

---

---

# Detachments of Hypergraphs I: The Berge–Johnson Problem

---

M. A. BAHMANIAN

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849-5310, USA  
(e-mail: mzb0004@auburn.edu)

*Received 1 September 2011; revised 20 January 2012; first published online 27 February 2012*

A detachment of a hypergraph is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices. For a given edge-coloured hypergraph  $\mathcal{F}$ , we prove that there exists a detachment  $\mathcal{G}$  such that the degree of each vertex and the multiplicity of each edge in  $\mathcal{F}$  (and each colour class of  $\mathcal{F}$ ) are shared fairly among the subvertices in  $\mathcal{G}$  (and each colour class of  $\mathcal{G}$ , respectively).

Let  $(\lambda_1, \dots, \lambda_m)K_{p_1, \dots, p_n}^{h_1, \dots, h_m}$  be a hypergraph with vertex partition  $\{V_1, \dots, V_n\}$ ,  $|V_i| = p_i$  for  $1 \leq i \leq n$  such that there are  $\lambda_i$  edges of size  $h_i$  incident with every  $h_i$  vertices, at most one vertex from each part for  $1 \leq i \leq m$  (so no edge is incident with more than one vertex of a part). We use our detachment theorem to show that the obvious necessary conditions for  $(\lambda_1, \dots, \lambda_m)K_{p_1, \dots, p_n}^{h_1, \dots, h_m}$  to be expressed as the union  $\mathcal{G}_1 \cup \dots \cup \mathcal{G}_k$  of  $k$  edge-disjoint factors, where for  $1 \leq i \leq k$ ,  $\mathcal{G}_i$  is  $r_i$ -regular, are also sufficient. Baranyai solved the case of  $h_1 = \dots = h_m$ ,  $\lambda_1 = \dots = \lambda_m = 1$ ,  $p_1 = \dots = p_m$ ,  $r_1 = \dots = r_k$ . Berge and Johnson (and later Brouwer and Tijdeman, respectively) considered (and solved, respectively) the case of  $h_i = i$ ,  $1 \leq i \leq m$ ,  $p_1 = \dots = p_m = \lambda_1 = \dots = \lambda_m = r_1 = \dots = r_k = 1$ . We also extend our result to the case where each  $\mathcal{G}_i$  is almost regular.

AMS 2010 *Mathematics subject classification*: 05C65, 05C15, 05C70, 05C51, 05B40, 05B05

## 1. Introduction

Intuitively speaking, a detachment of a hypergraph is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices. As the main result of this paper (see Theorem 4.1), we prove that for a given edge-coloured hypergraph  $\mathcal{F}$ , there exists a detachment  $\mathcal{G}$  such that the degree of each vertex and the multiplicity of each edge in  $\mathcal{F}$  (and each colour class of  $\mathcal{F}$ ) are shared fairly among the subvertices in  $\mathcal{G}$  (and each colour class of  $\mathcal{G}$ , respectively). This result is not only interesting by itself and generalizes various graph-theoretic results (see for example [1, 9, 12, 14, 15, 17, 18, 19]), but is also used to obtain extensions of existing results on edge-decompositions of hypergraphs by Bermond, Baranyai [2, 3], Berge and Johnson [4, 13], and Brouwer and Tijdeman [5, 6].

Given a set  $N$  of  $n$  elements, Berge and Johnson [4, 13] addressed the question of when do there exist disjoint partitions of  $N$ , each partition containing only subsets of  $h$  or fewer elements,

such that every subset of  $N$  having  $h$  or fewer elements is in exactly one partition. Here we state the problem in a more general setting with the hypergraph-theoretic notation.

Let  $(\lambda_1, \dots, \lambda_m)K_{p_1, \dots, p_n}^{h_1, \dots, h_m}$  be a hypergraph with vertex partition  $\{V_1, \dots, V_n\}$ ,  $|V_i| = p_i$  for  $1 \leq i \leq n$  such that there are  $\lambda_i$  edges of size  $h_i$  incident with every  $h_i$  vertices, at most one vertex from each part for  $1 \leq i \leq m$  (so no edge is incident with more than one vertex of a part). We use our detachment theorem to show that the obvious necessary conditions for  $(\lambda_1, \dots, \lambda_m)K_{p_1, \dots, p_n}^{h_1, \dots, h_m}$  to be expressed as the union  $\mathcal{G}_1 \cup \dots \cup \mathcal{G}_k$  of  $k$  edge-disjoint factors, where for  $1 \leq i \leq k$ ,  $\mathcal{G}_i$  is  $r_i$ -regular, are also sufficient. Baranyai [2, 3] solved the case of  $h_1 = \dots = h_m$ ,  $\lambda_1 = \dots = \lambda_m = 1$ ,  $p_1 = \dots = p_m$ ,  $r_1 = \dots = r_k$ . Berge and Johnson [4, 13] (and later Brouwer and Tijdeman [5, 6], respectively) considered (and solved, respectively) the case of  $h_i = i$ ,  $1 \leq i \leq m$ ,  $p_1 = \dots = p_m = \lambda_1 = \dots = \lambda_m = r_1 = \dots = r_k = 1$ . We also extend our result to the case where each  $\mathcal{G}_i$  is almost regular.

In the next two sections, we give more precise definitions along with terminology. In Section 4, we state our main result, followed by the proof in Section 5. In the last section, we show the usefulness of the main result on decompositions of various classes of hypergraphs. We defer the applications of the main result in solving embedding problems to a future paper.

### 2. Terminology and precise definitions

If  $x, y \in \mathbb{R}$  ( $\mathbb{R}$  is the set of real numbers), then  $\lfloor x \rfloor$  and  $\lceil x \rceil$  denote the integers such that  $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$ , and  $x \approx y$  means  $\lfloor y \rfloor \leq x \leq \lceil y \rceil$ . We observe that the relation  $\approx$  is transitive (but not symmetric) and for  $x, y \in \mathbb{R}$ , and  $n \in \mathbb{N}$  ( $\mathbb{N}$  is the set of positive integers),  $x \approx y$  implies  $x/n \approx y/n$ . These properties of  $\approx$  will be used in Section 5 without further explanation. For a multiset  $A$  and  $u \in A$ , let  $\mu_A(u)$  denote the multiplicity of  $u$  in  $A$ , and let  $|A| = \sum_{u \in A} \mu_A(u)$ . For multisets  $A_1, \dots, A_n$ , we define  $A = \bigcup_{i=1}^n A_i$  by  $\mu_A(u) = \sum_{i=1}^n \mu_{A_i}(u)$ . We may use abbreviations such as  $\{u^r\}$  for  $\underbrace{\{u, \dots, u\}}_r$ , for example

$$\{u^2, v, w^2\} \cup \{u, w^2\} = \{u^3, v, w^4\}.$$

For the purpose of this paper, a *hypergraph*  $\mathcal{G}$  is an ordered quintuple  $(V(\mathcal{G}), E(\mathcal{G}), H(\mathcal{G}), \psi, \phi)$  where  $V(\mathcal{G}), E(\mathcal{G}), H(\mathcal{G})$  are disjoint finite sets,  $\psi : H(\mathcal{G}) \rightarrow V(\mathcal{G})$  is a function and  $\phi : H(\mathcal{G}) \rightarrow E(\mathcal{G})$  is a surjection. Elements of  $V(\mathcal{G}), E(\mathcal{G}), H(\mathcal{G})$  are called *vertices*, *edges* and *hinges* of  $\mathcal{G}$ , respectively. A vertex  $v$  (edge  $e$ , respectively) and hinge  $h$  are said to be *incident* with each other if  $\psi(h) = v$  ( $\phi(h) = e$ , respectively). A hinge  $h$  is said to *attach* the edge  $\phi(h)$  to the vertex  $\psi(h)$ . In this manner, the vertex  $\psi(h)$  and the edge  $\phi(h)$  are said to be *incident* with each other. If  $e \in E(\mathcal{G})$ , and  $e$  is incident with  $n$  hinges  $h_1, \dots, h_n$  for some  $n \in \mathbb{N}$ , then the edge  $e$  is said to *join* (not necessarily distinct) vertices  $\psi(h_1), \dots, \psi(h_n)$ . If  $v \in V(\mathcal{G})$ , then the number of hinges incident with  $v$  (i.e.,  $|\psi^{-1}(v)|$ ) is called the *degree* of  $v$  and is denoted by  $d(v)$ . The number of (distinct) vertices incident with an edge  $e$ , denoted by  $|e|$ , is called the *size* of  $e$ . If, for all edges  $e$  of  $\mathcal{G}$ ,  $|e| \leq 2$  and  $|\phi^{-1}(e)| = 2$ , then  $\mathcal{G}$  is a *graph*.

