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RESEARCH ARTICLE

Optimal VIX-linked structure for the target benefit pension plan

Lv Chen[1], Danping Li[2], Yumin Wang[3] and Xiaobai Zhu[4]

[1]Academy of Statistics and Interdisciplinary Sciences, KLATASDS-MOE, East China Normal University, Shanghai, China, [2]School of Statistics, KLATASDS-MOE, East China Normal University, Shanghai, China, [3]Warren Centre for Actuarial Studies and Research, University of Manitoba, Winnipeg, Canada and [4]Department of Finance, CUHK Business School, Hong Kong, China

Corresponding author: Danping Li; Email: dpli@fem.ecnu.edu.cn

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Abstract

In this paper, we study the optimal VIX-linked target benefit (TB) pension design. By applying the dynamic programming approach, we show the optimal risk-sharing structure for the benefit payment exhibits a linear form that consists of three components: (1) a model-robust performance adjustment, (2) a counter-cyclical volatility adjustment that depends on the VIX index, and (3) a TB level that is partially indexed to the cost-of-living adjustment. Differences between our results and the previous literature are highlighted via both theoretical derivations and numerical illustrations.

1. Introduction and motivation

Recently, intergenerational risk-sharing pension plans have been gaining increasing attention from both the industry and academia. Unlike the traditional defined benefit (DB) and defined contribution (DC) pension plans, risk-sharing plans spread the risks over multiple generations, which ensures relatively stable retirement incomes while maintaining the sustainability of the pension fund. For example, the target benefit (TB) pension plan in Canada (a.k.a. the Defined Ambition plan in the UK) aims to provide retirees with a DB-type benefit but allows the benefit payments to be adjustable based on the performance of the pension assets. In this situation, the risks are shared across different generations rather than retained by each individual (e.g., DC plans) or transferred entirely to future generations (e.g., DB plans). While extensive studies have demonstrated the advantages of risk-sharing plans,[1] the risk-sharing structures from most of the existing literature lack either transparency (i.e., easy to understand) or theoretical justification.[2]

[1] Properly designed risk-sharing pension plans are shown to be welfare-enhancing for all participants (e.g., Gollier, 2008), sustainable for the pension asset (e.g., Chen *et al.*, 2017), adequate for the retirement income (e.g., Hardy *et al.*, 2020), affordable for the active workers (e.g., Cui *et al.*, 2011), and fair across generations (e.g., Bégin, 2020). The risk-sharing mechanism introduces additional complexity to the pension plan, but studies such as Cui *et al.* (2011), Khorasanee (2012), and Bégin (2020) make efforts in proposing more transparent designs than the current practice for a DB plan.

[2] For example, Cui *et al.* (2011), Khorasanee (2012), and Bégin (2020) presume a linear risk-sharing structure for the contributions and the benefits without theoretically justifying the form of the structure. Goecke (2013), Bovenberg and Mehlkopf (2014), Boes and Siegmann (2018), Bams *et al.* (2016), and Chen *et al.* (2017) propose more complicated non-linear risk-sharing structures that might undermine the transparency of the structures. Beetsma and Lans Bovenberg (2009) and Beetsma *et al.* (2012) explore the equilibrium strategies for risk-sharing pensions using a two-period model.

In this paper, we explore an optimal *VIX*[3] *-linked risk-sharing pension design*. In contrast to some of the existing literature, we apply the stochastic control approach to derive the optimal benefit payment strategy for a stylized TB plan *without pre-specifying* the risk-sharing structure. As for the specific analysis, we consider a TB plan where the retirement benefit is adjustable according to both the pension asset performance and the VIX index. The optimal TB design is constructed by maximizing the welfare function of retirees. Following the work of Pan (2002), we model the equity market using a stochastic volatility model with jumps to incorporate the volatility index. We also investigate the optimal future cost-of-living adjustment (COLA) by modeling the future inflation index using an Ornstein–Uhlenbeck (OU) process, which is similar to Luo (2017).

In summary, we find a *linear* risk-sharing structure for the benefit payment in terms of two components, that is, the performance adjustment and the volatility adjustment. Specifically, any surplus/deficit of the pension asset or any deviation of the volatility index (VIX) from a reference point will be evenly distributed to all retirees in the form of benefit promotion/reduction. Note that our linear risk-sharing structure is consistent with the proposal made by the relevant literature (see, e.g., Cui *et al.*, 2011 and Bégin, 2020). It is also worth mentioning that the linearity of the risk-sharing structure makes it easy to understand and implement in practice. Introducing the volatility adjustment does not undermine the transparency of the structure as it is still linear and the volatility adjustment depends on the VIX, which is observable and exogenous to the pension plan.

We present our findings in the following. First, we find that the volatility adjustment is *counter-cyclical* in the sense that it provides a hedging effect during the market crash, which indicates that incorporating the volatility adjustment can effectively stabilize retirees' future income. Specifically, we demonstrate that the volatility adjustment term is *non-positive*. This is in contrast with the *non-negative assumption* of this term in Bégin (2020), which is based on the changing cost of the embedded options in most collective pension plans.[4]

In addition, we show that the volatility adjustment is strictly negative and plays a notable role in the risk-sharing structure when the equity premia are significantly affected by the volatility value (a situation that is evidenced by Pan, 2002). In fact, our numerical experiments suggest that the optimal TB design proposed by Cui *et al.* (2011) that contains the performance adjustment only is a special case under our framework when the equity premia are assumed to be independent of the volatility. This observation also replicates the zero-valued volatility adjustment in Bégin (2020) where the equity premia are assumed to be a constant.

The reason for a non-positive volatility adjustment in our optimal benefit payment structure can be attributed to the fact that our objective is to maximize the expected utility of the current generation while the utility of the future generations is also considered. Indeed, a counter-cyclical (i.e., negative) volatility adjustment indicates that our benefit structure increases intergenerational risk-sharing, which is in the interest of the current generation. On the contrary, Bégin (2020) considers a frozen pension plan where no future generations are allowed. It is therefore natural to assume a non-negative volatility adjustment so that intergenerational risk-sharing is reduced since no other generations are considered.

Second, we find that the performance adjustment is independent of most of the economic assumptions, in particular the stochastic volatility and the jump-diffusion processes. This model-robust feature may facilitate the practical implementation of the risk-sharing pension plans without being overly concerned about the sensitivity of the model assumptions. In addition, we explicitly express the relationship between the risk aversion parameters and the performance adjustment and provide natural restrictions in selecting the risk aversion parameters regarding individuals' income risk and the pension fund's sustainability risk.

[3]The Chicago Board Options Exchange (CBOE) Volatility Index, known as the VIX by its ticker symbol, is a real-time index that represents the market's expectation for volatility (derived from S&P 500 index options) over the coming 30 days.

[4]Specifically, most collective pension plans include a protective put option and a written call option. When the VIX is large, these options are expensive and the generation should receive less benefit while the generation should receive more benefit when the VIX is low for the options are cheap.

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Third, we show that our model can explain the partial indexation of the benefit payment in the current market practice for DB pension plans. Indeed, we find that the optimal benefit payment strategy contains a third adjustment term representing the COLA. The TB level is partially indexed in the sense that it consists of both a fixed amount and a full COLA term. Our numerical analysis illustrates that a partial indexation is always preferred. This observation establishes a closer and more realistic link between the theoretical TB design and the existing DB plans.

To align with the literature, we also consider the risk-sharing structure with the contribution adjustment and prove that the optimal contribution schedule also preserves a linear form with both the performance and the volatility adjustment. We prove that the risk-sharing coefficients exhibit a constant multiplier relation between the benefit and the contribution (a similar finding in Zhu *et al.*, 2021 which minimizes the income instability and uses Wilkie's Economic Scenario Generator).

In the end, sensitivity tests over the salary risk, the jump risk, and the changes in the population structure have been examined. While the main findings remain valid for a large range of parameter values, our numerical analysis demonstrates that ignoring the jump risk leads to an underestimation of the volatility adjustment, which reduces the risk-hedging effect of the risk-sharing structure.

Risk-sharing plans have been an important topic among existing literature, see, for example, Cui *et al.* (2011), Gollier (2008), and related literature in Footnote 1. Nevertheless, the risk-sharing structures from most of the literature are either not transparent enough or based on some pre-specified forms that lack theoretical justification (see Footnote 1). In contrast, we search for the optimal risk-sharing pension design without pre-specifying the structure of the design. We obtain a transparent linear risk-sharing structure for the benefit payment, which is consistent with the presumed forms in Cui *et al.* (2011) and Bégin (2020). In other words, our results can provide a theoretical justification for the presumed linear risk-sharing structure proposed by the literature.

