On the generalizations of the theorems of Parseval and Riesz-Fischer. By Mr S. POLLARD, Trinity College.

[*Received* 14 June, *read* 25 July 1927.]

1. Any two real numbers p and q will be called *conjugate* if

$$\frac{1}{p}+\frac{1}{q}=1.$$

In what follows it is supposed that $p \geqslant 1$. This implies $q \geqslant 1$ and either $p=q=2$ or p and q are separated by 2.

The theorems in question are:

A. *If* $p \geqslant 2$ *and*

$$\int_{-\pi}^{\pi} |f(x)|^q \, dx < \infty, \tag{1·1}$$

then

$$\left|\frac{a_0}{\sqrt{2}}\right|^p + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p) < \infty,$$

where, for $n = 0, 1, 2, \ldots,$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt,$$

and

$$\left[\left|\frac{a_0}{\sqrt{2}}\right|^p + \sum_{n=1}^{\infty}(|a_n|^p + |b_n|^p)\right]^{\frac{1}{p}} \leqslant \left[\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^q \, dx\right]^{\frac{1}{q}}.$$

B. *If* $p \leqslant 2$ *and*

$$\left|\frac{a_0}{\sqrt{2}}\right|^p + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p) < \infty, \tag{1·2}$$

then there is a function $g(x)$ *such that*

$$\int_{-\pi}^{\pi} |g(x)|^q < \infty,$$

and, for $n = 0, 1, 2, \ldots,$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t) \sin nt \, dt. \tag{1·3}$$

Moreover

$$\left[\left|\frac{a_0}{\sqrt{2}}\right|^p + \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p)\right]^{\frac{1}{p}} \geqslant \left[\frac{1}{\pi}\int_{-\pi}^{\pi} |g(x)|^q \, dx\right]^{\frac{1}{q}}. \tag{1·4}$$

Theorem A generalizes Parseval's theorem and theorem B that of Riesz-Fischer*.

* E. W. Hobson, *Theory of Functions of a Real Variable*, 2 (1926), 599–606.

It will be observed that a distinction is drawn between the function which appears in the enunciation of B and that which appears in the enunciation of A. The two functions should not be confused.

2. As is well known these two theorems are of different types. Theorem A is of "function" type, i.e. is a direct theorem about the Fourier series of a given function; while theorem B is of "coefficient" type, i.e. is a theorem about trigonometrical series whose terms are given. It will be observed that, when the coefficients in theorem B are the Fourier coefficients of a given function $f(x)$, there is nothing in the theorem which allows us to identify this function with the function $g(x)$ given by the theorem. It is certainly true that $f(x)$ and $g(x)$ are essentially the same, but this is because of a special property of the trigonometrical orthogonal system which it does not share with all orthogonal systems.

The object of this note is to obtain a single theorem of "function" type which covers as much as possible of the ground covered by theorems A and B. It does not cover all the ground, but requires only to be supplemented by part of the ordinary Riesz-Fischer theorem, i.e. by the consequences of the usual theory of convergence on the mean.

It should be observed that the single theorem is designed in such a way that its hypothesis is as free as possible from restrictions. The only assumption made is that of integrability in the general Denjoy sense.

3. Consider:

C. *If $f(x)$ is integrable in the general Denjoy sense in $(-\pi, \pi)$ and, for $n = 0, 1, 2, \ldots$,*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt,$$

then

$$\left[\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^q\,dx\right]^{\frac{1}{q}}$$

bears to

$$\left[\left|\frac{a_0}{\sqrt{2}}\right|^p + \sum_{n=1}^{\infty}(|a_n|^p + |b_n|^p)\right]^{\frac{1}{p}}$$

the same relation of inequality that p bears to 2 in the sense that

$$(3{\cdot}1) \qquad \left[\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^q\,dx\right]^{\frac{1}{q}} \geqslant \left[\left|\frac{a_0}{\sqrt{2}}\right|^p + \sum_{n=1}^{\infty}(|a_n|^p + |b_n|^p)\right]^{\frac{1}{q}} \qquad (p \geqslant 2),$$

(3·2)
$$\left[\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^q\,dx\right]^{\frac{1}{q}} \leqslant \left[\left|\frac{a_0}{\sqrt{2}}\right|^p + \sum_{n=1}^{\infty}(|a_n|^p + |b_n|^p)\right]^{\frac{1}{p}} \qquad (p \leqslant 2).$$

D. *If*

(3·3)
$$\tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) < \infty$$

then the equations

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin nt\,dt\ (n = 0, 1, 2\ldots)$$

have a solution $g(t)$ *which is integrable.*

Of these D is part of the ungeneralized Riesz-Fischer theorem, and C is the proposed direct theorem.

