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ON A CLASS OF CRITICAL N-LAPLACIAN PROBLEMS

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Abstract We establish some existence results for a class of critical N-Laplacian problems in a bounded domain in $\mathbb{R}^N$. In the absence of a suitable direct sum decomposition of the underlying Sobolev space to which the classical linking theorem can be applied, we use an abstract linking theorem based on the $\mathbb{Z}_2$-cohomological index to obtain a non-trivial critical point.

Keywords: critical N-Laplacian problems; existence; critical points; linking; $\mathbb{Z}_2$-cohomological index

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1. Introduction

In this paper, we establish some existence results for the class of critical N-Laplacian problems

$$\begin{cases} -\Delta_N\, u = h(u)\, e^{\alpha\, |u|^{N'}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth-bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $N' = N/(N-1)$ is the Hölder conjugate of N, and h is a continuous function such that

$$\lim_{|t|\to\infty} h(t) = 0 \tag{1.2}$$

and

$$0 < \beta := \liminf_{|t|\to\infty}\, th(t) < \infty. \tag{1.3}$$

This problem is motivated by the Trudinger–Moser inequality

$$\sup_{\substack{u\in W_0^{1,N}(\Omega)\\ \|u\|\leq 1}} \int_\Omega e^{\alpha_N\, |u|^{N'}}\, dx < \infty, \tag{1.4}$$

where $W_0^{1,N}(\Omega)$ is the usual Sobolev space with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^N \, \mathrm{d}x\right)^{1/N},$$

$$\alpha_N = N\, \omega_{N-1}^{1/(N-1)},$$

and

$$\omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

is the area of the unit sphere in $\mathbb{R}^N$ (see Trudinger [14] and Moser [10]). Problem (1.1) is critical with respect to this inequality and hence lacks compactness. Indeed, the associated variational functional satisfies the Palais–Smale compactness condition only at energy levels below a certain threshold (see Proposition 2.1 in the next section).

In dimension $N = 2$, problem (1.1) is semilinear and has been extensively studied in the literature (see, e.g., [2–4, 6]). In dimensions $N \geq 3$, this problem is quasilinear and has been studied mainly when

$$G(t) := \int_0^t h(s)\, e^{\alpha\, |s|^{N'}} \mathrm{d}s \leq \lambda\, |t|^N \quad \text{for small } t \tag{1.5}$$

for some $\lambda \in (0, \lambda_1)$ (see, e.g., [1, 5, 8]). Here,

$$\lambda_1 = \inf_{u \in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^N \, \mathrm{d}x}{\displaystyle\int_\Omega |u|^N \, \mathrm{d}x} \tag{1.6}$$

is the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta_N\, u = \lambda\, |u|^{N-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The case $h(t) = \lambda\, |t|^{N-2}\, t$ with $\lambda > 0$, for which $\beta = \infty$, was recently studied in Yang and Perera [15]. The remaining case, where $N \geq 3$, $\lambda \geq \lambda_1$, and $\beta < \infty$, does not seem to have been studied in the literature. This case is covered in our results here, which are for large $\beta < \infty$ and allow $N \geq 3$ and $\lambda \geq \lambda_1$ in (1.5).

Let d be the radius of the largest open ball contained in Ω. Our first result is the following theorem.

Theorem 1.1. *Assume that $\alpha > 0$, h satisfies (1.2) and (1.3), and G satisfies*

$$G(t) \geq -\frac{1}{N}\, \sigma_0\, |t|^N \quad \textit{for } t \geq 0, \tag{1.8}$$

$$G(t) \leq \frac{1}{N}\, (\lambda_1 - \sigma_1)\, |t|^N \quad \textit{for } |t| \leq \delta \tag{1.9}$$

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for some $\sigma_0 \geq 0$ *and* $\sigma_1, \delta > 0$*. If*

$$\beta > \frac{1}{N\alpha^{N-1}} \left(\frac{N}{d}\right)^N e^{\sigma_0/(N-1)\,\kappa}, \tag{1.10}$$

where $\kappa = \dfrac{1}{N!} \left(\dfrac{N}{d}\right)^N$*, then problem* (1.1) *has a non-trivial solution.*

In particular, we have the following corollary for $\sigma_0 = 0$.

Corollary 1.2. *Assume that* $\alpha > 0$*,* h *satisfies* (1.2) *and* (1.3)*, and* G *satisfies*

$$G(t) \geq 0 \quad \text{for } t \geq 0,$$

$$G(t) \leq \frac{1}{N}\,(\lambda_1 - \sigma_1)\,|t|^N \quad \text{for } |t| \leq \delta$$

for some $\sigma_1, \delta > 0$*. If*

$$\beta > \frac{1}{N\alpha^{N-1}} \left(\frac{N}{d}\right)^N,$$

then problem (1.1) *has a non-trivial solution.*

Corollary 1.2 should be compared with Theorem 1 of do Ó [8], where this result is proved under the stronger assumption $h(t) \geq 0$ for $t \geq 0$.

To state our second result, let (λ_k) be the sequence of eigenvalues of problem (1.7) based on the $\mathbb{Z}_2$-cohomological index that was introduced in Perera [11] (see Proposition 2.3 in the next section). We have the following theorem.

Theorem 1.3. *Assume that* $\alpha > 0$*,* h *satisfies* (1.2) *and* (1.3)*, and* G *satisfies*

$$G(t) \geq \frac{1}{N}\,(\lambda_{k-1} + \sigma_0)\,|t|^N \quad \forall t, \tag{1.11}$$

$$G(t) \leq \frac{1}{N}\,(\lambda_k - \sigma_1)\,|t|^N \quad \text{for } |t| \leq \delta \tag{1.12}$$

for some $k \geq 2$ *and* $\sigma_0, \sigma_1, \delta > 0$*. Then there exists a constant* $c > 0$ *depending on* Ω*,* α*, and* k*, but not on* σ_0*,* σ_1*, or* δ*, such that if*

$$\beta > \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N e^{c/\sigma_0^{N-1}},$$

then problem (1.1) *has a non-trivial solution.*

Theorem 1.3 should be compared with Theorem 1.4 of de Figueiredo *et al.* [3, 4], where this result is proved in the case $N = 2$ under the additional assumption that $0 < 2G(t) \leq th(t)\,e^{\alpha t^2}$ for all $t \in \mathbb{R}\setminus\{0\}$. However, the linking argument used in [3, 4] is based on a splitting of $H_0^1(\Omega)$ that involves the eigenspaces of the Laplacian, and this argument does not extend to the case $N \geq 3$ where the N-Laplacian is a nonlinear operator and therefore has no linear eigenspaces. We will prove Theorem 1.3 using an

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abstract critical point theorem based on the $\mathbb{Z}_2$-cohomological index that was proved in Yang and Perera [15] (see § 2.4).

In the proofs of Theorems 1.1 and 1.3, the inner radius d of Ω comes into play when verifying that certain minimax levels are below the compactness threshold given in Proposition 2.1.