Thus a hypergraph, in the sense of our definition, is a generalization of a hypergraph as it is usually defined. In fact, if for every edge  $e$ ,  $|e| = |\phi^{-1}(e)|$ , then our definition is essentially the same as the usual definition. Here, for convenience, we imagine each edge of a hypergraph to be attached to the vertices which it joins by in-between objects called hinges. Readers from a graph

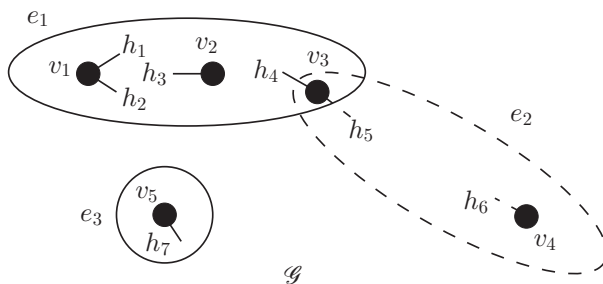


Figure 1. Representation of a hypergraph  $\mathcal{G}$ .

theory background may think of this as a bipartite multigraph with vertex bipartition  $\{V, E\}$ , in which the hinges form the edges. A hypergraph may be drawn as a set of points representing the vertices. A hyperedge is represented by a simple closed curve enclosing its incident vertices. A hinge is represented by a small line attached to the vertex incident with it (see Figure 1).

The set of hinges of  $\mathcal{G}$  which are incident with a vertex  $v$  (and an edge  $e$ , respectively), is denoted by  $H(v)$  ( $H(v, e)$ , respectively). Thus, if  $v \in V(\mathcal{G})$ , then  $H(v) = \psi^{-1}(v)$ , and  $|H(v)|$  is the degree  $d(v)$  of  $v$ . If  $U$  is a multi-subset of  $V(\mathcal{G})$ , and  $u \in V(\mathcal{G})$ , let  $E(U)$  denote the set of edges  $e$  with  $|\phi^{-1}(e)| = |U|$  joining vertices in  $U$ . More precisely,  $E(U) = \{e \in E(\mathcal{G}) \mid |H(v, e)| = \mu_U(v)\}$  for all  $v \in V(\mathcal{G})$ . For  $U_1, \dots, U_n \subset V$ , where for  $1 \leq i \leq n$  each  $U_i$  is a multiset, let  $E(U_1, \dots, U_n)$  denote  $E(\bigcup_{i=1}^n U_i)$ . We write  $m(U)$  for  $|E(U)|$  and call it the *multiplicity* of  $U$ . For simplicity,  $E(u^r, U)$  denotes  $E(\{u^r\}, U)$ , and  $m(u_1^{m_1}, \dots, u_r^{m_r})$  denotes  $m(\{u_1^{m_1}, \dots, u_r^{m_r}\})$ . The set of hinges that are incident with  $u$  and an edge in  $E(u^r, U)$  is denoted by  $H(u^r, U)$ .

**Example 2.1.** Let  $\mathcal{G} = (V, E, H, \psi, \phi)$ , with

$$V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \{e_1, e_2, e_3\}, \quad H = \{h_i, 1 \leq i \leq 7\},$$

such that

$$\psi(h_1) = \psi(h_2) = v_1, \quad \psi(h_3) = v_2, \quad \psi(h_4) = \psi(h_5) = v_3, \quad \psi(h_6) = v_4, \quad \psi(h_7) = v_5$$

and

$$\phi(h_1) = \phi(h_2) = \phi(h_3) = \phi(h_4) = e_1, \quad \phi(h_5) = \phi(h_6) = e_2, \quad \phi(h_7) = e_3.$$

We have:

- $|e_1| = 3, |e_2| = 2, |e_3| = 1,$
- $d(v_1) = d(v_3) = 2, d(v_2) = d(v_4) = d(v_5) = 1,$
- $H(v_1) = \{h_1, h_2\}, H(v_2) = \{h_3\}, H(v_3) = \{h_4, h_5\},$
- $H(v_3, e_1) = \{h_4\}, H(v_3, e_2) = \{h_5\}, H(v_3, e_3) = \emptyset,$
- $E(\{v_1, v_2, v_3\}) = \emptyset, E(\{v_1^2, v_2, v_3\}) = E(v_1^2, \{v_2, v_3\}) = \{e_1\},$
- $m(v_1, v_2, v_3) = 0, m(v_1^2, v_2, v_3) = 1,$
- $H(v_1^2, \{v_2, v_3\}) = \{h_1, h_2\}, H(v_1, \{v_2, v_3\}) = \emptyset, H(v_3, \{v_1^2, v_2\}) = \{h_4\}.$

A *k*-edge-colouring of  $\mathcal{G}$  is a mapping  $f : E(\mathcal{G}) \rightarrow C$ , where  $C$  is a set of  $k$  colours (we often use  $C = \{1, \dots, k\}$ ), and the edges of one colour form a *colour class*. The sub-hypergraph of  $\mathcal{G}$  induced by the colour class  $j$  is denoted by  $\mathcal{G}(j)$ . To avoid ambiguity, subscripts may be used to indicate the hypergraph in which hypergraph-theoretic notation should be interpreted: for example,  $d_{\mathcal{G}}(v)$ ,  $m_{\mathcal{G}}(v^2, w)$ ,  $H_{\mathcal{G}}(v)$ .

### 3. Amalgamations and detachments

If  $\mathcal{F} = (V, E, H, \psi, \phi)$  is a hypergraph and  $\Psi$  is a function from  $V$  onto a set  $W$ , then we shall say that the hypergraph  $\mathcal{G} = (W, E, H, \Psi \circ \psi, \phi)$  is an *amalgamation* of  $\mathcal{F}$  and that  $\mathcal{F}$  is a *detachment* of  $\mathcal{G}$ . Associated with  $\Psi$  is the *number function*  $g : W \rightarrow \mathbb{N}$  defined by  $g(w) = |\Psi^{-1}(w)|$ , for each  $w \in W$ ; being more specific, we may also say that  $\mathcal{F}$  is a *g-detachment* of  $\mathcal{G}$ . Intuitively speaking, a *g-detachment* of  $\mathcal{G}$  is obtained by splitting each  $u \in V(\mathcal{G})$  into  $g(u)$  vertices. Thus  $\mathcal{F}$  and  $\mathcal{G}$  have the same edges and hinges, and each vertex  $v$  of  $\mathcal{G}$  is obtained by identifying those vertices of  $\mathcal{F}$  which belong to the set  $\Psi^{-1}(v)$ . In this process, a hinge incident with a vertex  $u$  and an edge  $e$  in  $\mathcal{F}$  becomes incident with the vertex  $\Psi(u)$  and the edge  $e$  in  $\mathcal{G}$ .

