RISK REDISTRIBUTION GAMES WITH DUAL UTILITIES

BY

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ABSTRACT

This paper studies optimal risk redistribution between firms, such as institutional investors, banks or insurance companies. We consider the case where every firm uses dual utility (also called a distortion risk measure) to evaluate risk. We characterize optimal risk redistributions via four properties that need to be satisfied jointly. The characterized risk redistribution is unique under three conditions. Whereas we characterize risk redistributions by means of properties, we can also use some results to study competitive equilibria. We characterize uniqueness of the competitive equilibrium in markets with dual utilities. Finally, we identify two conditions that are jointly necessary and sufficient for the case that there exists a trade that is welfare-improving for all firms.

KEYWORDS

Dual utility, market games, risk sharing, competitive equilibria, no-trade.

1. INTRODUCTION

This paper characterizes optimal risk redistributions via cooperative market games. There is a relatively large literature that analyzes cooperative bargaining for optimal redistributions of risk, based on the seminal works of Borch (1962) and Shapley and Shubik (1969). This paper mainly differs in terms of the objective of the firms. We study optimal risk sharing in the context of dual utility instead of Von Neumann–Morgenstern expected utility. We characterize a specific risk redistribution in closed form using cooperative market games (Shapley and Shubik, 1969) and fuzzy games (Aubin, 1979, 1981). Moreover, we characterize the situations where trade is welfare-improving for all firms, and we characterize uniqueness of the competitive equilibrium.

Dual utility is originally introduced by Yaari (1987). It coincides with preferences given by minimizing a distortion risk measure (Wang *et al.*, 1997). Yaari (1987) characterizes dual utility by a modification of the independence axiom in expected utility theory. Instead of requiring independence with respect to

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probability mixtures of risks, he requires independence with respect to direct mixing the realizations of the risks. Dual utility has applications in both actuarial science and finance, being also related to coherent risk measures (Artzner *et al.*, 1999) such as the conditional Value-at-Risk (Acerbi and Tasche, 2002). Moreover, distortion risk measures are applied as premium principle as well by, e.g., Wang (1995, 1996, 2000), Chateauneuf *et al.* (1996), and Wang *et al.* (1997). It is also shown to be compatible with market-consistent valuation, using the Esscher–Girsanov transform by Goovaerts and Laeven (2008). Alternatively, we can interpret the preferences as if the firm is risk-neutral, and it faces a cost of holding a buffer. This buffer is then given by a distortion risk measure such as the conditional Value-at-Risk. These preferences are widely studied in the actuarial literature on optimal reinsurance contract design (see, e.g., Chi, 2012; Asimit *et al.*, 2013; Chi and Weng, 2013), and can be formulated as dual utilities.

In risk sharing, most papers study Pareto optimality or competitive equilibria. For instance, Ludkovski and Young (2009) and Boonen (2015) analyze risk redistributions in settings where every firm uses dual utility. They provide all Pareto optimal risk redistributions, and show that they are given by a particular tranching of the aggregate risk. We use in this paper a cooperative game-theoretic approach to characterize risk redistributions. In other words, we study the problem via a cooperative game in order to select specific risk redistributions. We show that there exists a correspondence from solutions of specific cooperative games to Pareto optimal risk redistributions. Such a correspondence is called an allocation rule. Allocation rules are popular in the game-theoretic literature (see, e.g., Aumann and Shapley, 1974; Aubin, 1979, 1981; Billera and Heath, 1982; Mirman and Tauman, 1982). To the best of our knowledge, we are the first to study the allocation rule *Aumann–Shapley value* (Aumann and Shapley, 1974) for the problem to redistribute risk. We characterize it via four properties. The main property requires stability, i.e, a risk redistribution should correspond to an element of the core of the cooperative market game. For market games, of which the game we discuss in this paper is a special case, the core is characterized by Peleg (1989).

If all firms use the same risk measure, the risk redistribution problem can be analyzed using the risk capital allocation game as in, e.g., Denault (2001) and Csóka *et al.* (2009). In risk capital allocation problems, the goal is to allocate the aggregate risk capital of a firm to its business units. The Aumann–Shapley value is very popular in the literature on risk capital allocation problems (see, e.g., Tasche, 1999; Denault, 2001; Myers and Read, 2001; Tsanakas and Barnett, 2003; Kalkbrener, 2005). As argued by e.g., Tasche (1999), the risk capital allocation problem is mainly designed for financial performance measurement. If the aim is to redistribute risk among firms as in this paper, the assumption that all firms use the same risk measure is however restrictive. In this paper, we relax the assumption that all firms use the same risk measure.