2. Preliminaries

2.1. A compactness result

Weak solutions of problem (1.1) coincide with critical points of the C^1-functional

$$E(u) = \frac{1}{N} \int_\Omega |\nabla u|^N \, \mathrm{d}x - \int_\Omega G(u) \, \mathrm{d}x, \quad u \in W_0^{1,N}(\Omega).$$

We recall that a $(\mathrm{PS})_c$ sequence of E is a sequence $(u_j) \subset W_0^{1,N}(\Omega)$ such that $E(u_j) \to c$ and $E'(u_j) \to 0$. Proofs of Theorem 1.1 and Theorem 1.3 will be based on the following compactness result.

Proposition 2.1. *Assume that* $\alpha > 0$ *and* h *satisfies* (1.2) *and* (1.3). *Then for all* $c \neq 0$ *satisfying*

$$c < \frac{1}{N} \left(\frac{\alpha_N}{\alpha} \right)^{N-1},$$

every $(PS)_c$ *sequence of* E *has a subsequence that converges weakly to a non-trivial solution of problem* (1.1).

Proof. Let $(u_j) \subset W_0^{1,N}(\Omega)$ be a $(\mathrm{PS})_c$ sequence of E. Then,

$$E(u_j) = \frac{1}{N} \|u_j\|^N - \int_\Omega G(u_j) \, \mathrm{d}x = c + \mathrm{o}(1) \tag{2.1}$$

and

$$E'(u_j)\, u_j = \|u_j\|^N - \int_\Omega u_j\, h(u_j)\, e^{\alpha |u_j|^{N'}} \, \mathrm{d}x = \mathrm{o}(\|u_j\|). \tag{2.2}$$

First, we show that (u_j) is bounded in $W_0^{1,N}(\Omega)$. Multiplying (2.1) by $2N$ and subtracting (2.2) gives

$$\|u_j\|^N + \int_\Omega \Big(u_j\, h(u_j)\, e^{\alpha |u_j|^{N'}} - 2NG(u_j) \Big) \, \mathrm{d}x = 2Nc + \mathrm{o}(\|u_j\| + 1),$$

so it suffices to show that $th(t)\, e^{\alpha |t|^{N'}} - 2NG(t)$ is bounded from below. Let $0 < \varepsilon < \beta/(2N+1)$. By (1.2) and (1.3), for some constant $C_\varepsilon > 0$,

$$|G(t)| \leq \varepsilon\, e^{\alpha |t|^{N'}} + C_\varepsilon \tag{2.3}$$

and

$$th(t)\, e^{\alpha |t|^{N'}} \geq (\beta - \varepsilon)\, e^{\alpha |t|^{N'}} - C_\varepsilon \tag{2.4}$$

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for all t. So

$$th(t)\, e^{\alpha\, |t|^{N'}} - 2NG(t) \ge [\beta - (2N+1)\,\varepsilon]\, e^{\alpha\, |t|^{N'}} - (2N+1)\, C_\varepsilon,$$

which is bounded from below.

Since (u_j) is bounded in $W_0^{1,N}(\Omega)$, a renamed subsequence converges to some u weakly in $W_0^{1,N}(\Omega)$, strongly in $L^p(\Omega)$ for all $p \in [1, \infty)$, and a.e.in Ω. We have

$$E'(u_j)\, v = \int_\Omega |\nabla u_j|^{N-2}\, \nabla u_j \cdot \nabla v\, \mathrm{d}x - \int_\Omega v\, h(u_j)\, e^{\alpha\, |u_j|^{N'}}\, \mathrm{d}x \to 0 \tag{2.5}$$

for all $v \in W_0^{1,N}(\Omega)$. By (1.2), given any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|h(t)\, e^{\alpha\, |t|^{N'}}| \le \varepsilon\, e^{\alpha\, |t|^{N'}} + C_\varepsilon \quad \forall t. \tag{2.6}$$

By (2.2),

$$\sup_j \int_\Omega u_j\, h(u_j)\, e^{\alpha\, |u_j|^{N'}}\, \mathrm{d}x < \infty,$$

which together with (2.4) gives

$$\sup_j \int_\Omega e^{\alpha\, |u_j|^{N'}}\, \mathrm{d}x < \infty. \tag{2.7}$$

For $v \in C_0^\infty(\Omega)$, it follows from (2.6) and (2.7) that the sequence $(v\, h(u_j)\, e^{\alpha\, |u_j|^{N'}})$ is uniformly integrable and hence

$$\int_\Omega v\, h(u_j)\, e^{\alpha\, |u_j|^{N'}}\, \mathrm{d}x \to \int_\Omega v\, h(u)\, e^{\alpha\, |u|^{N'}}\, \mathrm{d}x$$

by Vitali's convergence theorem, so it follows from (2.5) that

$$\int_\Omega |\nabla u|^{N-2}\, \nabla u \cdot \nabla v\, \mathrm{d}x - \int_\Omega v\, h(u)\, e^{\alpha\, |u|^{N'}}\, \mathrm{d}x = 0.$$

Then this holds for all $v \in W_0^{1,N}(\Omega)$ by density, so the weak limit u is a solution of problem (1.1).

Suppose that $u = 0$. Then

$$\int_\Omega G(u_j)\, \mathrm{d}x \to 0$$

since (2.3) and (2.7) imply that the sequence $(G(u_j))$ is uniformly integrable, so (2.1) gives $c \ge 0$ and

$$\|u_j\| \to (Nc)^{1/N}. \tag{2.8}$$

Let $Nc < \nu < (\alpha_N/\alpha)^{N-1}$. Then $\|u_j\| \le \nu^{1/N}$ for all $j \ge j_0$ for some j_0. Let $q = \alpha_N/\alpha\nu^{1/(N-1)} > 1$. By the Hölder inequality,

$$\left| \int_\Omega u_j\, h(u_j)\, e^{\alpha\, |u_j|^{N'}}\, \mathrm{d}x \right| \le \left(\int_\Omega |u_j\, h(u_j)|^p\, \mathrm{d}x \right)^{1/p} \left(\int_\Omega e^{q\alpha\, |u_j|^{N'}}\, \mathrm{d}x \right)^{1/q},$$

where $1/p + 1/q = 1$. The first integral on the right-hand side converges to zero since h is bounded and $u_j \to 0$ in $L^p(\Omega)$, and the second integral is bounded by (1.4) since

$q\alpha\,|u_j|^{N'} = \alpha_N\,|\widetilde{u}_j|^{N'}$, where $\widetilde{u}_j = u_j/\nu^{1/N}$ satisfies $\|\widetilde{u}_j\| \le 1$ for $j \ge j_0$, so

$$\int_\Omega u_j\,h(u_j)\,e^{\alpha\,|u_j|^{N'}}\,dx \to 0.$$

Then, $u_j \to 0$ by (2.2) and hence $c = 0$ by (2.8), contrary to assumption. So u is a nontrivial solution. □

2.2. $\mathbb{Z}_2$-cohomological index

The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [9] is defined as follows. Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander–Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} \sup\left\{m \ge 1 : f^*(\omega^{m-1}) \ne 0\right\}, & A \ne \emptyset \\ 0, & A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \ge 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \le m-1$, so $i(S^{m-1}) = m$.