There are quite a lot of other papers on amalgamations, and some highlights include [7], [8], [9], [10], [12], [14], [18] and [19].

### 4. Main result

A function  $g : V(\mathcal{G}) \rightarrow \mathbb{N}$  is said to be *simple* if

$$|H(v, e)| \leq g(v) \text{ for } v \in V(\mathcal{G}), e \in E(\mathcal{G}).$$

A hypergraph  $\mathcal{G}$  is said to be *simple* if  $g : V(\mathcal{G}) \rightarrow \mathbb{N}$  with  $g(v) = 1$  for  $v \in V(\mathcal{G})$  is simple. It is clear that for a hypergraph  $\mathcal{F}$  and a function  $g : V(\mathcal{F}) \rightarrow \mathbb{N}$ , there exists a simple *g-detachment* if and only if  $g$  is simple.

**Theorem 4.1.** *Let  $\mathcal{F}$  be a *k*-edge-coloured hypergraph and let  $g : V(\mathcal{F}) \rightarrow \mathbb{N}$  be a simple function. Then there exists a simple *g-detachment*  $\mathcal{G}$  (possibly with multiple edges) of  $\mathcal{F}$  with amalgamation function  $\Psi : V(\mathcal{G}) \rightarrow V(\mathcal{F})$ ,  $g$  being the number function associated with  $\Psi$ , such that:*

- (A1)  $d_{\mathcal{G}}(v) \approx d_{\mathcal{F}}(u)/g(u)$  for each  $u \in V(\mathcal{F})$  and each  $v \in \Psi^{-1}(u)$ ,
- (A2)  $d_{\mathcal{G}(j)}(v) \approx d_{\mathcal{F}(j)}(u)/g(u)$  for each  $u \in V(\mathcal{F})$ , each  $v \in \Psi^{-1}(u)$  and  $1 \leq j \leq k$ ,
- (A3)  $m_{\mathcal{G}}(U_1, \dots, U_r) \approx m_{\mathcal{F}}(u_1^{m_1}, \dots, u_r^{m_r}) / \prod_{i=1}^r \binom{g(u_i)}{m_i}$  for distinct  $u_1, \dots, u_r \in V(\mathcal{F})$  and  $U_i \subset \Psi^{-1}(u_i)$  with  $|U_i| = m_i \leq g(u_i)$  for  $1 \leq i \leq r$ , and
- (A4)  $m_{\mathcal{G}(j)}(U_1, \dots, U_r) \approx m_{\mathcal{F}(j)}(u_1^{m_1}, \dots, u_r^{m_r}) / \prod_{i=1}^r \binom{g(u_i)}{m_i}$  for distinct  $u_1, \dots, u_r \in V(\mathcal{F})$  and  $U_i \subset \Psi^{-1}(u_i)$  with  $|U_i| = m_i \leq g(u_i)$  for  $1 \leq i \leq r$  and  $1 \leq j \leq k$ .

A family  $\mathcal{A}$  of sets is *laminar* if, for every pair  $A, B$  of sets belonging to  $\mathcal{A}$ , either  $A \subset B$ , or  $B \subset A$ , or  $A \cap B = \emptyset$ . To prove the main result, we need the following lemma.

**Lemma 4.2 (Nash-Williams [18], Lemma 2).** *If  $\mathcal{A}, \mathcal{B}$  are two laminar families of subsets of a finite set  $S$ , and  $n \in \mathbb{N}$ , then there exists a subset  $A$  of  $S$  such that, for every  $P \in \mathcal{A} \cup \mathcal{B}$ ,  $|A \cap P| \approx |P|/n$ .*

**5. Proof of Theorem 4.1**

**5.1. Inductive construction of  $\mathcal{G}$**

Let  $\mathcal{F} = (V, E, H, \psi, \phi)$ . Let  $n = \sum_{v \in V} (g(v) - 1)$ . Initially we let  $\mathcal{F}_0 = \mathcal{F}$  and  $g_0 = g$ , and we let  $\Phi_0$  be the identity function from  $V$  into  $V$ . Now assume that

$$\mathcal{F}_0 = (V_0, E_0, H_0, \psi_0, \phi_0), \dots, \mathcal{F}_i = (V_i, E_i, H_i, \psi_i, \phi_i)$$

and  $\Phi_0, \dots, \Phi_i$  have been defined for some  $i \geq 0$ . Also assume that the simple functions  $g_0 : V_0 \rightarrow \mathbb{N}, \dots, g_i : V_i \rightarrow \mathbb{N}$  have been defined for some  $i \geq 0$ . Let  $\Psi_i = \Phi_0 \cdots \Phi_i$ . If  $i = n$ , we terminate the construction, letting  $\mathcal{G} = \mathcal{F}_n$  and  $\Psi = \Psi_n$ .

If  $i < n$ , we can select a vertex  $\alpha$  of  $\mathcal{F}_i$  such that  $g_i(\alpha) \geq 2$ . As we will see,  $\mathcal{F}_{i+1}$  is formed from  $\mathcal{F}_i$  by splitting off a vertex  $v_{i+1}$  from  $\alpha$  so that we end up with  $\alpha$  and  $v_{i+1}$ . Let

$$\begin{aligned} \mathcal{A}_i &= \{H_{\mathcal{F}_i}(\alpha)\} \\ &\cup \{H_{\mathcal{F}_i(1)}(\alpha), \dots, H_{\mathcal{F}_i(k)}(\alpha)\} \\ &\cup \{H_{\mathcal{F}_i(j)}(\alpha, e) : e \in E_{\mathcal{F}_i(j)}(\alpha), 1 \leq j \leq k\}, \end{aligned} \tag{5.1}$$

and let

$$\begin{aligned} \mathcal{B}_i &= \{H_{\mathcal{F}_i}(\alpha^t, U) : t \geq 1, U \subset V_i \setminus \{\alpha\}\} \\ &\cup \{H_{\mathcal{F}_i(j)}(\alpha^t, U) : t \geq 1, U \subset V_i \setminus \{\alpha\}, 1 \leq j \leq k\}. \end{aligned} \tag{5.2}$$

It is easy to see that both  $\mathcal{A}_i$  and  $\mathcal{B}_i$  are laminar families of subsets of  $H(\mathcal{F}_i, \alpha)$ . Therefore, by Lemma 4.2, there exists a subset  $Z_i$  of  $H(\mathcal{F}_i, \alpha)$  such that

$$|Z_i \cap P| \approx |P|/g_i(\alpha), \quad \text{for every } P \in \mathcal{A}_i \cup \mathcal{B}_i. \tag{5.3}$$

Let  $v_{i+1}$  be a vertex which does not belong to  $V_i$  and let  $V_{i+1} = V_i \cup \{v_{i+1}\}$ . Let  $\Phi_{i+1}$  be the function from  $V_{i+1}$  onto  $V_i$  such that  $\Phi_{i+1}(v) = v$  for every  $v \in V_i$  and  $\Phi_{i+1}(v_{i+1}) = \alpha$ . Let  $\mathcal{F}_{i+1}$  be the detachment of  $\mathcal{F}_i$  under  $\Phi_{i+1}$  such that  $V(\mathcal{F}_{i+1}) = V_{i+1}$ , and

$$H_{\mathcal{F}_{i+1}}(v_{i+1}) = Z_i, H_{\mathcal{F}_{i+1}}(\alpha) = H_{\mathcal{F}_i}(\alpha) \setminus Z_i. \tag{5.4}$$

In fact,  $\mathcal{F}_{i+1}$  is obtained from  $\mathcal{F}_i$  by splitting  $\alpha$  into two vertices  $\alpha$  and  $v_{i+1}$  in such a way that hinges which were incident with  $\alpha$  in  $\mathcal{F}_i$  become incident in  $\mathcal{F}_{i+1}$  with  $\alpha$  or  $v_{i+1}$  according to whether or not they belong to  $Z_i$ , respectively. Obviously,  $\Psi_i$  is an amalgamation function from  $\mathcal{F}_{i+1}$  into  $\mathcal{F}_i$ . Let  $g_{i+1}$  be the function from  $V_{i+1}$  into  $\mathbb{N}$ , such that  $g_{i+1}(v_{i+1}) = 1, g_{i+1}(\alpha) = g_i(\alpha) - 1$ , and  $g_{i+1}(v) = g_i(v)$  for every  $v \in V_i \setminus \{\alpha\}$ . This finishes the construction of  $\mathcal{F}_{i+1}$ .