Applying the stochastic control approach to study the optimal TB design is not new, see, for example, Wang *et al.* (2018), Wang and Lu (2019), Wang *et al.* (2019), Chen *et al.* (2023), Zhao and Wang (2022), and Rong *et al.* (2023). In particular, Wang *et al.* (2018) derive the risk-sharing design that minimizes retirees' income risk and confirms the optimality of the linear risk-sharing structure proposed by Cui *et al.* (2011). The aforementioned studies rely on a Geometric Brownian Motion to model the equity risk and thus only the performance adjustment is considered. Optimal control problems that are based on a stochastic volatility model with jumps are only studied in other contexts, such as portfolio optimization, see, for instance, Egloff *et al.* (2010), Jin and Zhang (2012; 2013), Escobar *et al.* (2015), and Hong and Jin (2022). Our paper can help fill this void for the literature regarding risk-sharing pension plans.

There are only limited studies regarding the VIX index in the actuarial context. Aside from the novel contribution of Bégin (2020) in the risk-sharing pension design, Cui *et al.* (2017) and Kouritzin and MacKay (2018) utilize the VIX index in the fee design of variable annuities. We contribute to the literature on actuarial applications of the VIX index. Moreover, the VIX index is associated with the cyclicality of the financial market. Our results, where the performance adjustment indicates a pro-cyclicality and the volatility adjustment exhibits a counter-cyclicality, partially align with the cyclical design of the risk-sharing pension in Chen *et al.* (2023).

The remainder of this paper is organized as follows. Section 2 presents the financial framework, the stylized pension plan, and the objective function studied in this paper. Section 3 presents our theoretical results. Section 4 provides the numerical analysis of our main findings. Section 5 examines the assumptions used in this study. Section 6 concludes.

2. Model

In this section, we introduce the models and assumptions used in this paper. To incorporate the volatility index into the risk-sharing design, a stochastic volatility model is applied. We follow a similar setup as in Bégin (2020) with some modifications to enhance the reasonableness of our framework and facilitate the derivation of the explicit solution. The assumptions on the population and pension provisions are standard as in the literature (e.g., Wang *et al.*, 2018; 2019; Chen *et al.*, 2021b, etc.).

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2.1 Financial market

Denote $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ as a complete filtered probability space, where $\mathbb{P}$ is a real-world probability measure and $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous and $\mathbb{P}$-complete filtration, representing all the information up to time t. The stochastic processes in this paper are supposed to be well defined in the probability space and the moments of random variables are made under the probability measure $\mathbb{P}$. In addition, we denote by $\mathbb{E}[\cdot]$ the expectation of the random variables under probability $\mathbb{P}$.

The financial market consists of two assets, one risk-free asset $S_0(t)$ and one risky asset $S(t)$. We assume that $S_0(t)$ is growing at a constant interest rate r, that is,

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r\mathrm{d}t,$$

and $S(t)$ is modeled by a general jump-diffusion model that allows for stochastic volatility and returns jumps (e.g., Bates, 2000; Pan, 2002 and Bégin, 2020),

$$\frac{\mathrm{d}S(t)}{S(t-)} = [r + \lambda \times \nu(t) + \theta \times \nu(t) \times (\mu - \mu^{\mathbb{Q}})]\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W_S(t) + \mathrm{d}\left(\sum_{n=1}^{N_t}(e^{Z_n} - 1)\right) - \mu \times \theta \times \nu(t)\mathrm{d}t,$$
$$\mathrm{d}\nu(t) = \kappa_\nu(\bar{\nu} - \nu(t))\mathrm{d}t + \sigma_\nu\sqrt{\nu(t)}\left[\rho_\nu \mathrm{d}W_S(t) + \sqrt{1-\rho_\nu^2}\mathrm{d}W_\nu(t)\right]. \tag{2.1}$$

Here, $(W_S(t), W_\nu(t), W_l(t))$ is a three-dimensional standard Brownian Motion and we assume that $W_S(t)$, $W_\nu(t)$, and $W_l(t)$ are mutually independent. The instantaneous variance is modeled by $\nu(t)$ which is assumed to be a mean-reverting process with a long-run mean $\bar{\nu}$, a mean-reverting rate of κ_ν, and a volatility coefficient of σ_ν, where $|\rho_\nu| \leq 1$ is a correlation coefficient between the volatility and the risky asset. We assume the parameters satisfy the condition $2\kappa_\nu\bar{\nu} \geq \sigma_\nu^2$ that ensures the positivity of $\nu(t)$ (see also Cox *et al.*, 1985; Chen *et al.*, 2018, and Chen *et al.*, 2021a for a similar condition). In addition, following Bégin (2020), we include a jump component that follows a Poisson process $\{N_t\}_{t\geq 0}$ with a jump intensity of $\theta\nu(t)$. The jump size process $\{Z_n\}_{n\in\mathbb{N}}$ follows a normal distribution with mean μ_z and standard deviation σ_z, and we define $\mu = \mathbb{E}(\exp(Z_n) - 1)$ as the expected jump return. The definition of the expected jump return under the risk-neutral measure $\mu^{\mathbb{Q}}$ is given in Section 2.2. Furthermore, we assume that the Poisson process $\{N_t\}_{t\geq 0}$ and the jump size process $\{Z_n\}_{n\in\mathbb{N}}$ are both independent of the Brownian motions.

The risk premium now consists of two components, $\lambda\nu(t)$ and $\theta \times \nu(t) \times (\mu - \mu^{\mathbb{Q}})$, corresponding to the premium for the return risks and the jump risks, respectively, and both are expressed as a constant multiplier of the volatility term $\nu(t)$. We refer the readers to Pan (2002) for a thorough discussion on the specification of the risk premium and model calibration.

Furthermore, we model the salary uncertainty $L(t)$ using an OU process, similar to Wang (2006; 2009) and Luo (2017), that is,

$$\mathrm{d}L(t) = \kappa_l(\bar{L}(t) - L(t))\mathrm{d}t + \sigma_l\sqrt{\nu(t)}\left[\rho_{lS}\mathrm{d}W_S(t) + \rho_{l\nu}\mathrm{d}W_\nu(t) + \sqrt{1-\rho_{lS}^2-\rho_{l\nu}^2}\mathrm{d}W_l(t)\right]. \tag{2.2}$$

Here, $\kappa_l > 0$ governs the speed of convergence to a deterministic function $\bar{L}(t)$ and we use an exponential function $\bar{L}(t) = \exp(\psi t)$ to roughly align with the average nominal salary growth over the past few decades.[5] $\sigma_l \times \sqrt{\nu(t)}$ is the volatility of the salary, and ρ_{lS} and $\rho_{l\nu}$ are the correlation coefficients between the salary and the risky asset and between the salary and the volatility, respectively. Without loss of generality, we unitize the salary such that $L(0) = 1$. Alternatively, $L(t)$ can be regarded as the inflation index, which reflects the average cost of living of retirees. In this paper, we do not differentiate the salary and the inflation risks.

[5] In the benchmark scenario, we use $\psi = 0.0274$ to align with the average wage index (AWI) in the USA from 2000 to 2020; see https://www.ssa.gov/oact/cola/awidevelop.html.

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2.2 Risk-neutral dynamics and volatility index

The estimate of the risk premium and the benchmark $\overline{\mathrm{VIX}^2}$ involves the specification of the risk-neutral dynamics. Due to the additional sources of uncertainty (i.e., the stochastic volatility and the random jump sizes), the market is *incomplete* in our setting. As a result, the state-price density is not *unique*. We follow the method from Pan (2002) by focusing on a candidate state-price density that prices the two important sources of risks: diffusive price shocks and jump risks. Our candidate state-price density is specified in Appendix B.1. Under the risk-neutral measure $\mathbb{Q}$ associated with the candidate state-price density, the dynamics of the risky asset are in the following.

$$\frac{\mathrm{d}S(t)}{S(t-)} = r\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W_S^{\mathbb{Q}}(t) + \mathrm{d}\left(\sum_{n=1}^{N_t^{\mathbb{Q}}}\left(e^{Z_n^{\mathbb{Q}}} - 1\right)\right) - \mu^{\mathbb{Q}}\theta\nu(t)\mathrm{d}t$$

$$\mathrm{d}\nu(t) = \kappa_\nu(\bar{\nu} - \nu(t))\mathrm{d}t + \sigma_\nu\sqrt{\nu(t)}\left[\rho_\nu\mathrm{d}W_S^{\mathbb{Q}}(t) + \sqrt{1-\rho_\nu^2}\mathrm{d}W_\nu^{\mathbb{Q}}(t)\right], \tag{2.3}$$

where $\mu^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}\left(\exp(Z_n^{\mathbb{Q}}) - 1\right)$. Here, $\left(W_S^{\mathbb{Q}}(t), W_\nu^{\mathbb{Q}}(t), W_l^{\mathbb{Q}}(t)\right)$, $\{N_t^{\mathbb{Q}}\}_{t\geq 0}$, and $\{Z_n^{\mathbb{Q}}\}_{n\in\mathbb{N}}$ are a three-dimensional standard Brownian Motion, the Poisson process with jump intensity of $\theta\cdot\nu(t)$, and the jump size process with mean $\mu_z^{\mathbb{Q}}$ and variance σ_z^2 under $\mathbb{Q}$, respectively.[6]

Given the risk-neutral dynamic, the volatility index that measures the market's expectation of the annualized volatility over the next 30 days has been derived explicitly by Lin (2007). Here, we present the results only and refer interested readers to Lin (2007) for details. The squared volatility index VIX_t^2 can be expressed as

$$\mathrm{VIX}_t^2 = a_{\mathrm{VIX}} \times \nu(t) + b_{\mathrm{VIX}},$$

where

$$a_{\mathrm{VIX}} = 100^2 \times \left(1 + 2\theta\cdot(\mu^{\mathbb{Q}} - \mu_z^{\mathbb{Q}})\right)\cdot\frac{1 - e^{\kappa_\nu\cdot\tau}}{\kappa_\nu\cdot\tau},$$

$$b_{\mathrm{VIX}} = 100^2 \times \left(1 + 2\theta\cdot(\mu^{\mathbb{Q}} - \mu_z^{\mathbb{Q}})\right)\cdot\bar{\nu}\cdot\left[1 - \frac{1 - e^{\kappa_\nu\cdot\tau}}{\kappa_\nu\cdot\tau}\right],$$

and $\tau = \frac{30}{365}$.

