A_n+T_n),\zeta_n(B_n+1)].$$

So

$$\mathrm{Jac}_{(\rho,A_n+T_n,B_n+T_n,\zeta_n)\mapsto(\rho,A_n,B_n,\zeta_n)}\circ\mathrm{Jac}_{\zeta_n(B_n+T_n-1),\ldots,\zeta_n(B_n)}\circ\mathrm{Jac}_{X_{<n}}(7.4)=$$
$$\langle\zeta_n(B_n-1),\ldots,-\zeta_n(A_n+T_n+1)\rangle\rtimes\pi_{M,>_\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)},\underline{l}_-,\underline{\eta}_-;(\rho,A_n,B_n,l_n,\eta_n,\zeta_n)).$$

Thirdly, $\mathrm{Jac}_{\zeta_n(A_n+T_n+1),\ldots,\zeta_n(A_n+2)}$ can only apply to

$$\langle\zeta_n(B_n-1),\ldots,-\zeta_n(A_n+T_n+1)\rangle$$

for the same reason as before, so

$$\begin{aligned}\mathrm{Jac}_{X_n'}\circ\mathrm{Jac}_{X_{<n}}(7.4)&=\langle\zeta_n(B_n-1),\ldots,-\zeta_n(A_n+1)\rangle\\&\quad\rtimes\pi_{M,>_\psi}^{\Sigma_0}(\psi_{\gg,-}^{(n-1)},\underline{l}_-,\underline{\eta}_-;(\rho,A_n,B_n,l_n,\eta_n,\zeta_n)).\end{aligned}$$

For $i>n$, $\mathrm{Jac}_{X_{>n}}$ can only apply to $\pi_{M,>_\psi}^{\Sigma_0}(\psi_{\gg,-}^{n-1},\underline{l}_-,\underline{\eta}_-;(\rho,A_n,B_n,l_n,\eta_n,\zeta_n))$ as $B_i>A_n+1$. Therefore,

$$\mathrm{Jac}_{X^c}\circ\mathrm{Jac}_{X_{>n}}\circ\mathrm{Jac}_{X_n'}\circ\mathrm{Jac}_{X_{<n}}(7.4)=\langle\zeta_n(B_n-1),\ldots,-\zeta_n(A_n+1)\rangle\rtimes\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta})\neq0.$$

Hence $\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta})\neq0$.

Next, we suppose $\pi_{M,>_\psi}^{\Sigma_0}(\psi,\underline{l},\underline{\eta})\neq0$. Let

$$\mathcal{C}_{X_{>n}}:=\times_{i>n}\begin{pmatrix}\zeta_i(B_i+T_i)&\cdots&\zeta_i(B_i+1)\\\vdots&&\vdots\\\zeta_i(A_i+T_i)&\cdots&\zeta_i(A_i+1)\end{pmatrix},$$
$$\mathcal{C}_{X_{<n}}:=\times_{i<n}\begin{pmatrix}\zeta_i(B_i+T_i)&\cdots&\zeta_i(B_i+1)\\\vdots&&\vdots\\\zeta_i(A_i+T_i)&\cdots&\zeta_i(A_i+1)\end{pmatrix},$$

where i increases, and

$$\mathcal{C}_{X_n}:=\begin{pmatrix}\zeta_n(B_n+T_n)&\cdots&\zeta_n(B_n+1)\\\vdots&&\vdots\\\zeta_n(A_n+T_n)&\cdots&\zeta_n(A_n+1)\end{pmatrix}.$$

Then

$$\begin{aligned}
&\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T_n+1, B_n+T_n-1, l_n+1, \eta_n, \zeta_n)) \\
&\quad \hookrightarrow \langle \zeta_n(B_n+T_n-1), \ldots, -\zeta_n(A_n+T_n+1)\rangle \times \mathcal{C}_{X_{<n}} \times \mathcal{C}_{X_n} \times \mathcal{C}_{X_{>n}} \times \mathcal{C}_{X^c} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \\
&\quad \cong \mathcal{C}_{X^c} \times \mathcal{C}_{X_{<n}} \times \langle \zeta_n(B_n+T_n-1), \ldots, \zeta_n B_n\rangle \\
&\qquad \times \mathcal{C}_{X_n} \times \mathcal{C}_{X_{>n}} \times \langle \zeta_n(B_n-1), \ldots, -\zeta_n(A_n+1)\rangle \\
&\qquad \times \langle -\zeta_n(A_n+2), \ldots, -\zeta_n(A_n+T_n+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).
\end{aligned}$$

By (4.1), we can take the dual of $\langle -\zeta_n(A_n+2), \ldots, -\zeta_n(A_n+T_n+1)\rangle$. Hence

$$\begin{aligned}
&\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+T_n+1, B_n+T_n-1, l_n+1, \eta_n, \zeta_n)) \\
&\quad \hookrightarrow \mathcal{C}_{X^c} \times \mathcal{C}_{X_{<n}} \times \langle \zeta_n(B_n+T_n-1), \ldots, \zeta_n B_n\rangle \\
&\qquad \times \mathcal{C}_{X_n} \times \mathcal{C}_{X_{>n}} \times \langle \zeta_n(B_n-1), \ldots, -\zeta_n(A_n+1)\rangle \\
&\qquad \times \langle \zeta_n(A_n+T_n+1), \ldots, \zeta_n(A_n+2)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \\
&\quad \cong \mathcal{C}_{X^c} \times \mathcal{C}_{X_{<n}} \times \langle \zeta_n(B_n+T_n-1), \ldots, \zeta_n B_n\rangle \times \mathcal{C}_{X_n} \times \langle \zeta_n(A_n+T_n+1), \ldots, \zeta_n(A_n+2)\rangle \\
&\qquad \times \mathcal{C}_{X_{>n}} \times \langle \zeta_n(B_n-1), \ldots, -\zeta_n(A_n+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+1, B_n-1, l_n+1, \eta_n, \zeta_n)) \\
&\quad \hookrightarrow \langle \zeta_n(B_n-1), \ldots, -\zeta_n(A_n+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).
\end{aligned}$$

This proves the proposition in case $t = 1$, except for the uniqueness and the statement about Jacquet modules.

In fact, the first part of the proposition follows easily from that of case $t = 1$. Moreover, we have

$$\begin{aligned}
&\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+t, B_n-t, l_n+t, \eta_n, \zeta_n)) \hookrightarrow \\
&\langle \zeta_n(B_n-t), \ldots, -\zeta_n(A_n+t)\rangle \times \cdots \times \langle \zeta_n(B_n-1), \ldots, -\zeta_n(A_n+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).
\end{aligned}$$

Then there exists an irreducible constituent τ of

$$\langle \zeta_n(B_n-t), \ldots, -\zeta_n(A_n+t)\rangle \times \cdots \times \langle \zeta_n(B_n-1), \ldots, -\zeta_n(A_n+1)\rangle$$

such that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+t, B_n-t, l_n+t, \eta_n, \zeta_n)) \hookrightarrow \tau \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).$$

We claim

$$\tau = \begin{pmatrix} \zeta_n(B_n-t) & \cdots & -\zeta_n(A_n+t) \\ \vdots & & \vdots \\ \zeta_n(B_n-1) & \cdots & -\zeta_n(A_n+1) \end{pmatrix}.$$

Otherwise, $\mathrm{Jac}_x \tau \neq 0$ for some x in $]\zeta_n(B_n-t), \zeta_n(B_n-1)]$, and hence

$$\mathrm{Jac}_x \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+t, B_n-t, l_n+t, \eta_n, \zeta_n)) \neq 0.$$

This means there exists $i < n$ such that $B_i > B_n - t$ and $\zeta_i = \zeta_n$, which contradicts to our assumption.

Finally, since $A_n \geqslant A_i$ for $i < n$, after we apply

$$\mathrm{Jac}_{\zeta_n(B_n-1),\ldots,-\zeta_n(A_n+1)} \circ \cdots \circ \mathrm{Jac}_{\zeta_n(B_n-t),\ldots,-\zeta_n(A_n+t)}$$

to the full-induced representation $\tau \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$, we get $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$. So

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+t, B_n-t, l_n+1, \eta_n, \zeta_n)) \hookrightarrow \tau \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$$

as the unique irreducible subrepresentation, and

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \circ_{i\in[1,t]}\mathrm{Jac}_{\zeta_n(B_n-i),\ldots,-\zeta_n(A_n+i)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_n+t, B_n-t, l_n+t, \eta_n, \zeta_n)).$$

So we have finished the proof. □

7.3. Change sign

We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$. We index the Jordan blocks in $\mathrm{Jord}_\rho(\psi)$ such that

$$(\rho, A_{i+1}, B_{i+1}, \zeta_{i+1}) >_\psi (\rho, A_i, B_i, \zeta_i).$$

Suppose there exists n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg \bigcup_{j=1}^{n}\{(\rho, A_j, B_j, \zeta_j)\}.$$

Moreover,

$$\text{for } 1 < i \leqslant n,\ A_1 \geqslant A_i,\ B_1 = 1/2 \text{ or } 0, \text{ and } \zeta_i \neq \zeta_1.$$

We define ψ_- by

$$\mathrm{Jord}(\psi_-) = \mathrm{Jord}(\psi)\backslash\{(\rho, A_1, B_1, \zeta_1)\}.$$

We denote the restriction of $(\underline{l}, \underline{\eta})$ to $\mathrm{Jord}(\psi_-)$ by $(\underline{l}_-, \underline{\eta}_-)$.

7.3.1. $B_1 = 0$.

Proposition 7.5. *For any* $(\underline{l}, \underline{\eta})$, $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ *if and only if*

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 0, l_1, \eta_1, -\zeta_1)) \neq 0.$$

Moreover,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 0, l_1, \eta_1, -\zeta_1)). \tag{7.5}$$

Proof. Let $\psi_\gg$ dominates ψ with discrete diagonal restriction such that $T_1 = 0$. It suffices to prove (7.5) for $\psi_\gg$. When $l_1 = 0$, (7.5) is clear. So we can further assume $l_1 \neq 0$. Let $\psi_{\gg,-}$ be obtained from $\psi_\gg$ by removing $(\rho, A_1, B_1, \zeta_1)$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 0, l_1, \eta_1, -\zeta_1)) \hookrightarrow \langle 0, \ldots, \zeta_1 A_1\rangle \times \langle -\zeta_1 1, \ldots, -\zeta_1(A_1-1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, -\zeta_1)),$$

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as the unique irreducible subrepresentation. On the other hand,

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \langle 0, \ldots, -\zeta_1 A_1\rangle \times \langle \zeta_1 1, \ldots, \zeta_1(A_1-1)\rangle \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1)) \\ &\hookrightarrow \rho \times \langle -\zeta_1 1 \ldots, -\zeta_1 A_1\rangle \times \langle \zeta_1 1, \ldots, \zeta_1(A_1-1)\rangle \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1)) \\ &\cong \rho \times \langle \zeta_1 1, \ldots, \zeta_1(A_1-1)\rangle \times \langle -\zeta_1 1 \ldots, -\zeta_1 A_1\rangle \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1)) \\ &\hookrightarrow \rho \times \langle \zeta_1 1, \ldots, \zeta_1(A_1-1)\rangle \times \langle -\zeta_1 1 \ldots, -\zeta_1(A_1-1)\rangle \times \rho|\,|^{-\zeta_1 A_1} \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1)).\end{aligned}$$

Since $\rho|\,|^{-\zeta_1 A_1} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1))$ is irreducible, we have

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \rho \times \langle \zeta_1 1, \ldots, \zeta_1(A_1-1)\rangle \times \langle -\zeta_1 1 \ldots, -\zeta_1(A_1-1)\rangle \times \rho|\,|^{\zeta_1 A_1} \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1)) \\ &\cong \rho \times \langle \zeta_1 1, \ldots, \zeta_1(A_1-1)\rangle \times \rho|\,|^{\zeta_1 A_1} \times \langle -\zeta_1 1 \ldots, -\zeta_1(A_1-1)\rangle \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1)).\end{aligned}$$

Since $\mathrm{Jac}_x \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) = 0$ for $x = \zeta_1 1, \zeta_1 A_1$, then

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) &\hookrightarrow \langle 0, \ldots, \zeta_1 A_1\rangle \times \langle -\zeta_1 1 \ldots, -\zeta_1(A_1-1)\rangle \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1)).\end{aligned}$$

By induction on l_1, we can assume

$$\begin{aligned}&\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, \zeta_1)) \\ &\quad = \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-2, 0, l_1-1, \eta_1, -\zeta_1)).\end{aligned}$$

Then we necessarily have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 0, l_1, \eta_1, -\zeta_1)).$$

This finishes the proof. □

7.3.2. $B_1 = 1/2$.