**5.2. Relations between  $\mathcal{F}_{i+1}$  and  $\mathcal{F}_i$**

The hypergraph  $\mathcal{F}_{i+1}$ , satisfies the following conditions:

- (B1)  $d_{\mathcal{F}_{i+1}}(\alpha) \approx d_{\mathcal{F}_i}(\alpha)g_{i+1}(\alpha)/g_i(\alpha)$ ,
- (B2)  $d_{\mathcal{F}_{i+1}}(v_{i+1}) \approx d_{\mathcal{F}_i}(\alpha)/g_i(\alpha)$ ,
- (B3)  $m_{\mathcal{F}_{i+1}}(v_{i+1}^s, \alpha^t, U) = 0$  for  $s \geq 2$ , and  $t \geq 0$ ,
- (B4)  $m_{\mathcal{F}_{i+1}}(\alpha^t, U) \approx m_{\mathcal{F}_i}(\alpha^t, U)(g_i(\alpha) - t)/g_i(\alpha)$  for each  $U \subset V_i \setminus \{\alpha\}$ , and  $g_i(\alpha) \geq t \geq 1$ ,
- (B5)  $m_{\mathcal{F}_{i+1}}(\alpha^t, v_{i+1}, U) \approx (t + 1)m_{\mathcal{F}_i}(\alpha^{t+1}, U)/g_i(\alpha)$  for each  $U \subset V_i \setminus \{\alpha\}$ , and  $t \geq 0$ .

**Proof.** Since  $H_{\mathcal{F}_i}(\alpha) \in \mathcal{A}_i$ , from (5.4) it follows that

$$\begin{aligned} d_{\mathcal{F}_{i+1}}(v_{i+1}) &= |H_{\mathcal{F}_{i+1}}(v_{i+1})| = |Z_i| = |Z_i \cap H_{\mathcal{F}_i}(\alpha)| \\ &\approx |H_{\mathcal{F}_i}(\alpha)|/g_i(\alpha) = d_{\mathcal{F}_i}(\alpha)/g_i(\alpha), \\ d_{\mathcal{F}_{i+1}}(\alpha) &= |H_{\mathcal{F}_{i+1}}(\alpha)| = |H_{\mathcal{F}_i}(\alpha)| - |Z_i| \\ &\approx d_{\mathcal{F}_i}(\alpha) - d_{\mathcal{F}_i}(\alpha)/g_i(\alpha) = (g_i(\alpha) - 1)d_{\mathcal{F}_i}(\alpha)/g_i(\alpha) \\ &= d_{\mathcal{F}_i}(\alpha)g_{i+1}(\alpha)/g_i(\alpha). \end{aligned}$$

This proves (B1) and (B2).

If  $t \geq 1$ ,  $U \subset V_i \setminus \{\alpha\}$ , and  $e \in E_{\mathcal{F}_i}(\alpha^t, U)$ , then for some  $j$ ,  $1 \leq j \leq k$ ,  $H_{\mathcal{F}_i(j)}(\alpha, e) \in \mathcal{A}_i$ , so

$$|Z_i \cap H_{\mathcal{F}_i(j)}(\alpha, e)| \approx |H_{\mathcal{F}_i(j)}(\alpha, e)|/g_i(\alpha) = t/g_i(\alpha) \leq 1,$$

where the inequality is implied by the fact that  $g_i$  is simple. Thus, either  $|Z_i \cap H_{\mathcal{F}_i(j)}(\alpha, e)| = 1$  and consequently  $e \in E_{\mathcal{F}_{i+1}}(\alpha^{t-1}, v_{i+1}, U)$ , or  $Z_i \cap H_{\mathcal{F}_i(j)}(\alpha, e) = \emptyset$  and consequently  $e \in E_{\mathcal{F}_{i+1}}(\alpha^t, U)$ . Therefore

$$m_{\mathcal{F}_{i+1}}(v_{i+1}^s, \alpha^r, U) = 0,$$

for  $r \geq 1$ , and  $s \geq 2$ . This proves (B3). Moreover, since  $H_{\mathcal{F}_i}(\alpha^t, U) \in \mathcal{B}_i$ , we have

$$\begin{aligned} m_{\mathcal{F}_{i+1}}(\alpha^{t-1}, v_{i+1}, U) &= |Z_i \cap H_{\mathcal{F}_i}(\alpha^t, U)| \\ &\approx |H_{\mathcal{F}_i}(\alpha^t, U)|/g_i(\alpha) \\ &= tm_{\mathcal{F}_i}(\alpha^t, U)/g_i(\alpha), \\ m_{\mathcal{F}_{i+1}}(\alpha^t, U) &\approx m_{\mathcal{F}_i}(\alpha^t, U) - |H_{\mathcal{F}_i}(\alpha^t, U)|/g_i(\alpha) \\ &= m_{\mathcal{F}_i}(\alpha^t, U) - tm_{\mathcal{F}_i}(\alpha^t, U)/g_i(\alpha) \\ &= m_{\mathcal{F}_i}(\alpha^t, U)(g_i(\alpha) - t)/g_i(\alpha). \end{aligned}$$

This proves (B4) and (B5). □

Let us fix  $j \in \{1, \dots, k\}$ . It is enough to replace  $\mathcal{F}_i$  with  $\mathcal{F}_i(j)$  in the statement and the proof of (B1)–(B5) to obtain companion conditions, say (C1)–(C5), for each colour class.

**5.3. Relations between  $\mathcal{F}_i$  and  $\mathcal{F}$**

Recall that  $\Psi_i = \Phi_0 \cdots \Phi_i$ , that  $\Phi_0 : V \rightarrow V$ , and that  $\Phi_i : V_i \rightarrow V_{i-1}$  for  $i > 0$ . Therefore  $\Psi_i : V_i \rightarrow V$  and thus  $\Psi_i^{-1} : V \rightarrow V_i$ . Now we use (B1)–(B5) to prove that the hypergraph  $\mathcal{F}_i$  satisfies the following conditions for  $0 \leq i \leq n$ :

- (D1)  $d_{\mathcal{F}_i}(v)/g_i(v) \approx d_{\mathcal{F}}(u)/g(u)$  for each  $u \in V$  and each  $v \in \Psi_i^{-1}(u)$ ,
- (D2)  $m_{\mathcal{F}_i}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)/\prod_{j=1}^r \binom{g_i(u_j)}{a_j} \approx m_{\mathcal{F}}(u_1^{m_1}, \dots, u_r^{m_r})/\prod_{j=1}^r \binom{g(u_j)}{m_j}$  for distinct vertices  $u_1, \dots, u_r \in V$ ,  $a_j \geq 0$ ,  $U_j \subset \Psi_i^{-1}(u_j) \setminus \{u_j\}$  with  $1 \leq m_j = a_j + |U_j| \leq g(u_j)$ ,  $1 \leq j \leq r$  if  $g_i(u_j) \geq a_j$ ,  $1 \leq j \leq r$ .

