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Generalized eigenfunction method for floating bodies

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We consider the time domain problem of a floating body in two dimensions, constrained to move in heave and pitch only, subject to the linear equations of water waves. We show that using the acceleration potential, we can write the equations of motion as an abstract wave equation. From this we derive a generalized eigenfunction solution in which the time domain problem is solved using the frequency-domain solutions. We present numerical results for two simple cases and compare our results with an alternative time domain method.

Key words: surface gravity waves, wave scattering, wave–structure interactions

1. Introduction

We present here a solution in the time domain using the generalized eigenfunction method for floating bodies which allows the solution for a given body and free-surface initial displacement (and initial velocity) to be computed from the frequency-domain solutions driven by an incident wave. The numerical accuracy of the method is determined only by the accuracy of the solutions in the frequency domain and is independent of time. As well as giving a numerical method to solve problems in the time domain which is independent of time, the generalized eigenfunction expansion gives a simple method to derive incident wave forcings which give a specified wave profile and body motion at a prescribed time and it also allows approximate solutions to be derived and the long-time asymptotic behaviour to be studied.

The generalized eigenfunction method was derived first for the Schrödinger equation (Povzner 1953; Ikebe 1960) and then extended to various wave problems, e.g. to the wave equation by Wilcox (1975). It has been applied to linear water waves by Friedman & Shinbrot (1967) and Beale (1977) and more recently by Hazard & Lenoir (2002) and Hazard & Loret (2007) for floating bodies, Meylan (2002), Hazard & Meylan (2007) and Peter & Meylan (2010) for elastic bodies and Meylan & Eatock Taylor (2009) and Meylan (2009) for fixed bodies. However, much of the emphasis in these papers has been on the theory and only the last five included any numerical computations.

Several other methods have been proposed to solve the linear water-wave problem for a floating body in the time domain. The classical method uses an integro-differential equation and the Cummins' decomposition (which involves a convolution of the impulse response function) (see Cummins 1962) to obtain the motion of the body. We refer to this method as the memory effect method. The equation for the impulse response function can be found by a Fourier transform of the frequency-domain solution (Mei 1989, chap. 7, (11.23)) but the equation of motion for the body

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involves a convolution integral in time (Mei 1989, chap. 7, (11.32)). For the case of a floating elastic plate, Meylan & Sturova (2009) provide a detailed comparison of the memory effect method with the generalized eigenfunction method. The time-dependent Green function method can also be used (Wehausen & Laitone 1960), and time-stepping method based on the boundary element method with damping zones has been used by McIver, McIver & Zhang (2003). All of these methods require a time-stepping procedure, so that the solutions become less accurate with time. Each method has its own particular disadvantages (and advantages). For example, in the time-dependent Green function method, finding a form of the free-surface Green function which is suitable for computation can be a problem in itself (although some innovative methods have been devised, Clément 1998), while the boundary element approach requires the introduction of damping zones to reduce reflections from the artificial bounding surfaces. Numerous numerical methods and computer codes have been developed to solve the frequency-domain problem, so time-domain solution methods incorporating the frequency-domain solution are considered preferable. The memory effect method is one such method; however, we argue that it is more logical to directly calculate the solution using the generalized eigenfunction method without using time-stepping. Furthermore, our method calculates the fluid motion and the body motion. Knowledge of the fluid motion may be important, especially if it is large compared with the body motion.

This work is an extension of Meylan (2009), where the problem of fixed bodies was considered, to the case of a floating body. The theory behind this calculation was developed by Hazard & Loret (2007) but in a very abstract and general setting (without any numerical examples) and only for infinite depth. In this work, we consider a body which can move in heave and pitch only (so in general it must be constrained in some way). The inclusion of surge is possible but only as it arises as a consequence of the motion in heave and pitch and this is discussed by Hazard & Loret (2007). The reason for this difficulty in including surge motion is that there is no restoring force. In § 2, we present the equations of motion for a floating body. In § 3, we rewrite these equations, using the acceleration potential, as an abstract wave equation (an equation in which the second time derivative acting on some variable plus a positive operator acting on the same variable is equal to zero). In § 4, we show that we can then expand the solutions in eigenfunctions using the self-adjointness property of the abstract wave equation (we show symmetry for this operator formally in the Appendix). In § 5, we present some results for simple body geometries and further results are included as supplementary movies available at journals.cambridge.org/flm. Section 6 provides a summary.

2. Initial-value problem for a freely floating structure

Consider a surface-piercing structure, constrained to move in heave and pitch, floating in an inviscid and incompressible fluid of constant finite depth h. The motion of the fluid is assumed to be irrotational and, along with the amplitude of the structure motion, sufficiently small so as to permit its description within the framework of the linearized water-wave theory. Attention is restricted to two dimensions, and Cartesian coordinates $\boldsymbol{x} = (x, z)$ are chosen with z directed vertically upwards and with the origin in the mean free surface. The floating-body problem involves a dynamic coupling of the motion of the body and surrounding fluid. A full solution of the problem requires the simultaneous solution of the fluid motion, described by the velocity potential $\Phi(\boldsymbol{x}, t)$, and the body motion, described by the displacement $\boldsymbol{\xi}(t) = (\xi_3, \xi_5)$. Only heave and pitch body motions are considered here so that the displacement is two-component vector and the indexing is standard in the study of the motion of ships and floating bodies.