The following proposition summarizes the basic properties of the cohomological index (see Fadell and Rabinowitz [9]).

Proposition 2.2. *The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:*

(i) Definiteness: $i(A) = 0$ if and only if $A = \emptyset$.

(ii) Monotonicity: If there is an odd continuous map from A to B (in particular, if $A \subset B$), then $i(A) \le i(B)$. Thus, equality holds when the map is an odd homeomorphism.

(iii) Dimension: $i(A) \le \dim W$.

(iv) Continuity: If A is closed, then there is a closed neighbourhood $N \in \mathcal{A}$ of A such that $i(N) = i(A)$. When A is compact, N may be chosen to be a δ-neighbourhood $N_\delta(A) = \{u \in W : dist\,(u, A) \le \delta\}$.

(v) Subadditivity: If A and B are closed, then $i(A \cup B) \le i(A) + i(B)$.

(vi) Stability: If SA is the suspension of $A \ne \emptyset$, obtained as the quotient space of $A \times [-1, 1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, then $i(SA) = i(A) + 1$.

(vii) Piercing property: If A, A_0 and A_1 are closed, and $\varphi : A \times [0, 1] \to A_0 \cup A_1$ is a continuous map such that $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in A \times [0, 1]$, $\varphi(A \times [0, 1])$ is closed, $\varphi(A \times \{0\}) \subset A_0$ and $\varphi(A \times \{1\}) \subset A_1$, then $i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \ge i(A)$.

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(viii) Neighborhood of zero: *If* U *is a bounded closed symmetric neighbourhood of* 0, *then* $i(\partial U) = \dim W$.

2.3. Eigenvalues

Eigenvalues of problem (1.7) coincide with critical values of the functional

$$\Psi(u) = \frac{1}{\displaystyle\int_\Omega |u|^N \, \mathrm{d}x}, \quad u \in S = \left\{ u \in W_0^{1,N}(\Omega) : \int_\Omega |\nabla u|^N \, \mathrm{d}x = 1 \right\}.$$

We have the following proposition (see Perera [11] and Perera *etal*.[12, Proposition 3.52 and Proposition 3.53]).

Proposition 2.3. *Let* $\mathcal{F}$ *denote the class of symmetric subsets of* S *and set*

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$

Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$ *is a sequence of eigenvalues of problem* (1.7). *Moreover, if* $\lambda_{k-1} < \lambda_k$, *then*

$$i(\Psi^{\lambda_{k-1}}) = i(S \backslash \Psi_{\lambda_k}) = k - 1,$$

where $\Psi^a = \{u \in S : \Psi(u) \leq a\}$ *and* $\Psi_a = \{u \in S : \Psi(u) \geq a\}$ *for* $a \in \mathbb{R}$.

We will also need the following result of Degiovanni and Lancelotti [7, Theorem 2.3].

Proposition 2.4. *If* $\lambda_{k-1} < \lambda_k$, *then* $\Psi^{\lambda_{k-1}}$ *contains a compact symmetric set* C *of index* $k - 1$ *that is bounded in* $C^1(\overline{\Omega})$.

2.4. An abstract critical point theorem

We will use the following abstract critical point theorem proved in Yang and Perera [15, Theorem 2.2] to prove Theorem 1.3. This result generalizes the linking theorem of Rabinowitz [13].

Theorem 2.5. *Let* E *be a* C^1*-functional defined on a Banach space* W *and let* A_0 *and* B_0 *be disjoint non-empty closed symmetric subsets of the unit sphere* $S = \{u \in W : \|u\| = 1\}$ *such that*

$$i(A_0) = i(S \backslash B_0) < \infty. \tag{2.9}$$

Assume that there exist $R > \rho > 0$ *and* $\omega \in S \backslash A_0$ *such that*

$$\sup E(A) \leq \inf E(B), \quad \sup E(X) < \infty,$$

where

$$A = \{sv : v \in A_0, \, 0 \leq s \leq R\} \cup \{R\,\pi((1-t)\,v + t\omega) : v \in A_0, \, 0 \leq t \leq 1\},$$
$$B = \{\rho u : u \in B_0\},$$
$$X = \{sv + t\omega : v \in A_0, \, s, t \geq 0, \, \|sv + t\omega\| \leq R\},$$

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and $\pi : W \setminus \{0\} \to S$, $u \mapsto u / \|u\|$ *is the radial projection onto* S. *Let*

$$\Gamma = \{\gamma \in C(X, W) : \gamma(X) \text{ is closed and } \left.\gamma\right|_A = id_A\},$$

and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} E(u).$$

Then, $\inf E(B) \le c \le \sup E(X)$, *and* E *has a* $(PS)_c$ *sequence.*

2.5. Moser sequence

For $j \ge 2$, let

$$\omega_j(x) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log j)^{(N-1)/N}, & |x| \le d/j \\ \dfrac{\log(d/|x|)}{(\log j)^{1/N}}, & d/j < |x| < d \\ 0, & |x| \ge d \end{cases} \tag{2.10}$$

(see Moser [10]).

Proposition 2.6. *We have*

$$\int_\Omega \omega_j^m \, dx = \frac{m! \, \omega_{N-1}^{1-m/N} d^N}{N^{m+1} (\log j)^{m/N}} \left[1 - \frac{1}{j^N} \sum_{l=1}^{m} \frac{(N \log j)^{m-l}}{(m-l)!}\right], \quad m = 1, \ldots, N \tag{2.11}$$

and

$$\int_\Omega |\nabla \omega_j|^m \, dx = \begin{cases} \dfrac{\omega_{N-1}^{1-m/N} d^{N-m}}{(N-m)(\log j)^{m/N}} \left(1 - \dfrac{1}{j^{N-m}}\right), & m = 1, \ldots, N-1 \\ 1, & m = N. \end{cases} \tag{2.12}$$

Proof. We have

$$\int_\Omega \omega_j^m \, dx = \frac{\omega_{N-1}^{1-m/N} d^N}{(\log j)^{m/N}} \left[I_m + \frac{(\log j)^m}{N j^N}\right],$$

where

$$I_m = \int_{1/j}^{1} (-\log s)^m \, s^{N-1} \, ds.$$

We have

$$I_1 = \frac{1}{N^2} \left[1 - \frac{1}{j^N} (N \log j + 1)\right],$$

and integrating by parts gives the recurrence relation

$$I_m = \frac{m}{N} I_{m-1} - \frac{(\log j)^m}{N j^N}, \quad m \ge 2.$$

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So

$$I_m = \frac{m!}{N^{m+1}} \left[1 - \frac{1}{j^N} \sum_{l=0}^{m} \frac{(N \log j)^{m-l}}{(m-l)!} \right],$$

and (2.11) follows. The integral in (2.12) is easily evaluated. □

2.6. A limit calculation

We will need the following limit in the proof of Theorem 1.1.