**Proof.** The proof is by induction. Recall that  $\mathcal{F}_0 = \mathcal{F}$ , and  $g_0(u) = g(u)$  for  $u \in V$ . Thus, (D1) and (D2) are trivial for  $i = 0$ . Now we will show that if  $\mathcal{F}_i$  satisfies the conditions (D1) and (D2) for some  $i < n$ , then  $\mathcal{F}_{i+1}$  satisfies these conditions by replacing  $i$  with  $i + 1$ ; we denote the corresponding conditions for  $\mathcal{F}_{i+1}$  by (D1)' and (D2)'.

Let  $u \in V$ . If  $g_{i+1}(u) = g_i(u)$ , then (D1)' is obviously true. So we just check (D1)' in the case where  $u = \alpha$ . By (B1) and (D1) we have  $d_{\mathcal{F}_{i+1}}(\alpha)/g_{i+1}(\alpha) \approx d_{\mathcal{F}_i}(\alpha)/g_i(\alpha) \approx d_{\mathcal{F}}(\alpha)/g(\alpha)$ . Moreover, from (B2) and (D1) it follows that  $d_{\mathcal{F}_{i+1}}(v_{i+1}) \approx d_{\mathcal{F}_i}(\alpha)/g_i(\alpha) \approx d_{\mathcal{F}}(\alpha)/g(\alpha)$ . Since in forming  $\mathcal{F}_{i+1}$  no edge is detached from  $v_r$  for each  $v_r \in \Psi_i^{-1}(\alpha) \setminus \{\alpha\}$ , we have  $d_{\mathcal{F}_{i+1}}(v_r) = d_{\mathcal{F}_i}(v_r)$ . Therefore  $d_{\mathcal{F}_{i+1}}(v_r) = d_{\mathcal{F}_i}(v_r) \approx d_{\mathcal{F}}(\alpha)/g(\alpha)$  for each  $v_r \in \Psi_i^{-1}(\alpha) \setminus \{\alpha\}$ . This proves (D1)'. Let  $u_1, \dots, u_r$  be distinct vertices in  $V$ . If  $g_{i+1}(u_j) = g_i(u_j)$  for  $1 \leq j \leq r$ , then (D2)' is clearly true. Therefore, in order to prove (D2)', without loss of generality we may assume that  $g_{i+1}(u_1) = g_i(u_1) - 1$  (so  $\alpha = u_1$  and  $v_{i+1} \in \Psi_i^{-1}(u_1)$ ). First, note that for integers  $a, b$  we always have  $(a - b) \binom{a}{b} = a \binom{a-1}{b} = (b + 1) \binom{a}{b+1}$ . If  $v_{i+1} \notin U_1$ , we have

$$\begin{aligned} \frac{m_{\mathcal{F}_{i+1}}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)}{\prod_{j=1}^r \binom{g_{i+1}(u_j)}{a_j}} &\stackrel{(B4)}{\approx} \frac{m_{\mathcal{F}_i}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)(g_i(u_1) - a_1)/g_i(u_1)}{\binom{g_i(u_1)-1}{a_1} \prod_{j=2}^r \binom{g_i(u_j)}{a_j}} \\ &= \frac{m_{\mathcal{F}_i}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)(g_i(u_1) - a_1)/g_i(u_1)}{(g_i(u_1) - a_1)/g_i(u_1) \binom{g_i(u_1)}{a_1} \prod_{j=2}^r \binom{g_i(u_j)}{a_j}} \\ &= \frac{m_{\mathcal{F}_i}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)}{\prod_{j=1}^r \binom{g_i(u_j)}{a_j}} \\ &\stackrel{(D2)}{\approx} \frac{m_{\mathcal{F}}(u_1^{m_1}, \dots, u_r^{m_r})}{\prod_{j=1}^r \binom{g(u_j)}{m_j}}. \end{aligned}$$

If  $v_{i+1} \in U_1$ , we have

$$\begin{aligned} \frac{m_{\mathcal{F}_{i+1}}(u_1^{a_1}, U_1, \dots, u_r^{a_r}, U_r)}{\prod_{j=1}^r \binom{g_{i+1}(u_j)}{a_j}} &\stackrel{(B5)}{\approx} \frac{m_{\mathcal{F}_i}(u_1^{a_1+1}, U_1 \setminus \{v_{i+1}\}, \dots, u_r^{a_r}, U_r)(a_1 + 1)/g_i(u_1)}{\binom{g_i(u_1)-1}{a_1} \prod_{j=2}^r \binom{g_i(u_j)}{a_j}} \\ &= \frac{m_{\mathcal{F}_i}(u_1^{a_1+1}, U_1 \setminus \{v_{i+1}\}, \dots, u_r^{a_r}, U_r)}{g_i(u_1)/(a_1 + 1) \binom{g_i(u_1)-1}{a_1} \prod_{j=2}^r \binom{g_i(u_j)}{a_j}} \\ &= \frac{m_{\mathcal{F}_i}(u_1^{a_1+1}, U_1 \setminus \{v_{i+1}\}, \dots, u_r^{a_r}, U_r)}{\binom{g_i(u_1)}{a_1+1} \prod_{j=2}^r \binom{g_i(u_j)}{a_j}} \\ &\stackrel{(D2)}{\approx} \frac{m_{\mathcal{F}}(u_1^{m_1}, \dots, u_r^{m_r})}{\prod_{j=1}^r \binom{g(u_j)}{m_j}}. \end{aligned}$$

This proves (D2)'. □

Let us fix  $j \in \{1, \dots, k\}$ . It is enough to replace  $\mathcal{F}$  with  $\mathcal{F}(j)$ ,  $\mathcal{F}_i$  with  $\mathcal{F}_i(j)$ ,  $\mathcal{F}_{i+1}$  with  $\mathcal{F}_{i+1}(j)$ , and (Bi) with (Ci) for  $i = 1, 2, 4, 5$ , in the statement and the proof of (D1) and (D2) to obtain companion conditions, say (E1) and (E2), for each colour class.

**5.4.  $\mathcal{G}$  satisfies (A1)–(A4)**

Recall that  $\mathcal{G} = \mathcal{F}_n$  and  $g_n(u) = 1$  for every  $u \in V$ , therefore when  $i = n$ , (D1) implies (A1). Moreover, if we let  $i = n$  in (D2), we have  $a_j \in \{0, 1\}$  for  $1 \leq j \leq r$  and thus  $\prod_{j=1}^r (a_j^{g_i(u_j)}) = \prod_{j=1}^r (a_j^1) = 1$ . This proves (A3). By a similar argument, one can prove (A2) and (A4), and this completes the proof of Theorem 4.1.

**6. Corollaries**

For a matrix  $A$ , let  $A_j$  denote the  $j$ th column of  $A$ , and let  $s(A)$  denote the sum of all the elements of  $A$ . Let  $R = [r_1 \dots r_k]^T$  (or  $R^T = [r_i]_{1 \times k}$ ),  $\Lambda = [\lambda_1 \dots \lambda_m]^T$  and  $H = [h_1 \dots h_m]^T$  be three column vectors with  $r_i, \lambda_i \in \mathbb{N}$ , and  $h_i \in \{1, \dots, n\}$  for  $1 \leq i \leq m$ , such that  $h_1, \dots, h_m$  are distinct. Let  $\Lambda K_n^H$  denote a hypergraph with vertex set  $V$ ,  $|V| = n$ , such that there are  $\lambda_i$  edges of size  $h_i$  incident with every  $h_i$  vertices for  $1 \leq i \leq m$ . A hypergraph  $\mathcal{G}$  is said to be  $k$ -regular if every vertex has degree  $k$ . A  $k$ -factor of  $\mathcal{G}$  is a  $k$ -regular spanning sub-hypergraph of  $\mathcal{G}$ . An  $R$ -factorization is a partition (decomposition)  $\{F_1, \dots, F_k\}$  of  $E(\mathcal{G})$  in which  $F_i$  is an  $r_i$ -factor for  $1 \leq i \leq k$ . Note that  $\Lambda K_n^H$  is  $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$ -regular. We show that the obvious necessary conditions for the existence of an  $R$ -factorization of  $\Lambda K_n^H$  are also sufficient.