4. LEMMA. *If* $f(x)$ *and* $g(x)$ *are integrable in* $(-\pi, \pi)$ *in the general Denjoy sense and have the same Fourier coefficients, then**

(4·1)
$$f(x) \sim g(x) \qquad (-\pi \leqslant x \leqslant \pi).$$

Write

$$F(x) = \int_{-\pi}^{x}\{f(t) - \tfrac{1}{2}a_0\}\,dt, \qquad G(x) = \int_{-\pi}^{x}\{g(t) - \tfrac{1}{2}a_0\}\,dt,$$

so that $F(-\pi)$, $F(\pi)$, $G(-\pi)$, $G(\pi)$ all vanish and integration by parts shows that $F(x)$ and $G(x)$ have the same Fourier coefficients.

Now $F(x)$ and $G(x)$ are continuous functions and therefore, by Fejér's theorem, can be obtained from their Fourier series by the process of summation by arithmetic means. Since the Fourier series are the same it follows that $F(x)$ and $G(x)$ are the same.

Denjoy and others have, however, shown that a function which is integrable in the general Denjoy sense can be obtained almost everywhere from its indefinite integral by the process of asymptotic differentiation. Thus, provided that a function cannot have at a given point two different asymptotic differential coefficients,

$$f(x) - \tfrac{1}{2}a_0 \sim g(x) - \tfrac{1}{2}a_0 \qquad (-\pi \leqslant x \leqslant \pi),$$

which is (4·1).

To investigate the point just raised suppose that, if possible, $F(x)$ has at a point ξ two different asymptotic differential coefficients Δ_1, Δ_2. Then there are sets E_1, E_2 of unit density at ξ such that

$$\frac{F(x) - F(\xi)}{x - \xi} \to \Delta_1 \qquad (x \to \xi \text{ in } E_1),$$
$$\to \Delta_2 \qquad (x \to \xi \text{ in } E_2).$$

* By (4·1) is to be understood that $f(x) = g(x)$ throughout $-\pi \leqslant x \leqslant \pi$ except possibly at the points of a set of measure zero.

Since E_1 and E_2 are of unit density at ξ, corresponding to each positive number ϵ there is a positive number η such that

$$mE_1I, \quad mE_2I > (1-\epsilon)\, mI$$

for all intervals I enclosing ξ of length not greater than η. Thus

$$mE_1E_2I > (1-2\epsilon)\, mI,$$

which shows that E_1E_2 has unit density at ξ.

This is impossible unless ξ is a limit point of E_1E_2, and we now have $\dfrac{F(x)-F(\xi)}{x-\xi}$ tending to both Δ_1 and Δ_2 as $x \to \xi$ in E_1E_2, which cannot be the case.

5. Suppose $p \geqslant 2$. Then unless (1·1) holds, (3·1) goes without saying, and if (1·1) holds, (3·1) is a consequence of A.

Now suppose $p \leqslant 2$. Unless (1·2) holds, (3·2) goes without saying. If (1·2) holds, then there is a function $g(x)$ such that (1·3) holds. The functions $f(x)$ and $g(x)$ have the same Fourier coefficients and, by the lemma, must coincide almost everywhere. And now (3·2) follows from (1·4).

Theorem C is thus completely established.

It will be observed that, as (1·2) with $p \leqslant 2$ implies (3·3), the special case D of B is sufficient to establish the existence of the function $g(x)$.

6. It is easily seen that C and D are in effect A and B together with the equivalence of $f(x)$ and $g(x)$. So that C and D state a little more than A and B.

In view of this a certain amount of care has to be exercised. For C and D do not extend to the case of an arbitrary orthogonal system in the same way that A and B do.

The difference is most easily illustrated in the case $p=2$, when A and B are known to hold, with appropriate modifications, for all complete systems of normalized orthogonal functions. The same is true of D, but not of C.

For, with the aid of a construction due to Banach*, it is possible, given any integrable function $f(x)$ such that

$$\int_a^b f^2(t)\, dt = \infty, \qquad \int_a^b f(t)\, dt \neq 0,$$

to obtain a complete system of normalized orthogonal functions

(6·1) $$\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_n(x),\ \ldots,$$

* S. Banach, *Proc. London Math. Soc.* (2) 21 (1923), 95–97. Banach assumes that $f(x)$ is positive and so absolutely integrable. It is interesting to verify that his construction can be extended to the case of conditionally integrable functions.

with (a, b) as basic interval, such that, if

$$c_n = \int_a^b f(t)\,\phi_n(t)\,dt,$$

then, for every n,

(6·2) $$c_n = 0,$$

so that the analogue of (3·2) cannot hold.