Proposition 2.7. *We have*

$$\lim_{n \to \infty} \int_0^1 n e^{-n\,(t - t^{N'})}\, \mathrm{d}t = N.$$

Proof. Let $f_n(t) = n e^{-n\,(t - t^{N'})}$ and set $t_0 = (N')^{-1/(N'-1)}$. For $t \neq t_0$,

$$f_n(t) = g_n(t) - \frac{d}{\mathrm{d}t} \left(\frac{e^{-n\,(t - t^{N'})}}{1 - N'\, t^{N'-1}} \right), \tag{2.13}$$

where

$$g_n(t) = \frac{N'(N'-1)\, t^{N'-2}\, e^{-n\,(t - t^{N'})}}{(1 - N'\, t^{N'-1})^2}.$$

Fix δ so small that $0 < \delta < t_0 < 1 - \delta < 1$ and write

$$\int_0^1 f_n(t)\, \mathrm{d}t = \int_0^\delta f_n(t)\, \mathrm{d}t + \int_\delta^{1-\delta} f_n(t)\, \mathrm{d}t + \int_{1-\delta}^1 f_n(t)\, \mathrm{d}t. \tag{2.14}$$

By (2.13),

$$\int_0^\delta f_n(t)\, \mathrm{d}t = \int_0^\delta g_n(t)\, \mathrm{d}t - \frac{e^{-n\,(\delta - \delta^{N'})}}{1 - N'\, \delta^{N'-1}} + 1. \tag{2.15}$$

For all $t \in (0, \delta)$, $g_n(t) \to 0$ as $n \to \infty$ and $|g_n(t)| \leq N'(N'-1)\, t^{N'-2}/(1 - N'\, \delta^{N'-1})^2$, so $\int_0^\delta g_n(t)\, \mathrm{d}t \to 0$ by the dominated convergence theorem. So $\int_0^1 f_n(t)\, \mathrm{d}t \to 1$ by (2.15). A similar calculation shows that $\int_{1-\delta}^1 f_n(t)\, \mathrm{d}t \to N - 1$. On the other hand, it is easily seen that $\int_\delta^{1-\delta} f_n(t)\, \mathrm{d}t \to 0$. So $\int_0^1 f_n(t)\, \mathrm{d}t \to N$ by (2.14). □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by showing that the functional E has the mountain pass geometry with the mountain pass level $c \in (0, (1/N)(\alpha_N/\alpha)^{N-1})$ and applying Proposition 2.1.

Lemma 3.1. *There exists a $\rho > 0$ such that*

$$\inf_{\|u\| = \rho} E(u) > 0.$$

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Proof. Since (1.2) implies that h is bounded, there exists a constant $C_\delta > 0$ such that

$$|G(t)| \leq C_\delta \, |t|^{N+1} \, e^{\alpha \, |t|^{N'}} \quad \text{for } |t| > \delta,$$

which together with (1.9) gives

$$\int_\Omega G(u) \, \mathrm{d}x \leq \frac{1}{N} \, (\lambda_1 - \sigma_1) \int_\Omega |u|^N \, \mathrm{d}x + C_\delta \int_\Omega |u|^{N+1} \, e^{\alpha \, |u|^{N'}} \, \mathrm{d}x. \tag{3.1}$$

By (1.6),

$$\int_\Omega |u|^N \, \mathrm{d}x \leq \frac{\rho^N}{\lambda_1}, \tag{3.2}$$

where $\rho = \|u\|$. By the Hölder inequality,

$$\int_\Omega |u|^{N+1} \, e^{\alpha \, |u|^{N'}} \, \mathrm{d}x \leq \left(\int_\Omega |u|^{2\,(N+1)} \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega e^{2\alpha \, |u|^{N'}} \, \mathrm{d}x \right)^{1/2}. \tag{3.3}$$

The first integral on the right-hand side is bounded by $C\rho^{2\,(N+1)}$ for some constant $C > 0$ by the Sobolev embedding theorem. Since $2\alpha \, |u|^{N'} = 2\alpha \, \rho^{N'} \, |\widetilde{u}|^{N'}$, where $\widetilde{u} = u/\rho$ satisfies $\|\widetilde{u}\| = 1$, the second integral is bounded when $\rho^{N'} \leq \alpha_N / 2\alpha$ by (1.4). So combining (3.1)–(3.3) gives

$$\int_\Omega G(u) \, \mathrm{d}x \leq \frac{1}{N} \left(1 - \frac{\sigma_1}{\lambda_1} \right) \rho^N + \mathrm{O}(\rho^{N+1}) \quad \text{as } \rho \to 0.$$

Then,

$$E(u) \geq \frac{1}{N} \, \frac{\sigma_1}{\lambda_1} \, \rho^N + \mathrm{O}(\rho^{N+1}),$$

and the desired conclusion follows from this for sufficiently small $\rho > 0$. □

We may assume without loss of generality that $B_d(0) \subset \Omega$. Let (ω_j) be the sequence of functions defined in (2.10).

Lemma 3.2. *We have*

(i) $E(t\omega_j) \to -\infty$ *as* $t \to \infty$ *for all* $j \geq 2$,

(ii) $\exists j_0 \geq 2$ *such that*

$$\sup_{t \geq 0} \, E(t\omega_{j_0}) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha} \right)^{N-1}.$$

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Proof. (i) Fix $0 < \varepsilon < \beta$. By (1.3), $\exists M_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha\, |t|^{N'}} > (\beta - \varepsilon)\, e^{\alpha\, |t|^{N'}} \quad \text{for } |t| > M_\varepsilon. \tag{3.4}$$

Since $e^{\alpha\, |t|^{N'}} > \alpha^{2N-2}\, t^{2N}/(2N-2)!$ for all t, then there exists a constant $C_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha\, |t|^{N'}} \geq \frac{1}{(2N-2)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} - C_\varepsilon\, |t| \tag{3.5}$$

and

$$G(t) \geq \frac{2N-1}{(2N)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} - C_\varepsilon\, |t| \tag{3.6}$$

for all t. Since $\|\omega_j\| = 1$ and $\omega_j \geq 0$, then

$$E(t\omega_j) \leq \frac{t^N}{N} - \frac{2N-1}{(2N)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} \int_\Omega \omega_j^{2N}\, \mathrm{d}x + C_\varepsilon\, t \int_\Omega \omega_j\, \mathrm{d}x,$$

and the conclusion follows.