**Theorem 6.1.**  $\Lambda K_n^H$  is  $R$ -factorizable if and only if  $s(R) = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$ , and there exists a non-negative integer matrix  $A = [a_{ij}]_{k \times m}$  such that  $AH = nR$ , and  $s(A_j) = \lambda_j \binom{n}{h_j}$  for  $1 \leq j \leq m$ .

**Proof.** To prove the necessity, suppose that  $\Lambda K_n^H$  is  $R$ -factorizable. Since each  $r_i$ -factor is an  $r_i$ -regular spanning sub-hypergraph for  $1 \leq i \leq k$ , and  $\Lambda K_n^H$  is  $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$ -regular, we must have  $s(R) = \sum_{i=1}^k r_i = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$ . Let  $a_{ij}$  be the number of edges (counting multiplicities) of size  $h_j$  contributing to the  $i$ th factor for  $1 \leq i \leq k, 1 \leq j \leq m$ . Since for  $1 \leq j \leq m$ , each edge of size  $h_j$  contributes  $h_j$  to the sum of the degrees of the vertices in an  $r_i$ -factor for  $1 \leq i \leq k$ , we must have  $\sum_{j=1}^m a_{ij} h_j = nr_i$  for  $1 \leq i \leq k$  and  $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j}$  for  $1 \leq j \leq m$ .

To prove the sufficiency, let  $\mathcal{F}$  be a hypergraph consisting of a single vertex  $v$  with  $m_{\mathcal{F}(v^{h_j})} = \lambda_j \binom{n}{h_j}$  for  $1 \leq j \leq m$ . Note that  $\mathcal{F}$  is an amalgamation of  $\Lambda K_n^H$ . Now we colour the edges of  $\mathcal{F}$  so that  $m_{\mathcal{F}(i)}(v^{h_j}) = a_{ij}$  for  $1 \leq i \leq k, 1 \leq j \leq m$ . This can be done, because

$$\sum_{i=1}^k m_{\mathcal{F}(i)}(v^{h_j}) = \sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} = m_{\mathcal{F}}(v^{h_j}) \quad \text{for } 1 \leq j \leq m.$$

Moreover,

$$d_{\mathcal{F}(i)}(v) = \sum_{j=1}^m a_{ij} h_j = nr_i \quad \text{for } 1 \leq i \leq k.$$

Let  $g : V(\mathcal{F}) \rightarrow \mathbb{N}$  be a function so that  $g(v) = n$ . Since for  $1 \leq i \leq m, h_i \leq n$ ,  $g$  is simple. By Theorem 4.1, there exists a simple  $g$ -detachment  $\mathcal{G}$  of  $\mathcal{F}$  with  $n$  vertices, say  $v_1, \dots, v_n$ , such that by (A2),  $d_{\mathcal{G}(i)}(v_j) \approx d_{\mathcal{F}(i)}(v)/g(v) = nr_i/n = r_i$  for  $1 \leq i \leq k, 1 \leq j \leq n$ , and by (A3), for



each  $U \subset \{v_1, \dots, v_n\}$  with  $|U| = h_j$ ,  $m_{\mathcal{G}}(U) \approx m_{\mathcal{F}}(v^{h_j})/\binom{n}{h_j} = \lambda_j \binom{n}{h_j} / \binom{n}{h_j} = \lambda_j$  for  $1 \leq j \leq m$ . Therefore  $\mathcal{G} \cong \Lambda K_n^H$ , and the  $i$ th colour class induces an  $r_i$ -factor for  $1 \leq i \leq k$ .  $\square$

In particular, if  $m = 1$ ,  $h := h_1$ ,  $\lambda_1 = 1$ ,  $r := r_1 = \dots = r_k$ , then Theorem 6.1 implies Baran-yai’s theorem: the complete  $h$ -uniform hypergraph  $K_n^h$  is  $r$ -factorizable if and only if  $h \mid rn$  and  $r \mid \binom{n-1}{h-1}$ .

Now let  $h_i \geq 2$  for  $1 \leq i \leq m$ , and let  $\Lambda K_{p_1, \dots, p_n}^H$  be a hypergraph with vertex partition  $\{V_1, \dots, V_n\}$ ,  $|V_i| = p_i$  for  $1 \leq i \leq n$  such that there are  $\lambda_i$  edges of size  $h_i$  incident with every  $h_i$  vertices, at most one vertex from each part for  $1 \leq i \leq m$  (so no edge is incident with more than one vertex of a part). If  $p_1 = \dots = p_n := p$ , we denote  $\Lambda K_{p_1, \dots, p_n}^H$  by  $\Lambda K_{n \times p}^H$ .

**Theorem 6.2.**  $\Lambda K_{p_1, \dots, p_n}^H$  is  $R$ -factorizable if and only if  $p_1 = \dots = p_n := p$ ,  $s(R) = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$ , and there exists a non-negative integer matrix  $A = [a_{ij}]_{k \times m}$  such that  $AH = nRp$ , and  $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$  for  $1 \leq j \leq m$ .

**Proof.** To prove the necessity, suppose that  $\Lambda K_{p_1, \dots, p_n}^H$  is  $R$ -factorizable (so it is regular). Let  $u$  and  $v$  be two vertices from two different parts, say the  $a$ th and  $b$ th parts, respectively. Since  $d(u) = d(v)$ , we have

$$\begin{aligned} \sum_{1 \leq j \leq m} \lambda_j \sum_{\substack{1 \leq i_1 < \dots < i_{h_j-1} \leq n \\ a \notin \{i_1, \dots, i_{h_j-1}\}}} p_{i_1} \cdots p_{i_{h_j-1}} &= \sum_{1 \leq j \leq m} \lambda_j \sum_{\substack{1 \leq i_1 < \dots < i_{h_j-1} \leq n \\ b \notin \{i_1, \dots, i_{h_j-1}\}}} p_{i_1} \cdots p_{i_{h_j-1}} \\ \iff \sum_{1 \leq j \leq m} \lambda_j \left( \sum_{\substack{1 \leq i_1 < \dots < i_{h_j-1} \leq n \\ a \notin \{i_1, \dots, i_{h_j-1}\}}} p_{i_1} \cdots p_{i_{h_j-1}} - \sum_{\substack{1 \leq i_1 < \dots < i_{h_j-1} \leq n \\ b \notin \{i_1, \dots, i_{h_j-1}\}}} p_{i_1} \cdots p_{i_{h_j-1}} \right) &= 0 \\ \iff \sum_{1 \leq j \leq m} \lambda_j \left( p_b \sum_{1 \leq i_1 < \dots < i_{h_j-2} \leq n} p_{i_1} \cdots p_{i_{h_j-2}} - p_a \sum_{1 \leq i_1 < \dots < i_{h_j-2} \leq n} p_{i_1} \cdots p_{i_{h_j-2}} \right) &= 0 \\ \iff (p_b - p_a) \sum_{1 \leq j \leq m} \lambda_j \sum_{1 \leq i_1 < \dots < i_{h_j-2} \leq n} p_{i_1} \cdots p_{i_{h_j-2}} &= 0 \\ \iff p_b = p_a. \end{aligned}$$

Therefore,  $p_1 = \dots = p_n := p$ . So  $\Lambda K_{n \times p}^H$  is  $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$ -regular, and we must have  $s(R) = \sum_{i=1}^k r_i = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$ . Moreover, there must exist non-negative integers  $a_{ij}$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq m$ , such that  $\sum_{j=1}^m a_{ij} h_j = n p r_i$  for  $1 \leq i \leq k$  and  $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} p^{h_j}$  for  $1 \leq j \leq m$ . We note that  $a_{ij}$  is in fact the number of edges (counting multiplicities) of size  $h_j$  contributing to the  $i$ th factor.