The benchmark used for volatility adjustment in Bégin (2020) is the asymptotic mean of VIX_t^2 under the risk-neutral measure, that is,

$$\overline{\mathrm{VIX}^2} = \lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}[\mathrm{VIX}_t^2] = a_{\mathrm{VIX}} \times \bar{\nu} + b_{\mathrm{VIX}}.$$

2.3 Population and pension plan provision

In this paper, we consider a continuous population where all employees join the work at age A and retire at age R with a maximum attainable age of ω. We assume there is no other pre-exit of the pension plan except death. Denote $n_y(t)$ as the number of members aged y at time t, which is assumed to be a deterministic function of time t. We use the standard actuarial notation ${}_tp_y$ to represent the probability for a person aged y that survives for the next t years. We do not consider the longevity risk and assume the mortality risk is fully diversifiable, that is, $n_y(t) = n_A(t-(y-A)) \times {}_{y-A}p_A$.

The pension plan is assumed to be fully funded by the existing members, where a fixed contribution rate of c applies to all active members. The benefit is quoted in terms of the instantaneous replacement

[6] Pan (2002) has discussed the difficulty in identifying the risk premium for jump-timing uncertainty. To facilitate our sensitivity test in later chapters, we try to keep our model as simple as possible and set the risk premium for jump-timing risk and salary risk to zero. In addition, Pan (2002) shows that introducing a volatility-risk premium in addition to the jump-risk premium will not result in any significant improvement in the goodness of fit to the options data. Therefore, we only include premiums for return and jump risks.

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rate $b(t)$, which is not a guaranteed amount.[7] Later in this paper, we will include a discussion on the contribution adjustment (CA) scheme and the hybrid scheme that have been studied in Cui *et al.* (2011) and Bégin (2020). The aggregate contribution and the aggregate benefit at time t are defined as

$$\text{Aggregate contribution}(t) = \int_A^R n_y(t) \times c \times L(t) \times e^{\eta_a(y-A)}\mathrm{d}y = c \times L(t) \times \mathcal{A}(t),$$

$$\text{Aggregate benefit}(t) = \int_A^\omega n_y(t) \times b(t) \times L(t) \times e^{\eta_a(R-A)}\mathrm{d}y = b(t) \times L(t) \times \mathcal{R}(t),$$

where $\eta_a \geq 0$ is the promotional adjustment such that employees with longer career life generally have a higher salary. When $\eta_a = 0$, the aggregate contribution and benefits are simply multipliers of the population sizes of the active workers $\mathcal{A}(t)$ and the retirees $\mathcal{R}(t)$, respectively.

2.4 Pension asset and liability

Given the financial models and the aggregate contributions and benefits, the pension asset $X(t)$ has the following dynamics:

$$\begin{aligned} \mathrm{d}X(t) &= (X(t) - \pi(t))\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi(t)\frac{\mathrm{d}S(t)}{S(t-)} + c \times L(t) \times \mathcal{A}(t)\mathrm{d}t - b(t) \times L(t) \times \mathcal{R}(t)\mathrm{d}t \\ &= \left[rX(t) + \pi(t)\nu(t)(\lambda - \theta\mu^{\mathbb{Q}}) + (c\mathcal{A}(t) - b(t)\mathcal{R}(t)) \times L(t)\right]\mathrm{d}t \\ &\quad + \pi(t)\sqrt{\nu(t)}\mathrm{d}W_S(t) + \pi(t)\mathrm{d}\left(\sum_{n=1}^{N_t}(e^{Z_n} - 1)\right), \end{aligned} \tag{2.4}$$

with the initial asset level $X(0) = x_0$. The amount invested in the risky asset $\pi(t)$ and the replacement rate $b(t)$ are control variables that are dynamically decided by the pension sponsor.

In this paper, we use the traditional unit credit (TUC) method to valuate the actuarial liability (AL). Since the benefit level is not guaranteed, we apply the TUC to a benchmark benefit level $\hat{b}$ such that

$$\begin{aligned} \mathrm{AL}(t) &= \int_A^\omega n_y(t) \times \underbrace{\frac{\min(R, y) - A}{R - A} \times \hat{b}}_{=\text{ Benefit Accrual}} \times L(t) \times e^{\eta_a(\min(R,y)-A)} \times \int_{\max(0,R-y)}^{\omega-y} {}_sp_y \times e^{-\phi s}\mathrm{d}s\mathrm{d}y \\ &= \mathcal{H}(t) \times L(t), \end{aligned}$$

where ϕ is the actuarial discount rate and $\mathcal{H}(t)$ represents the AL value at time t in the real term.

2.5 Objective

We search for a welfare-enhancing risk-sharing design. We follow the standard approach to define the welfare function as the aggregate discounted expected utility function for all members during a planning horizon of T years. To avoid the situation where the interests of the future generations beyond the time T are sacrificed for the benefits of the current and nearby generations, we include a penalty term that depends on the terminal asset level $X(T)$.

[7] This is a stylized version of the Canadian target benefit plan. Cui *et al.* (2011) and Bégin (2020) name such a design as the "benefit adjustment" (BA) scheme.

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The objective is to maximize the welfare function by controlling the investment and the benefit payment strategies. Define the value function $H(t, x, \nu, l)$ as

$$H(t,x,\nu,l)=\sup_{(\pi,b)\in\Pi}\mathbb{E}_{t,x,\nu,l}\left[\int_t^T e^{-\zeta(s-t)}\times\mathcal{R}(s)\times U(b(s)\times L(s);\gamma_r)\mathrm{d}s+\varrho\times e^{-\zeta(T-t)}\times U(X(T);\gamma_T)\right], \tag{2.5}$$

where $E_{t,x,\nu,l} := E[\,\cdot\,|t, X(t)=x, \nu(t)=\nu, L(t)=l]$, ζ is the discount rate for the time preference, ϱ is the preference weight given to the utility function of the terminal asset level, and Π is the admissible set defined in Appendix B.2. We adopt an exponential utility function such that $U(x;\gamma)=-\frac{1}{\gamma}\exp(-\gamma x)$, where γ is the risk aversion parameter. Note that we specify different values of the risk aversion parameters for the individual retirement benefit and the asset level at the terminal time, namely γ_r and γ_T.

3. Optimal risk-sharing structure

We present the main results and their economic implications in this section. The details of the derivation are given in Appendix B.3.

3.1. Optimal benefit payment structure

To solve Problem (2.5), we use the Hamiltonian–Jacobi–Bellman approach and derive the explicit solution. The explicit solution and the optimal benefit payment strategy are summarized in the following theorem.

Theorem 3.1. *The optimal instantaneous replacement rate $b^*(t)$ is*

$$b^*(t)=-\frac{1}{\gamma_r L(t)}\ln\frac{\gamma_T\varrho A(t)}{\gamma_r}+\frac{\gamma_T}{\gamma_r L(t)}[A(t)X(t)+\bar{A}(t)\nu(t)+\hat{A}(t)L(t)+\tilde{A}(t)], \tag{3.1}$$

the optimal investment strategy $\pi^*(t)$ satisfies the following equation:

$$\pi^*(t)\cdot\gamma_T A(t)-\theta\mathbb{E}[\mathrm{e}^{-\gamma_T A(t)\pi^*(t)\cdot(\mathrm{e}^{Z_n}-1)}(\mathrm{e}^{Z_n}-1)]=(\lambda-\theta\mu^{\mathbb{Q}})-\sigma_\nu\rho_\nu\gamma_T\bar{A}(t)-\sigma_l\rho_{lS}\gamma_T\hat{A}(t), \tag{3.2}$$

and the corresponding value function $H(t, x, \nu, l)$ has an exponential form such that

$$H(t,x,\nu,l)=-\frac{\varrho}{\gamma_T}\mathrm{e}^{-\gamma_T[A(t)x+\bar{A}(t)\nu+\hat{A}(t)l+\tilde{A}(t)]}.$$