We characterize existence of the Aumann–Shapley value by means of a property that describes a strict ordering of the aggregate risk. We make use of a result of Aubin (1981) to also characterize uniqueness of the competitive equilibrium as well. This characterization extends the result of Boonen (2015) by providing necessary conditions for uniqueness of the equilibrium. For competitive equilibria, a key and debatable assumption is that firms cannot influence the underlying prices by individual transactions. In this paper, we characterize the equilibrium risk redistribution, if unique, by means of four properties. These properties provide an alternative motivation of applying equilibrium prices in markets where the firms do not act as price-takers. For pricing longevity risk securities, Zhou *et al.* (2015) use the equilibrium prices in settings where there are just two firms. They call this a tâtonnement approach.

Distorted probabilities are also used to model rank-dependent utilities, as introduced by Quiggin (1982, 1991, 1992). In line with De Castro and Chateauneuf (2011) for rank-dependent utilities with strictly concave utility functions, we characterize no-trade for the problem with dual utilities. If utility functions are linear, as in this paper, the characterization of De Castro and Chateauneuf (2011) does not apply, and we provide an alternative characterization. This characterization is based on two properties that need to hold jointly.

This paper contributes to the literature in four ways. First, we characterize existence and uniqueness of the risk redistributions corresponding to the Aumann–Shapley value. Second, we characterize this risk redistribution by means of four desirable properties. Our third contribution is that we extend the main result of Boonen (2015) in the context of competitive equilibria. We derive that there are two conditions that are jointly necessary and sufficient to have a unique competitive equilibrium. Finally, we characterize when there are opportunities to trade.

This paper is set out as follows. Section 2 introduces dual utility and the risk redistribution problem. Section 3 provides preliminary results on Pareto optimality. Section 4 provides our main results. We use a cooperative game-theoretic approach to characterize a risk redistribution via a fuzzy core criterion. Section 5 shows a link between our main results and competitive equilibria. Section 6 characterizes no-trade. Section 7 provides a numerical illustration of our results with the conditional Value-at-Risk and Section 8 concludes.

2. Basic setting with dual utilities and risk redistribution problems

In this section, we briefly introduce dual utility and risk redistribution problems with dual utilities.

2.1. Dual utility

Let Ω be a finite state space and $\mathbb{P}$ the physical probability measure on the power set 2^{Ω} such that $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. Moreover, denote $\mathbb{R}^{\Omega}$ as the space of all real valued stochastic variables on Ω that are realized at a well-defined future reference time. These stochastic variables are referred to as risks. We interpret

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a realization of a risk as a future loss. We assume the state space to be finite for simplicity, but also because we refer in this paper to some results of Boonen (2015) that only hold for a finite state space.

Dual theory is developed and characterized by Yaari (1987) by a modification of the independence axiom in the Von Neumann–Morgenstern expected utility theory. Wang *et al.* (1997) define a distortion risk measure $\rho : \mathbb{R}^{\Omega} \rightarrow \mathbb{R}$ as minus dual utility, which leads to the following definition of dual utility:

$$U(Y) = -\rho(Y) = -E_{\mathbb{Q}_{g^{\rho},Y}}[Y], \qquad \text{for all } Y \in \mathbb{R}^{\Omega}, \tag{1}$$

where $\mathbb{Q}_{g^{\rho},Y} : 2^{\Omega} \rightarrow (0, 1]$ is the additive mapping such that

$$\mathbb{Q}_{g^{\rho},Y}(\{\omega\}) = g^{\rho}(\mathbb{P}(Y \geq Y(\{\omega\}))) - g^{\rho}(\mathbb{P}(Y > Y(\{\omega\}))), \qquad \text{for all } \omega \in \Omega, \tag{2}$$

for a distortion function g^{ρ}, where we define g^{ρ} as a distortion function if it is continuous, concave, increasing, and such that $g^{\rho}(0) = 0$ and $g^{\rho}(1) = 1$. Here, we explicitly assume concavity of the distortion function g^{ρ}. Concavity of the distortion function g^{ρ} is equivalent to aversion to mean-preserving spreads (Yaari, 1987). Preferences given by maximizing a risk-reward trade-off where the risk is measured by a distortion risk measure, can be formulated as dual utility. This setting is discussed by, e.g., De Giorgi and Post (2008). For notational convenience, we focus on minimizing a distortion risk measure instead of maximizing dual utility. For every risk $Y \in \mathbb{R}^{\Omega}$, we refer to $\rho(Y)$ as risk capital.

Wang *et al.* (1997) show that every distortion risk measure is coherent. Coherence is later formally introduced by Artzner *et al.* (1999) as a risk measure satisfying the four axioms *Subadditivity*, *Monotonicity*, *Positive Homogeneity*, and *Translation Invariance*. Here, *Subadditivity* of a risk measure implies that the aggregate risk capital weakly decreases if risks are pooled. It also implies that there is no incentive for a firm to split its risk into pieces and evaluate them separately. Artzner *et al.* (1999) show that a risk measure ρ is coherent if and only if there exists a set of probability measures Q such that

$$\rho(Y) = \sup\left\{E_{\mathbb{Q}}[Y] : \mathbb{Q} \in Q\right\}, \qquad \text{for all } Y \in \mathbb{R}^{\Omega}. \tag{3}$$

A representation of the set Q in (3) for distortion risk measures is given by (Denneberg, 1994):

$$Q(g^{\rho}) = \left\{\mathbb{Q} \in \mathcal{P}(\Omega) : \mathbb{Q}(A) \leq g^{\rho}(\mathbb{P}(A)) \quad \text{for all } A \in 2^{\Omega}\right\}, \tag{4}$$

where $\mathcal{P}(\Omega)$ is the set of probability measures on Ω.