Proposition 7.6. *For any $(\underline{l}, \underline{\eta})$, one can construct*

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) := \begin{cases} \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_1+1, 1/2, l_1+1, -\eta_1, -\zeta_1)) & \textit{if } \eta_1 = +1, \\ \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_1+1, 1/2, l_1, -\eta_1, -\zeta_1)) & \textit{if } \eta_1 = -1. \end{cases}$$

In case $l_1 = (A_1 + \frac{1}{2})/2$, *we fix* $\eta_1 = -1$. *Then*

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0 \quad \textit{if and only if} \quad \pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0.$$

Moreover,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$$

as the unique irreducible subrepresentation, and

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = Jac_{-\zeta_1 1/2,\ldots,-\zeta_1(A_1+1)}\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*).$$

Proof. Let us choose $\psi_{\gg}$ dominating ψ with discrete diagonal restriction, and we require $T_1 = 0$. Then it determines $\psi^*_{\gg}$ which dominates ψ^*. We assume the proposition for $\psi_{\gg}$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_{\gg}, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}).$$

Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0$, then after we apply

$$\circ_{i>1}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)}$$

(i decreases) and Jac_{X^c} to the full-induced representation

$$\langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \tag{7.6}$$

we should get something nonzero. Since

$$-\zeta_1 1/2 \quad \text{and} \quad \zeta_1(A_1+1) \notin [\zeta_i(A_i+T_i), \zeta_i(B_i+1)]$$

for $i > 1$, $\circ_{i>1}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)}$ and Jac_{X^c} can only apply to $\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta})$. Therefore,

$$\begin{aligned}&\circ_{i>1}\mathrm{Jac}_{(\rho,A_i+T_i,B_i+T_i,\zeta_i)\mapsto(\rho,A_i,B_i,\zeta_i)} \circ \mathrm{Jac}_{X^c}\ (7.6)\\ &\quad = \langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0.\end{aligned}$$

This shows $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, let

$$\mathcal{C}_{X_i} := \begin{pmatrix} \zeta_i(B_i+T_i) & \cdots & \zeta_i(B_i+1) \\ \vdots & & \vdots \\ \zeta_i(A_i+T_i) & \cdots & \zeta_i(A_i+1)\end{pmatrix},$$

then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_{\gg}, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle \times (\times_{i>1}\mathcal{C}_{X_i}) \times \mathcal{C}_{X^c} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}),$$

where i increases. For $i > n$, we have $B_i > A_1 + 1$, so $\mathcal{C}_{X_i}$ and $\langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle$ are interchangeable. For $i < n$, we have $A_1 \geqslant A_i$ and $\zeta_i = -\zeta_1$, so

$$[\zeta_i(B_i+1), \zeta_i(A_i+1)] \subseteq [-\zeta_1 1/2, -\zeta_1(A_1+1)].$$

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It follows $\mathcal{C}_{X_i}$ and $\langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle$ are also interchangeable. Therefore,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \hookrightarrow (\times_{i>1}\mathcal{C}_{X_i}) \times \mathcal{C}_{X^c} \times \langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).$$

This implies $\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0$, and

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).$$

To see $\pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*)$ is the unique irreducible subrepresentation, it suffices to check that

$$\mathrm{Jac}_{-\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)}(\langle -\zeta_1 1/2, \ldots - \zeta_1(A_1+1)\rangle \times \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})) = \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}).$$

As a consequence, we also get

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \mathrm{Jac}_{-\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)} \pi^{\Sigma_0}_{M,>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*).$$

To complete the proof, we still need to show the proposition for $\psi_\gg$, and we leave it to the next lemma. □

Lemma 7.7. *Proposition 7.6 holds for $\psi_\gg$.*

Proof. It is clear that $\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \neq 0$ and $\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \neq 0$ in this case. So we only need to show

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta})$$

as the unique irreducible subrepresentation, and

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = \mathrm{Jac}_{-\zeta_1 1/2, \ldots, -\zeta_1(A_1+1)} \pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*).$$

Let $\psi_{\gg,-}$ be obtained from $\psi_\gg$ by removing $(\rho, A_1, B_1, \zeta_1)$. When $l_1 = 0$ and $\eta_1 = -1$, the lemma is clear. When $A_1 = 1/2$, then necessarily $l_1 = 0$. In this case, if $\eta_1 = +1$, then

$$\begin{aligned}
\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) &\hookrightarrow \langle -\zeta_1 1/2, \ldots, \zeta_1 3/2\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-) \\
&\hookrightarrow \rho|\,|^{-\zeta_1 1/2} \times \rho|\,|^{\zeta_1 1/2} \times |\,|^{\zeta_1 3/2} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-) \\
&\cong \rho|\,|^{-\zeta_1 1/2} \times \rho|\,|^{\zeta_1 1/2} \times |\,|^{-\zeta_1 3/2} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-) \\
&\cong \rho|\,|^{-\zeta_1 1/2} \times |\,|^{-\zeta_1 3/2} \times \rho|\,|^{\zeta_1 1/2} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-).
\end{aligned}$$

There exists an irreducible constituent σ of $\rho|\,|^{\zeta_1 1/2} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-)$ such that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \rho|\,|^{-\zeta_1 1/2} \rtimes |\,|^{-\zeta_1 3/2} \rtimes \sigma.$$

Since $\mathrm{Jac}_{-\zeta_1 3/2} \pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) = 0$, we must have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, -\zeta_1 3/2\rangle \rtimes \sigma.$$

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Suppose $\mathrm{Jac}_{-\zeta_1 1/2}\sigma \neq 0$, then there exists an irreducible constituent σ' of $\mathrm{Jac}_{-\zeta_1 1/2}\sigma$ such that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, -\zeta_1 3/2\rangle \times \rho||^{-\zeta_1 1/2} \rtimes \sigma' \cong \rho||^{-\zeta_1 1/2} \times \langle -\zeta_1 1/2, -\zeta_1 3/2\rangle \rtimes \sigma'.$$

This implies $\mathrm{Jac}_{-\zeta_1 1/2, -\zeta_1 1/2}\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \neq 0$, which is impossible. Therefore, we must have $\mathrm{Jac}_{\zeta_1 1/2}\sigma \neq 0$. In particular, this means $\sigma = \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta})$. So

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, -\zeta_1 3/2\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}).$$

To see $\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*)$ is the unique irreducible subrepresentation, it suffices to check that

$$\mathrm{Jac}_{-\zeta_1 1/2, -\zeta_1 3/2}(\langle -\zeta_1 1/2, -\zeta_1 3/2\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta})) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}).$$

As a consequence,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = \mathrm{Jac}_{-\zeta_1 1/2, -\zeta_1 3/2}\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*).$$

Next we would like to prove this lemma by induction on A_1. Let $A_1 > 1/2$. Suppose $\eta_1 = +1$, then

$$\begin{aligned}
\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) &\hookrightarrow \langle -\zeta_1 1/2, \ldots, \zeta_1(A_1+1)\rangle \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1, 3/2, l_1, -\eta_1, -\zeta_1)) \\
&\hookrightarrow \langle -\zeta_1 1/2, \ldots, \zeta_1(A_1+1)\rangle \times \langle -\zeta_1 3/2, \ldots, -\zeta_1 A_1\rangle \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-1, 1/2, l_1, -\eta_1, -\zeta_1)) \\
&\hookrightarrow \rho||^{-\zeta_1 1/2} \times \langle -\zeta_1 3/2, \ldots, -\zeta_1 A_1\rangle \times \langle \zeta_1 1/2, \ldots, \zeta_1 A_1\rangle \times \rho||^{\zeta_1(A_1+1)} \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-1, 1/2, l_1, -\eta_1, -\zeta_1)) \\
&\cong \rho||^{-\zeta_1 1/2} \times \langle -\zeta_1 3/2, \ldots, -\zeta_1 A_1\rangle \times \langle \zeta_1 1/2, \ldots, \zeta_1 A_1\rangle \times \rho||^{-\zeta_1(A_1+1)} \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-1, 1/2, l_1, -\eta_1, -\zeta_1)) \\
&\cong \rho||^{-\zeta_1 1/2} \times \langle -\zeta_1 3/2, \ldots, -\zeta_1 A_1\rangle \times \rho||^{-\zeta_1(A_1+1)} \times \langle \zeta_1 1/2, \ldots, \zeta_1 A_1\rangle \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-1, 1/2, l_1, -\eta_1, -\zeta_1)).
\end{aligned}$$

There exists an irreducible constituent σ of

$$\langle \zeta_1 1/2, \ldots, \zeta_1 A_1\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-1, 1/2, l_1, -\eta_1, -\zeta_1))$$

such that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \rho||^{-\zeta_1 1/2} \times \langle -\zeta_1 3/2, \ldots, -\zeta_1 A_1\rangle \times \rho||^{-\zeta_1(A_1+1)} \rtimes \sigma.$$

Since $\mathrm{Jac}_x\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) = 0$ for $x \in [-\zeta_1 3/2, -\zeta_1(A_1+1)]$, then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_\gg, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \ldots - \zeta_1(A_1+1)\rangle \rtimes \sigma. \tag{7.7}$$

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If we apply $\mathrm{Jac}_{-\zeta_1 1/2,\dots-\zeta_1(A_1+1)}$ to the full-induced representation in (7.7), then $\mathrm{Jac}_{-\zeta_1(A_1+1)}$ can only apply to $\langle -\zeta_1 1/2, \dots - \zeta_1(A_1+1)\rangle$. As a consequence, we must have the whole Jacquet functor $\mathrm{Jac}_{-\zeta_1 1/2,\dots-\zeta_1(A_1+1)}$ applied to $\langle -\zeta_1 1/2, \dots - \zeta_1(A_1+1)\rangle$, and hence

$$\mathrm{Jac}_{-\zeta_1 1/2,\dots-\zeta_1(A_1+1)} \pi^{\Sigma_0}_{M,>_\psi}(\psi^*_{\gg}, \underline{l}^*, \underline{\eta}^*) = \sigma,$$

which is irreducible. Therefore,

$$\sigma \hookrightarrow \langle \zeta_1 1/2, \dots, \zeta_1 A_1\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg,-}, \underline{l}_-, \underline{\eta}_-; (\rho, A_1-1, 1/2, l_1, -\eta_1, -\zeta_1)). \tag{7.8}$$

By induction, $\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta})$ is the unique irreducible subrepresentation of the induced representation in (7.8), so it has to be equal to σ. Hence

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_{\gg}, \underline{l}^*, \underline{\eta}^*) \hookrightarrow \langle -\zeta_1 1/2, \dots - \zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}).$$

To see $\pi^{\Sigma_0}_{M,>_\psi}(\psi^*_{\gg}, \underline{l}^*, \underline{\eta}^*)$ is the unique irreducible subrepresentation, it suffices to check that

$$\mathrm{Jac}_{-\zeta_1 1/2,\dots-\zeta_1(A_1+1)}(\langle -\zeta_1 1/2, \dots - \zeta_1(A_1+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta})) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}).$$

As a consequence,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \mathrm{Jac}_{-\zeta_1 1/2,\dots,-\zeta_1(A_1+1)} \pi^{\Sigma_0}_{M,>_\psi}(\psi^*_{\gg}, \underline{l}^*, \underline{\eta}^*).$$

Suppose $\eta_1 = -1$, we can also assume $l_1 \neq 0$, then the proof is the same. □

8. General procedure

The three operations ('Pull', 'Expand', and 'Change sign') introduced in the previous section allow us to develop a procedure to find the combinatorial conditions for the nonvanishing of $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$.

8.1. Step one

We choose an admissible order $>_\psi$, and we also fix a self-dual unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$. We index the Jordan blocks in $\mathrm{Jord}_\rho(\psi)$ such that

$$(\rho, A_i, B_i, \zeta_i) >_\psi (\rho, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

We choose n such that for $i > n$,

$$(\rho, A_i, B_i, \zeta_i) \gg_2 \bigcup_{j=1}^{n} \{(\rho, A_j, B_j, \zeta_j)\},$$

and the Jordan blocks for $i > n$ are in 'good shape' (see Remark 5.4). Then for $i \leqslant n$, let us choose (ρ, A, B, ζ) so that A is maximal. We consider the set

$$\{(\rho, A_i, B_i, \zeta_i) \text{ for } i \leqslant n : [A_i, B_i] \subsetneq [A, B] \text{ and } \zeta_i = \zeta\}. \tag{8.1}$$

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If this set is nonempty, we take (ρ, A', B', ζ') such that A' is maximal within the set. We can rearrange the order $>_\psi$ for $i \leqslant n$, so that

$$(\rho, A_n, B_n, \zeta_n) = (\rho, A, B, \zeta) \quad \text{and} \quad (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1}) = (\rho, A', B', \zeta').$$

Then we can 'Pull' the pairs $(\rho, A_n, B_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})$ using Proposition 7.1. Suppose the set (8.1) is empty, but there exists (ρ, A', B', ζ') such that

$$[A', B'] = [A, B] \quad \text{and} \quad \zeta' = \zeta,$$

then we can again rearrange the order $>_\psi$ for $i \leqslant n$, so that

$$(\rho, A_n, B_n, \zeta_n) = (\rho, A, B, \zeta) \quad \text{and} \quad (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1}) = (\rho, A', B', \zeta').$$

And we can 'Pull' the pairs $(\rho, A_n, B_n, \zeta_n), (\rho, A_{n-1}, B_{n-1}, \zeta_{n-1})$ using Proposition 7.3.

8.2. Step two

Following Step one, we suppose the set

$$\{(\rho, A_i, B_i, \zeta_i) \text{ for } i \leqslant n : [A_i, B_i] \subseteq [A, B] \text{ and } \zeta_i = \zeta\} \backslash \{(\rho, A, B, \zeta)\} \tag{8.2}$$

is empty. We can still rearrange the order $>_\psi$ for $i \leqslant n$ such that

$$(\rho, A_n, B_n, \zeta_n) = (\rho, A, B, \zeta).$$

Then we 'Expand' $[A_n, B_n]$, and use Proposition 7.4.

8.3. Step three

Following Step two, let us denote the 'Expansion' of $[A_n, B_n]$ by $[A_n^*, B_n^*]$. The set (8.1) becomes

$$\{(\rho, A_i, B_i, \zeta_i) \text{ for } i < n : [A_i, B_i] \subsetneq [A_n^*, B_n^*] \text{ and } \zeta_i = \zeta_n\}.$$

If this set is nonempty, then we are back to Step one. If this set is empty, then by our definition of 'Expand', it is necessary that $B_n^* = 1/2$ or 0, and $\zeta_i \neq \zeta_n$ for all $i < n$. In this case, we can change the order for $i \leqslant n$ again by switching $(\rho, A_n^*, B_n^*, \zeta_n)$ with $(\rho, A_i, B_i, \zeta_i)$ one by one as i goes from $n-1$ to 1. Then we can 'Change sign' of $(\rho, A_n^*, B_n^*, \zeta_n)$, and use Proposition 7.5 or Proposition 7.6. After that, we are back to Step one again.