The motion of the fluid is governed by Laplace's equation subject to various boundary conditions on each enclosing surface. Therefore, the velocity potential Φ satisfies Laplace's equation

$$\Delta\Phi = 0, \quad \boldsymbol{x} \in \Omega, \tag{2.1a}$$

where Ω is the fluid domain and the bed condition

$$\partial_n \Phi = 0, \quad z = -h, \tag{2.1b}$$

where ∂_n is the outward normal derivative. The free-surface elevation $\zeta(x, t)$ is related to Φ through the linearized free-surface conditions

$$\partial_t \Phi = -g\zeta, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{2.1c}$$

where g is the acceleration due to gravity and $\partial\Omega_F$ is the free surface, and

$$\partial_t \zeta = \partial_n \Phi, \quad \boldsymbol{x} \in \partial\Omega_F. \tag{2.1d}$$

The motion of the fluid is coupled to the motion of the structure by the boundary condition on the structure surface

$$\partial_n \Phi = \boldsymbol{n}\cdot\partial_t \boldsymbol{\xi} = \partial_t \xi_3 n_3 + \partial_t \xi_5 n_5, \quad \boldsymbol{x} \in \partial\Omega_B, \tag{2.1e}$$

where $\partial\Omega_B$ is the body surface and $\boldsymbol{n} = (n_3, n_5)$ are the heave and pitch components of the generalized normal vector on the body, respectively. The motions in heave and pitch are determined by

$$\begin{pmatrix} M & -Mx^c \\ -Mx^c & I_M \end{pmatrix} \partial_t^2 \boldsymbol{\xi} = -\rho \int_{\partial\Omega_B} \partial_t \Phi \boldsymbol{n} \, \mathrm{d}S - \rho g \begin{pmatrix} W & I_1^A \\ I_1^A & I_W \end{pmatrix} \boldsymbol{\xi}, \tag{2.1f}$$

where M is the mass of the body, I_M is the moment of inertia of the rigid body about the axis through the centre of rotation of the body (perpendicular to the x–z axis), x_c is the x-coordinate of the centre of mass, W is the waterplane area, I_W is the pitch restoring coefficient defined as the sum $I_{11}^A + I_3^V$. The moment quantities I_1^A, I_{11}^A and I_W are the first and second moments of the waterplane and the moment of the static submerged volume about the z-plane through the centre of rotation, respectively – the formula for these moment quantities can be found in Mei (1989). The only assumption regarding the geometry of the body is that the z-coordinate of the centre of mass and centre of rotation coincides. However, cross-coupling of the modes of motion (where motion in pitch influences motion in heave and *vice versa*) is possible. If the body is symmetric about the z-axis and the centre of mass and centre of rotation lie on the same central axis (it is assumed that the centre of rotation lies at the origin on the free-surface), then $I_1^A = 0$ and $x^c = 0$ so that the equations for heave and pitch decouple and become

$$\boldsymbol{M}\partial_t^2 \boldsymbol{\xi} = -\rho \int_{\partial\Omega_B} \partial_t \Phi \boldsymbol{n} \, \mathrm{d}S + \rho g \boldsymbol{W}\boldsymbol{\xi}, \tag{2.2}$$

where

$$\boldsymbol{M} = \begin{pmatrix} M & 0 \\ 0 & I_M \end{pmatrix} \quad \text{and} \quad \boldsymbol{W} = \begin{pmatrix} W & 0 \\ 0 & I_W \end{pmatrix}. \tag{2.3}$$

The absence of coupling between modes results in a significantly simplified problem and we will focus on this case. The general case is discussed by Hazard & Loret (2007).

3. Spectral formulation

The introduction of the acceleration potential $\Psi = \partial_t \Phi$ allows, after some further manipulations, the floating-body problem to be written in the form of an abstract wave equation. It is possible, for the case of a fixed body, to write an abstract wave equation using the velocity potential (Meylan 2009), but for the floating body we require the acceleration potential. In the case of a floating body with a vertical axis of symmetry moving in both heave and pitch, the boundary-value problem in non-dimensionalized form (where we remove gravity and the density but allow the length scale to be arbitrary) is

$$\Delta\Psi = 0, \quad \boldsymbol{x} \in \Omega, \tag{3.1a}$$

$$\partial_n \Psi = 0, \quad z = -h, \tag{3.1b}$$

$$\Psi = -\zeta, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{3.1c}$$

$$\partial_n \Psi = -\boldsymbol{n} \cdot \left[\boldsymbol{M}^{-1} \left\{ \int_{\partial\Omega_B} \Psi \boldsymbol{n} \, \mathrm{d}S + \boldsymbol{W}\boldsymbol{\xi} \right\} \right], \quad \boldsymbol{x} \in \partial\Omega_B, \tag{3.1d}$$

$$\partial_t^2 \zeta = \partial_n \Psi, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{3.1e}$$

and

$$\partial_t^2 \boldsymbol{\xi} = \mathscr{P} \partial_n \Psi, \quad \boldsymbol{x} \in \partial\Omega_B, \tag{3.1f}$$

respectively, where $\mathscr{P}$ is a projection operator which gives the n_3 and n_5 components of the normal velocity on the body from the normal of the velocity potential on the body surface (which must be a sum of the two rigid-body modes of motion).