7. Let $f(x)$ be as described and write

$$\gamma = \int_a^b f(t)\,dt,$$

so that $\gamma \neq 0$.

Take $$\psi_1(x) = \left(\frac{1}{b-a}\right)^{\frac{1}{2}},$$

$$\psi_{2r}(x) = \left(\frac{2}{b-a}\right)^{\frac{1}{2}} \cos\left(2r\pi\frac{x-a}{b-a}\right),$$

$$\psi_{2r+1}(x) = \left(\frac{2}{b-a}\right)^{\frac{1}{2}} \sin\left(2r\pi\frac{x-a}{b-a}\right) \qquad (r \geqslant 1),$$

so that the functions $\psi_r(x)$ are the trigonometrical orthogonal system adjusted to the interval (a, b) and normalized.

Since each $\psi_n(x)$ is of bounded variation, the product $f(x)\,\psi_n(x)$ is integrable. Write

$$\gamma_n = \int_a^b f(t)\,\psi_n(t)\,dt,$$

so that, by theorem C,

(7·1) $$\sum_{n=1}^{\infty} \gamma_n^2 = \int_a^b f^2(t)\,dt = \infty.$$

Take now $$\chi_n(x) = \psi_n(x) - \frac{\gamma_n}{\gamma},$$

so that

(7·2) $$\int_a^b f(t)\,\chi_n(t)\,dt = \int_a^b f(t)\,\psi_n(t)\,dt - \frac{\gamma_n}{\gamma}\int_a^b f(t)\,dt$$
$$= \gamma_n - \gamma_n = 0.$$

If possible, let the functions $\chi_n(x)$ form an incomplete system, so that there is a normalized function $F(x)$ such that

$$\int_a^b F(t)\,\chi_n(t)\,dt = 0$$

for every n.

This gives

(7·3) $$\Gamma_n = \int_a^b F(t)\,\psi_n(t)\,dt = \frac{\gamma_n}{\gamma}\int^b F(t)\,dt.$$

https://doi.org/10.1017/S0305004100013724 Published online by Cambridge University Press

Now, by theorem C,

$$\sum_{n=1}^{\infty} \Gamma_n^2 = \int_a^b F^2(t)\,dt = 1. \tag{7·4}$$

Thus, by (7·3)

$$\left\{\int_a^b F(t)\,dt\right\}^2 \sum_{n=1}^{\infty} \gamma_n^2 = \gamma^2,$$

which is impossible, in view of (7·1), as

$$\int_a^b F(t)\,dt \neq 0,$$

for otherwise, by (7·3), all the coefficients Γ_n would vanish, in defiance of (7·4).

It follows that no function such as $F(x)$ can exist and the system $\{\chi_n(t)\}$ must be complete, but not necessarily orthogonal.

The first function $\chi_1(x)$ is easily seen to vanish, since

$$\psi_1(x) - \frac{\gamma_1}{\gamma} = \left(\frac{1}{b-a}\right)^{\frac{1}{2}} - \frac{1}{\gamma}\int_a^b f(t)\left(\frac{1}{b-a}\right)^{\frac{1}{2}} dt$$

$$= \left(\frac{1}{b-a}\right)^{\frac{1}{2}} - \left(\frac{1}{b-a}\right)^{\frac{1}{2}} \frac{1}{\gamma}\gamma = 0;$$

but for all the others

$$\int_a^b \chi_n^2(t)\,dt = \int_a^b \psi_n^2(t)\,dt - \frac{2\gamma_n}{\gamma}\int_a^b \psi_n(t)\,dt + \frac{\gamma_n^2}{\gamma^2}\int_a^b 1dt$$

$$= 1 + (b-a)\frac{\gamma_n^2}{\gamma^2} \neq 0,$$

and the system

$$\chi_2(x),\ \chi_3(x),\ \ldots,\ \chi_n(x),\ \ldots$$

can be converted, in the usual way, into a complete normalized orthogonal system (6·1).

Since $\phi_n(x)$ is a finite linear combination of the functions $\chi_n(x)$, c_n vanishes in view of (7·2). Thus (6·2) holds, as had to be proved.

It may be observed that, since the system $\chi_n(x)$ is complete it must contain functions whose norm relative to (a, b) does not vanish and so is capable of yielding a normalized orthogonal system. The analysis given, however, shows that $\chi_1(x)$ is the only function whose norm vanishes.