Here $A(t)$, $\bar{A}(t)$, $\hat{A}(t)$, and $\tilde{A}(t)$ are the solution of the following system of differential equations:

$$\begin{cases} A_t(t)=-rA(t)+\dfrac{\mathcal{R}(t)\gamma_T A^2(t)}{\gamma_r} \\ \hat{A}_t(t)=-c\mathcal{A}(t)A(t)+\kappa_l\hat{A}(t)+\dfrac{\mathcal{R}(t)\gamma_T A(t)\hat{A}(t)}{\gamma_r} \\ \bar{A}_t(t)=-\pi^*(t)(\lambda-\theta\mu^{\mathbb{Q}})A(t)+\kappa_\nu\bar{A}(t)+\dfrac{1}{2}(\pi^*)^2\gamma_T A^2(t)+\dfrac{1}{2}\sigma_\nu^2\gamma_T\bar{A}^2(t)+\dfrac{1}{2}\sigma_l^2\gamma_T\hat{A}^2(t) \\ \qquad\qquad +\pi^*(t)\sigma_\nu\rho_\nu\gamma_T A(t)\bar{A}(t)+\pi^*(t)\sigma_l\rho_{lS}\gamma_T A(t)\hat{A}(t)+\sigma_l\sigma_\nu(\rho_{lS}\rho_\nu+\rho_{l\nu}\sqrt{1-\rho_\nu^2})\gamma_T\bar{A}(t)\hat{A}(t) \\ \qquad\qquad +\dfrac{\theta}{\gamma_T}\mathbb{E}[\mathrm{e}^{-\gamma_T A(t)\pi^*(t)\cdot(\mathrm{e}^{Z_n}-1)}-1]+\dfrac{\mathcal{R}(t)\gamma_T A(t)\bar{A}(t)}{\gamma_r} \\ \tilde{A}_t(t)=-\dfrac{\zeta}{\gamma_T}-\kappa_\nu\bar{\nu}\bar{A}(t)-\kappa_l\bar{L}(t)\varrho\hat{A}(t)-\dfrac{\mathcal{R}(t)A(t)}{\gamma_r}\ln(\varrho A(t))+\dfrac{\mathcal{R}(t)\gamma_T A(t)\tilde{A}(t)}{\gamma_r}+\dfrac{\mathcal{R}(t)A(t)}{\gamma_r}, \end{cases} \tag{3.3}$$

with terminal conditions $A(T)=1$ and $\bar{A}(T)=\hat{A}(T)=\tilde{A}(T)=0$.[8]

[8] Note that all the functions in the theorem should also depend on the choice of T (e.g., $b^*(t;T)$). Since T can be omitted without any confusion, we suppress it to keep our notation simple.

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The solution for the optimal investment strategy aligns well with other stochastic control studies using an exponential utility function (e.g., Merton, 1969; Yang and Zhang, 2005, and Wang, 2007). More importantly, we focus on the risk-sharing structure of the benefit payment. Rearranging Equation (3.1), we have the following important remark.

Remark 1. The optimal benefit payment can be written in the following structure:

$$\text{benefit}(t) = b^*(t) \cdot L(t) = \bar{b}(t) + \beta_A(t)\frac{X(t) - \xi_A(t) \times \text{AL}(t)}{\mathcal{R}(t)} - \beta_{\text{VIX}}(t)\frac{\text{VIX}_t^2 - \xi_{\text{VIX}}(t) \times \overline{\text{VIX}^2}}{\mathcal{R}(t)}, \tag{3.4}$$

where the risk-sharing parameters $\beta_A(t)$ and β_{VIX} are defined as

$$\beta_A(t) = \frac{\gamma_T \mathcal{R}(t) A(t)}{\gamma_r}, \qquad \beta_{\text{VIX}}(t) = \frac{-\gamma_T \mathcal{R}(t)\bar{A}(t)}{\gamma_r a_{\text{VIX}}},$$

and the values for $\xi_A(t)$ and $\xi_{\text{VIX}}(t)$ can be arbitrarily chosen such that the TB level $\bar{b}(t)$ is defined as

$$\bar{b}(t) = -\frac{1}{\gamma_r}\ln\left(\frac{\gamma_T}{\gamma_r}\varrho A(t)\right) + \frac{\gamma_T}{\gamma_r}\tilde{A}(t) - \frac{\gamma_T \bar{A}(t)}{\gamma_r a_{\text{VIX}}}(b_{\text{VIX}} - \xi_{\text{VIX}}(t) \times \overline{\text{VIX}^2}) + \frac{\gamma_T}{\gamma_r}\left(A(t)\xi_A(t)\mathcal{H}(t) + \hat{A}(t)\right)L(t).$$

In general, the structure of the optimal benefit payment (a.k.a. Equation (3.4)) coincides with the proposal made by Bégin (2020), except that all the risk-sharing parameters (i.e., $\beta_A(t)$ and $\beta_{\text{VIX}}(t)$) in Equation (3.4) are deterministic functions of time t instead of constants when the planning time horizon is finite or when the population structure changes in the future. Therefore, the implication of the optimal structure is straightforward, that is, the actual benefit $(b(t) \cdot L(t))$ deviates from the target level $\bar{b}(t)$ when the asset level is different from a certain threshold $(\xi_A(t))$ of the AL or when the VIX index is different from a certain threshold (ξ_{VIX}) of the benchmark $\overline{\text{VIX}^2}$. To further align with the proposal in Bégin (2020) and simplify the following numerical analysis, we set $\xi_A(t) = \xi_{\text{VIX}} = 1$ to represent that the risk-sharing threshold is 100% of the AL and the volatility benchmark. We keep the terms $\xi_A(t)$ and $\xi_{\text{VIX}}(t)$ in Equation (3.4) to stress that these thresholds are allowed to be different from 100%.[9] In what follows, we examine each of the three components in Equation (3.4), namely the performance adjustment β_A, the volatility adjustment β_{VIX}, and the TB level $\bar{b}$.

3.2. Performance adjustment β_A

The performance adjustment $\beta_A(t)$ can be explicitly expressed as

$$\beta_A(t) = \frac{\mathcal{R}(t)}{\frac{\gamma_r}{\gamma_T}e^{-(T-t)r} + \int_t^T e^{-(s-t)r}\mathcal{R}(s)\mathrm{d}s}, \tag{3.5}$$

which is independent of most of the economic parameters and guaranteed to be positive. In fact, it is only affected by the choice of the risk-free rate r, the ratio between the risk aversion parameters $\frac{\gamma_T}{\gamma_r}$, and the size of the retired population $\mathcal{R}(t)$. The effect of the risk aversion ratio $\frac{\gamma_T}{\gamma_r}$ becomes insignificant for a sufficiently long horizon. In particular, we have

$$\lim_{T\to\infty}\beta_A(t;T) = \frac{\mathcal{R}(t)}{\int_t^\infty e^{-(s-t)r}\mathcal{R}(s)\mathrm{d}s}. \tag{3.6}$$

We can observe from Equation (3.6) that β_A only depends on the relative population size of the current retirees compared with that of the future retirees. If we have a fast-growing population, then future generations have more ability in absorbing the pension deficit. Therefore, it is optimal to stabilize the retirement income for current retirees with more deficit transferred to the future, which results in a smaller β_A. Notice that the discount rate for the time preference (or intergenerational discount rate) ζ is

[9] A threshold different from 100% is also common in the practice. For example, the threshold is 130% for full indexation for the pension plan in the Netherlands, and 80% funding level in the USA is often considered fully funded (American Academy of Actuaries, 2012).

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not involved in Equation (3.6), which means that the sponsor does not need to worry about the issues of intergenerational fairness when determining the performance adjustment.

It is important to emphasize that the performance adjustment β_A is determined independently of the stochastic volatility assumption. We can show that $\beta_A(t)$ still has the same formula when the risky asset is assumed to follow a geometric Brownian motion as Merton (1969).[10] This suggests that the performance adjustment term is model-robust. The volatility adjustment can be regarded as a supplementary adjustment on top of the performance adjustment.

3.3. Volatility adjustment β_{VIX}

We first show that the volatility adjustment parameter $\beta_{\text{VIX}}(t)$ is guaranteed to be *non-positive* when the jump and salary risks are excluded. This result is summarized in the following remark.