For the spectral formulation, we write the evolution equations (3.1) as

$$\partial_t^2 \begin{pmatrix} \zeta \\ \boldsymbol{\xi} \end{pmatrix} + \mathscr{A} \begin{pmatrix} \zeta \\ \boldsymbol{\xi} \end{pmatrix} = 0, \tag{3.2}$$

where $\mathscr{A}$ consists of four components, i.e.

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{pmatrix}, \tag{3.3}$$

with the Dirichlet-to-Neumann operators $\mathscr{A}_{11}$, $\mathscr{A}_{12}$, $\mathscr{A}_{21}$ and $\mathscr{A}_{22}$ being defined as follows. The operators A_{11} and A_{21} map in the following way:

$$\left. \begin{aligned} \mathscr{A}_{11} &: \zeta \rightarrow -\partial_n \Psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_F, \\ \mathscr{A}_{21} &: \zeta \rightarrow -\mathscr{P}\partial_n \Psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_B, \end{aligned} \right\} \tag{3.4}$$

where Ψ is the solution of (3.1*a*), (3.1*b*) and the condition of zero-body displacement, i.e.

$$\left. \begin{aligned} \Psi &= -\zeta, \quad \boldsymbol{x} \in \partial\Omega_F, \\ \partial_n \Psi &= -\boldsymbol{n} \cdot \left[\boldsymbol{M}^{-1} \left\{ \int_{\partial\Omega_B} \Psi \boldsymbol{n} \, \mathrm{d}S \right\} \right] \quad \boldsymbol{x} \in \partial\Omega_B. \end{aligned} \right\} \tag{3.5}$$

Similarly, the operators A_{12} and A_{22} map in the following way:

$$\left. \begin{aligned} \mathscr{A}_{12} &: \boldsymbol{\xi} \rightarrow -\partial_n \Psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_F, \\ \mathscr{A}_{22} &: \boldsymbol{\xi} \rightarrow -\mathscr{P}\partial_n \Psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_B, \end{aligned} \right\} \tag{3.6}$$

where in this case Ψ corresponds to a solution of (3.1a), (3.1b) and a zero free-surface elevation boundary condition, i.e.

$$\left.\begin{aligned} \Psi &= 0, \quad \boldsymbol{x} \in \partial\Omega_F, \\ \partial_n \Psi &= -\boldsymbol{n}\cdot\left[\boldsymbol{M}^{-1}\left\{\int_{\partial\Omega_B} \Psi \boldsymbol{n}\,\mathrm{d}S + \boldsymbol{W}\boldsymbol{\xi}\right\}\right] \quad \boldsymbol{x} \in \partial\Omega_B. \end{aligned}\right\} \tag{3.7}$$

The operator $\mathscr{A}$ is self-adjoint with respect to the inner product

$$\left\langle \begin{pmatrix} \zeta \\ \boldsymbol{\xi} \end{pmatrix}, \begin{pmatrix} \zeta' \\ \boldsymbol{\xi}' \end{pmatrix} \right\rangle_{\mathscr{A}} = \langle \zeta, \zeta' \rangle_{\partial\Omega_F} + \boldsymbol{\xi}\cdot(\boldsymbol{W}\boldsymbol{\xi}')^*, \tag{3.8}$$

where the star denotes conjugation and

$$\langle \zeta, \zeta' \rangle_{\partial\Omega_F} = \int_{\partial\Omega_F} \zeta \zeta'^{\star}\,\mathrm{d}x. \tag{3.9}$$

In the Appendix, we show that the evolution operator $\mathscr{A}$ in (3.2) is symmetric and we assume this implies self-adjointness.

4. Solution as expansion in the frequency-domain solutions

The frequency-domain problem is a solution to the time-dependent equations, assuming that all quantities are proportional to $\exp(-\mathrm{i}\omega t)$ and the solution is driven by an incident wave. The frequency-domain equations are

$$-\omega^2 \begin{pmatrix} \zeta \\ \boldsymbol{\xi} \end{pmatrix} + \mathscr{A} \begin{pmatrix} \zeta \\ \boldsymbol{\xi} \end{pmatrix} = 0, \tag{4.1}$$

and the boundary conditions at infinity. If we write these equations in terms of the frequency-domain potential $\Phi = \mathrm{Re}\{\phi \mathrm{e}^{-\mathrm{i}\omega t}\}$, we obtain the familiar non-dimensionalized frequency-domain equations for a floating body:

$$\Delta\phi = 0, \quad \boldsymbol{x} \in \Omega, \tag{4.2a}$$

$$\partial_n \phi = 0, \quad z = -h, \tag{4.2b}$$

$$-\mathrm{i}\omega\phi = -\zeta, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{4.2c}$$

$$-\mathrm{i}\omega\zeta = \partial_n\phi, \quad \boldsymbol{x} \in \partial\Omega_F, \tag{4.2d}$$

$$-\omega^2 \boldsymbol{M}\boldsymbol{\xi} = \mathrm{i}\omega \int_{\partial\Omega_B} \phi \boldsymbol{n}\,\mathrm{d}S + \boldsymbol{W}\boldsymbol{\xi}. \tag{4.2e}$$

The frequency-domain solutions are driven by an incident wave of the form $\zeta = \exp(\mathrm{i}(\kappa k x - \omega t)$, where $\kappa = 1$ for wave incident from $x \to -\infty$ and $\kappa = -1$ for wave incident from $x \to \infty$ (note that we have normalized so the wave has unit amplitude in displacement). The wavenumber k is a function of frequency ω and *vice versa* through the non-dimensional dispersion relation $\omega^2 = kh\tanh(kh)$. We denote the frequency-domain solutions by

$$\begin{pmatrix} \zeta_\kappa(x,k) \\ \boldsymbol{\xi}_\kappa(k) \end{pmatrix}. \tag{4.3}$$

Note that we are specifically excluding here the possibility of a trapped mode, and the theory which follows would have to be amended to include this possibility.