8.4. Step four

By the above three steps, we end up with a collection of parameters $\{\psi^\star\}$ such that $\mathrm{Jord}_\rho(\psi^\star)$ is in 'good shape' (cf. Propositions 7.1 and 7.3). Then we can change ρ and repeat all the previous steps to $\{\psi^\star\}$.

Appendix A. Proof of Proposition 5.2

In this appendix, we give the proof of Proposition 5.2, which also includes a proof of Lemma 5.1. We proceed by induction. So let us suppose the proposition holds when

$(A_1 - B_1) + (A_2 - B_2) < L$ for some positive integer L. Note when $(A_1 - B_1) + (A_2 - B_2) = 0$, this is clear (cf. Theorem 1.1). When $(A_1 - B_1) + (A_2 - B_2) = L$, we first prove the proposition, except for the necessity of condition (5.1) in the case that $A_2 = A_1$ and $B_2 = B_1$. This remaining case is actually part of Lemma 5.1 and will be treated in the end.

A.1. Proof of Proposition 5.2

We first show the necessity of the condition (5.1). So let us suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, and we take the following two reduction steps.

- *First reduction*: we assume $A_2 > A_1$ and $l_2 \neq 0$.
  Let us define $\psi_\gg$ by shifting $(\rho, A_2, B_2, \zeta_2)$ to $(\rho, A_2 + T, B_2 + T, \zeta_2)$, such that $\psi_\gg$ has discrete diagonal restriction and the natural order is the same as $>_\psi$. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \langle \zeta(B_2 + T), \ldots, -\zeta(A_2 + T)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 + T - 1, B_2 + T + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)).$$

  Note

$$\mathrm{Jac}_{(\rho, A_2+T, B_2+T, \zeta)\mapsto(\rho, A_2, B_2, \zeta)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0.$$

  So after applying $\mathrm{Jac}_{(\rho, A_2+T, B_2+T, \zeta)\mapsto(\rho, A_2, B_2, \zeta)}$ to the full-induced representation above, we have

$$\langle \zeta B_2, \ldots, -\zeta A_2\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - 1, B_2 + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)),$$

  which is again nonzero. In particular,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - 1, B_2 + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \neq 0.$$

  By our assumption, $A_2 - 1 \geqslant A_1$, $B_2 + 1 > B_1$ and $l_2 - 1 \geqslant 0$, so we get by induction assumption

$$\begin{cases} \eta_2 = (-1)^{A_1 - B_1}\eta_1 & \Rightarrow (A_2 - 1) - (l_2 - 1) \geqslant A_1 - l_1, (B_2 + 1) + (l_2 - 1) \geqslant B_1 + l_1, \\ \eta_2 \neq (-1)^{A_1 - B_1}\eta_1 & \Rightarrow (B_2 + 1) + (l_2 - 1) > A_1 - l_1. \end{cases}$$

  This gives the condition (5.1).
- *Second reduction*: we assume $B_2 > B_1$ and $l_1 \neq 0$.
  We choose $\psi_\gg$ as in the previous step. Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \langle \zeta B_1, \ldots, -\zeta A_1\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 + T, B_2 + T, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)).$$

  Note

$$\mathrm{Jac}_{(\rho, A_2+T, B_2+T, \zeta)\mapsto(\rho, A_2, B_2, \zeta)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0.$$

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So after applying $\mathrm{Jac}_{(\rho,A_2+T,B_2+T,\zeta)\mapsto(\rho,A_2,B_2,\zeta)}$ to the full-induced representation above, we have

$$\langle \zeta B_1, \ldots, -\zeta A_1\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1-1, B_1+1, l_1-1, \eta_1, \zeta)),$$

which is again nonzero. In particular,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1-1, B_1+1, l_1-1, \eta_1, \zeta)) \neq 0.$$

By our assumption, $A_2 > A_1 - 1$, $B_2 \geqslant B_1 + 1$ and $l_1 - 1 \geqslant 0$, so we get by induction assumption

$$\begin{cases} \eta_2 = (-1)^{A_1-B_1}\eta_1 & \Rightarrow A_2 - l_2 \geqslant (A_1-1)-(l_1-1),\ B_2 + l_2 \geqslant (B_1+1)+(l_1-1), \\ \eta_2 \neq (-1)^{A_1-B_1}\eta_1 & \Rightarrow B_2 + l_2 > (A_1-1)-(l_1-1). \end{cases}$$

This again gives the condition (5.1).

After these two steps, we are reduced to the following cases:

- *Case 1*: $A_2 = A_1$ and $B_2 = B_1$.
  This is the remaining case, which will be treated in the end.
- *Case 2*: $A_2 = A_1$, $B_2 > B_1$ and $l_1 = 0$.
  In this case, the condition (5.1) becomes

$$\eta_2 = (-1)^{A_1-B_1}\eta_1 \quad \text{and} \quad l_2 = 0.$$

Note

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1, B_2, 0, (-1)^{B_2-B_1}\eta_1, \zeta), (\rho, B_2-1, B_1, 0, \eta_1, \zeta)).$$

Applying the induction assumption to $(\rho, A_2, B_2, l_2, \eta_2, \zeta)$ and $(\rho, A_1, B_2, 0, (-1)^{B_2-B_1}\eta_1, \zeta)$, we get

$$\eta_2 = (-1)^{A_1-B_2}\cdot(-1)^{B_2-B_1}\eta_1 = (-1)^{A_1-B_1}\eta_1,$$

and $l_2 = 0$. This is exactly what we want.

- *Case 3*: $A_2 > A_1$, $B_2 = B_1$ and $l_2 = 0$.
  In this case, the condition (5.1) becomes

$$\eta_2 = (-1)^{A_1-B_1}\eta_1 \quad \text{and} \quad l_1 = 0.$$

Note

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+T, A_1+T+1, 0, (-1)^{A_1-B_2+1}\eta_2, \zeta), (\rho, A_1+T, B_2+T, 0, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)).$$

Since

$$\mathrm{Jac}_{(\rho,A_2+T,B_2+T,\zeta)\mapsto(\rho,A_2,B_2,\zeta)} = \mathrm{Jac}_{(\rho,A_2+T,A_1+T+1,\zeta)\mapsto(\rho,A_2,A_1+1,\zeta)} \circ \mathrm{Jac}_{(\rho,A_1+T,B_2+T,\zeta)\mapsto(\rho,A_1,B_2,\zeta)},$$

then

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$$\begin{aligned}&\mathrm{Jac}_{(\rho,A_1+T,B_2+T,\zeta)\mapsto(\rho,A_1,B_2,\zeta)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})\\&\quad=\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2+T,A_1+T+1,0,(-1)^{A_1-B_2+1}\eta_2,\zeta),\\&\qquad(\rho,A_1,B_2,0,\eta_2,\zeta),(\rho,A_1,B_1,l_1,\eta_1,\zeta))\neq 0.\end{aligned}$$

Applying the induction assumption to $(\rho, A_1, B_2, 0, \eta_2, \zeta)$ and $(\rho, A_1, B_1, l_1, \eta_1, \zeta)$, we get exactly

$$\eta_2=(-1)^{A_1-B_1}\eta_1 \quad\text{and}\quad l_1=0.$$

- *Case 4*: $A_2 > A_1$, $B_2 > B_1$ and $l_2 = l_1 = 0$.
  If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, the condition is automatically satisfied.
  If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$. We can suppose $B_2 \leqslant A_1$, and let $T = A_1 - B_2 + 1$. One observes

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})=\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2+T,B_1,0,\eta_1,\zeta)).$$

  So $\mathrm{Jac}_{\zeta(A_1+1)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})=0$. Therefore,

$$\mathrm{Jac}_{(\rho,A_2+T,B_2+T,\zeta)\mapsto(\rho,A_2,B_2,\zeta)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})=0.$$

Next we would like to show the sufficiency of condition (5.1) by computing $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})$ directly. We take $\psi_\gg$ to be defined as before.

- Suppose $l_1 = l_2 = 0$. If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then $B_2 > A_1$ and there is nothing to prove. So let us also assume $\eta_2 = (-1)^{A_1-B_1}\eta_1$.

  (1) $A_2 - B_2 \leqslant A_1 - B_1$.

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})\hookrightarrow&\begin{pmatrix}\zeta(B_2+T)&\cdots&-\zeta A_1\\\vdots&&\vdots\\\zeta(A_2+T)&\cdots&-\zeta(A_1-A_2+B_2)\end{pmatrix}\\&\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_1-A_2+B_2-1,B_1,0,\eta_1,\zeta)).\end{aligned}$$

  It is clear that $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})\neq 0$.

  (2) $A_2 - B_2 > A_1 - B_1$.

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta})\hookrightarrow&\begin{pmatrix}\zeta(B_2+T)&\cdots&-\zeta A_1\\\vdots&&\vdots\\\zeta(B_2+A_1-B_1+T)&\cdots&-\zeta B_1\end{pmatrix}\\&\times\begin{pmatrix}\zeta(B_2+A_1-B_1+T+1)&\cdots&\zeta(B_2+A_1-B_1+2)\\\vdots&&\vdots\\\zeta(A_2+T)&\cdots&\zeta(A_2+1)\end{pmatrix}\\&\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2,B_2+A_1-B_1+1,0,-\eta_1,\zeta)).\end{aligned}$$

  It is again clear that $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})\neq 0$.

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- Suppose $l_1 \neq 0$ or $l_2 \neq 0$.

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta(B_2+T) & \cdots & -\zeta(A_2+T) \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1+T) & \cdots & -\zeta(A_2-l_2+1+T) \end{pmatrix}$$
$$\times \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1) \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2-l_2+T, B_2+l_2+T, 0, \eta_2, \zeta),$$
$$(\rho, A_1-l_1, B_1+l_1, 0, \eta_1, \zeta)).$$

From our previous discussion, we know

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2-l_2, B_2+l_2, 0, \eta_2, \zeta), (\rho, A_1-l_1, B_1+l_1, 0, \eta_1, \zeta)) \neq 0,$$

so

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2-l_2+T, B_2+l_2+T, 0, \eta_2, \zeta),$$
$$(\rho, A_1-l_1, B_1+l_1, 0, \eta_1, \zeta))$$
$$\hookrightarrow \begin{pmatrix} \zeta(B_2+l_2+T) & \cdots & \zeta(B_2+l_2+1) \\ \vdots & & \vdots \\ \zeta(A_2-l_2+T) & \cdots & \zeta(A_2-l_2+1) \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2-l_2, B_2+l_2, 0, \eta_2, \zeta), (\rho, A_1-l_1, B_1+l_1, 0, \eta_1, \zeta)).$$

Therefore,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta(B_2+T) & \cdots & \zeta(B_2+1) \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1+T) & \cdots & \zeta(B_2+l_2) \end{pmatrix}$$
$$\times \underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta(A_2+T) \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1) & \cdots & -\zeta(A_2-l_2+1+T) \end{pmatrix}}_{I}$$
$$\times \underbrace{\begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1) \end{pmatrix}}_{II}$$
$$\times \underbrace{\begin{pmatrix} \zeta(B_2+l_2+T) & \cdots & \zeta(B_2+l_2+1) \\ \vdots & & \vdots \\ \zeta(A_2-l_2+T) & \cdots & \zeta(A_2-l_2+1) \end{pmatrix}}_{III}$$

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$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)).$$

Since $[\zeta B_2, -\zeta(A_2 + T)] \supseteq [\zeta B_1, -\zeta A_1]$, (I) and (II) are interchangeable. Also note $B_2 + l_2 + 1 > B_1 + l_1$, so we can interchange (II) and (III). It is clear that (I) and (III) are interchangeable too. As a result,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta(B_2+T) & \cdots & \zeta(B_2+1) \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1+T) & \cdots & \zeta(B_2+l_2) \end{pmatrix}$$
$$\times \underbrace{\begin{pmatrix} \zeta(B_2+l_2+T) & \cdots & \zeta(B_2+l_2+1) \\ \vdots & & \vdots \\ \zeta(A_2-l_2+T) & \cdots & \zeta(A_2-l_2+1) \end{pmatrix}}_{III}$$
$$\times \underbrace{\begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1) \end{pmatrix}}_{II}$$
$$\times \underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta(A_2+1) & \cdots & -\zeta(A_2+T) \\ \vdots & & \vdots & & \vdots \\ \zeta(B_2+l_2-1) & \cdots & -\zeta(A_2-l_2+2) & \cdots & -\zeta(A_2-l_2+1+T) \end{pmatrix}}_{I}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)).$$

By Proposition 4.6,

$$\underbrace{\begin{pmatrix} -\zeta(A_2+1) & \cdots & -\zeta(A_2+T) \\ \vdots & & \vdots \\ -\zeta(A_2-l_2+2) & \cdots & -\zeta(A_2-l_2+1+T) \end{pmatrix}}_{IV}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2, B_2 + l_2, 0, \eta_2, \zeta), (\rho, A_1 - l_1, B_1 + l_1, 0, \eta_1, \zeta)),$$

is irreducible. So we can take the dual of (IV) (see (4.1)). Moreover, $(IV)^\vee$ is interchangeable with

$$\underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta A_2 \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1) & \cdots & -\zeta(A_2-l_2+1) \end{pmatrix}}_{I_-}$$

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and (*II*). Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta(B_2+T) & \cdots & \zeta(B_2+1) \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1+T) & \cdots & \zeta(B_2+l_2) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta(B_2+l_2+T) & \cdots & \zeta(B_2+l_2+1) \\ \vdots & & \vdots \\ \zeta(A_2-l_2+T) & \cdots & \zeta(A_2-l_2+1) \end{pmatrix}}_{III} \times \underbrace{\begin{pmatrix} \zeta(A_2-l_2+1+T) & \cdots & \zeta(A_2-l_2+2) \\ \vdots & & \vdots \\ \zeta(A_2+T) & \cdots & \zeta(A_2+1) \end{pmatrix}}_{(IV)^\vee} \times \underbrace{\begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1) \end{pmatrix}}_{II} \times \underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta A_2 \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1) & \cdots & -\zeta(A_2-l_2+1) \end{pmatrix}}_{I_-} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2-l_2, B_2+l_2, 0, \eta_2, \zeta), (\rho, A_1-l_1, B_1+l_1, 0, \eta_1, \zeta)).$$

It follows $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, and

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1) \end{pmatrix}}_{II} \times \underbrace{\begin{pmatrix} \zeta B_2 & \cdots & -\zeta A_2 \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1) & \cdots & -\zeta(A_2-l_2+1) \end{pmatrix}}_{I_-} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2-l_2, B_2+l_2, 0, \eta_2, \zeta), (\rho, A_1-l_1, B_1+l_1, 0, \eta_1, \zeta)).$$

Finally, one just needs to observe (*II*) and (I_-) are interchangeable.