To prove the sufficiency, let  $\Lambda^p = [p^{h_i} \lambda_i]_{1 \times m}^T$ , and let  $\mathcal{F} = \Lambda^p K_n^H$  with vertex set  $V = \{v_1, \dots, v_n\}$ . Note that  $\mathcal{F}$  is an amalgamation of  $\Lambda K_{n \times p}^H$ . By Theorem 6.1,  $\mathcal{F}$  is  $pR$ -factorizable.

Therefore, we can colour the edges of  $\mathcal{F}$  so that

$$d_{\mathcal{F}(i)}(v) = pr_i \quad \text{for } v \in V, 1 \leq i \leq k.$$

Let  $g : V \rightarrow \mathbb{N}$  be a function so that  $g(v) = p$  for  $v \in V$ . Since  $p \geq 1$ ,  $g$  is simple. By Theorem 4.1, there exists a simple  $g$ -detachment  $\mathcal{G}$  of  $\mathcal{F}$  with  $np$  vertices, say  $v_i$  is detached to  $v_{i1}, \dots, v_{ip}$  for  $1 \leq i \leq n$ , such that by (A2),  $d_{\mathcal{G}(i)}(v_{ab}) \approx d_{\mathcal{F}(i)}(v_a)/g(v_a) = pr_i/p = r_i$  for  $1 \leq i \leq k$ ,  $1 \leq a \leq n$ ,  $1 \leq b \leq p$ , and by (A3),  $m_{\mathcal{G}}(v_{a_1 b_1}, \dots, v_{a_{h_j} b_{h_j}}) \approx m_{\mathcal{F}}(v_{a_1}, \dots, v_{a_{h_j}})/p^{h_j} = p^{h_j} \lambda_j / p^{h_j} = \lambda_j$  for  $1 \leq j \leq m$ ,  $1 \leq a_1 < \dots < a_{h_j} \leq n$ ,  $1 \leq b_1, \dots, b_{h_j} \leq p$ . Therefore  $\mathcal{G} \cong \Lambda K_{n \times p}^H$ , and the  $i$ th colour class induces an  $r_i$ -factor for  $1 \leq i \leq k$ .  $\square$

In particular, if  $m = 1$ ,  $h := h_1$ ,  $\lambda_1 = 1$ ,  $r := r_1 = \dots = r_k$ , then Theorem 6.2 implies another one of Baranyai’s theorems: the complete  $h$ -uniform  $n$ -partite hypergraph  $K_{n \times p}^h$  is  $r$ -factorizable if and only if  $h \mid npr$  and  $r \mid \binom{n-1}{h-1} p^{h-1}$ .

Let  $J_k^T = [1 \dots 1]_{1 \times k}$ . For two column vectors  $Q = [q_1 \dots q_k]^T$ ,  $R = [r_1 \dots r_k]^T$ , if  $q_i \leq r_i$  for  $1 \leq i \leq k$ , we say that  $Q \leq R$ . For a hypergraph  $\mathcal{G}$ , a  $(q, r)$ -factor is a spanning sub-hypergraph in which

$$q \leq d(v) \leq r \quad \text{for each } v \in V(\mathcal{G}).$$

A  $(Q, R)$ -factorization is a partition  $\{F_1, \dots, F_k\}$  of  $E(\mathcal{G})$  in which  $F_i$  is a  $(q_i, r_i)$ -factor for  $1 \leq i \leq k$ . An almost  $k$ -factor of  $\mathcal{G}$  is  $(k - 1, k)$ -factor. An almost  $R$ -factorization is an  $(R - J_k, R)$ -factorization. The proofs of the following theorems are very similar to those of Theorem 6.1 and 6.2.

**Theorem 6.3.**  $\Lambda K_n^H$  is  $(Q, R)$ -factorizable if and only if  $s(Q) \leq \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} \leq s(R)$ , and there exists a non-negative integer matrix  $A = [a_{ij}]_{k \times m}$  such that  $nQ \leq AH \leq nR$ , and  $s(A_j) = \lambda_j \binom{n}{h_j}$  for  $1 \leq j \leq m$ .

**Proof.** To prove the necessity, suppose that  $\Lambda K_n^H$  is  $(Q, R)$ -factorizable. Because  $\Lambda K_n^H$  is  $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$ -regular, we must have

$$s(Q) = \sum_{i=1}^k q_i \leq \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} \leq \sum_{i=1}^k r_i = s(R).$$

Since for  $1 \leq j \leq m$ , each edge of size  $h_j$  contributes  $h_j$  to the the sum of the degrees of the vertices in each  $(q_i, r_i)$ -factor, for  $1 \leq i \leq k$ , there must exist non-negative integers  $a_{ij}$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq m$ , such that  $nq_i \leq \sum_{j=1}^m a_{ij} h_j \leq nr_i$  for  $1 \leq i \leq k$  and  $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j}$  for  $1 \leq j \leq m$ .

To prove the sufficiency, let  $\mathcal{F}$  be a hypergraph consisting of a single vertex  $v$  with  $m_{\mathcal{F}}(v^{h_j}) = \lambda_j \binom{n}{h_j}$  for  $1 \leq j \leq m$ . Note that  $\mathcal{F}$  is an amalgamation of  $\Lambda K_n^H$ . Now we colour the edges of  $\mathcal{F}$  so that  $m_{\mathcal{F}(i)}(v^{h_j}) = a_{ij}$  for  $1 \leq i \leq k$ ,  $1 \leq j \leq m$ . This can be done, because

$$\sum_{i=1}^k m_{\mathcal{F}(i)}(v^{h_j}) = \sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} = m_{\mathcal{F}}(v^{h_j}) \quad \text{for } 1 \leq j \leq m.$$

Moreover,

$$nq_i \leq d_{\mathcal{F}(i)}(v) = \sum_{j=1}^m a_{ij}h_j \leq nr_i \quad \text{for } 1 \leq i \leq k.$$

Let  $g : V(\mathcal{F}) \rightarrow \mathbb{N}$  be a function so that  $g(v) = n$ . Since for  $1 \leq i \leq m$ ,  $h_i \leq n$ ,  $g$  is simple. By Theorem 4.1, there exists a simple  $g$ -detachment  $\mathcal{G}$  of  $\mathcal{F}$  with  $n$  vertices, say  $v_1, \dots, v_n$ , such that by (A2),  $q_i = nq_i/n \leq d_{\mathcal{G}(i)}(v_j) \leq nr_i/n = r_i$  for  $1 \leq i \leq k$ ,  $1 \leq j \leq n$ , and by (A3), for each  $U \subset \{v_1, \dots, v_n\}$  with  $|U| = h_j$ ,  $m_{\mathcal{G}}(U) \approx m_{\mathcal{F}}(v^{h_j})/\binom{n}{h_j} = \lambda_j \binom{n}{h_j} / \binom{n}{h_j} = \lambda_j$  for  $1 \leq j \leq m$ . Therefore  $\mathcal{G} \cong \Lambda K_n^H$ , and the  $i$ th colour class induces a  $(q_i, r_i)$ -factor for  $1 \leq i \leq k$ .  $\square$

**Theorem 6.4.**  $\Lambda K_n^H$  is almost  $R$ -factorizable if and only if  $s(R) - k \leq \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} \leq s(R)$ , and there exists a non-negative integer matrix  $A = [a_{ij}]_{k \times m}$  such that  $n(R - J_k) \leq AH \leq nR$ , and  $s(A_j) = \lambda_j \binom{n}{h_j}$  for  $1 \leq j \leq m$ .