The normalized frequency-domain solutions satisfy the following orthogonality condition:

$$\left\langle \begin{pmatrix} \zeta_{\kappa_1}(x,k) \\ \boldsymbol{\xi}_{\kappa_1}(k) \end{pmatrix}, \begin{pmatrix} \zeta_{\kappa_2}(x,k') \\ \boldsymbol{\xi}_{\kappa_2}(k') \end{pmatrix} \right\rangle_{\mathscr{A}} = 2\pi\delta_{\kappa_1\kappa_2}\delta(k-k'). \tag{4.4}$$

The proof of this is given for infinite depth by Hazard & Loret (2007). This result is very general and it is common to all generalized eigenfunction expansions that the normalization with and without the scatter is the same. The normalization without the scatterer (the floating body in our case) is trivial and follows from the Fourier transform.

The time-dependent equations are expanded in terms of the frequency-domain solutions as follows:

$$\begin{pmatrix} \zeta(x,t) \\ \boldsymbol{\xi}(t) \end{pmatrix} = \int_{\mathbb{R}^+} \sum_{\kappa\in\{-1,1\}} \left\{ f_\kappa(\omega)\cos(\omega t) + g_\kappa(\omega)\frac{\sin(\omega t)}{\omega} \right\} \begin{pmatrix} \zeta_\kappa(x,k) \\ \boldsymbol{\xi}_\kappa(k) \end{pmatrix} \mathrm{d}\omega, \tag{4.5}$$

where f_κ and g_κ are determined by the initial conditions. The initial free-surface elevation, free-surface velocity, body displacement and body velocity are specified as follows:

$$\begin{pmatrix} \zeta(x,0) \\ \boldsymbol{\xi}(0) \end{pmatrix} = \begin{pmatrix} \zeta_0(x) \\ \boldsymbol{\xi}_0 \end{pmatrix}, \quad \begin{pmatrix} \partial_t\zeta(x,0) \\ \partial_t\boldsymbol{\xi}(0) \end{pmatrix} = \begin{pmatrix} \partial_t\zeta_0(x) \\ \mathbf{v}_0 \end{pmatrix}. \tag{4.6}$$

Taking the inner product of the initial elevation/displacement equations with respect to a generalized eigenfunction (frequency-domain solution) gives

$$\left\langle \begin{pmatrix} \zeta_0(x) \\ \boldsymbol{\xi}_0 \end{pmatrix}, \begin{pmatrix} \zeta_\kappa(x,k) \\ \boldsymbol{\xi}_\kappa(k) \end{pmatrix} \right\rangle_{\mathscr{A}} = 2\pi f_\kappa(\omega)\frac{\mathrm{d}\omega}{\mathrm{d}k}, \tag{4.7}$$

and similarly for the velocity condition

$$\left\langle \begin{pmatrix} \partial_t\zeta_0(x) \\ \mathbf{v}_0 \end{pmatrix}, \begin{pmatrix} \zeta_\kappa(x,k) \\ \boldsymbol{\xi}_\kappa(k) \end{pmatrix} \right\rangle_{\mathscr{A}} = 2\pi g_\kappa(\omega)\frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{4.8}$$

5. Results

We consider two bodies, a half-immersed circle of radius 0.5 centred at the origin and a rigid plate of negligible submergence which occupies $-0.5 \leqslant x \leqslant 0.5$. We assume that we have non-dimensionalized so that $\rho = g = 1$ and the water depth is $h = 2$. The mass M of both the circle and the plate is taken to be $\pi/8$, which is the correct weight for the circle using Archimedes, principle (but not for the plate which has negligible submergence). The moments of inertia for the circle are zero (it moves in heave only), while for the plate $I_M = \pi/96$ and $I_W = 1/12$. The wetted area is $W = 1$ for both cases. We solve the frequency-domain problem by a simple boundary element solution method which we do not describe here (details can be found at www.wikiwaves.org). For our calculations, we use 500 frequency points evenly spaced between 0 and 20π. We discretize the surfaces into panels of size 0.005. We begin with a comparison of the time-domain solution calculated by a time-stepping method as described by McIver *et al.* (2003) for the half-submerged circle with an initial displacement of $\xi_3 = 0.5$, and all other initial quantities zero. We show in the comparison only the heave component in figure 1. The solution for the circle with the initial displacement of $\xi_3 = 0.5$ (there is no pitch motion for a circle) and zero initial surface displacement is shown in figure 2. Note that this is the

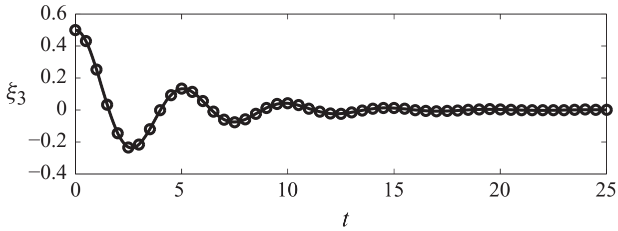

FIGURE 1. The displacement ξ of a circle of radius 0.5 which is released from an initial displacement of 0.5. The solid line is calculated using the generalized eigenfunction method and the circles denote the solution based on McIver *et al.* (2003).