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A.2. The remaining case

We show the necessity of condition (5.1) in the remaining case, i.e., $A_2 = A_1$ and $B_2 = B_1$. Let us define $\psi_{\gg}$ by shifting $(\rho, A_2, B_2, \zeta_2)$ to $(\rho, A_2 + T, B_2 + T, \zeta_2)$, such that $\psi_{\gg}$ has discrete diagonal restriction and it admits the same order $>_\psi$. Suppose $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$, we first want to show

$$\begin{cases} \eta_2 = (-1)^{A_1 - B_1}\eta_1 & \Rightarrow |l_1 - l_2| \leqslant 1, \\ \eta_2 \neq (-1)^{A_1 - B_1}\eta_1 & \Rightarrow l_1 + l_2 + 1 > A_1 - B_1. \end{cases} \tag{A 1}$$

Let us consider the following situations.

(1) If $l_1 = l_2 = 0$, it is clear that one must have $\eta_2 = (-1)^{A_1 - B_1}\eta_1$.

(2) If $l_1 \neq 0$, then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \langle \zeta B_1, \ldots - \zeta A_1 \rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ (\rho, A_2 + T, B_2 + T, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)).$$

Note

$$\mathrm{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0.$$

So after applying $\mathrm{Jac}_{(\rho, A_2+T, B_2+T, \zeta) \mapsto (\rho, A_2, B_2, \zeta)}$ to the full-induced representation above, we have

$$\langle \zeta B_1, \ldots - \zeta A_1 \rangle \\ \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)),$$

which is again nonzero. In particular,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2, B_2, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)) \neq 0.$$

Here we only need the weak fact that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 + 1, B_2 + 1, l_2, \eta_2, \zeta), (\rho, A_1 - 1, B_1 + 1, l_1 - 1, \eta_1, \zeta)) \neq 0.$$

By our induction assumption, we can conclude

$$\begin{cases} \eta_2 = (-1)^{(A_1-1)-(B_1+1)}\eta_1 & \Rightarrow (A_2 + 1) - l_2 \geqslant (A_1 - 1) - (l_1 - 1), \\ & \quad (B_2 + 1) + l_2 \geqslant (B_1 + 1) + (l_1 - 1); \\ \eta_2 \neq (-1)^{(A_1-1)-(B_1+1)}\eta_1 & \Rightarrow (B_2 + 1) + l_2 > (A_1 - 1) - (l_1 - 1). \end{cases}$$

In the first case, we get $\eta_2 = (-1)^{A_1 - B_1}\eta_1$ and $-1 \leqslant l_2 - l_1 \leqslant 1$. In the second case, we have $\eta_2 \neq (-1)^{A_1 - B_1}\eta_1$ and $l_1 + l_2 + 1 > A_1 - B_2 = A_1 - B_1$.

(3) If $l_2 \neq 0$, then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \langle \zeta(B_2 + T), \ldots - \zeta(A_2 + T) \rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ (\rho, A_2 + T - 1, B_2 + T + 1, l_2 - 1, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)).$$

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Note

$$\mathrm{Jac}_{(\rho,A_2+T,B_2+T,\zeta)\mapsto(\rho,A_2,B_2,\zeta)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta}) = \pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) \neq 0.$$

So after applying $\mathrm{Jac}_{(\rho,A_2+T,B_2+T,\zeta)\mapsto(\rho,A_2,B_2,\zeta)}$ to the full-induced representation above, we have

$$\begin{aligned}&\langle \zeta B_2, \ldots - \zeta A_2\rangle\\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2-1,B_2+1,l_2-1,\eta_2,\zeta),(\rho,A_1,B_1,l_1,\eta_1,\zeta)),\end{aligned}$$

which is again nonzero. In particular,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2-1,B_2+1,l_2-1,\eta_2,\zeta),(\rho,A_1,B_1,l_1,\eta_1,\zeta)) \neq 0.$$

Here we only need the weak fact that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2,B_2+2,l_2-1,\eta_2,\zeta),(\rho,A_1,B_1,l_1,\eta_1,\zeta)) \neq 0.$$

By our induction assumption, we can conclude

$$\begin{cases}\eta_2 = (-1)^{A_1-B_1}\eta_1 & \Rightarrow A_2-(l_2-1) \geqslant A_1-l_1,\\ & \quad (B_2+2)+(l_2-1) \geqslant B_1+l_1;\\ \eta_2 \neq (-1)^{A_1-B_1}\eta_1 & \Rightarrow (B_2+2)+(l_2-1) > A_1-l_1.\end{cases}$$

In the first case, we get $\eta_2 = (-1)^{A_1-B_1}\eta_1$ and $-1 \leqslant l_2-l_1 \leqslant 1$. In the second case, we have $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$ and $l_1+l_2+1 > A_1-B_2 = A_1-B_1$.

Now we assume (A 1). If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, we still need to eliminate the case $|l_1-l_2| = 1$. If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, we need to eliminate the following cases:

(1) $|l_1-l_2| = 1$, $\max\{l_1,l_2\} = (A_1-B_1+1)/2$.

(2) $l_1 = l_2 = (A_1-B_1)/2$.

To simplify the notations, we let $A = A_1 = A_2$ and $B = B_1 = B_2$.

A.2.1. Case: $l_1 = l_2+1$. Let us denote l_2 by l. Since $A-l_1+1 > B+l_1-1$, then $A-l > B+l$.

(1) $A-l > B+l+1$.

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta}) \hookrightarrow &\begin{pmatrix}\zeta B_1 & \cdots & -\zeta A_1\\ \vdots & & \vdots\\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1)\end{pmatrix}\\ &\times\begin{pmatrix}\zeta(B_2+T) & \cdots & -\zeta(A_2+T)\\ \vdots & & \vdots\\ \zeta(B_2+l_2-1+T) & \cdots & -\zeta(A_2-l_2+1+T)\end{pmatrix}\\ &\times\begin{pmatrix}\zeta(B_2+l_2+T) & \cdots & -\zeta(A_1-l_1)\\ \vdots & & \vdots\\ \zeta(A_2-l_2-2+T) & \cdots & -\zeta(B_1+l_1)\end{pmatrix}\end{aligned}$$

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$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2 - l_2 + T, A_2 - l_2 - 1 + T, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta))$$

$$\hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix}}_{*-1}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2}$$

$$\times \underbrace{\begin{pmatrix} -\zeta(A+1) & \cdots & -\zeta(A+T) \\ \vdots & & \vdots \\ -\zeta(A-l+2) & \cdots & -\zeta(A-l+1+T) \end{pmatrix}}_{I}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B+l+T) & \cdots & -\zeta(A-l-1) \\ \vdots & & \vdots \\ \zeta(A-l-2+T) & \cdots & -\zeta(B+l+1) \end{pmatrix}}_{II}$$

$$\times \underbrace{\begin{pmatrix} \zeta(A-l-1+T) & \cdots & \zeta(A-l) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+1) \end{pmatrix}}_{III}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

We can interchange (I) with (II) and (III). Note

$$(I) \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta))$$

is irreducible (see Proposition 4.6), so we can also take dual of (I) (see (4.1)). Moreover, $(*-1)$ and $(*-2)$ are interchangeable. Therefore,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2}$$

$$\times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix}}_{*-1}$$

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$$\times \underbrace{\begin{pmatrix} \zeta(B+l+T) & \cdots & -\zeta(A-l-1) \\ \vdots & & \vdots \\ \zeta(A-l-2+T) & \cdots & -\zeta(B+l+1) \end{pmatrix}}_{II}$$

$$\times \underbrace{\begin{pmatrix} \zeta(A-l-1+T) & \cdots & \zeta(A-l) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+1) \end{pmatrix}}_{III}$$

$$\times \underbrace{\begin{pmatrix} \zeta(A-l+1+T) & \cdots & \zeta(A-l+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}}_{(I)^\vee}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

We can 'combine' (*II*) with (*III*), for otherwise $\mathrm{Jac}_{\zeta(A-l+1+T)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \neq 0$, which is impossible. Here we have used the fact $A-l > B+l+1$, in order to switch $\rho||^{\zeta(A-l+1+T)}$ with $(*-2)$. For the same kind of reason, we can 'combine' (*III*) with $(I)^\vee$. Consequently,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2}$$

$$\times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix}}_{*-1}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B+l+T) & \cdots & \zeta(B+l+1) & \cdots & -\zeta(A-l-1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-2+T) & \cdots & \zeta(A-l-1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+1) & & \end{pmatrix}}_{IV}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

We can further 'combine' $(*-1)$ with (*IV*), for otherwise $\mathrm{Jac}_{\zeta(A-l)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \neq 0$, which is again impossible. Here we have used the fact that

$$\rho||^{-\zeta(A-l)} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)) \quad \text{(A 2)}$$

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is irreducible (we prove it in end of this case), and $A-l > B+l+1$. As a result,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2}$$

$$\times \underbrace{\begin{pmatrix} & & \zeta B & \cdots & -\zeta A \\ & & \vdots & & \vdots \\ & & \zeta(B+l) & \cdots & -\zeta(A-l) \\ \zeta(B+l+T) & \cdots & \zeta(B+l+1) & \cdots & -\zeta(A-l-1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-2+T) & \cdots & \zeta(A-l-1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+1) & & \end{pmatrix}}_{(*-1)+IV}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

Hence

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$
$$\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix}$$
$$\times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l-1) & \cdots & -\zeta(B+l+1) \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

If we apply $\mathrm{Jac}_{(\rho,A+1,B+1,\zeta)\mapsto(\rho,A,B,\zeta)}$ to the full-induced representation above, we should get zero. This means $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = 0$.
To complete the discussion of this case, we still need to show (A 2) is irreducible. We use the criterion of Lemma 4.5. Since

$$\mathrm{Jac}_{-\zeta(A-l)}\ (\mathrm{A}\ 2) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)),$$

we see (A 2) has a unique subrepresentation σ, and σ is multiplicity free in $s.s.$ (A 2).
Since

$$\mathrm{Jac}_{\zeta(A-l)}\ (\mathrm{A}\ 2) = \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)),$$

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it suffices to show $\mathrm{Jac}_{\zeta(A-l)}\sigma \neq 0$. Note $A-l-2 > B+l-1$, so

$$\begin{aligned}\sigma &\hookrightarrow \rho||^{-\zeta(A-l)} \times \rho||^{\zeta(A-l-1)} \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l, 0, (-1)^{A_2-B_2}\eta_2, \zeta), \\ &\qquad (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)) \\ &\cong \rho||^{\zeta(A-l-1)} \times \rho||^{-\zeta(A-l)} \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l, 0, (-1)^{A_2-B_2}\eta_2, \zeta), \\ &\qquad (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)) \\ &\cong \rho||^{\zeta(A-l-1)} \times \rho||^{\zeta(A-l)} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l, 0, (-1)^{A_2-B_2}\eta_2, \zeta), \\ &\qquad (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).\end{aligned}$$

The last isomorphism does not follow from Lemma 4.4 exactly, but one can prove it using the same argument there together with [5, Proposition 2.7]. If

$$\begin{aligned}\sigma &\hookrightarrow \langle \zeta(A-l), \zeta(A-l-1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l, 0, (-1)^{A_2-B_2}\eta_2, \zeta), \\ &\qquad (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)),\end{aligned}$$

then it is clear that $\mathrm{Jac}_{\zeta(A-l)}\sigma \neq 0$. Otherwise, we have

$$\begin{aligned}\sigma &\hookrightarrow \langle \zeta(A-l-1), \zeta(A-l)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ &\qquad (\rho, A-l, A-l, 0, (-1)^{A_2-B_2}\eta_2, \zeta), \\ &\qquad (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)) \\ &\hookrightarrow \langle \zeta(A-l-1), \zeta(A-l)\rangle \times \rho||^{\zeta(A-l)} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ &\qquad (\rho, A-l-1, A-l-1, 0, (-1)^{A_2-B_2}\eta_2, \zeta), \\ &\qquad (\rho, A-l-2, A-l-2, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)) \\ &\cong \rho||^{\zeta(A-l)} \times \langle \zeta(A-l-1), \zeta(A-l)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; \\ &\qquad (\rho, A-l-1, A-l-2, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).\end{aligned}$$

So we again have $\mathrm{Jac}_{\zeta(A-l)}\sigma \neq 0$. This finishes the proof.