(ii) Set

$$H_j(t) = E(t\omega_j) = \frac{t^N}{N} - \int_\Omega G(t\omega_j)\, \mathrm{d}x, \quad t \geq 0.$$

If the conclusion is false, then it follows from (i) that for all $j \geq 2$, $\exists t_j > 0$ such that

$$H_j(t_j) = \frac{t_j^N}{N} - \int_\Omega G(t_j\omega_j)\, \mathrm{d}x = \sup_{t \geq 0}\, H_j(t) \geq \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}, \tag{3.7}$$

$$H_j'(t_j) = t_j^{N-1} - \int_\Omega \omega_j\, h(t_j\omega_j)\, e^{\alpha\, t_j^{N'}\, \omega_j^{N'}}\, \mathrm{d}x = 0. \tag{3.8}$$

Since $G(t) \geq -C_\varepsilon\, t$ for all $t \geq 0$ by (3.6), (3.7) gives

$$t_j^N \geq t_0^N - N\delta_j\, t_j, \tag{3.9}$$

where

$$t_0 = \left(\frac{\alpha_N}{\alpha}\right)^{(N-1)/N}$$

and

$$\delta_j = C_\varepsilon \int_\Omega \omega_j\, \mathrm{d}x \to 0 \quad \text{as } j \to \infty \tag{3.10}$$

by Proposition 2.6. First, we will show that $t_j \to t_0$.

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By (3.9) and the Young's inequality,

$$(1+\nu)\, t_j^N \ge t_0^N - \frac{N-1}{\nu^{1/(N-1)}}\, \delta_j^{N'} \quad \forall \nu > 0,$$

which together with (3.10) gives

$$\liminf_{j \to \infty} t_j \ge t_0. \tag{3.11}$$

Write (3.8) as

$$t_j^N = \int_{\{t_j \omega_j > M_\varepsilon\}} t_j \omega_j\, h(t_j \omega_j)\, e^{\alpha\, t_j^{N'} \omega_j^{N'}}\, \mathrm{d}x + \int_{\{t_j \omega_j \le M_\varepsilon\}} t_j \omega_j\, h(t_j \omega_j)\, e^{\alpha\, t_j^{N'} \omega_j^{N'}}\, \mathrm{d}x =: I_1 + I_2. \tag{3.12}$$

Set $r_j = d e^{-M_\varepsilon\, (\omega_{N-1} \log j)^{1/N} / t_j}$. Since $\liminf t_j > 0$, for all sufficiently large j, $d/j < r_j < d$ and $t_j \omega_j(x) > M_\varepsilon$ if and only if $|x| < r_j$. So, (3.4) gives

$$I_1 \ge (\beta - \varepsilon) \int_{\{|x| < r_j\}} e^{\alpha\, t_j^{N'} \omega_j^{N'}}\, \mathrm{d}x = (\beta - \varepsilon) \left(\int_{\{|x| \le d/j\}} e^{\alpha\, t_j^{N'} \omega_j^{N'}}\, \mathrm{d}x \right. \tag{3.13}$$

$$\left. + \int_{\{d/j < |x| < r_j\}} e^{\alpha\, t_j^{N'} \omega_j^{N'}}\, \mathrm{d}x \right) =: (\beta - \varepsilon)\, (I_3 + I_4). \tag{3.14}$$

We have

$$I_3 = \frac{\omega_{N-1}}{N} \left(\frac{d}{j} \right)^N e^{\alpha\, t_j^{N'} \log j / \omega_{N-1}^{1/(N-1)}} = \frac{\omega_{N-1}}{N}\, d^N j^{\alpha\, (t_j^{N'} - t_0^{N'}) / \omega_{N-1}^{1/(N-1)}}. \tag{3.15}$$

Since $t h(t)\, e^{\alpha\, |t|^{N'}} \ge -C_\varepsilon\, t$ for all $t \ge 0$ by (3.5),

$$I_2 \ge -C_\varepsilon\, t_j \int_{\{t_j \omega_j \le M_\varepsilon\}} \omega_j\, \mathrm{d}x \ge -\delta_j\, t_j. \tag{3.16}$$

Combining (3.12)–(3.16) and noting that $I_4 \ge 0$ gives

$$t_j^N \ge (\beta - \varepsilon)\, \frac{\omega_{N-1}}{N}\, d^N j^{\alpha\, (t_j^{N'} - t_0^{N'}) / \omega_{N-1}^{1/(N-1)}} - \delta_j\, t_j.$$

It follows from this that

$$\limsup_{j \to \infty} t_j \le t_0,$$

which together with (3.11) shows that $t_j \to t_0$.

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Next, we estimate I_4. We have

$$\begin{aligned}
I_4 &= \int_{\{d/j<|x|<r_j\}} e^{\alpha\, t_j^{N'} [\log(d/|x|)]^{N'}/(\omega_{N-1}\log j)^{1/(N-1)}}\, \mathrm{d}x \\
&= \omega_{N-1} \left(\int_{d/j}^{d} e^{\alpha\, t_j^{N'} [\log(d/r)]^{N'}/(\omega_{N-1}\log j)^{1/(N-1)}}\, r^{N-1}\, \mathrm{d}r \right. \\
&\quad \left. - \int_{r_j}^{d} e^{\alpha\, t_j^{N'} [\log(d/r)]^{N'}/(\omega_{N-1}\log j)^{1/(N-1)}}\, r^{N-1}\, \mathrm{d}r \right) \\
&= \omega_{N-1}\, d^N \Big(\log j \int_0^1 e^{-Nt\,[1-(t_j/t_0)^{N'} t^{1/(N-1)}]\log j}\, \mathrm{d}t \\
&\quad - \int_{s_j}^{1} s^{N-1}\, e^{\alpha\, t_j^{N'}(-\log s)^{N'}/(\omega_{N-1}\log j)^{1/(N-1)}}\, \mathrm{d}s\Big),
\end{aligned} \tag{3.17}$$

where $t = \log(d/r)/\log j$, $s = r/d$, and $s_j = r_j/d = e^{-M_\varepsilon\,(\omega_{N-1}\log j)^{1/N}/t_j} \to 0$. For $s_j < s < 1$, $\alpha\, t_j^{N'}(-\log s)^{N'}/(\omega_{N-1}\log j)^{1/(N-1)}$ is bounded by $\alpha M_\varepsilon^{N'}$ and goes to zero as $j \to \infty$, so the last integral converges to

$$\int_0^1 s^{N-1}\, \mathrm{d}s = \frac{1}{N}.$$

So, combining (3.12)–(3.17) and letting $j \to \infty$ gives

$$t_0^N \geq (\beta - \varepsilon)\, \frac{\omega_{N-1}}{N}\, d^N (L_1 + L_2 - 1),$$

where

$$\begin{aligned}
L_1 &= \liminf_{j\to\infty}\, e^{-n\,[1-(t_j/t_0)^{N'}]}, \\
L_2 &= \liminf_{j\to\infty} \int_0^1 n e^{-n\,[t-(t_j/t_0)^{N'} t^{N'}]}\, \mathrm{d}t,
\end{aligned}$$

and $n = N\log j \to \infty$. Letting $\varepsilon \to 0$ in this inequality gives

$$\beta \leq \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N \frac{1}{L_1 + L_2 - 1}. \tag{3.18}$$