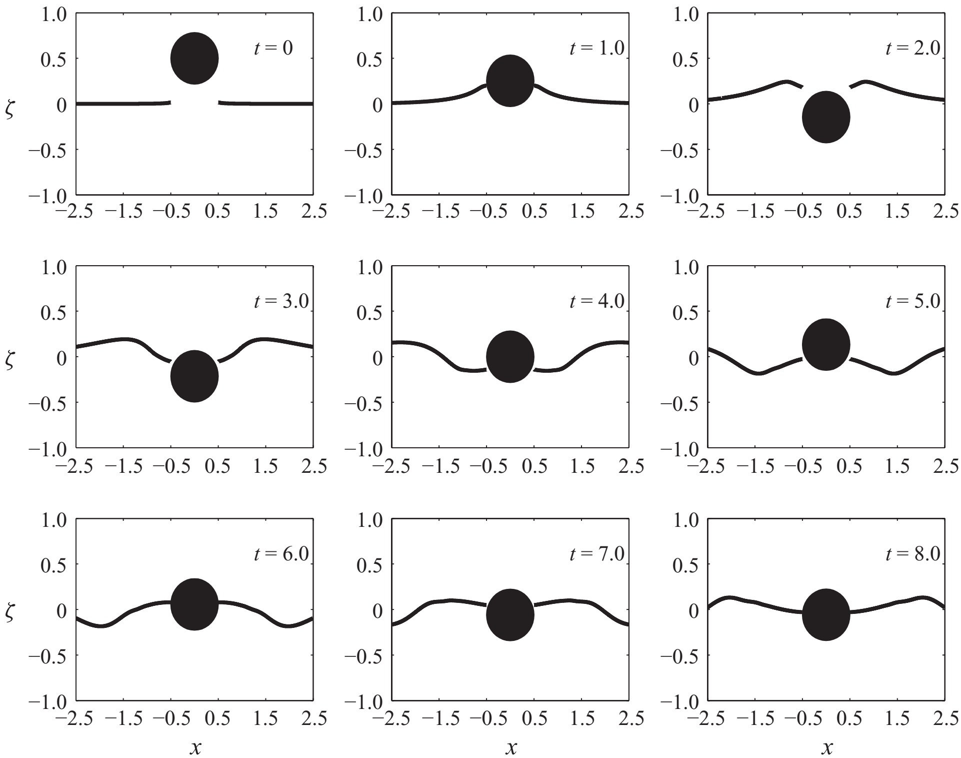

FIGURE 2. The displacement ξ of a circle of radius 0.5 which is released from an initial displacement of 0.5 at $t = 0$ for the times shown. This is also shown in supplementary movie 1.

exact problem solved by Ursell (1964) and Maskell & Ursell (1970), who found the solution by a Laplace transform for this simple geometry. One test of the accuracy is how well the initial solution when calculated from the Fourier-type integral matches the initial condition. For the circle, the initial surface displacement calculated from (4.5) with $t = 0$ is 0.499908. Note, however, the calculation of the free surface may not necessarily be as accurate.

The solution for the dock with initial conditions $\xi_3 = 0.5$ and $\xi_5 = \pi/16$ with zero initial surface displacement is shown in figure 3. These solutions are also shown in supplementary movies 1 and 2. Movies 3 and 4 are the equivalent problem except the surface has an initial displacement of $\zeta = \exp(-4(x+3)^2)$ as well. The movies also show the negative time solution and illustrate that the generalized eigenfunction expansion could be used in a wave tank to generate a prescribed motion of the body (and fluid surface) at a given time.