(2) $A-l = B+l+1$.
Following the previous discussion, we find (*II*) is 'missing', but we can still 'combine' (*III*) and $(I)^\vee$.

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1)\end{pmatrix}}_{*-2}$$

$$\times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l)\end{pmatrix}}_{*-1}$$

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$$\times \underbrace{\begin{pmatrix} \zeta(A-l-1+T) & \cdots & \zeta(A-l) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}}_{IV}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

Hence

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$
$$\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix}$$
$$\times \begin{pmatrix} \zeta(A-l) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

We claim the induced representation above has a unique irreducible subrepresentation. It is clear that for any irreducible subrepresentation σ, one has

$$\sigma \hookrightarrow \langle \zeta B, \ldots -\zeta A\rangle \times \langle \zeta(B+1), \ldots, -\zeta A\rangle \times \cdots \times \langle \zeta(B+l-1), \ldots, -\zeta(A-l+1)\rangle$$
$$\times \langle \zeta(B+l), \ldots, -\zeta(A-l+1)\rangle \times \langle \zeta(B+l), \ldots, -\zeta(A-l)\rangle$$
$$\times \begin{pmatrix} \zeta(A-l) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

So the corresponding Jacquet module of σ under

$$\begin{aligned} \mathrm{Jac}_X := \ & \mathrm{Jac}_{\zeta(A-l),\ldots,\zeta(A+1)} \circ \mathrm{Jac}_{\zeta(B+l),\ldots,-\zeta(A-l)} \circ \mathrm{Jac}_{\zeta(B+l),\ldots,-\zeta(A-l+1)} \circ \\ & \mathrm{Jac}_{\zeta(B+l-1),\ldots,-\zeta(A-l+1)} \circ \cdots \circ \mathrm{Jac}_{\zeta(B+1),\ldots,-\zeta A} \circ \mathrm{Jac}_{\zeta B,\ldots,-\zeta A} \end{aligned}$$

contains the irreducible representation $\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta))$. On the other hand, we can also apply Jac_X to the full-induced representation

$$\begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \end{pmatrix}$$
$$\times \begin{pmatrix} \zeta(A-l) \\ \vdots \\ \zeta(A+1) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)),$$

and we get $\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta))$. This proves our claim. As a result,

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$

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$$\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l) \\ \zeta(A-l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A-l, A-l-1, 0, (-1)^{A_2-B_2-1}\eta_2, \zeta)).$$

Therefore, if we apply $\mathrm{Jac}_{(\rho,A+1,B+1,\zeta)\mapsto(\rho,A,B,\zeta)}$ to the full-induced representation above, we should get zero. This means $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = 0$.

A.2.2. Case: $l_2 = l_1 + 1$. Let us denote l_1 by l. Since $A - l_2 + 1 > B + l_2 - 1$, then $A - l > B + l$.

(1) $l \neq 0$ and $A - l > B + l + 1$.

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1) \end{pmatrix}$$
$$\times \begin{pmatrix} \zeta(B_2+T) & \cdots & -\zeta(A_2+T) \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1+T) & \cdots & -\zeta(A_2-l_2+1+T) \end{pmatrix}$$
$$\times \begin{pmatrix} \zeta(B_2+l_2+T) & \cdots & -\zeta(A_1-l_1) \\ \vdots & & \vdots \\ \zeta(A_2-l_2+T) & \cdots & -\zeta(B_1+l_1+2) \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B_1+l_1+1, B_1+l_1, 0, \eta_1, \zeta))$$
$$\hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$
$$\times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots \\ \zeta(B+l+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{I}$$
$$\times \underbrace{\begin{pmatrix} -\zeta(A+2) & \cdots & -\zeta(A+T) \\ \vdots & & \vdots \\ -\zeta(A-l+2) & \cdots & -\zeta(A-l+T) \end{pmatrix}}_{II}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B+l+1+T) & \cdots & -\zeta(A-l) \\ \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & -\zeta(B+l+2) \end{pmatrix}}_{III}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)).$$

We first interchange (*II*) and (*III*), then take dual of (*II*) (see (4.1)).

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$
$$\times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots \\ \zeta(B+l+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{I}$$
$$\times \underbrace{\begin{pmatrix} \zeta(B+l+1+T) & \cdots & -\zeta(A-l) \\ \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & -\zeta(B+l+2) \end{pmatrix}}_{III}$$
$$\times \underbrace{\begin{pmatrix} \zeta(A-l+T) & \cdots & \zeta(A-l+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix}}_{(II)^\vee}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)).$$

Since $\mathrm{Jac}_{\zeta(B+l+1+T)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = 0$, we can 'combine' (*I*) and (*III*). For the same kind of reason, we can further 'combine' them with $(II)^\vee$. So

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$

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$$\times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+2) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{*-2}$$

$$\rtimes\, \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta))$$

$$\hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-}$$

$$\times \underbrace{\begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+2) \end{pmatrix}}_{IV} \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{V}$$

$$\rtimes\, \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).$$

We interchange $(*-1)$ and $(*-2)_-$, also (IV) and (V). Then we take the dual of (IV).

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-}$$

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$$\times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{V} \times \underbrace{\begin{pmatrix} \zeta(B+l+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{(IV)^\vee}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).$$

We can 'combine' $(*-1)$ and (V) for $\mathrm{Jac}_{\zeta(B+l)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = 0$. We can also 'combine' (V) and $(IV)^\vee$ for $\mathrm{Jac}_{\zeta(B+l+2)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = 0$. Here we have used the fact that $A-l > B+l+1$. So

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+1) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-}$$

$$\times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix}}_{(*-1)_+}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).$$

Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$

$$\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix}$$

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$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).$$

Therefore, if we apply $\mathrm{Jac}_{(\rho,A+1,B+1,\zeta)\mapsto(\rho,A,B,\zeta)}$ to the full-induced representation above, we should get zero. This means $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = 0$.

(2) $l \neq 0$ and $A - l = B + l + 1$.
It follows from the previous discussion that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$

$$\times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-l-1+T) & \cdots & \zeta(A-l+1) & \cdots & -\zeta(B+l+2) \\ \zeta(A-l+T) & \cdots & \zeta(A-l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{*-2}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)).$$

Since $(*-1)$ and $(*-2)$ are interchangeable, then we have

$$\begin{aligned} &\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\ &\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+2) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix} \\ &\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)). \end{aligned}$$

It follows

$$\begin{aligned} &\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\ &\hookrightarrow \langle \zeta B, \ldots - \zeta A \rangle \times \langle \zeta(B+1), \ldots, -\zeta(A+1) \rangle \\ &\quad \times \cdots \times \langle \zeta(B+l-1), \ldots, -\zeta(A-l+1) \rangle \\ &\quad \times \langle \zeta(A-l-1), \ldots, -\zeta(B+l+3) \rangle \times \langle \zeta(A-l), \ldots, -\zeta(B+l+2) \rangle \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)). \end{aligned}$$

Therefore,

$$\mathrm{Jac}_X \, \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \neq 0, \tag{A 3}$$

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where

$$\begin{aligned}\mathrm{Jac}_X :=& \mathrm{Jac}_{\zeta(A-l),\ldots,-\zeta(B+l+2)} \circ \mathrm{Jac}_{\zeta(A-l-1),\ldots,-\zeta(B+l+3)} \circ \\ & \mathrm{Jac}_{\zeta(B+l-1),\ldots,-\zeta(A-l+1)} \circ \cdots \circ \mathrm{Jac}_{\zeta(B+1),\ldots,-\zeta(A+1)} \circ \mathrm{Jac}_{\zeta B,\ldots,-\zeta A}.\end{aligned}$$

On the other hand, we can rewrite

$$\begin{aligned}&\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\ &\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \underbrace{\begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+2) \end{pmatrix}}_{IV} \\ &\quad \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix} \\ &\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)) \\ &\cong \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix} \\ &\quad \times \underbrace{\begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+2) \end{pmatrix}}_{IV} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)).\end{aligned}$$

So there exits an irreducible constituent σ of

$$(IV) \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)),$$

such that

$$\begin{aligned}&\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) \\ &\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix} \rtimes \sigma.\end{aligned}$$

We claim $\mathrm{Jac}_{\zeta(B+l+2)}\sigma = 0$. Otherwise, we necessarily have

$$\mathrm{Jac}_{\zeta(B+l+2)}\sigma = \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta))$$

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which is irreducible by Proposition 4.6. Then

$$\sigma \hookrightarrow \rho||^{\zeta(B+l+2)} \times \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta))$$
$$\cong \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \times \rho||^{\zeta(B+l+2)}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+1, B+l, 0, \eta_1, \zeta)).$$

It is necessary that

$$\sigma \hookrightarrow \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta),$$
$$(\rho, B+l, B+l, 0, \eta_1, \zeta)).$$

In particular

$$\mathrm{Jac}_{-\zeta(B+l+2)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta),$$
$$(\rho, B+l, B+l, 0, \eta_1, \zeta)) = 0.$$

Now we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$
$$\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}$$
$$\times \begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+l+3) \end{pmatrix} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta),$$
$$(\rho, B+l, B+l, 0, \eta_1, \zeta)).$$

If we apply

$$\mathrm{Jac}_{X'} := \mathrm{Jac}_{\zeta(A-l-1),\ldots,-\zeta(B+l+3)} \circ \mathrm{Jac}_{\zeta(B+l-1),\ldots,-\zeta(A-l+1)} \circ \cdots \circ$$
$$\mathrm{Jac}_{\zeta(B+1),\ldots,-\zeta(A+1)} \circ \mathrm{Jac}_{\zeta B,\ldots,-\zeta A},$$

to the full-induced representation above, we get

$$\langle \zeta(A-l), \ldots, -\zeta(B+l+1) \rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta),$$
$$(\rho, B+l, B+l, 0, \eta_1, \zeta)).$$

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Here we have used the fact that $A-l=B+l+1$. Note

$$\mathrm{Jac}_X = \mathrm{Jac}_{\zeta(A-l),\ldots,-\zeta(B+l+2)} \circ \mathrm{Jac}_{X'}$$

and

$$\mathrm{Jac}_{\zeta(A-l),\ldots,-\zeta(B+l+2)} \langle \zeta(A-l), \ldots, -\zeta(B+l+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l+2, B+l+2, 0, -\eta_1, \zeta), (\rho, B+l, B+l, 0, \eta_1, \zeta)) = 0.$$

This contradicts to (A 3). So we have shown our claim.
For $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta})$ being nonzero, there necessarily exits $x \in [B+1, A+1]$ such that $\mathrm{Jac}_{\zeta x,\ldots,\zeta(A+1)}\sigma \neq 0$. By our claim, $\mathrm{Jac}_{\zeta x}\sigma \neq 0$ implies $x = B+l$. It follows

$$\sigma \hookrightarrow \langle \zeta(B+l), \ldots, \zeta(A+1)\rangle \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).$$

Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$
$$\begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$
$$\times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{VI} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).$$

Since

$$\mathrm{Jac}_{\zeta(B+l),\zeta(B+1)} \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta)) = 0,$$

we can 'combine' $(*-1)$ and (VI). So

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$
$$\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(A-l) & \cdots & -\zeta(B+l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l-1, 0, \eta_1, \zeta)).$$

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If we apply $\mathrm{Jac}_{(\rho,A+1,B+1,\zeta)\mapsto(\rho,A,B,\zeta)}$ to the full-induced representation above, we should get zero. This means $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})=0$.

(3) $l=0$ and $A>B+1$.
From the previous discussion in (1), we have

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(A+1) \\ \vdots & & \vdots & & \vdots \\ \zeta(A-1+T) & \cdots & \zeta(A+1) & \cdots & -\zeta(B+2) \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)}$$

$$\rtimes\, \pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+1,B,0,\eta_1,\zeta))$$

$$\hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-1+T) & \cdots & \zeta(A+1) & \cdots & -\zeta(B+1) \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-}$$

$$\times \underbrace{\begin{pmatrix} -\zeta(A+1) \\ \vdots \\ -\zeta(B+2) \end{pmatrix}}_{IV} \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+1,B,0,\eta_1,\zeta)).$$

There exists an irreducible constituent σ of

$$(IV)\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+1,B,0,\eta_1,\zeta))$$

such that

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \vdots \\ \zeta(A-1+T) & \cdots & \zeta(A+1) & \cdots & -\zeta(B+1) \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_-} \rtimes \sigma.$$

We claim $\mathrm{Jac}_x\sigma=0$ for $x\in[\zeta(B+1),\zeta(A+1)]$. This is clear when $x\neq\zeta(B+2)$. If $\mathrm{Jac}_{\zeta(B+2)}\sigma\neq 0$, then $\mathrm{Jac}_{\zeta(B+2)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta})\neq 0$, and we get a contradiction. Here we have used the fact that $A>B+1$. Note

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2+1,B_2+1,l_2,\eta_2,\zeta),(\rho,A_1,B_1,l_1,\eta_1,\zeta))$$

$$\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta A & \cdots & -\zeta(B+1) \end{pmatrix} \rtimes \sigma.$$

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If we apply $\mathrm{Jac}_{(\rho,A+1,B+1,\zeta)\mapsto(\rho,A,B,\zeta)}$ to the full-induced representation above, we should get zero. This means $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta})=0$.