By (3.7), (1.8), and Proposition 2.6,

$$t_j^N - t_0^N \geq N \int_\Omega G(t_j\omega_j)\, \mathrm{d}x \geq -\sigma_0\, t_j^N \int_\Omega \omega_j^N\, \mathrm{d}x \geq -\frac{\sigma_0\, t_j^N}{\kappa n},$$

so

$$\left(\frac{t_j}{t_0}\right)^{N'} \geq \left(1 + \frac{\sigma_0}{\kappa n}\right)^{-1/(N-1)} \geq 1 - \frac{\sigma_0}{(N-1)\,\kappa n}.$$

This gives

$$L_1 \geq e^{-\sigma_0/(N-1)\,\kappa}$$

and

$$L_2 \geq \lim_{n\to\infty} \int_0^1 n e^{-n\,(t-t^{N'})-\sigma_0\,t^{N'}/(N-1)\,\kappa}\,\mathrm{d}t \geq N e^{-\sigma_0/(N-1)\,\kappa}$$

by Proposition 2.7. So (3.18) gives

$$\beta \leq \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N \frac{1}{N e^{-\sigma_0/(N-1)\,\kappa} - (1 - e^{-\sigma_0/(N-1)\,\kappa})} \leq \frac{1}{N\alpha^{N-1}} \left(\frac{N}{d}\right)^N e^{\sigma_0/(N-1)\,\kappa},$$

contradicting (1.10). □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let j_0 be as in Lemma 3.2 (ii). By Lemma 3.2 (i), $\exists R > \rho$ such that $E(R\omega_{j_0}) \leq 0$, where ρ is as in Lemma 3.1. Let

$$\Gamma = \left\{\gamma \in C([0,1], W_0^{1,N}(\Omega)) : \gamma(0) = 0,\, \gamma(1) = R\omega_{j_0}\right\}$$

be the class of paths joining the origin to $R\omega_{j_0}$, and set

$$c := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} E(u).$$

By Lemma 3.1, $c > 0$. Since the path $\gamma_0(t) = tR\omega_{j_0}$, $t \in [0,1]$ is in Γ,

$$c \leq \max_{u\in\gamma_0([0,1])} E(u) \leq \sup_{t\geq 0} E(t\omega_{j_0}) < \frac{1}{N}\left(\frac{\alpha_N}{\alpha}\right)^{N-1}.$$

If there are no $(\mathrm{PS})_c$ sequences of E, then E satisfies the $(\mathrm{PS})_c$ condition vacuously and hence has a critical point u at the level c by the mountain pass theorem. Then u is a solution of problem (1.1) and u is non-trivial since $c > 0$. So we may assume that E has a $(\mathrm{PS})_c$ sequence. Then this sequence has a subsequence that converges weakly to a non-trivial solution of problem (1.1) by Proposition 2.1. □

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 using Theorem 2.5. We take A_0 to be the set C in Proposition 2.4 and $B_0 = \Psi_{\lambda_k}$. Since $i(S\backslash B_0) = k - 1$ by Proposition 2.3, (2.9) holds.

Lemma 4.1. *There exists a* $\rho > 0$ *such that* $\inf E(B) > 0$, *where* $B = \{\rho u : u \in B_0\}$.

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Proof. As in the proof of Lemma 3.1, there exists a constant $C_\delta > 0$ such that

$$|G(t)| \le C_\delta\, |t|^{N+1}\, e^{\alpha\, |t|^{N'}} \quad \text{for } |t| > \delta,$$

which together with (1.12) gives

$$G(t) \le \frac{1}{N}\,(\lambda_k - \sigma_1)\, |t|^N + C_\delta\, |t|^{N+1}\, e^{\alpha\, |t|^{N'}} \quad \forall t. \tag{4.1}$$

For $u \in B_0$ and $\rho > 0$,

$$\int_\Omega |\rho u|^N\, \mathrm{d}x \le \frac{\rho^N}{\lambda_k} \tag{4.2}$$

and

$$\int_\Omega |\rho u|^{N+1}\, e^{\alpha\, |\rho u|^{N'}}\, \mathrm{d}x \le \rho^{N+1} \left(\int_\Omega |u|^{2\,(N+1)}\, \mathrm{d}x\right)^{1/2} \left(\int_\Omega e^{2\alpha\, \rho^{N'} |u|^{N'}}\, \mathrm{d}x\right)^{1/2}. \tag{4.3}$$

The first integral on the right-hand side of (4.3) is bounded by the Sobolev embedding theorem, and the second integral is bounded when $\rho^{N'} \le \alpha_N/2\alpha$ by (1.4). So, combining (4.1)–(4.3) gives

$$\int_\Omega G(\rho u)\, \mathrm{d}x \le \frac{1}{N}\left(1 - \frac{\sigma_1}{\lambda_k}\right) \rho^N + \mathrm{O}(\rho^{N+1}) \quad \text{as } \rho \to 0.$$

Then,

$$E(\rho u) \ge \frac{1}{N}\, \frac{\sigma_1}{\lambda_k}\, \rho^N + \mathrm{O}(\rho^{N+1}),$$

and the desired conclusion follows from this for sufficiently small ρ. □

We may assume without loss of generality that $B_d(0) \subset \Omega$. Let (ω_j) be the sequence of functions defined in (2.10).

Lemma 4.2. *We have*

(i) $E(sv) \le 0 \;\; \forall v \in A_0,\ s \ge 0$,

(ii) for all $j \ge 2$,

$$\sup\{E(R\,\pi((1-t)\,v + t\omega_j)) : v \in A_0,\ 0 \le t \le 1\} \to -\infty \text{ as } R \to \infty,$$

(iii) $\exists j_0 \ge 2$ *such that*

$$\sup\{E(sv + t\omega_{j_0}) : v \in A_0,\ s, t \ge 0\} < \frac{1}{N}\left(\frac{\alpha_N}{\alpha}\right)^{N-1}.$$

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Proof. (i) By (1.11),

$$E(u) \leq \frac{1}{N} \left[\int_\Omega |\nabla u|^N \, \mathrm{d}x - (\lambda_{k-1} + \sigma_0) \int_\Omega |u|^N \, \mathrm{d}x\right]. \tag{4.4}$$

For $v \in A_0$ and $s \geq 0$,

$$\int_\Omega |sv|^N \, \mathrm{d}x \geq \frac{s^N}{\lambda_{k-1}}$$

since $A_0 \subset \Psi^{\lambda_{k-1}}$, so (4.4) gives

$$E(sv) \leq -\frac{1}{N} \frac{\sigma_0}{\lambda_{k-1}} s^N \leq 0.$$

(ii) Fix $0 < \varepsilon < \beta$. As in the proof of Lemma 3.2 (i), $\exists M_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha\, |t|^{N'}} > (\beta - \varepsilon)\, e^{\alpha\, |t|^{N'}} \quad \text{for } |t| > M_\varepsilon \tag{4.5}$$

and there exists a constant $C_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha\, |t|^{N'}} \geq \frac{1}{(2N-2)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} - C_\varepsilon\, |t| \tag{4.6}$$

and

$$G(t) \geq \frac{2N-1}{(2N)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} - C_\varepsilon\, |t| \tag{4.7}$$

for all t. Let $A_1 = \{\pi((1-t)\, v + t\omega_j) : v \in A_0,\ 0 \leq t \leq 1\}$. For $u \in A_1$ and $R > 0$, (4.7) gives

$$E(Ru) \leq \frac{R^N}{N} - \frac{2N-1}{(2N)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, R^{2N} \int_\Omega |u|^{2N} \, \mathrm{d}x + C_\varepsilon R \int_\Omega |u| \, \mathrm{d}x.$$

The set A_1 is compact since A_0 is compact, so the first integral on the right-hand side is bounded away from zero on A_1. Since the second integral is bounded, the desired conclusion follows.