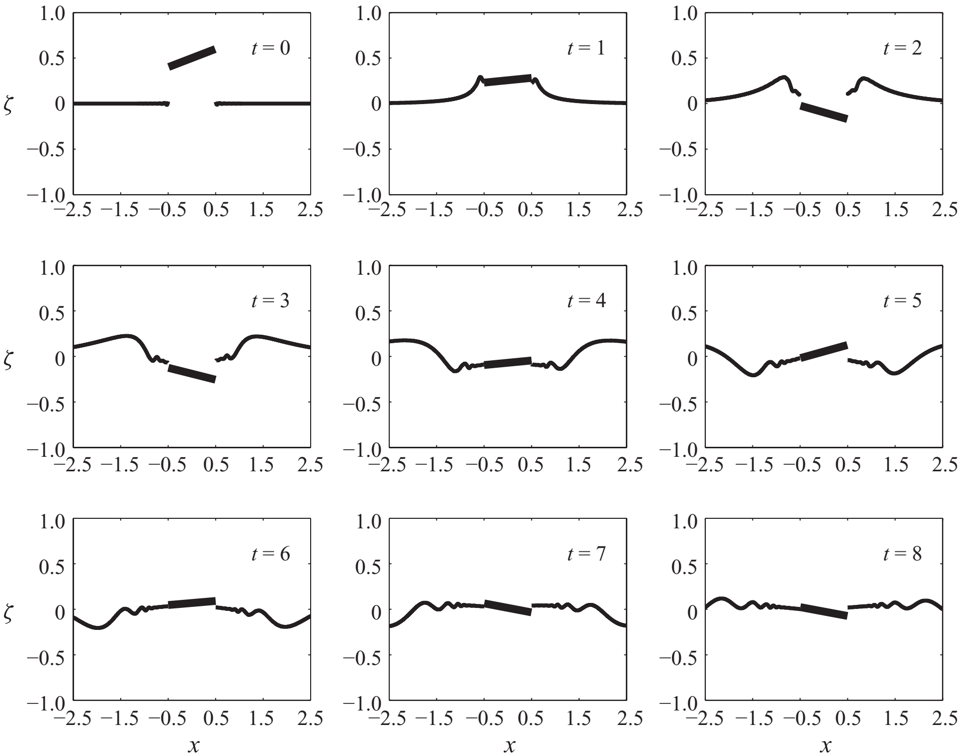

FIGURE 3. The displacement ξ of a dock of length 2 which is released from a displacement of 0.5 and a pitch of $\pi/16$ at $t = 0$ for the times shown. This is also shown in supplementary movie 1.

6. Summary

We have shown how the generalized eigenfunction method can be used to calculate the response of a floating body and have presented some example calculations. We hope that the development of this method will be useful both theoretically and practically. For example, we may be able to derive general formula for the asymptotic behaviour of a floating body (which were given for a semi-submerged cylinder by Ursell 1964 and Maskell & Ursell 1970). It would also be useful to make a clear connection between this method and the standard solution in the time domain by the memory effect (or impulse response function) method Mei (1989), as was given for a floating elastic plate by Meylan & Sturova (2009).

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Supplementary movies are available at journals.cambridge.org/flm.

Appendix. Symmetry of the evolution operator $\mathscr{A}$

The symmetry of the operator $\mathscr{A}$ was shown by Hazard & Loret (2007) but we present here a derivation of this result for the symmetric body case to make the steps in this as clear as possible. The method here is similar to that used by Peter & Meylan

(2010). We need to show that

$$\left\langle \begin{pmatrix} \zeta \\ \boldsymbol{\xi} \end{pmatrix}, \mathscr{A} \begin{pmatrix} \zeta' \\ \boldsymbol{\xi}' \end{pmatrix} \right\rangle_{\mathscr{A}} = \left\langle \mathscr{A} \begin{pmatrix} \zeta \\ \boldsymbol{\xi} \end{pmatrix}, \begin{pmatrix} \zeta' \\ \boldsymbol{\xi}' \end{pmatrix} \right\rangle_{\mathscr{A}}. \tag{A 1}$$

We expand the inner product on the left using the matrix structure of $\mathscr{A}$ as

$$\left\langle \begin{pmatrix} \zeta \\ \boldsymbol{\xi} \end{pmatrix}, \mathscr{A} \begin{pmatrix} \zeta' \\ \boldsymbol{\xi}' \end{pmatrix} \right\rangle_{\mathscr{A}} = \langle \zeta, \mathscr{A}_{11}\zeta' \rangle_{\partial\Omega_F} + \langle \zeta, \mathscr{A}_{12}\boldsymbol{\xi}' \rangle_{\partial\Omega_F} + \boldsymbol{\xi}\cdot(\mathbf{W}\mathscr{A}_{21}\zeta')^* + \boldsymbol{\xi}\cdot(\mathbf{W}\mathscr{A}_{22}\boldsymbol{\xi})'^* \tag{A 2}$$

and consider each term on the right-hand side individually.

Using Green's second identity, we have

$$\langle \zeta, \mathscr{A}_{11}\zeta' \rangle_{\partial\Omega_F} = \langle -\Psi, -\partial_n\Psi' \rangle_{\partial\Omega_F} = \langle -\partial_n\Psi, -\Psi' \rangle_{\partial\Omega_F} + \langle \Psi, \partial_n\Psi' \rangle_{\partial\Omega_B} - \langle \partial_n\Psi, \Psi' \rangle_{\partial\Omega_B}, \tag{A 3}$$

where we have introduced the notation

$$\langle \Psi, \partial_n\Psi' \rangle_{\partial\Omega_B} = \int_{\partial\Omega_B} \Psi\Psi'^* \,\mathrm{d}S. \tag{A 4}$$

It follows that

$$\langle \Psi, \partial_n\Psi' \rangle_{\partial\Omega_B} - \langle \partial_n\Psi, \Psi' \rangle_{\partial\Omega_B} = 0, \tag{A 5}$$

by substitution of the boundary condition

$$\partial_n\Psi = -\boldsymbol{n}\cdot\left(\boldsymbol{M}^{-1}\int_{\partial\Omega_B} \Psi\boldsymbol{n}\,\mathrm{d}S\right) = -\boldsymbol{n}\cdot(\boldsymbol{M}^{-1}\langle \Psi, \boldsymbol{n} \rangle_{\partial\Omega_B}), \quad \boldsymbol{x} \in \partial\Omega_B, \tag{A 6}$$