(4) $l=0$ and $A=B+1$.
We can further simplify from the previous case (3) that

$$\begin{aligned}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) &\hookrightarrow \underbrace{\begin{pmatrix}\zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(A+1)\\ \zeta(A+T) & \cdots & \zeta(A+2)\end{pmatrix}}_{(*-2)}\\ &\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+1,B,0,\eta_1,\zeta))\\ &\hookrightarrow \underbrace{\begin{pmatrix}\zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta(B+1)\\ \zeta(B+1+T) & \cdots & \zeta(B+3)\end{pmatrix}}_{(*-2)_-}\\ &\times \rho||^{-\zeta(B+2)}\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+1,B,0,\eta_1,\zeta)).\end{aligned}$$

So

$$\begin{aligned}&\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2+1,B_2+1,l_2,\eta_2,\zeta),(\rho,A_1,B_1,l_1,\eta_1,\zeta))\\ &\quad\hookrightarrow \langle\zeta(B+1),\ldots,-\zeta(B+1)\rangle\times\rho||^{-\zeta(B+2)}\\ &\qquad\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+1,B,0,\eta_1,\zeta)).\end{aligned}$$

There exists an irreducible constituent σ of

$$\rho||^{-\zeta(B+2)}\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+1,B,0,\eta_1,\zeta))$$

such that

$$\begin{aligned}&\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,A_2+1,B_2+1,l_2,\eta_2,\zeta),(\rho,A_1,B_1,l_1,\eta_1,\zeta))\\ &\quad\hookrightarrow\langle\zeta(B+1),\ldots,-\zeta(B+1)\rangle\rtimes\sigma.\end{aligned}$$

We claim $\mathrm{Jac}_x\sigma=0$ for $x\in[\zeta(B+1),\zeta(B+2)]$. It is clear $\mathrm{Jac}_{\zeta(B+1)}\sigma=0$. Suppose $\mathrm{Jac}_{\zeta(B+2)}\sigma\neq 0$, then

$$\sigma\hookrightarrow\rho||^{\zeta(B+2)|}\rtimes\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+1,B,0,\eta_1,\zeta))$$

as the unique irreducible subrepresentation. So

$$\sigma=\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+2,B+2,-\eta_1,\zeta),(\rho,B,B,\eta_1,\zeta)).$$

This implies $\mathrm{Jac}_{-\zeta(B+2)}\sigma=0$. In particular,

$$\mathrm{Jac}_{\zeta(B+1),\ldots,-\zeta(B+2)}(\langle\zeta(B+1),\ldots,-\zeta(B+1)\rangle\rtimes\sigma)=0.$$

On the other hand,

$$\begin{aligned}\mathrm{Jac}_{\zeta(B+1),\ldots,-\zeta(B+2)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;&(\rho,A_2+1,B_2+1,l_2,\eta_2,\zeta),\\ &(\rho,A_1,B_1,l_1,\eta_1,\zeta))\neq 0.\end{aligned}$$

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So we get a contradiction. As a result,

$$\mathrm{Jac}_{\zeta(B+1),\zeta(B+2)}(\langle\zeta(B+1),\ldots,-\zeta(B+1)\rangle \rtimes \sigma) = 0,$$

and hence $\pi^{\Sigma_0}_{M,>_\psi}(\psi,\underline{l},\underline{\eta}) = 0$.

A.2.3. Case: $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$ and $l_1 = l_2 = (A_1-B_1)/2$. Let $l = l_1 = l_2 \neq 0$, then $A-l = B+l$.

$$\begin{aligned}
\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg,\underline{l},\underline{\eta}) &\hookrightarrow \begin{pmatrix} \zeta B_1 & \cdots & -\zeta A_1 \\ \vdots & & \vdots \\ \zeta(B_1+l_1-1) & \cdots & -\zeta(A_1-l_1+1) \end{pmatrix} \\
&\quad \times \begin{pmatrix} \zeta(B_2+T) & \cdots & -\zeta(A_2+T) \\ \vdots & & \vdots \\ \zeta(B_2+l_2-1+T) & \cdots & -\zeta(A_2-l_2+1+T) \end{pmatrix} \\
&\quad \times \langle\zeta(B_2+l_2+T),\ldots,\zeta(B_2+l_2+2)\rangle \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B_2+l_2+1,B_2+l_2+1,\eta_2,\zeta), \\
&\qquad (\rho,B_1+l_1,B_1+l_1,\eta_1,\zeta)) \\
&\hookrightarrow \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \\
&\quad \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \\
&\quad \times \underbrace{\begin{pmatrix} -\zeta(A+1) & \cdots & -\zeta(A+T) \\ \vdots & & \vdots \\ -\zeta(A-l+2) & \cdots & -\zeta(A-l+1+T) \end{pmatrix}}_{I} \\
&\quad \times \underbrace{\langle\zeta(B+l+T),\ldots\zeta(B+l+2)\rangle}_{II} \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{III} \\
&\quad \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+l,B+l,-\eta_1,\zeta), \\
&\qquad (\rho,B+l-1,B+l-1,\eta_1,\zeta)).
\end{aligned}$$

We interchange (I) with (II) and (III), then we take dual of (I).

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$$
\begin{aligned}
\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg}, \underline{l}, \underline{\eta}) \hookrightarrow\ & \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \\
& \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \\
& \times \underbrace{\langle \zeta(B+l+T), \dots \zeta(B+l+2) \rangle}_{II} \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{III} \\
& \times \underbrace{\begin{pmatrix} \zeta(A-l+1+T) & \cdots & \zeta(A-l+2) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+1) \end{pmatrix}}_{(I)^\vee} \\
& \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), \\
& \quad (\rho, B+l-1, B+l-1, \eta_1, \zeta)) \\
\hookrightarrow\ & \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1} \\
& \times \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2} \\
& \times \underbrace{\langle \zeta(B+l+T), \dots \zeta(B+l+2) \rangle}_{II} \times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{III} \\
& \times \underbrace{\begin{pmatrix} \zeta(A-l+1+T) & \cdots & \zeta(A-l+3) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix}}_{(I)^\vee_-} \times \underbrace{\begin{pmatrix} \zeta(A-l+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{IV} \\
& \rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), \\
& \quad (\rho, B+l-1, B+l-1, \eta_1, \zeta)).
\end{aligned}
$$

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Note $(*-1)$ is interchangeable with $(*-2)$, (II) and $(I)^{\vee}_{-}$. And $(I)^{\vee}_{-}$ is also interchangeable with (III). So

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1+T) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-2}$$
$$\times \underbrace{\langle \zeta(B+l+T),\ldots\zeta(B+l+2)\rangle}_{II}$$
$$\times \underbrace{\begin{pmatrix} \zeta(A-l+1+T) & \cdots & \zeta(A-l+3) \\ \vdots & & \vdots \\ \zeta(A+T) & \cdots & \zeta(A+2) \end{pmatrix}}_{(I)^{\vee}_{-}}$$
$$\times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$
$$\times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{III} \times \underbrace{\begin{pmatrix} \zeta(A-l+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{IV}$$
$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-,\underline{l}_-,\underline{\eta}_-;(\rho,B+l,B+l,-\eta_1,\zeta),$$
$$(\rho,B+l-1,B+l-1,\eta_1,\zeta)).$$

Since $\mathrm{Jac}_{\zeta(B+l+T)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) = \mathrm{Jac}_{\zeta(A-l+1+T)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) = 0$, we can 'combine' $(*-2)$, (II) and $(I)^{\vee}_{-}$.

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_{\gg},\underline{l},\underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \\ \zeta(B+l-1+T) & \cdots & \zeta(B+l+1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l+T) & \cdots & \zeta(B+l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_+}$$
$$\times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \end{pmatrix}}_{*-1}$$

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$$\times \underbrace{\begin{pmatrix} \zeta(B+l) \\ \zeta(B+l+1) \end{pmatrix}}_{III} \times \underbrace{\begin{pmatrix} \zeta(A-l+2) \\ \vdots \\ \zeta(A+1) \end{pmatrix}}_{IV}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta)).$$

Since $\mathrm{Jac}_{\zeta(B+l)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = \mathrm{Jac}_{\zeta(A-l+2)}\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) = 0$, we can 'combine' $(*-1)$, (III) and (IV).

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \underbrace{\begin{pmatrix} \zeta(B+T) & \cdots & \zeta(B+2) & \cdots & -\zeta A \\ \vdots & & \vdots & & \\ \zeta(B+l-1+T) & \cdots & \zeta(B+l+1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l+T) & \cdots & \zeta(B+l+2) & & \\ \vdots & & \vdots & & \\ \zeta(A+T) & \cdots & \zeta(A+2) & & \end{pmatrix}}_{(*-2)_+}$$

$$\times \underbrace{\begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix}}_{(*-1)_+}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta)).$$

Then

$$\pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, A_2+1, B_2+1, l_2, \eta_2, \zeta), (\rho, A_1, B_1, l_1, \eta_1, \zeta))$$

$$\hookrightarrow \begin{pmatrix} \zeta(B+1) & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l) & \cdots & -\zeta(A-l+1) \end{pmatrix} \times \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+l-1) & \cdots & -\zeta(A-l+1) \\ \zeta(B+l) & & \\ \vdots & & \\ \zeta(A+1) & & \end{pmatrix}$$

$$\rtimes \pi^{\Sigma_0}_{M,>_\psi}(\psi_-, \underline{l}_-, \underline{\eta}_-; (\rho, B+l, B+l, -\eta_1, \zeta), (\rho, B+l-1, B+l-1, \eta_1, \zeta)).$$

Therefore, if we apply $\mathrm{Jac}_{(\rho, A+1, B+1, \zeta)\mapsto(\rho, A, B, \zeta)}$ to the full-induced representation above, we should get zero. This means $\pi^{\Sigma_0}_{M,>_\psi}(\psi, \underline{l}, \underline{\eta}) = 0$.

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Appendix B. Example

In this appendix, we want to demonstrate how our procedure (cf. § 8) works in a simple example. We fix ρ and choose $\psi \in \bar{\Psi}(G)$, such that

$$\mathrm{Jord}(\psi) = \{(\rho, A_3, B_3, \zeta_3), (\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\}.$$

We also assume

- $A_i, B_i \in \mathbb{Z}$ for $i = 1, 2, 3$;
- $A_3 \geqslant A_2 \geqslant A_1$ and $B_3 \geqslant B_2 \geqslant B_1$;
- $\zeta_3 = \zeta_1 = +1$ and $\zeta_2 = -1$.

We put an order $>_\psi$ on $\mathrm{Jord}(\psi)$ such that

$$(\rho, A_3, B_3, \zeta_3) >_\psi (\rho, A_2, B_2, \zeta_2) >_\psi (\rho, A_1, B_1, \zeta_1).$$

We would like to find all $(\underline{l}, \underline{\eta})$ such that $\pi^{\Sigma_0}_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

B.1. Results

First of all, we have

$$0 \leqslant l_i \leqslant [(A_i - B_i + 1)/2],$$

and

$$\prod_{i=1}^{3} \eta_i^{A_i - B_i + 1} (-1)^{[(A_i - B_i + 1)/2] + l_i} = 1$$

(cf. (2.1)). Next, we formulate the necessary and sufficient conditions as linear constraints on $\{l_i\}$ for each selection of signs $\{\eta_i\}$.

(1) If $\eta_3 = (-1)^{(A_1 - B_1) + (A_2 - B_2)} \eta_1$ and $\eta_2 = (-1)^{A_1 - B_1} \eta_1$, then

$$\begin{cases} l_3 + l_1 > A_1 - B_3 \\ -B_2 \leqslant l_2 - l_1 \leqslant A_2 - (A_1 - B_1) \\ (A_1 - B_1) - B_3 + 1 \leqslant l_3 - l_2 + 2l_1 \leqslant A_3 + (A_1 - B_1) - (A_2 - B_2) + 1. \end{cases}$$

(2) If $\eta_3 = (-1)^{(A_1 - B_1) + (A_2 - B_2)} \eta_1$ and $\eta_2 \neq (-1)^{A_1 - B_1} \eta_1$, then

$$\begin{cases} l_3 + l_1 > A_1 - B_3 \\ l_1 + l_2 > (A_1 - B_1) - B_2 \\ l_3 + l_2 + 2l_1 > (A_1 - B_1) + (A_2 - B_2) - B_3 + 1. \end{cases}$$

(3) If $\eta_3 \neq (-1)^{(A_1 - B_1) + (A_2 - B_2)} \eta_1$, $\eta_2 = (-1)^{A_1 - B_1} \eta_1$, and

$$l_3 - l_1 < (A_3 - B_3)/2 - (A_1 - B_1) + l_1$$

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then

$$\begin{cases} -(B_3 - B_1) \leqslant l_3 - l_1 \leqslant A_3 - A_1 \\ -B_2 \leqslant l_2 - l_1 \leqslant A_2 - (A_1 - B_1) \\ l_3 + l_2 - 2l_1 > (A_2 - B_2) - (A_1 - B_1) - B_3 - 1. \end{cases}$$

(4) If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, $\eta_2 = (-1)^{A_1-B_1}\eta_1$, and

$$l_3 - l_1 \geqslant (A_3 - B_3)/2 - (A_1 - B_1) + l_1$$

then

$$\begin{cases} -(B_3 - B_1) \leqslant l_3 - l_1 \leqslant A_3 - A_1 \\ -B_2 \leqslant l_2 - l_1 \leqslant A_2 - (A_1 - B_1) \\ (A_1 - B_1) - A_3 \leqslant -l_3 - l_2 + 2l_1 \leqslant (A_1 - B_1) - (A_2 - B_2) + B_3. \end{cases}$$