Remark 2. If the jump and salary risks are not considered, then

$$\beta_{\text{VIX}}(t) = \begin{cases} -\dfrac{\gamma_T}{\gamma_r}\dfrac{\mathcal{R}(t)}{a_{\text{VIX}}}\dfrac{\nu_1 - \nu_1 \mathrm{e}^{-\frac{\sigma_\nu^2(1-\rho_\nu^2)\gamma_T(\nu_1-\nu_2)}{2}(T-t)}}{1 - \frac{\nu_1}{\nu_2}\mathrm{e}^{-\frac{\sigma_\nu^2(1-\rho_\nu^2)\gamma_T(\nu_1-\nu_2)}{2}(T-t)}}, & \rho_\nu \neq \pm 1, \\ -\dfrac{\mathcal{R}(t)}{a_{\text{VIX}}}\dfrac{\lambda^2}{2\gamma_r}\displaystyle\int_t^T \mathrm{e}^{-\int_0^w (\lambda\sigma_\nu + \kappa_\nu + \beta_A(s))\cdot(1+\mathbb{1}_{s\le t})\mathrm{d}s}\mathrm{d}w, & \rho_\nu = 1,\ \lambda\sigma_\nu\rho_\nu + \kappa_\nu \neq -\beta_A(t), \\ -\dfrac{\mathcal{R}(t)}{a_{\text{VIX}}}\dfrac{\lambda^2}{2\gamma_r}\displaystyle\int_t^T \mathrm{e}^{-\int_0^w (-\lambda\sigma_\nu + \kappa_\nu + \beta_A(s))\cdot(1+\mathbb{1}_{s\le t})\mathrm{d}s}\mathrm{d}w, & \rho_\nu = -1,\ \lambda\sigma_\nu\rho_\nu + \kappa_\nu \neq -\beta_A(t), \\ -\dfrac{\mathcal{R}(t)}{a_{\text{VIX}}}\dfrac{\lambda^2}{2\gamma_r}(T-t), & \rho_\nu = \pm 1,\ \lambda\sigma_\nu\rho_\nu + \kappa_\nu = -\beta_A(t), \end{cases} \tag{3.7}$$

where

$$\nu_{1,2} = \frac{-\lambda\sigma_\nu\rho_\nu - \kappa_\nu - \beta_A(t) \pm \sqrt{(\lambda\sigma_\nu\rho_\nu + \kappa_\nu + \beta_A(t))^2 + \lambda^2\sigma_\nu^2(1-\rho_\nu^2)}}{\sigma_\nu^2(1-\rho_\nu^2)\gamma_T}.$$

It is clear from Remark 2 that

$$\beta_{\text{VIX}}(t) \le 0$$

by observing that $\nu_1 \ge 0 \ge \nu_2$. The derivation of Equation (3.7) can be found in Appendix C.2. We can also show that $\beta_{\text{VIX}}(t)$ is non-positive numerically when both the jump and the salary risks are included in Section 4 (see Figures 2 and 3).

The non-positivity of $\beta_{\text{VIX}}(t)$ indicates that the actual benefit will be increased when the volatility index is higher than the long-term benchmark (i.e., $\text{VIX}^2 > \overline{\text{VIX}^2}$) if all other items are kept unchanged in Equation (3.4). It can be explained by the positive choices in the risk premium λ, where a higher risk premium increases the equity's investment return and raises the benefit level further. Interestingly, the findings imply that retirees will receive compensation for potential benefit reductions during a market crash where the value of the pension asset is likely to decline sharply and the volatility index is typically quite high. This observation suggests a *counter-cyclical* risk-sharing policy and is in line with Chen *et al.* (2023). Notice that the non-positive values of $\beta_{\text{VIX}}(t)$ are the direct result of our optimization problem. This is in contrast with the non-negative assumption made in Bégin (2020). While Bégin (2020) sets the assumption due to financial fairness, our welfare-maximizing design provides a different interpretation on the volatility adjustment $\beta_{\text{VIX}}(t)$.

[10] The solution of the optimal risk-sharing strategy without stochastic volatility is given in the Appendix E. We omit the details of the proof as it is a special case of our general result.

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3.4. Partial indexation of TB $\bar{b}$

The TB level $\bar{b}(t)$ in Remark 1 can be decomposed in the following:

$$\bar{b}(t) = \bar{b}_F(t) + \bar{b}_L(t) \times L(t),$$

where

$$\bar{b}_F(t) = -\frac{1}{\gamma_r} \ln\left(\frac{\gamma_T}{\gamma_r}\varrho A(t)\right) + \frac{\gamma_T}{\gamma_r}\tilde{A}(t) - \frac{\gamma_T \bar{A}(t)}{\gamma_r a_{\mathrm{VIX}}}\left(b_{\mathrm{VIX}} - \xi_{\mathrm{VIX}}(t) \times \overline{\mathrm{VIX}^2}\right)$$

represents the fixed amount of the TB that is independent of the salary risk, and

$$\bar{b}_L(t) = \frac{\gamma_T}{\gamma_r}\left(A(t)\xi_A(t)\mathcal{H}(t) + \hat{A}(t)\right)$$

stands for the variable amount that reflects the COLA. The benefit is fully indexed to a wage index when $\bar{b}_F = 0$, and no indexation is applied when $\bar{b}_L = 0$. It is evident that both $\bar{b}_F$ and $\bar{b}_L$ are strictly positive when $\xi_A = \xi_{\mathrm{VIX}} = 1$. This indicates that the optimal retirement benefit adopts a partial COLA, which aligns well with the market practice for many existing DB plans.[11] In contrast, most of the literature finds either full indexation or no indexation.

3.5. Hybrid scheme with adjustable contribution rate c(t)

In this section, we include a short discussion on the optimal CA. Specifically, we now set the contribution rate as an additional control variable $c(t)$ and the objective is still to maximize the welfare function. Active members' utility is based on their disposable income after the contribution, and an exponential utility function with risk aversion parameter γ_a is used. Here, we summarize the results in the following remark; the details of the problem formulation along with the derivation are deferred to Appendix D.

Remark 3. The optimal contribution $c(t) \cdot L(t)$ and benefit payment $b(t) \cdot L(t)$ can be written in a similar structure as in Bégin (2020),

$$\mathrm{Contribution}(t) = \bar{c}(t) - \alpha_A(t)\frac{X(t) - \mathrm{AL}(t)}{\mathcal{A}(t)} + \alpha_{\mathrm{VIX}}(t)\frac{\mathrm{VIX}_t^2 - \overline{\mathrm{VIX}^2}}{\mathcal{A}(t)},$$

$$\mathrm{Benefit}(t) = \bar{b}(t) + \beta_A(t)\frac{X(t) - \mathrm{AL}(t)}{\mathcal{R}(t)} - \beta_{\mathrm{VIX}}(t)\frac{\mathrm{VIX}_t^2 - \overline{\mathrm{VIX}^2}}{\mathcal{R}(t)},$$

where the risk-sharing parameters α and β and the target contribution $\bar{c}(t)$ and TB $\bar{b}(t)$ are defined in Appendix D with the same interpretation and the same structure as β in Remark 2.

The remark justifies the linear risk-sharing proposals made by Cui *et al.* (2011) and Bégin (2020) on the optimal hybrid design. Most observations on the TB plan remain valid for the hybrid plan and more importantly, the relationship of the risk-sharing parameters between the contribution and the benefit exhibits the following constant ratio:

$$\frac{\alpha_A(t)}{\beta_A(t)} = \frac{\alpha_{\mathrm{VIX}}(t)}{\beta_{\mathrm{VIX}}(t)} = \frac{\gamma_r \mathcal{A}(t)}{\gamma_a \mathcal{R}(t)}.$$

This indicates the values for α and β differ only due to the difference in the population size and risk aversion between active workers and retirees. In particular, when $\gamma_r = \gamma_a$, the amount of risk borne by any group is proportional to the group population size.[12] In the usual circumstance where $\mathcal{A}(t) > \mathcal{R}(t)$ and $\gamma_a < \gamma_r$, active workers will carry relative higher risks than retirees. Moreover, the constant ratio relationship implies that α_{VIX} should be non-positive, in contrast with the non-negative assumption used

[11] For example, the pension plan at the University of Waterloo has a 50% indexation, and the pension plan at the University of Columbia has a 75% indexation.

[12] Zhu *et al.* (2021)have documented a similar finding with an aim to minimize the expected future income instability.

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in Bégin (2020). In the end, when $\gamma_a = \gamma_r$ it can be shown that

$$\text{contribution}(t) + \text{benefit}(t) = L(t).$$

This means that the after-contribution income for the active worker is the same as the retirement income for the retirees, which implies a 100% replacement rate. Clearly, this is not the general consensus for a pension design as a replacement rate around 60% is often deemed to be reasonable for a lifetime worker and therefore it would be reasonable to set $\gamma_a \approx 60\% \times \gamma_r$.

4. Numerical analysis

In this section, we conduct numerical analysis on the risk-sharing structure of the optimal benefit payment, namely the performance adjustment (i.e., β_A), the volatility adjustment (i.e., β_{VIX}), and the cost-of-living adjustment (i.e., $\bar{b}$).

Due to a large number of notations, we summarize all notations in Table A.1. For each of the parameters, we also include the benchmark values. The economic model parameters are based on the calibration in Pan (2002). In what follows, we briefly discuss the benchmark values for some important parameters. First, we set the benchmark replacement rate for liability valuation (a.k.a. $\hat{b}$) to be 65% of the salary to match with an actuarially fair DB plan.[13] Second, γ_r, the risk aversion coefficient of the individual benefit, is assumed to be 50. This value corresponds to a relative risk aversion of 32.79, which is the upper bound of the calibrated values for the relative risk aversion of an exponential utility function using S&P 500 option data by Bliss and Panigirtzoglou (2004).[14] We choose the upper bound to reflect the fact that pensioners have a higher risk aversion than the general population (see, for instance, Van Rooij *et al.*, 2007). More discussions on the choices of values for γ_r, γ_T, ϱ, and ζ will be given in the following numerical analysis. Last, the actuarial discount rate ϕ is set to be 2.5% to align with the practice in the solvency liability valuation.