**Proof.** It is enough to take  $Q = R - J_k$  in Theorem 6.3.  $\square$

**Theorem 6.5.**  $\Lambda K_{n \times p}^H$  is  $(Q, R)$ -factorizable if and only if  $s(Q) \leq \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leq s(R)$ , and there exists a non-negative integer matrix  $A = [a_{ij}]_{k \times m}$  such that  $npQ \leq AH \leq npR$ , and  $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$  for  $1 \leq j \leq m$ .

**Proof.** To prove the necessity, suppose that  $\Lambda K_{n \times p}^H$  is  $(Q, R)$ -factorizable. Since  $\Lambda K_{n \times p}^H$  is  $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$ -regular, we must have

$$s(Q) = \sum_{i=1}^k q_i \leq \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leq \sum_{i=1}^k r_i = s(R).$$

Moreover, there must exist non-negative integers  $a_{ij}$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq m$ , such that  $npq_i \leq \sum_{j=1}^m a_{ij}h_j \leq npr_i$  for  $1 \leq i \leq k$  and  $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} p^{h_j}$  for  $1 \leq j \leq m$ .

To prove the sufficiency, let  $\Lambda^p = [p^{h_i} \lambda_i]_{1 \times m}^T$ , and let  $\mathcal{F} = \Lambda^p K_n^H$  with vertex set  $V = \{v_1, \dots, v_n\}$ . Note that  $\mathcal{F}$  is an amalgamation of  $\Lambda K_{n \times p}^H$ . By Theorem 6.3,  $\mathcal{F}$  is  $(pQ, pR)$ -factorizable. Therefore, we can colour the edges of  $\mathcal{F}$  so that

$$pq_i \leq d_{\mathcal{F}(i)}(v) \leq pr_i \quad \text{for } v \in V, 1 \leq i \leq k.$$

Let  $g : V \rightarrow \mathbb{N}$  be a function so that  $g(v) = p$  for  $v \in V$ . Since  $p \geq 1$ ,  $g$  is simple. By Theorem 4.1, there exists a simple  $g$ -detachment  $\mathcal{G}$  of  $\mathcal{F}$  with  $np$  vertices, say  $v_i$  is detached to  $v_{i1}, \dots, v_{ip}$  for  $1 \leq i \leq n$ , such that by (A2),  $q_i = pq_i/p \leq d_{\mathcal{G}(i)}(v_{ab}) \leq pr_i/p = r_i$  for  $1 \leq i \leq k$ ,  $1 \leq a \leq n$ ,  $1 \leq b \leq p$ , and by (A3),  $m_{\mathcal{G}}(v_{a_1 b_1}, \dots, v_{a_{h_j} b_{h_j}}) \approx m_{\mathcal{F}}(v_{a_1}, \dots, v_{a_{h_j}}) / p^{h_j} = p^{h_j} \lambda_j / p^{h_j} = \lambda_j$  for  $1 \leq j \leq m$ ,  $1 \leq a_1 < \dots < a_{h_j} \leq n$ ,  $1 \leq b_1, \dots, b_{h_j} \leq p$ . Therefore  $\mathcal{G} \cong \Lambda K_{n \times p}^H$ , and the  $i$ th colour class induces a  $(p_i, r_i)$ -factor for  $1 \leq i \leq k$ .  $\square$

**Theorem 6.6.**  $\Lambda K_{n \times p}^H$  is almost  $R$ -factorizable if and only if

$$s(R) - k \leq \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leq s(R),$$

and there exists a non-negative integer matrix  $A = [a_{ij}]_{k \times m}$  such that  $np(R - J_k) \leq AH \leq npR$ , and  $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$  for  $1 \leq j \leq m$ .

**Proof.** It is enough to take  $Q = R - J_k$  in Theorem 6.5. □

### Acknowledgement

The author wishes to thank the referee and professor D. G. Hoffman for very carefully reading this manuscript and for many suggestions.

### References

- [1] Bahmanian, M. A. and Rodger, C. A. Multiply balanced edge colorings of multigraphs. *J. Graph Theory*. doi:10.1002/jgt.20617.
- [2] Baranyai, Z. (1975) On the factorization of the complete uniform hypergraph. In *Infinite and Finite Sets I: Colloq., Keszthely, 1973, dedicated to P. Erdős on his 60th birthday*, Vol. of *Colloq. Math. Soc. Janos Bolyai*, North-Holland, pp. 91–108.
- [3] Baranyai, Z. (1979) The edge-coloring of complete hypergraphs I. *J. Combin. Theory Ser. B* **26** 276–294.
- [4] Berge, C. and Johnson, E. L. (1977) Coloring the edges of a hypergraph and linear programming techniques. In *Studies in Integer Programming: Proc. Workshop, Bonn, 1975. Ann. Discrete Math.* **1** 65–78.
- [5] Brouwer, A. E. (1977) On the edge-colouring property for the hereditary closure of a complete uniform hypergraph. Mathematisch Centrum, Amsterdam, Afdeling Zuivere Wiskunde. No. ZW 95/77.
- [6] Brouwer, A. E. and Tijdeman, R. (1981) On the edge-colouring problem for unions of complete uniform hypergraphs. *Discrete Math.* **34** 241–260.
- [7] Ferencak, M. N. and Hilton, A. J. W. (2002) Outline and amalgamated triple systems of even index. *Proc. London Math. Soc.* (3) **84** 1–34.
- [8] Hilton, A. J. W. (1980) The reconstruction of Latin squares with applications to school timetabling and to experimental design. *Math. Programming Study* **13** 68–77.
- [9] Hilton, A. J. W. (1984) Hamiltonian decompositions of complete graphs. *J. Combin. Theory Ser. B* **36** 125–134.
- [10] Hilton, A. J. W. (1987) Outlines of Latin squares. In *Combinatorial Design Theory*, Vol. 149 of *North-Holland Math. Stud.*, North-Holland, pp. 225–241.
- [11] Hilton, A. J. W., Johnson, M., Rodger, C. A. and Wantland, E. B. (2003) Amalgamations of connected  $k$ -factorizations. *J. Combin. Theory Ser. B* **88** 267–279.
- [12] Hilton, A. J. W. and Rodger, C. A. (1986) Hamiltonian decompositions of complete regular  $s$ -partite graphs. *Discrete Math.* **58** 63–78.
- [13] Johnson, E. L. (1978) On the edge-coloring property for the closure of the complete hypergraphs. In *Algorithmic Aspects of Combinatorics: Conf., Vancouver Island, BC, 1976. Ann. Discrete Math.* **2** 161–171.
- [14] Johnson, M. (2007) Amalgamations of factorizations of complete graphs. *J. Combin. Theory Ser. B* **97** 597–611.
- [15] Leach, C. D. and Rodger, C. A. (2001) Non-disconnecting disentanglements of amalgamated 2-factorizations of complete multipartite graphs. *J. Combin. Designs* **9** 460–467.
- [16] Leach, C. D. and Rodger, C. A. (2003) Hamilton decompositions of complete multipartite graphs with any 2-factor leave. *J. Graph Theory* **44** 208–214.

- [17] Leach, C. D. and Rodger, C. A. (2004) Hamilton decompositions of complete graphs with a 3-factor leave. *Discrete Math.* **279** 337–344.
- [18] Nash-Williams, C. St. J. A. (1987) Amalgamations of almost regular edge-colourings of simple graphs. *J. Combin. Theory Ser. B* **43** 322–342.
- [19] Rodger, C. A. and Wantland, E. B. (1995) Embedding edge-colorings into 2-edge-connected  $k$ -factorizations of  $K_{kn+1}$ . *J. Graph Theory* **19** 169–185.