(5) If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, and

$$l_3 - l_1 < (A_3 - B_3)/2 - (A_1 - B_1) + l_1$$

then

$$\begin{cases} -(B_3 - B_1) \leqslant l_3 - l_1 \leqslant A_3 - A_1 \\ l_1 + l_2 > (A_1 - B_1) - B_2 \\ -(A_1 - B_1) - B_3 - 1 \leqslant l_3 - l_2 - 2l_1 \leqslant A_3 - (A_1 - B_1) - (A_2 - B_2) - 1. \end{cases}$$

(6) If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, and

$$l_3 - l_1 \geqslant (A_3 - B_3)/2 - (A_1 - B_1) + l_1$$

then

$$\begin{cases} -(B_3 - B_1) \leqslant l_3 - l_1 \leqslant A_3 - A_1 \\ l_1 + l_2 > (A_1 - B_1) - B_2 \\ -l_3 + l_2 + 2l_1 > (A_1 - B_1) + (A_2 - B_2) - A_3. \end{cases}$$

Example B.1. Let $[A_3, B_3] = [40, 10]$, $[A_2, B_2] = [37, 7]$ and $[A_1, B_1] = [8, 4]$, i.e.,

$$\psi = \rho \otimes \nu_{51} \otimes \nu_{31} \oplus \rho \otimes \nu_{31} \otimes \nu_{45} \oplus \rho \otimes \nu_{13} \otimes \nu_5.$$

First, we have $0 \leqslant l_1 \leqslant 2, 0 \leqslant l_2 \leqslant 15, 0 \leqslant l_3 \leqslant 15$, and $(-1)^{l_1+l_2+l_3}\eta_1\eta_2\eta_3 = 1$. Note $B_3 > A_1$, so the conditions from (B 1) are always satisfied. Also note $B_2 > A_1 - B_1$, then the conditions from (B 2) are always satisfied too. Therefore, we can simplify the nonvanishing conditions as follows:

(1) If $\eta_3 = \eta_1$ and $\eta_2 = \eta_1$, then

$$-5 \leqslant l_3 - l_2 + 2l_1 \leqslant 15.$$

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(2) If $\eta_3 = \eta_1$ and $\eta_2 \neq \eta_1$, then

$$l_3 + l_2 + 2l_1 > 25.$$

(3) If $\eta_3 \neq \eta_1$, $\eta_2 = \eta_1$, and

$$l_3 - l_1 < 11 + l_1$$

then

$$l_3 + l_2 - 2l_1 > 15.$$

(4) If $\eta_3 \neq \eta_1$, $\eta_2 = \eta_1$, and

$$l_3 - l_1 \geqslant 11 + l_1$$

then

$$-36 \leqslant -l_3 - l_2 + 2l_1 \leqslant -16.$$

(5) If $\eta_3 \neq \eta_1$, $\eta_2 \neq \eta_1$, and

$$l_3 - l_1 < 11 + l_1$$

then

$$-15 \leqslant l_3 - l_2 - 2l_1 \leqslant 5.$$

(6) If $\eta_3 \neq \eta_1$, $\eta_2 \neq \eta_1$, and

$$l_3 - l_1 \geqslant 11 + l_1$$

then

$$-l_3 + l_2 + 2l_1 > -6.$$

To find the size of $\Pi_\psi^{\Sigma_0}$ is equivalent to counting integral points in certain polytopes for each of the above six cases. By running a simple computer program, we can get $|\Pi_\psi^{\Sigma_0}| = 1651$.

B.2. Deduction

First, we 'Expand' $[A_3, B_3]$ to $[A_3^*, B_3^*]$ such that $B_3^* = B_1$, and we denote the resulting parameter by ψ^*. Then $\pi_{>_\psi}^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \pi_{>_\psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*)$, where

$$l_1^* = l_1, \quad l_2^* = l_2, \quad l_3^* = l_3 + (B_3 - B_1),$$

and

$$\eta_1^* = \eta_1, \quad \eta_2^* = \eta_2, \quad \eta_3^* = \eta_3.$$

Next, we change the order $>_\psi$ to $>'_\psi$:

$$(\rho, A_3, B_3, \zeta_3) >'_\psi (\rho, A_1, B_1, \zeta_1) >'_\psi (\rho, A_2, B_2, \zeta_2).$$

So $\pi_{>_\psi}^{\Sigma_0}(\psi^*, \underline{l}^*, \underline{\eta}^*) = \pi_{>'_\psi}^{\Sigma_0}(\psi^*, \underline{l}'^*, \underline{\eta}'^*)$, where

$$l_1'^* = l_1^*, \quad l_2'^* = l_2^*, \quad l_3'^* = l_3^*,$$

and

$$\eta_1'^* = (-1)^{A_2 - B_2 + 1}\eta_1^*, \quad \eta_2'^* = (-1)^{A_1 - B_1 + 1}\eta_2^*, \quad \eta_3'^* = \eta_3^*.$$

Then we can 'Pull' $[A_3^*, B_3^*]$, $[A_1, B_1]$. As a consequence, we have $\pi_{>'_\psi}^{\Sigma_0}(\psi^*, \underline{l}'^*, \underline{\eta}'^*) \neq 0$ if and only if all the three conditions below are satisfied.

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(1) $\pi^{\Sigma_0}_{>'_{\psi}}(\psi_1^*, \underline{l}'^*, \underline{\eta}'^*) \neq 0$,

(2) $\pi^{\Sigma_0}_{>'_{\psi}}(\psi_2^*, \underline{l}'^*, \underline{\eta}'^*) \neq 0$,

(3) $\pi^{\Sigma_0}_{>''_{\psi}}(\psi_3^*, \underline{l}''^*, \underline{\eta}''^*) \neq 0$.

From each of these cases, we get some explicit conditions on $(\underline{l}, \underline{\eta})$.

Case (1): ψ_1^* is obtained from ψ^* by shifting both $[A_3^*, B_3^*]$ and $[A_1, B_1]$ away by T_1, so that $(\rho, A_1+T_1, B_1+T_1, \zeta_1) \gg (\rho, A_2, B_2, \zeta_2)$. Since $\mathrm{Jord}(\psi_1^*)$ is in 'good shape', we have $\pi^{\Sigma_0}_{>'_{\psi}}(\psi_1^*, \underline{l}'^*, \underline{\eta}'^*) \neq 0$ if and only if

$$\begin{cases} \eta_3'^* = (-1)^{A_1-B_1}\eta_1'^* & \Rightarrow 0 \leqslant l_3'^* - l_1'^* \leqslant (A_3^*-B_3^*)-(A_1-B_1), \\ \eta_3'^* \neq (-1)^{A_1-B_1}\eta_1'^* & \Rightarrow l_3'^* + l_1'^* > A_1 - B_1. \end{cases} \tag{B 1}$$

Now we want to translate these conditions to that on $(\underline{l}, \underline{\eta})$. Note

$$\eta_3'^* = \eta_3^* = \eta_3, \quad \eta_1'^* = (-1)^{A_2-B_2+1}\eta_1^* = (-1)^{A_2-B_2+1}\eta_1,$$

and

$$l_3'^* = l_3^* = l_3 + (B_3 - B_1), \quad l_1'^* = l_1^* = l_1.$$

So we get the following conditions from (B 1).

- If $\eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, then

$$0 \leqslant l_3 + (B_3 - B_1) - l_1 \leqslant (A_3 + (B_3 - B_1) - B_1) - (A_1 - B_1),$$

  which implies

$$-(B_3 - B_1) \leqslant l_3 - l_1 \leqslant A_3 - A_1.$$

- If $\eta_3 = (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$, then

$$l_3 + (B_3 - B_1) + l_1 > A_1 - B_1,$$

  which implies

$$l_3 + l_1 > A_1 - B_3.$$

Case (2): ψ_2^* is obtained from ψ^* by shifting $[A_3^*, B_3^*]$ away by T_2, so that $(\rho, A_3^*+T_2, B_3^*+T_2, \zeta_3) \gg \{(\rho, A_2, B_2, \zeta_2), (\rho, A_1, B_1, \zeta_1)\}$. Note $\pi^{\Sigma_0}_{>'_{\psi}}(\psi_2^*, \underline{l}'^*, \underline{\eta}'^*) = \pi^{\Sigma_0}_{>_{\psi}}(\psi_2^*, \underline{l}^*, \underline{\eta}^*)$. So we can 'Expand' $[A_2, B_2]$ to $[A_2^*, B_2^*]$ such that $B_2^* = 0$, and we denote the resulting parameter by ψ_2^{**}. Then $\pi^{\Sigma_0}_{>_{\psi}}(\psi_2^*, \underline{l}^*, \underline{\eta}^*) = \pi^{\Sigma_0}_{>_{\psi}}(\psi_2^{**}, \underline{l}^{**}, \underline{\eta}^{**})$, where

$$l_1^{**} = l_1^*, \quad l_2^{**} = l_2^* + B_2, \quad l_3^{**} = l_3^*,$$

and

$$\eta_1^{**} = \eta_1^*, \quad \eta_2^{**} = \eta_2^*, \quad \eta_3^{**} = \eta_3^*.$$

Finally, we can change the order again

$$\pi^{\Sigma_0}_{>_{\psi}}(\psi_2^{**}, \underline{l}^{**}, \underline{\eta}^{**}) = \pi^{\Sigma_0}_{>'_{\psi}}(\psi_2^{**}, \underline{l}'^{**}, \underline{\eta}'^{**}),$$

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where

$$l_1^{'**} = l_1^{**}, \quad l_2^{'**} = l_2^{**}, \quad l_3^{'**} = l_3^{**},$$

and

$$\eta_1^{'**} = (-1)^{A_2-B_2+1}\eta_1^{**}, \quad \eta_2^{'**} = (-1)^{A_1-B_1+1}\eta_2^{**}, \quad \eta_3^{'**} = \eta_3^{**}.$$

Then

$$\pi^{\Sigma_0}_{>'_\psi}(\psi_2^{**}, \underline{l}^{'**}, \underline{\eta}^{'**}) = \pi^{\Sigma_0}_{>'_\psi}(\psi_{2-}^{**}, \underline{l}_-^{'**}, \underline{\eta}_-^{'**}; (\rho, A_2^*, B_2^*, \eta_2^{'**}, -\zeta_2)) \neq 0,$$

if and only if

$$\begin{cases} \eta_1^{'**} = (-1)^{A_2-B_2}\eta_2^{'**} & \Rightarrow 0 \leqslant l_2^{'**} - l_1^{'**} \leqslant (A_2^* - B_2^*) - (A_1 - B_1), \\ \eta_1^{'**} \neq (-1)^{A_2-B_2}\eta_2^{'**} & \Rightarrow l_2^{'**} + l_1^{'**} > A_1 - B_1. \end{cases} \tag{B 2}$$

Now we want to translate these conditions to that on $(\underline{l}, \underline{\eta})$. Note

$$\eta_1^{'**} = (-1)^{A_2-B_2+1}\eta_1^{**} = (-1)^{A_2-B_2+1}\eta_1^{*} = (-1)^{A_2-B_2+1}\eta_1$$
$$\eta_2^{'**} = (-1)^{A_1-B_1+1}\eta_2^{**} = (-1)^{A_1-B_1+1}\eta_2^{*} = (-1)^{A_1-B_1+1}\eta_2$$

and

$$l_2^{'**} = l_2^{**} = l_2^* + B_2 = l_2 + B_2, \qquad l_1^{'**} = l_1^{**} = l_1^* = l_1.$$

So we get the following conditions from (B 2).

- If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, then

  $$0 \leqslant l_2 + B_2 - l_1 \leqslant (A_2 + B_2 - 0) - (A_1 - B_1),$$

  which implies

  $$-B_2 \leqslant l_2 - l_1 \leqslant A_2 - (A_1 - B_1).$$

- If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

  $$l_2 + B_2 + l_1 > A_1 - B_1,$$

  which implies

  $$l_2 + l_1 > (A_1 - B_1) - B_2.$$

Case (3): ψ_3^* is obtained from ψ^* by shifting $[A_1, B_1]$ away by T_3, so that $(\rho, A_1 + T_3, B_1 + T_3, \zeta_1) \gg \{(\rho, A_3^*, B_3^*, \zeta_3), (\rho, A_2, B_2, \zeta_2)\}$. The order $>''_\psi$ is given by

$$(\rho, A_1, B_1, \zeta_1) >''_\psi (\rho, A_3, B_3, \zeta_3) >''_\psi (\rho, A_2, B_2, \zeta_2).$$

And $\pi^{\Sigma_0}_{>'_\psi}(\psi^*, \underline{l}^{'*}, \underline{\eta}^{'*}) = \pi^{\Sigma_0}_{>''_\psi}(\psi^*, \underline{l}^{''*}, \underline{\eta}^{''*})$, where $(\underline{l}^{''*}, \underline{\eta}^{''*}) = S^+(\underline{l}^{'*}, \underline{\eta}^{'*})$. In particular,

$$\eta_2^{''*} = \eta_2^{'*}, \qquad l_2^{''*} = l_2^{'*}.$$

Then we can 'Expand' $[A_3^*, B_3^*]$ to $[A_3^{**}, B_3^{**}]$ such that $B_3^{**} = 0$, and we denote the resulting parameter by ψ_3^{**}. It follows $\pi^{\Sigma_0}_{>''_\psi}(\psi_3^*, \underline{l}^{''*}, \underline{\eta}^{''*}) = \pi^{\Sigma_0}_{>''_\psi}(\psi_3^*, \underline{l}^{''**}, \underline{\eta}^{''**})$, where

$$l_1^{''**} = l_1^{''*}, \quad l_2^{''**} = l_2^{''*}, \quad l_3^{''**} = l_3^{''*} + B_1,$$