4.1. Performance adjustment β_A

Due to the decreasing relative risk aversion of the exponential utility function, it is appropriate to set different values for the risk aversion parameters of the retirement income and the terminal asset level because they are significantly different in scale.[15] However, the choices of γ_r and γ_T are not straightforward because (1) they represent two different quantities that are not comparable, with the former one focusing on an individual's income adequacy while the latter one on the overall pension solvency and (2) both the retirement income and the terminal asset level are random. Therefore, we use an alternative approach to provide a new interpretation between γ_r and γ_T. Recall that $\beta_A(t)$ represents the percentage of deficit/surplus that will be distributed to the retirees and that $\beta_A(T) = \frac{\gamma_T \mathcal{R}(T)}{\gamma_r}$. The choice of $\frac{\gamma_T}{\gamma_r}$ has a natural upper limit of $\frac{1}{\mathcal{R}(T)}$ so that the pension fund cannot distribute more than 100% of the deficit to the retirees. Although the choices of γ_r and γ_T under this approach deviate from their original definitions with respect to utility functions, they reconcile with the objective of balancing the conflicting interests between different generations. The value of β_A (and hence the ratio of γ_T over γ_r) represents the level of risk bored by the current generations and thus indirectly reflects the risk transfer to the future.

Figure 1 displays the effect of $\frac{\gamma_T}{\gamma_r}$ (expressed in term of $\beta_A(T)$ within the range 1–100%) on $\beta_A(t)$. We select $\beta_A(T) = 3\%$ in our benchmark scenario to match the optimal value obtained in Bégin (2020). Note

[13] An actuarial fair DB plan satisfies

$$c \times \int_A^R e^{-\phi(t-E)} \times {}_{x-A}p_A \,\mathrm{d}x = \hat{b} \times \int_R^\omega e^{-\phi(t-A)} \times {}_{x-A}p_A \,\mathrm{d}x.$$

[14] Bliss and Panigirtzoglou (2004) find that the relative risk aversion values are between 0.65 and 32.79.

[15] Cautions must be made for scaling under the exponential utility, see, for example, Mania and Schweizer (2005).

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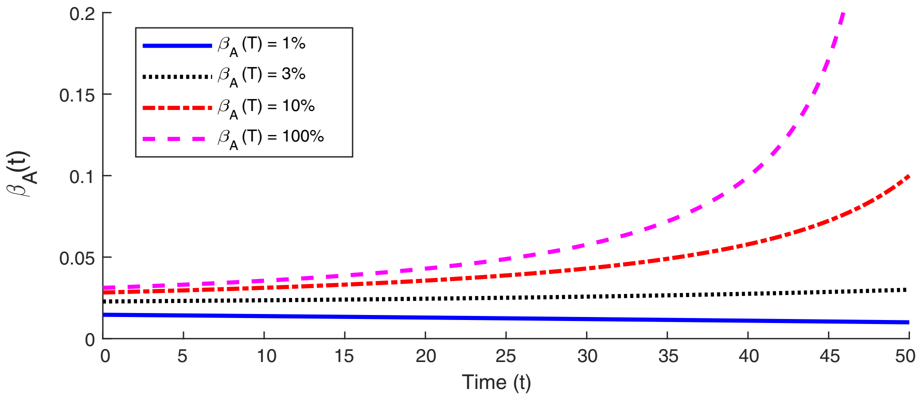

Figure 1. *$\beta_A(t)$ under different $\beta_A(T) = \frac{\gamma_T}{\gamma_r}\mathcal{R}(T)$.*

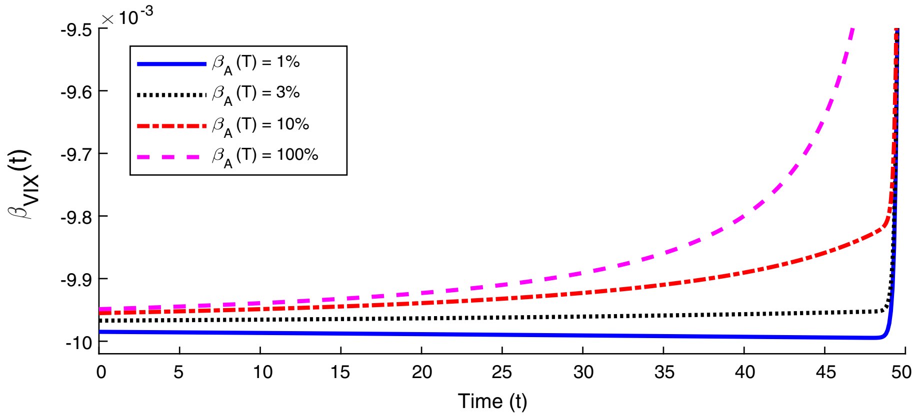

Figure 2. *$\beta_{VIX}(t)$ under different $\beta_A(T) = \frac{\gamma_T}{\gamma_r}\mathcal{R}(T)$.*

that this value also provides the most time-stable performance adjustment as shown in the graph. This figure demonstrates that β_A is relatively stable for all four cases (at least for the initial period of roughly twenty years). The shape of $\beta_A(t)$ is usually flat over the initial period of time and increases/decreases sharply when the time approaches T (with speed controlled by the risk-free rate and the population growth rate of the retirees).

4.2. Volatility adjustment β_{VIX}

Figure 2 demonstrates the values of the volatility adjustment across time under different values of γ_T (expressed in terms of $\beta_A(T)$). First, we can observe that β_{VIX} appears to be more stable over a longer period of time than $\beta_A(t)$ for all four cases. It stays flat for over 35 years before a sharp convergence to 0 near maturity. Second, the volatility adjustment is robust with respect to different choices of γ_T within a reasonable range, which is similar to the performance adjustment. Third, the volatility adjustment β_{VIX} is negative and significantly differs away from zero for all the cases, which echoes Remark 2 where the jump and salary risks are not considered. This indicates that the volatility adjustment in fact possesses the features of counter-cyclical risk-sharing policies.

Figure 3 displays the value of β_{VIX} with different risk premia (λ) and different correlation coefficients between the volatility and the risky asset (ρ_v) at time $t = 0$. We can observe that the effect of volatility

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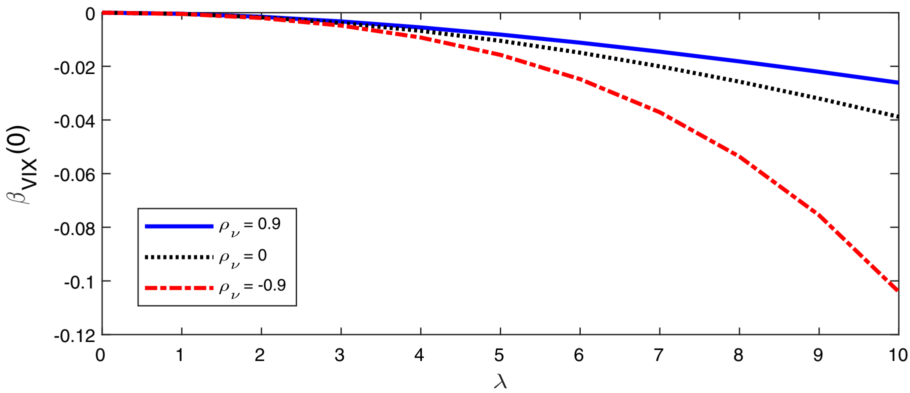

Figure 3. *$\beta_{VIX}(0)$ under different λ.*

adjustment is most significant (i.e., largest in the absolute term $|\beta_{\text{VIX}}|$) when the risk premia are high and ρ_ν is highly negative, which highlights the hedging effect of the volatility adjustment term during the bear market. When we move from negative correlation coefficients to positive, we can observe that the hedging effect of the volatility adjustment diminishes since the high-risk premia will offset the market crash. In addition, the higher the risk premia λ, the better the expected investment returns will be achieved, and thus more surplus are expected to be available for distribution to the retirees.

With a non-negative constraint, Bégin (2020) obtain obtain an optimal benefit structure that does not have a volatility adjustment ($\beta_{\text{VIX}} = 0$).[16] We deem that this is partly because of misspecification in the risk premia. If we adopt the same assumption as Bégin (2020) such that the risk premia are independent of the volatility term (i.e., $\lambda = 0$), then we would also obtain a zero volatility adjustment as shown in Figure 3. However, as demonstrated by Pan (2002), investors seek a higher expected return on riskier assets, and therefore a large positive λ is more consistent with the market expectation. In such a case, the optimal risk-sharing design should always contain a negative β_{VIX} in its benefit adjustment (BA).