(iii) If the conclusion is false, then it follows from (i) and (ii) that for all $j \geq 2$, there exist $v_j \in A_0$, $s_j \geq 0$, $t_j > 0$ such that

$$E(s_j v_j + t_j \omega_j) = \sup\, \{E(sv + t\omega_j) : v \in A_0,\ s, t \geq 0\} \geq \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}.$$

Set $u_j = s_j v_j + t_j \omega_j$. Then

$$E(u_j) = \frac{1}{N}\, \|u_j\|^N - \int_\Omega G(u_j) \, \mathrm{d}x \geq \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}. \tag{4.8}$$

Moreover, $\tau u_j \in \{sv + t\omega_j : v \in A_0,\ s, t \geq 0\}$ for all $\tau \geq 0$ and $E(\tau u_j)$ attains its maximum at $\tau = 1$, so

$$\frac{\partial}{\partial \tau}\, E(\tau u_j)\bigg|_{\tau=1} = E'(u_j)\, u_j = \|u_j\|^N - \int_\Omega u_j\, h(u_j)\, e^{\alpha\, |u_j|^{N'}} \, \mathrm{d}x = 0. \tag{4.9}$$

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Since $\|v_j\| = \|\omega_j\| = 1$ and $G(t) \geq 0$ for all t by (1.11), (4.8) gives

$$s_j + t_j \geq t_0,$$

where

$$t_0 = \left(\frac{\alpha_N}{\alpha}\right)^{(N-1)/N}.$$

First, we show that $s_j \to 0$ and $t_j \to t_0$ as $j \to \infty$.

Combining (4.8) with (1.11) gives

$$\|s_j v_j + t_j \omega_j\|^N \geq (\lambda_{k-1} + \sigma_0) \int_\Omega |s_j v_j + t_j \omega_j|^N \,\mathrm{d}x + t_0^N.$$

Set $\tau_j = s_j / t_j$. Then,

$$\|\tau_j v_j + \omega_j\|^N \geq (\lambda_{k-1} + \sigma_0) \int_\Omega |\tau_j v_j + \omega_j|^N \,\mathrm{d}x + (\frac{t_0}{t_j})^N. \tag{4.10}$$

Since (v_j) is bounded in $C^1(\overline{\Omega})$, Proposition 2.6 gives

$$\|\tau_j v_j + \omega_j\|^N \leq \int_\Omega (\tau_j |\nabla v_j| + |\nabla \omega_j|)^N \,\mathrm{d}x = \tau_j^N \int_\Omega |\nabla v_j|^N \,\mathrm{d}x + \int_\Omega |\nabla \omega_j|^N \,\mathrm{d}x$$
$$+ \sum_{m=1}^{N-1} \binom{N}{m} \tau_j^{N-m} \int_\Omega |\nabla v_j|^{N-m} |\nabla \omega_j|^m \,\mathrm{d}x \leq \tau_j^N + 1 + c_1 \sum_{m=1}^{N-1} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}$$

and

$$\int_\Omega |\tau_j v_j + \omega_j|^N \,\mathrm{d}x \geq \int_\Omega (\tau_j |v_j| - \omega_j)^N \,\mathrm{d}x = \tau_j^N \int_\Omega |v_j|^N \,\mathrm{d}x$$
$$+ \sum_{m=1}^{N} (-1)^m \binom{N}{m} \tau_j^{N-m} \int_\Omega |v_j|^{N-m} \omega_j^m \,\mathrm{d}x \geq \frac{\tau_j^N}{\lambda_{k-1}} - c_2 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}$$

for some constants $c_1, c_2 > 0$. So (4.10) gives

$$\frac{\sigma_0}{\lambda_{k-1}} \tau_j^N + \left(\frac{t_0}{t_j}\right)^N \leq 1 + c_3 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}} \tag{4.11}$$

for some constant $c_3 > 0$, which implies that (τ_j) is bounded and

$$\liminf_{j \to \infty} t_j \geq t_0. \tag{4.12}$$

Next, combining (4.9) with (4.5) and (4.6) gives

$$\|u_j\|^N = \int_{\{|u_j| > M_\varepsilon\}} u_j\, h(u_j)\, e^{\alpha |u_j|^{N'}} \,\mathrm{d}x + \int_{\{|u_j| \leq M_\varepsilon\}} u_j\, h(u_j)\, e^{\alpha |u_j|^{N'}} \,\mathrm{d}x$$
$$\geq (\beta - \varepsilon) \int_{\{|u_j| > M_\varepsilon\}} e^{\alpha |u_j|^{N'}} \,\mathrm{d}x - C_\varepsilon \int_{\{|u_j| \leq M_\varepsilon\}} |u_j| \,\mathrm{d}x. \tag{4.13}$$

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For $|x| \leq d/j$,

$$|u_j| \geq t_j\omega_j - s_j\,|v_j| \geq \frac{t_j}{\omega_{N-1}^{1/N}}\left[(\log j)^{(N-1)/N} - c_4\tau_j\right]$$

for some constant $c_4 > 0$, and the last expression is greater than M_ε for all sufficiently large j since (τ_j) is bounded and $\liminf t_j > 0$. So

$$\int_{\{|u_j|>M_\varepsilon\}} e^{\alpha\,|u_j|^{N'}}\,\mathrm{d}x \geq e^{\alpha\, t_j^{N'}[(\log j)^{(N-1)/N}-c_4\tau_j]^{N'}/\omega_{N-1}^{1/(N-1)}} \int_{\{|x|\leq d/j\}} \mathrm{d}x$$
$$= \frac{\omega_{N-1}\,d^N}{N}\, j^{\alpha\,[t_j^{N'}(1-c_4\tau_j/(\log j)^{(N-1)/N})^{N'}-t_0^{N'}]/\omega_{N-1}^{1/(N-1)}}$$

for large j. On the contrary,

$$\int_{\{|u_j|\leq M_\varepsilon\}} |u_j|\,\mathrm{d}x \leq \int_\Omega (s_j\,|v_j| + t_j\omega_j)\,\mathrm{d}x \leq c_5\,t_j\left[\tau_j + \frac{1}{(\log j)^{1/N}}\right]$$

for some constant $c_5 > 0$ by Proposition 2.6. So, (4.13) gives

$$(\beta-\varepsilon)\,j^{\alpha\,[t_j^{N'}(1-c_4\tau_j/(\log j)^{(N-1)/N})^{N'}-t_0^{N'}]/\omega_{N-1}^{1/(N-1)}} \leq \frac{N t_j^N(\tau_j+1)^N}{\omega_{N-1}\,d^N}$$
$$+ c_6\,t_j\left[\tau_j + \frac{1}{(\log j)^{1/N}}\right] \tag{4.14}$$

for some constant $c_6 > 0$. Since (τ_j) is bounded, it follows from this that

$$\limsup_{j\to\infty} t_j \leq t_0,$$

which together with (4.12) shows that $t_j \to t_0$. Then (4.11) implies that $\tau_j \to 0$, so $s_j = \tau_j\,t_j \to 0$.