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and

$$\eta_1''^{**} = \eta_1''^{*}, \quad \eta_2''^{**} = \eta_2''^{*}, \quad \eta_3''^{**} = \eta_3''^{*}.$$

We change the order $>''_\psi$ to $>'''_\psi$:

$$(\rho, A_1, B_1, \zeta_1) >'''_\psi (\rho, A_2, B_2, \zeta_2) >'''_\psi (\rho, A_3, B_3, \zeta_3),$$

then

$$\pi^{\Sigma_0}_{>''_\psi}(\psi_2^{**}, \underline{l}''^{**}, \underline{\eta}''^{**}) = \pi^{\Sigma_0}_{>'''_\psi}(\psi_2^{**}, \underline{l}'''^{**}, \underline{\eta}'''^{**}),$$

where

$$l_1'''^{**} = l_1''^{**}, \quad l_2'''^{**} = l_2''^{**}, \quad l_3'''^{**} = l_3''^{**},$$

and

$$\eta_1'''^{**} = \eta_1''^{**}, \quad \eta_2'''^{**} = (-1)^{A_3-B_3+1}\eta_2''^{**}, \quad \eta_3'''^{**} = (-1)^{A_2-B_2+1}\eta_3''^{**}.$$

Then

$$\pi^{\Sigma_0}_{>'''_\psi}(\psi_3^{**}, \underline{l}'''^{**}, \underline{\eta}'''^{**}) = \pi^{\Sigma_0}_{>'''_\psi}(\psi_{3-}^{**}, \underline{l}_-'''^{**}, \underline{\eta}_-'''^{**}; (\rho, A_3^{**}, B_3^{**}, \eta_3'''^{**}, -\zeta_3)) \neq 0,$$

if and only if

$$\begin{cases} \eta_2'''^{**} = (-1)^{A_3-B_3}\eta_3'''^{**} & \Rightarrow 0 \leqslant l_3'''^{**} - l_2'''^{**} \leqslant (A_3^{**} - B_3^{**}) - (A_2 - B_2), \\ \eta_2'''^{**} \neq (-1)^{A_3-B_3}\eta_3'''^{**} & \Rightarrow l_3'''^{**} + l_2'''^{**} > A_2 - B_2. \end{cases} \tag{B 3}$$

Now we want to translate these conditions to that on $(\underline{l}, \underline{\eta})$. Note

$$\begin{aligned} \eta_2'''^{**} &= (-1)^{A_3-B_3+1}\eta_2''^{**} = (-1)^{A_3-B_3+1}\eta_2''^{*} = (-1)^{A_3-B_3+1}\eta_2'^{*} \\ &= (-1)^{A_3-B_3+1}(-1)^{A_1-B_1+1}\eta_2^{*} = (-1)^{(A_3-B_3)+(A_1-B_1)}\eta_2 \\ \eta_3'''^{**} &= (-1)^{A_2-B_2+1}\eta_3''^{**} \end{aligned}$$

and

$$\begin{aligned} l_2'''^{**} &= l_2''^{**} = l_2''^{*} = l_2'^{*} = l_2^{*} = l_2 \\ l_3'''^{**} &= l_3''^{**} = l_3''^{*} + B_1. \end{aligned}$$

To proceed further, we need to use the formula for $(\underline{l}''^{*}, \underline{\eta}''^{*}) = S^+(\underline{l}'^{*}, \underline{\eta}'^{*})$.

- If $\eta_3'^{*} \neq (-1)^{A_1-B_1}\eta_1'^{*}$, then $\eta_1''^{*} = (-1)^{A_3-B_3}\eta_3''^{*}$ and

$$\begin{cases} l_1'^{*} = l_1''^{*} \\ l_3'^{*} - l_3''^{*} = (A_1 - B_1 - 2l_1'^{*}) + 1 \\ \eta_1''^{*} = (-1)^{A_3-B_3}\eta_1'^{*}. \end{cases}$$

It follows

$$\begin{aligned} &\eta_3'^{*} \neq (-1)^{A_1-B_1}\eta_1'^{*} \\ \Rightarrow\quad &\eta_3^{*} \neq (-1)^{A_1-B_1}(-1)^{A_2-B_2+1}\eta_1^{*} \end{aligned}$$

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$$\Rightarrow \quad \eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)+1}\eta_1$$
$$\Rightarrow \quad \eta_3 = (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1.$$

We also have

$$\begin{aligned} & (-1)^{A_3-B_3}\eta_3^{''*} = \eta_1^{''*} = (-1)^{A_3-B_3}\eta_1^{'*} \\ \Rightarrow \quad & \eta_3^{''*} = \eta_1^{'*} \\ \Rightarrow \quad & \eta_3^{''*} = (-1)^{A_2-B_2+1}\eta_1^{*} \\ \Rightarrow \quad & \eta_3^{''*} = (-1)^{A_2-B_2+1}\eta_1 \end{aligned}$$

and

$$\begin{aligned} l_3^{''*} &= l_3^{'*} - (A_1 - B_1 - 2l_1^{'*}) - 1 \\ &= l_3^{*} - (A_1 - B_1 - 2l_1^{*}) - 1 \\ &= l_3 + (B_3 - B_1) - (A_1 - B_1 - 2l_1) - 1 \\ &= l_3 + B_3 - A_1 + 2l_1 - 1. \end{aligned}$$

So we get the following conditions from (B 3).

– If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, then

$$0 \leqslant (l_3 + B_3 - A_1 + 2l_1 - 1) + B_1 - l_2 \leqslant (A_3 + B_3 - 0) - (A_2 - B_2)$$

which implies

$$(A_1 - B_1) - B_3 + 1 \leqslant l_3 - l_2 + 2l_1 \leqslant A_3 + (A_1 - B_1) - (A_2 - B_2) + 1.$$

– If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

$$(l_3 + B_3 - A_1 + 2l_1 - 1) + B_1 + l_2 > A_2 - B_2$$

which implies

$$l_3 + l_2 + 2l_1 > (A_1 - B_1) + (A_2 - B_2) - B_3 + 1.$$

• If $\eta_3^{'*} = (-1)^{A_1-B_1}\eta_1^{'*}$ and

$$l_3^{'*} - l_1^{'*} < (A_3^{*} - B_3^{*})/2 - (A_1 - B_1) + l_1^{'*},$$

then $\eta_1^{''*} \neq (-1)^{A_3-B_3}\eta_3^{''*}$ and

$$\begin{cases} l_1^{'*} = l_1^{''*} \\ l_3^{''*} - l_3^{'*} = (A_1 - B_1 - 2l_1^{'*}) + 1 \\ \eta_1^{''*} = (-1)^{A_3-B_3}\eta_1^{'*}. \end{cases}$$

It follows

$$\eta_3^{'*} = (-1)^{A_1-B_1}\eta_1^{'*}$$

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$$\Rightarrow \quad \eta_3^* = (-1)^{A_1-B_1}(-1)^{A_2-B_2+1}\eta_1^*$$
$$\Rightarrow \quad \eta_3 = (-1)^{(A_1-B_1)+(A_2-B_2)+1}\eta_1$$
$$\Rightarrow \quad \eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1$$

and

$$\begin{aligned} & l_3^{'*} - l_1^{'*} < (A_3^* - B_3^*)/2 - (A_1 - B_1) + l_1^{'*} \\ \Rightarrow \quad & l_3^* - l_1^* < (A_3 - B_3)/2 + (B_3 - B_1) - (A_1 - B_1) + l_1^* \\ \Rightarrow \quad & l_3 + (B_3 - B_1) - l_1 < (A_3 - B_3)/2 + (B_3 - B_1) - (A_1 - B_1) + l_1 \\ \Rightarrow \quad & l_3 - l_1 < (A_3 - B_3)/2 - (A_1 - B_1) + l_1. \end{aligned}$$

We also have

$$\begin{aligned} & (-1)^{A_3-B_3}\eta_3^{''*} \neq \eta_1^{''*} = (-1)^{A_3-B_3}\eta_1^{'*} \\ \Rightarrow \quad & \eta_3^{''*} = -\eta_1^{'*} \\ \Rightarrow \quad & \eta_3^{''*} = -(-1)^{A_2-B_2+1}\eta_1^* \\ \Rightarrow \quad & \eta_3^{''*} = (-1)^{A_2-B_2}\eta_1 \end{aligned}$$

and

$$\begin{aligned} l_3^{''*} &= l_3^{'*} + (A_1 - B_1 - 2l_1^{'*}) + 1 \\ &= l_3^* + (A_1 - B_1 - 2l_1^*) + 1 \\ &= l_3 + (B_3 - B_1) + (A_1 - B_1 - 2l_1) + 1 \\ &= l_3 - 2l_1 + A_1 + B_3 - 2B_1 + 1. \end{aligned}$$

So we get the following conditions from (B 3).

– If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

$$0 \leqslant (l_3 - 2l_1 + A_1 + B_3 - 2B_1 + 1) + B_1 - l_2 \leqslant (A_3 + B_3 - 0) - (A_2 - B_2)$$

which implies

$$-(A_1 - B_1) - B_3 - 1 \leqslant l_3 - l_2 - 2l_1 \leqslant A_3 - (A_1 - B_1) - (A_2 - B_2) - 1.$$

– If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, then

$$(l_3 - 2l_1 + A_1 + B_3 - 2B_1 + 1) + B_1 + l_2 > A_2 - B_2$$

which implies

$$l_3 + l_2 - 2l_1 > (A_2 - B_2) - (A_1 - B_1) - B_3 - 1.$$

- If $\eta_3^{'*} = (-1)^{A_1-B_1}\eta_1^{'*}$ and

$$l_3^{'*} - l_1^{'*} \geqslant (A_3^* - B_3^*)/2 - (A_1 - B_1) + l_1^{'*},$$

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then $\eta_1^{''*} = (-1)^{A_3-B_3}\eta_3^{''*}$ and

$$\begin{cases} l_1^{'*} = l_1^{''*} \\ (l_3^{''*} - l_1^{''*}) + (l_3^{'*} - l_1^{'*}) = (A_3^* - B_3^*) - (A_1 - B_1) \\ \eta_1^{''*} = (-1)^{A_3-B_3}\eta_1^{'*}. \end{cases}$$

It follows

$$\begin{aligned} & \eta_3^{'*} = (-1)^{A_1-B_1}\eta_1^{'*} \\ \Rightarrow \quad & \eta_3^{*} = (-1)^{A_1-B_1}(-1)^{A_2-B_2+1}\eta_1^{*} \\ \Rightarrow \quad & \eta_3 = (-1)^{(A_1-B_1)+(A_2-B_2)+1}\eta_1 \\ \Rightarrow \quad & \eta_3 \neq (-1)^{(A_1-B_1)+(A_2-B_2)}\eta_1 \end{aligned}$$

and

$$\begin{aligned} & l_3^{'*} - l_1^{'*} \geqslant (A_3^* - B_3^*)/2 - (A_1 - B_1) + l_1^{'*} \\ \Rightarrow \quad & l_3^{*} - l_1^{*} \geqslant (A_3 - B_3)/2 + (B_3 - B_1) - (A_1 - B_1) + l_1^{*} \\ \Rightarrow \quad & l_3 + (B_3 - B_1) - l_1 \geqslant (A_3 - B_3)/2 + (B_3 - B_1) - (A_1 - B_1) + l_1 \\ \Rightarrow \quad & l_3 - l_1 \geqslant (A_3 - B_3)/2 - (A_1 - B_1) + l_1. \end{aligned}$$

We also have

$$\begin{aligned} & (-1)^{A_3-B_3}\eta_3^{''*} = \eta_1^{''*} = (-1)^{A_3-B_3}\eta_1^{'*} \\ \Rightarrow \quad & \eta_3^{''*} = \eta_1^{'*} \\ \Rightarrow \quad & \eta_3^{''*} = (-1)^{A_2-B_2+1}\eta_1^{*} \\ \Rightarrow \quad & \eta_3^{''*} = (-1)^{A_2-B_2+1}\eta_1 \end{aligned}$$

and

$$\begin{aligned} l_3^{''*} &= l_1^{''*} - (l_3^{'*} - l_1^{'*}) + (A_3^* - B_3^*) - (A_1 - B_1) \\ &= 2l_1^{'*} - l_3^{'*} + (A_3 - B_3) + 2(B_3 - B_1) - (A_1 - B_1) \\ &= 2l_1^{*} - l_3^{*} + (A_3 - B_3) + 2(B_3 - B_1) - (A_1 - B_1) \\ &= 2l_1 - l_3 - (B_3 - B_1) + (A_3 - B_3) + 2(B_3 - B_1) - (A_1 - B_1) \\ &= 2l_1 - l_3 + (A_3 - B_3) + (B_3 - B_1) - (A_1 - B_1) \\ &= 2l_1 - l_3 + A_3 - A_1. \end{aligned}$$

So we get the following conditions from (B 3).

– If $\eta_2 = (-1)^{A_1-B_1}\eta_1$, then

$$0 \leqslant (2l_1 - l_3 + A_3 - A_1) + B_1 - l_2 \leqslant (A_3 + B_3 - 0) - (A_2 - B_2)$$

which implies

$$(A_1 - B_1) - A_3 \leqslant -l_3 - l_2 + 2l_1 \leqslant (A_1 - B_1) - (A_2 - B_2) + B_3.$$

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– If $\eta_2 \neq (-1)^{A_1-B_1}\eta_1$, then

$$(2l_1 - l_3 + A_3 - A_1) + B_1 + l_2 > A_2 - B_2$$

which implies

$$-l_3 + l_2 + 2l_1 > (A_1 - B_1) + (A_2 - B_2) - A_3.$$

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