Figure 4 illustrates the historical BA based on the optimal $\beta_A(t)$ and β_{VIX} using the monthly S&P 500 data from 2006 to 2014. The equity market has done poorly during this time (i.e., behind inflation), and BAs have been persistently negative. All terms are expressed in real values (by dividing the average wage index) and we adopt a 50-50 investment strategy in risky and risk-free assets in this illustration. We can observe that the performance adjustment is dominating the overall adjustment and the volatility adjustment becomes significant only during the 2008 financial crisis. The volatility adjustment is clearly counter-cyclical to provide a hedge on top of the performance adjustment, which is pro-cyclical. Indeed, we find that the variation in the benefit adjustment (i.e., the standard deviation of $\text{Adjustment}_{t+\Delta} - \text{Adjustment}_t$) is reduced by 13% when the volatility adjustment is introduced and the adjustment volatility (i.e., the standard deviation of BA) is reduced by 9%.

4.3. Partial indexation of the optimal benefit

To measure the partial indexation level of the TB, recall that given $L(0) = 1$ we have

$$\bar{b}(t) = \bar{b}_F(t) + \bar{b}_L(t) \cdot L(t) = \left(\bar{b}_F(t) + \bar{b}_L(t)\right) \cdot \left[1 + \frac{\bar{b}_L(t)}{\bar{b}_F(t) + \bar{b}_L(t)} \cdot (L(t) - 1)\right].$$

[16] Bégin (2020) obtained a zero volatility adjustment in the benefit payment for both a target benefit plan and a hybrid plan. However, the optimal contribution adjustment in a hybrid plan has a positive volatility adjustment term.

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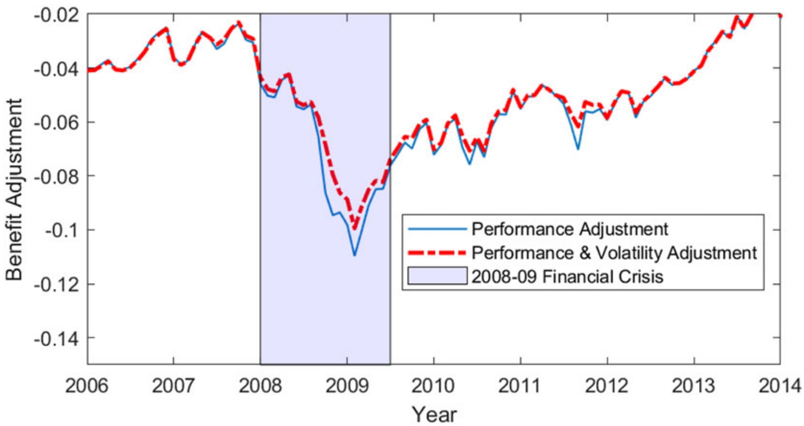

Figure 4. *Sample of historical benefit adjustments.*

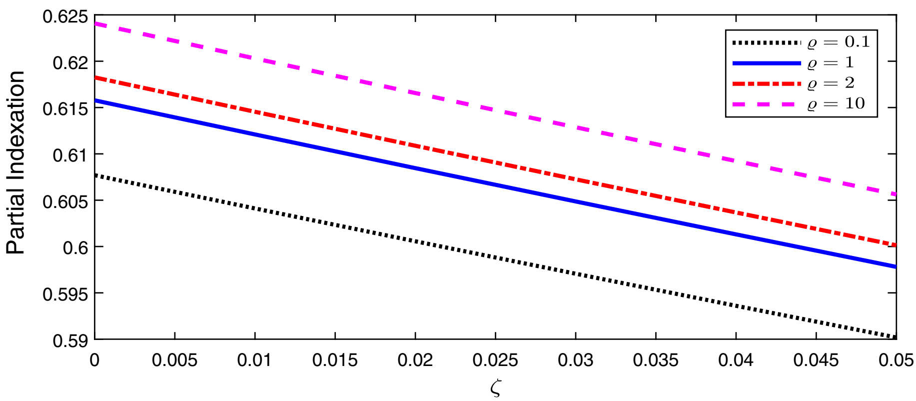

Figure 5. *Partial indexation* $\frac{\bar{b}_L \times L(t)}{\bar{b}_F + \bar{b}_L \times L(t)}$ *for* $t = 0$.

Since $L(t) - 1$ is the cumulative inflation growth up to time t, we define $\mathcal{I}(t) = \frac{\bar{b}_L(t)}{\bar{b}_F(t) + \bar{b}_L(t)}$ as the partial indexation rate.[17]

Figure 5 plots the levels of partial indexation at time $t = 0$ over different ϱ (weight given to the terminal asset) and ζ (time preference). Since ϱ and ζ only appear in the ordinary differential equation (ODE) of $\tilde{A}(t)$ and thus do not affect the risk-sharing parameters β or $\bar{b}_L$, Figure 5 also illustrates the sensitivity of $\bar{b}_F$ with respect to ϱ and ζ. We first find that the indexation level is within the acceptable range for DB practice, that is, between 59% and 63%, and is comparatively constant. Second, the higher the terminal penalty ϱ is, the relatively lesser weight is assigned to the generations up to time T, and therefore the retirement income level will be lower. Since $\bar{b}_L$ is unaffected by ϱ, this implies a lower $\bar{b}_F$ and thus a higher indexation level. The same argument applies to the time preference rate since a lower ζ assigns relatively lower weights to the current generation. Due to the insignificant effects of ζ and ϱ, we set $\zeta = 0$ and $\varrho = 1$ for the numerical analysis later on.

[17] This definition is based on the cumulative COLA, slightly different from practice as partial indexation is often applied to the growth rate of the inflation index (or the price index) during each time period. The partial indexation adopted in this paper is an approximation to the real world, and if the planning horizon T is long enough where $\bar{b}_F(t) \approx \bar{b}_F$ and $\bar{b}_L(t) \approx \bar{b}_L$, inflation indexation on the cumulative basis would be close to the on-going basis.

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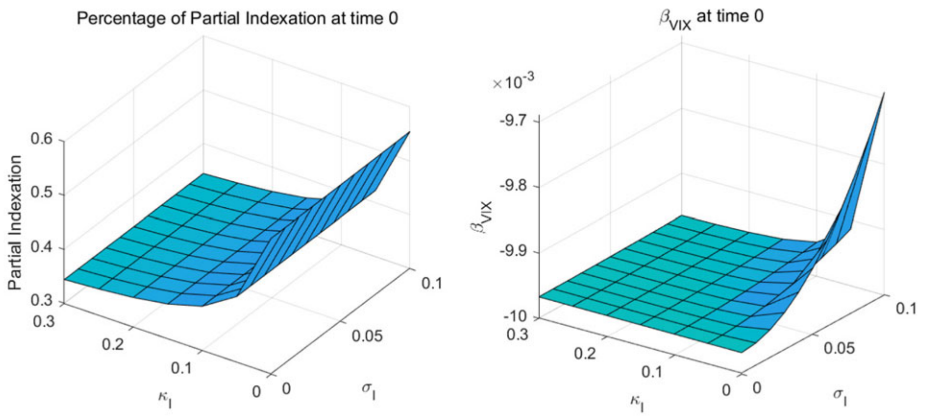

Figure 6. *Impact of salary risk on partial indexation $\frac{\bar{b}_F}{\bar{b}_F+\bar{b}_L}$ and β_{VIX}.*

We emphasize that the values of the TB level $\bar{b}(t)$ depend on the sponsors' subjective choices of the risk-sharing thresholds ξ_A and ξ_{VIX}. A lower TB level is anticipated as a result of lower criteria because they imply that surplus payouts are triggered more frequently. There is no such thing as a free lunch, so it is unrealistic to anticipate having both a bigger surplus distribution and a higher TB level.

5. Sensitivity analysis

This section provides sensitivity tests over economic and demographic parameters such as the salary risk, the jump risk, and changes in population structure.

5.1. Salary uncertainty

Figure 6 presents the effects of the salary risk on the percentage of partial indexation and the volatility adjustment β_{VIX} over different values of the mean-reverting speed κ_l for the salary process and different values of the salary volatility σ_l. It can be observed from the left panel that the mean-reverting speed κ_l is crucial in determining the partial indexation level while the salary volatility is rather insignificant. We can also observe that the partial indexation stays within a practical range for different values of κ_l and β_{VIX} (i.e., approximately between 35% and 55%).

The partial indexation level decreases as the mean-reverting speed κ_l increases. This might be because the salary L is more likely to be close to the long-term mean $\overline{L}$ under a larger κ_l and thus indicates that the benefit payment is less related to the wage index. The impact of the salary volatility σ_l on the partial indexation is rather insignificant compared with κ_l. In addition, the volatility adjustment β_{VIX} is shown to be stable if the salary index is close to what is anticipated (when κ_l is large or when σ_l is small).