Now, we show that there exists a constant $c > 0$ depending only on Ω, α, and k such that

$$\beta \leq \frac{1}{\alpha^{N-1}}\left(\frac{N}{d}\right)^N e^{c/\sigma_0^{N-1}}. \tag{4.15}$$

The right-hand side of (4.14) goes to $(N/d)^N/\alpha^{N-1}$ as $j\to\infty$. If $\beta \leq (N/d)^N/\alpha^{N-1}$, then we may take any $c > 0$, so suppose $\beta > (N/d)^N/\alpha^{N-1}$. Then for $\varepsilon < \beta - (N/d)^N/\alpha^{N-1}$ and all sufficiently large j, (4.14) gives $j^{\alpha\,[t_j^{N'}(1-c_4\tau_j/(\log j)^{(N-1)/N})^{N'}-t_0^{N'}]/\omega_{N-1}^{1/(N-1)}} \leq 1$, so

$$\frac{t_0}{t_j} \geq 1 - \frac{c_4\tau_j}{(\log j)^{(N-1)/N}}.$$

Combining this with (4.11) gives

$$\frac{\sigma_0}{\lambda_{k-1}}\,\tau_j^N - \frac{N c_4\tau_j}{(\log j)^{(N-1)/N}} \leq c_3\sum_{m=1}^{N}\frac{\tau_j^{N-m}}{(\log j)^{m/N}},$$

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so

$$\sigma_0 \tau_j^N \le c_7 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}$$

for some constant $c_7 > 0$. Set $\widetilde{\tau}_j = \tau_j (\log j)^{1/N}$. Then

$$\sigma_0 \widetilde{\tau}_j^N \le c_7 \sum_{m=1}^{N} \widetilde{\tau}_j^{N-m}. \tag{4.16}$$

We claim that

$$\widetilde{\tau}_j \le \frac{c_8}{\sigma_0} \tag{4.17}$$

for some constant $c_8 > 0$. Taking σ_0 smaller in (1.11) if necessary, we may assume that $\sigma_0 \le 1$. So if $\widetilde{\tau}_j < 1$, then (4.17) holds with $c_8 = 1$, so suppose $\widetilde{\tau}_j \ge 1$. Then (4.16) gives (4.17) with $c_8 = N c_7$. Now (4.11) gives

$$\left(\frac{t_0}{t_j}\right)^N \le 1 + \frac{c_3}{\log j} \sum_{m=1}^{N} \widetilde{\tau}_j^{N-m} \le 1 + \frac{c_9}{\sigma_0^{N-1} \log j}$$

for some constant $c_9 > 0$, so

$$\left(\frac{t_0}{t_j}\right)^{N'} \le \left(1 + \frac{c_9}{\sigma_0^{N-1} \log j}\right)^{1/(N-1)} \le 1 + \frac{c_9}{\sigma_0^{N-1} \log j}.$$

Then,

$$\begin{aligned} t_j^{N'} \left[1 - \frac{c_4 \tau_j}{(\log j)^{(N-1)/N}}\right]^{N'} - t_0^{N'} &= t_j^{N'} \left[\left(1 - \frac{c_4 \widetilde{\tau}_j}{\log j}\right)^{N'} - \left(\frac{t_0}{t_j}\right)^{N'}\right] \\ &\ge t_j^{N'} \left[(1 - \frac{c_{10}}{\sigma_0 \log j})^{N'} - \left(1 + \frac{c_9}{\sigma_0^{N-1} \log j}\right)\right] \ge -t_j^{N'} \left(\frac{N' c_{10}}{\sigma_0 \log j} + \frac{c_9}{\sigma_0^{N-1} \log j}\right) \\ &\ge -\frac{c_{11}}{\sigma_0^{N-1} \log j} \end{aligned}$$

for some constants $c_{10}, c_{11} > 0$, so

$$j^{\alpha\, [t_j^{N'} (1 - c_4 \tau_j/(\log j)^{(N-1)/N})^{N'} - t_0^{N'}]/\omega_{N-1}^{1/(N-1)}} \ge j^{-c/\sigma_0^{N-1} \log j} = e^{-c/\sigma_0^{N-1}}$$

for some constant $c > 0$. Combining this with (4.14) and passing to the limit gives

$$(\beta - \varepsilon)\, e^{-c/\sigma_0^{N-1}} \le \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N,$$

and letting $\varepsilon \to 0$ gives (4.15). □

We are now ready to prove Theorem 1.3.

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Proof of Theorem 1.3. Let $j_0 \geq 2$ be as in Lemma 4.2 (iii). By Lemma 4.2 (ii), $\exists R > \rho$ such that

$$\sup \{E(R\,\pi((1-t)\,v + t\omega_{j_0})) : v \in A_0,\, 0 \leq t \leq 1\} \leq 0, \tag{4.18}$$

where $\rho > 0$ is as in Lemma 4.1. Let

$$A = \{sv : v \in A_0,\, 0 \leq s \leq R\} \cup \{R\,\pi((1-t)\,v + t\omega_{j_0}) : v \in A_0,\, 0 \leq t \leq 1\},$$
$$X = \{sv + t\omega_{j_0} : v \in A_0,\, s, t \geq 0,\, \|sv + t\omega_{j_0}\| \leq R\}.$$

Combining Lemma 4.2 (i), (4.18), and Lemma 4.1 gives

$$\sup E(A) \leq 0 < \inf E(B), \tag{4.19}$$

while Lemma 4.2 (iii) gives

$$\sup E(X) \leq \sup \{E(sv + t\omega_{j_0}) : v \in A_0,\, s, t \geq 0\} < \frac{1}{N}\left(\frac{\alpha_N}{\alpha}\right)^{N-1}. \tag{4.20}$$

Let

$$\Gamma = \{\gamma \in C(X, W) : \gamma(X) \text{ is closed and } \gamma|_A = id_A\},$$

and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} E(u).$$

By Theorem 2.5, $\inf E(B) \leq c \leq \sup E(X)$, and E has a $(\mathrm{PS})_c$ sequence. By (4.19) and (4.20),

$$0 < c < \frac{1}{N}\left(\frac{\alpha_N}{\alpha}\right)^{N-1},$$

so a subsequence of this $(\mathrm{PS})_c$ sequence converges weakly to a non-trivial solution of problem (1.1) by Proposition 2.1. □

Competing interests declaration

The authors declare no competing interests.

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