(which applies to both Ψ and Ψ' in this case) and by changing the order of integration. Therefore,

$$\langle \zeta, \mathscr{A}_{11}\zeta' \rangle_{\partial\Omega_F} = \langle \mathscr{A}_{11}\zeta, \zeta' \rangle_{\partial\Omega_F}. \tag{A 7}$$

The second and third terms involving the operators $\mathscr{A}_{12}$ and $\mathscr{A}_{21}$ require similar manipulations which result in the interchange of the roles of the operators. In the case of the second term $\langle \zeta, \mathscr{A}_{12}\boldsymbol{\xi}' \rangle$, the free-surface elevation ζ corresponds to an acceleration potential satisfying (3.5) whereas ζ' corresponds to Ψ satisfying (3.7). To begin with, we show that

$$\begin{aligned} \langle \zeta, \mathscr{A}_{12}\boldsymbol{\xi}' \rangle_{\partial\Omega_F} &= \langle -\Psi, -\partial_n\Psi' \rangle_{\partial\Omega_F} \\ &= \langle \partial_n\Psi, \Psi' \rangle_{\partial\Omega_F} - \langle \Psi, \partial_n\Psi' \rangle_{\partial\Omega_B} + \langle \partial_n\Psi, \Psi' \rangle_{\partial\Omega_B} \\ &= -\langle \Psi, \partial_n\Psi' \rangle_{\partial\Omega_B} + \langle \partial_n\Psi, \Psi' \rangle_{\partial\Omega_B}, \end{aligned} \tag{A 8}$$

by the application of the definition of the operator $\mathscr{A}_{12}$, the use of Green's second identity and finally the simplification of the resultant term using the free-surface condition $\Psi' = 0$. The inner product $\langle \partial_n\Psi, \Psi' \rangle_{\partial\Omega_B}$ is rearranged by substituting for $\partial_n\Psi$ using (A 6) and swapping the order of integration to give

$$\begin{aligned} \langle \partial_n\Psi, \Psi' \rangle_{\partial\Omega_B} &= -\int_{\partial\Omega_B} \boldsymbol{n}(x)\cdot\left(\boldsymbol{M}^{-1}\int_{\partial\Omega_B(\hat{x})} \Psi(\hat{x})\boldsymbol{n}(\hat{x})\,\mathrm{d}S(\hat{x})\right)\Psi'^*(x)\,\mathrm{d}S(x) \\ &= \int_{\partial\Omega_B(\hat{x})} \Psi(\hat{x})\left(-\boldsymbol{n}(\hat{x})\cdot\boldsymbol{M}^{-1}\int_{\partial\Omega_B} \Psi'^*(x)\boldsymbol{n}(x)\,\mathrm{d}S(x)\right)\mathrm{d}S(\hat{x}). \end{aligned} \tag{A 9}$$

An application of the body boundary condition in (3.7) to the term in parentheses gives

$$\begin{aligned}\langle \partial_n \Psi, \Psi' \rangle_{\partial\Omega_B} &= \langle \Psi, \partial_n \Psi' + \boldsymbol{n}.\boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{\xi}' \rangle_{\partial\Omega_B} \\ &= \langle \Psi, \partial_n \Psi' \rangle_{\partial\Omega_B} + \langle \Psi, \boldsymbol{n} \rangle_{\partial\Omega_B}.\boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{\xi}'^{*},\end{aligned} \tag{A 10}$$

where $\langle \Psi, \boldsymbol{n} \rangle_{\partial\Omega_B} \cdot \boldsymbol{M}^{-1}$ corresponds to the fluid velocity components on the body, i.e. $\partial_t^2 \boldsymbol{\xi}'^{*}$. Therefore, a combination of the results from (A 8) and (A 10) and the definition of $\mathscr{A}_{21}$ (see (3.4) and (3.5)) gives

$$\langle \zeta, \mathscr{A}_{12}\boldsymbol{\xi}' \rangle_{\partial\Omega_F} = \mathscr{A}_{21}\zeta \cdot (\boldsymbol{W}\boldsymbol{\xi}')^{*}. \tag{A 11}$$

The third term in the inner product (A 2) is obtained using an inversion of the procedure used for the second term. Therefore, the first step is to use the definition of the operator $\mathscr{A}_{21}$ to introduce the normal velocity term on the body, i.e.

$$\boldsymbol{\xi} \cdot (\boldsymbol{W}\mathscr{A}_{21}\zeta')^{*} = \boldsymbol{\xi} \cdot (\boldsymbol{W}\boldsymbol{M}^{-1}\langle \boldsymbol{n}, \Psi' \rangle_{\partial\Omega_B}). \tag{A 12}$$

Notice that the role of the terms (and those related) Ψ and Ψ' has been interchanged as they are associated with opposite operators compared with the previous term. By taking the conjugate of the relation (A 10) and swapping the primed and unprimed terms, we get

$$\langle \Psi, \partial_n \Psi' \rangle_{\partial\Omega_B} = \langle \partial_n \Psi, \Psi' \rangle_{\partial\Omega_B} + \boldsymbol{\xi} \cdot \boldsymbol{W}\boldsymbol{M}^{-1}\langle \boldsymbol{n}, \Psi' \rangle_{\partial\Omega_B}. \tag{A 13}$$