5.2. Jump risk

Figure 7 presents the effect of jump risks on $\beta_{\text{VIX}}(t)$ and $\bar{b}_F(t)$ via the jump intensity θ and the expected jump size μ_z. Overall, we can observe the monotone decreasing relationships between θ and both $\beta_{\text{VIX}}(t)$ and $\bar{b}_F(t)$ in the two panels, while the expected jump size μ_z has only insignificant impact. In the left panel, the higher jump risk premia allow the sponsor to set a higher overall TB level, and thus a higher $\bar{b}_F$. Since $\bar{b}_L$ is unaffected, the level of partial indexation will be lower. In the right panel, the volatility adjustment is significantly affected by the amount of jump risk premia. Specifically, a higher θ represents both a higher frequency of downward shock and a higher jump risk premia, which in either case leads to a

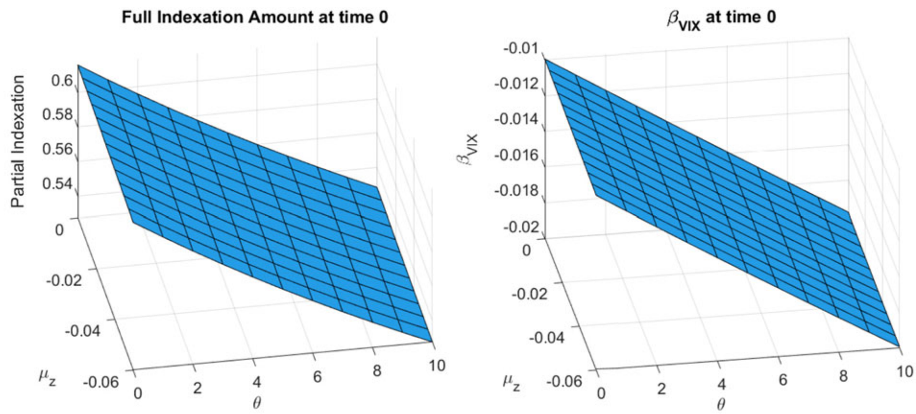

Figure 7. *Impact of jump risk on partial indexation* $\frac{\bar{b}_F}{\bar{b}_F+\bar{b}_L}$ *and* β_{VIX}.

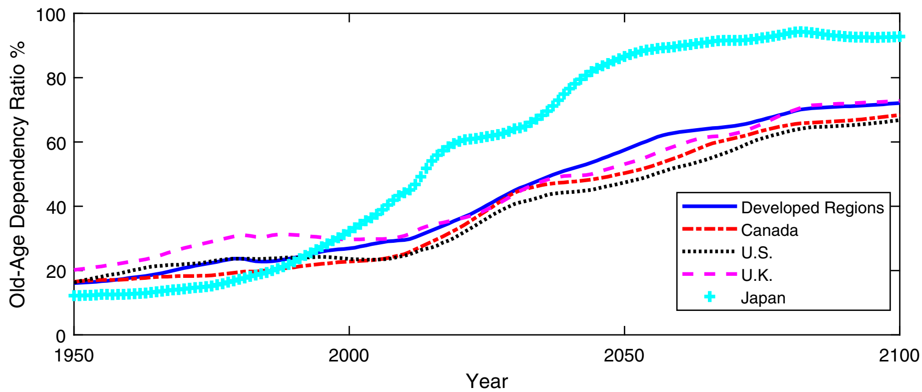

Figure 8. *United Nation Projection in old-age dependency ratio.*

higher volatility adjustment. This observation aligns with our interpretation on the volatility adjustment in the previous section. Therefore, ignoring the relationship between the jump risk and the stochastic volatility would significantly underestimate the volatility adjustment in the optimal benefit structure.

5.3. Dependency ratio

Most optimal control literature on pension plans assume a stationary population, which clearly does not match with the current experience of the aging society. We examine the impact of the aging population on both the performance adjustment and the volatility adjustment in this section.

Based on the projection of United Nations (2022), the population growth for developed regions is virtually zero. Therefore, to study the impact of the changing population structure, we set $\mathcal{A}(t) = \mathcal{A}$ and solely focus on the improvement in the old age mortality rate. Figure 8 plots the projections of the old-age dependency ratio[18] for the selected developed countries. Although the projection values are significantly different across different countries, they generally are of an S-shape, which means most of the countries are experiencing a rapidly growing aging population and the growth will eventually slow

[18] Old-age dependency ratio here represents the ratio between the population aged over 65 and the population aged between 25 and 64.

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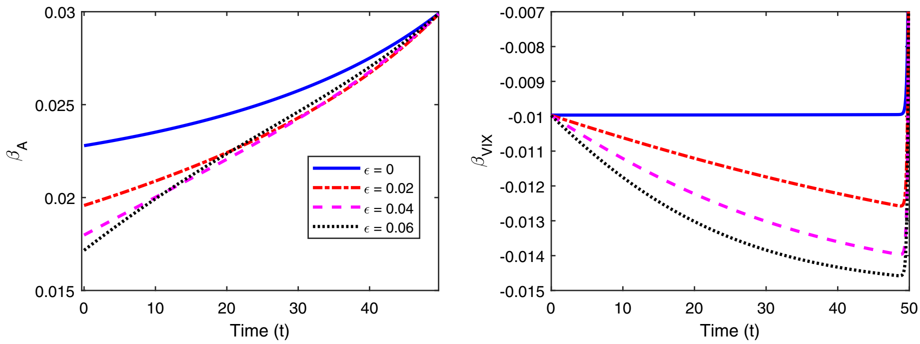

Figure 9. *United Nation Projection in old-age dependency ratio.*

down. It is therefore natural to use a Sigmoid function[19] to approximate the dynamic of the dependency ratio. Define the old-age dependency ratio as $\mathcal{D}(t) = \frac{\mathcal{R}(t)}{\mathcal{A}(t)}$ with ODE

$$\mathrm{d}\mathcal{D}(t) = \epsilon \times \left(\frac{\bar{\mathcal{D}} - \mathcal{D}(t)}{\bar{\mathcal{D}}} \right) \times \mathcal{D}(t)\mathrm{d}t,$$

where ϵ is the base growth rate and $\bar{\mathcal{D}}$ is the maximum dependency ratio can be achieved.

Figure 9 displays the effects of population growth rate on both the performance adjustment β_A and the volatility adjustment β_{VIX}, where $\bar{D} = 73\%$ is based on the UN's projection on Canadian data. From the left panel, we can observe a higher growth rate (corresponding to a larger retiree population throughout time) leads to a lower performance adjustment at initial periods of time (i.e., approximately 15–20 years) since the initial population size is relatively small. As time t increases, when the growth of the retiree population slows down, the relative position of the population at time t may become smaller for large ϵ and the population experiences a larger performance adjustment. Notice also that as long as $\epsilon > 0$, the difference is rather insignificant in the sense that β_A is rather stable once the long-term projection of the dependency ratio can be accurately estimated.

The right panel displays a significantly different trend in volatility adjustment under different population dynamics, compared to the performance adjustment. However, if we scale the volatility adjustment by the number of retirees $\left(\frac{\beta_{\text{VIX}}(t)}{\mathcal{R}(t)}\right)$ to examine the volatility adjustment for each individual, then the difference is much less significant. This is not surprising since the volatility index VIX_t^2 only appears in measuring the utility of each retirement benefit not in the terminal penalty. Thus, β_A is rather stable in terms of the aggregate adjustment, while β_{VIX} is more stable in terms of the individual adjustment.

6. Conclusion and future work

This paper studies the optimal risk-sharing pension plan by modeling the equity market using a stochastic volatility model with jumps and the inflation index using an OU process. We obtain an explicit solution for the benefit payment strategy, which consists of a performance adjustment, a volatility adjustment, and a cost-of-living adjustment.

We show that the performance adjustment is model-robust and illustrate its relationship with the risk aversion parameters regarding income and sustainability risk. We also find the volatility adjustment exhibits a counter-cyclical feature that acts as an income risk hedge during the financial crisis. Last, we demonstrate that the optimal TB level is partially indexed to the COLA.

[19] Sigmoid function has a long history in population modeling, for example, Berkson (1944) and Yin *et al.* (2003) among many others.

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Many aspects of the risk-sharing pension plans can be considered in future work. First, the downside deviation of the retirement income should be treated differently from the upside rewards and reflected in the objective function due to the counter-cyclical feature of the optimal pension design. Second, stochastic interest rate models may be considered to explain the time-invariant performance adjustment strategy when the risk-sharing coefficient equals the risk-free rate and the population is stationary. Third, regime-switching models may be considered to avoid a volatile structure using VIX as the only indicator when the reference point $\overline{\mathrm{VIX}^2}$ is high, that is, when the pension design can be regarded as having a regime-dependent benefit structure.

In addition, an important topic that we cannot address in the scope of this paper is the intergenerational fairness for risk-sharing pension plans. Several studies demonstrate the actuarial fairness across different generations for an initially fully funded risk-sharing plan. However, this leads to the commitment problem when the fund is severely underfunded such that members may prefer to opt out of the plan, threatening the sustainability of the fund. To properly address intergenerational fairness without threatening the sustainability of the pension fund, a more complicated pension design may be considered (e.g., including a vesting period or age-dependent risk-sharing structure) and the involvement of a third party may be necessary (e.g., risk-sharing of the pension sponsor or intervention from the government).

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