It is then straightforward to show that

$$\boldsymbol{\xi} \cdot (\boldsymbol{W}\mathscr{A}_{21}\zeta')^{*} = \langle \partial_n \Psi, \Psi' \rangle_{\partial\Omega_B} - \langle \Psi, \partial_n \Psi' \rangle_{\partial\Omega_B} + \langle \Psi, \partial_n \Psi' \rangle_{\partial\Omega_F}, \tag{A 14}$$

because $\Psi = 0$ for $\boldsymbol{x} \in \partial\Omega_F$. Therefore, from Green's identity and the definition of the operator $\mathscr{A}_{12}$, we get

$$\begin{aligned}\boldsymbol{\xi} \cdot (\boldsymbol{W}\mathscr{A}_{21}\zeta')^{*} &= \langle \partial_n \Psi, \Psi' \rangle_{\partial\Omega_F} \\ &= \langle \mathscr{A}_{12}\boldsymbol{\xi}, \zeta' \rangle.\end{aligned} \tag{A 15}$$

In the fourth term $\boldsymbol{\xi} \cdot (\boldsymbol{W}(\mathscr{A}_{22}\boldsymbol{\xi}')^{*}$ of the inner product expansion, the two displacement vectors ($\boldsymbol{\xi}$,$\boldsymbol{\xi}'$) correspond to acceleration potentials with $\Psi = \Psi' = 0$ on the free surface. Therefore, from Green's theorem

$$\langle \Psi, \partial_n \Psi' \rangle_{\partial\Omega_B} = \langle \partial_n \Psi, \Psi' \rangle_{\partial\Omega_B}. \tag{A 16}$$

The substitution of the body boundary condition into both terms gives, after some cancellation,

$$\langle \Psi, \boldsymbol{n} \rangle_{\partial\Omega_B} \cdot \boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{\xi} = \langle \boldsymbol{n}, \Psi' \rangle_{\partial\Omega_B} \cdot \boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{\xi}'^{*}. \tag{A 17}$$

Using this relation, we obtain the desired result as follows:

$$\begin{aligned}\boldsymbol{\xi} \cdot (\boldsymbol{W}(\mathscr{A}_{22}\boldsymbol{\xi}')^{*} &= \boldsymbol{\xi} \cdot \boldsymbol{W}(-\partial_t^2 \boldsymbol{\xi}')^{*} \\ &= \boldsymbol{\xi} \cdot \boldsymbol{W}\left[\boldsymbol{M}'(\langle \boldsymbol{n}, \Psi' \rangle_{\partial\Omega_B} + \boldsymbol{W}\boldsymbol{\xi}'^{*})\right] \\ &= \left[\boldsymbol{M}'(\langle \Psi, \boldsymbol{n} \rangle_{\partial\Omega_B} + \boldsymbol{W}\boldsymbol{\xi})\right] \cdot \boldsymbol{\xi} \quad \text{from (A 17)} \\ &= \mathscr{A}_{22}\boldsymbol{\xi} \cdot (\boldsymbol{W}\boldsymbol{\xi}'^{*}).\end{aligned} \tag{A 18}$$

The combination of all these results gives

$$\begin{aligned}\left\langle \begin{pmatrix}\zeta\\ \boldsymbol{\xi}\end{pmatrix}, \mathscr{A}\begin{pmatrix}\zeta'\\ \boldsymbol{\xi}'\end{pmatrix}\right\rangle_{\mathscr{A}} &= \left\langle \mathscr{A}\begin{pmatrix}\zeta\\ \boldsymbol{\xi}\end{pmatrix}, \begin{pmatrix}\zeta'\\ \boldsymbol{\xi}'\end{pmatrix}\right\rangle_{\mathscr{A}} \\ &= \langle \zeta, \mathscr{A}_{11}\zeta'\rangle_{\partial\Omega_F} + \langle \zeta, \mathscr{A}_{12}\boldsymbol{\xi}'\rangle_F + \boldsymbol{\xi}\boldsymbol{\cdot}(\boldsymbol{W}(\mathscr{A}_{21}\zeta'))^* + \boldsymbol{\xi}\boldsymbol{\cdot}(\boldsymbol{W}(\mathscr{A}_{22}\boldsymbol{\xi}'))^* \\ &= \langle \mathscr{A}_{11}\zeta, \zeta'\rangle_F + \mathscr{A}_{21}\zeta\boldsymbol{\cdot}(\boldsymbol{W}\boldsymbol{\xi}')^* + \langle \mathscr{A}_{12}\boldsymbol{\xi}, \zeta'\rangle_F + \mathscr{A}_{22}\boldsymbol{\xi}\boldsymbol{\cdot}(\boldsymbol{W}\boldsymbol{\xi}')^* \\ &= \left\langle \mathscr{A}\begin{pmatrix}\zeta\\ \boldsymbol{\xi}\end{pmatrix}, \begin{pmatrix}\zeta'\\ \boldsymbol{\xi}'\end{pmatrix}\right\rangle,\end{aligned} \tag{A 19}$$

thus proving the symmetry of the operator $\mathscr{A}$ under the energy inner product.

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