Distortion risk measures are characterized by Wang *et al.* (1997). A mapping is a distortion risk measure if and only if it is coherent and it satisfies the axioms *Conditional State Independence* and *Comonotonic Additivity*. It can be shown that the class of distortion risk measures is equal to the class of spectral risk measures (Acerbi, 2002). There exists a wide literature on spectral risk measures as well (see, e.g., Kasuoka, 2001).

2.2. The risk redistribution problem

The finite collection of firms is given by $N = \{1, \ldots, n\}$. The risk redistribution problem with dual utilities is defined as follows.

Definition 2.1. *A* risk redistribution problem with dual utilities *is a tuple* $(X_i, \rho_i)_{i\in S}$*, where* $S \subseteq N$*,*

- $X_i \in \mathbb{R}^\Omega$ *is the risk held by firm* $i \in S$*;*
- $\rho_i : \mathbb{R}^\Omega \to \mathbb{R}$ *is the distortion risk measure that firm* $i \in S$ *is endowed with. The corresponding distortion function is denoted by* g_i*.*[1]

The class of risk redistribution problems with dual utilities is denoted by $\mathcal{R}_S$*.*

In the sequel, we refer to a risk redistribution problem with dual utilities as a risk redistribution problem. We assume that there is common knowledge about the risks and preferences of all firms. We define the aggregate risk by $X = \sum_{i\in N} X_i$ and, without loss of generality, we order the state space $\Omega = \{\omega_1, \ldots, \omega_p\}$ such that $X(\omega_1) \geq \cdots \geq X(\omega_p)$.

For a risk redistribution problem, we aim to redistribute the aggregate risk among firms. The objective of a firm is to minimize its risk capital (and, hence, maximize dual utility). We allow for all forms of risk redistributions, as long as the aggregate risk is redistributed. The set of feasible risk redistributions of a risk redistribution problem $R \in \mathcal{R}_S$, with $S \subseteq N$, is given by

$$\mathcal{F}(R) = \left\{ (\widetilde{X}_i)_{i\in S} \in (\mathbb{R}^\Omega)^S : \sum_{i\in S} \widetilde{X}_i = \sum_{i\in S} X_i \right\}. \tag{5}$$

3. Preliminary results on Pareto optimality

In this section, we summarize some results of Jouini *et al.* (2008), Ludkovski and Young (2009) and Boonen (2015) that we need in this paper. A risk redistribution is called Pareto optimal if there does not exist another feasible redistribution that is weakly better for all firms, and strictly better for at least one firm. The set of Pareto optimal risk redistributions of a risk redistribution problem $R \in \mathcal{R}_S$, with $S \subseteq N$, is denoted by $\mathcal{PO}(R)$.

We next define risk measures ρ^*_S, $S \subseteq N$ that play a central role in this paper. The function $g^*_S : [0, 1] \to [0, 1]$ with $S \subseteq N$ is given by $g^*_S(x) = \min\{g_i(x) : i \in S\}$ for all $x \in [0, 1]$. Moreover, ρ^*_S is the risk measure as defined in (1) with $g^{\rho^*_S} = g^*_S$. The function g^*_S is continuous, concave, increasing and such that $g^*_S(0) = 0$ and $g^*_S(1) = 1$. Therefore, the risk measure ρ^*_S is a distortion risk measure.

As shown by Jouini *et al.* (2008), we get the Pareto optimal risk redistributions by minimizing the aggregate risk capital, i.e., by minimizing the value of $\sum_{i\in S} \rho_i(\widetilde{X}_i)$ over all $(\widetilde{X}_i)_{i\in S} \in \mathcal{F}(R)$, where $S \subseteq N$ and $R \in \mathcal{R}_S$. From Boonen

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(2015, Proposition 3.6 therein), it follows that for all $S \subseteq N$ and $R \in \mathcal{R}_S$, we have $(\widetilde{X}_i)_{i\in S} \in \mathcal{PO}(R)$ if and only if $(\widetilde{X}_i)_{i\in S} \in \mathcal{F}(R)$ and

$$\sum_{i\in S} \rho_i(\widetilde{X}_i) = \rho_S^*\left(\sum_{i\in S} X_i\right). \tag{6}$$

We denote $e_\Omega \in \mathbb{R}^\Omega$ as the risk with realization one in every state $\omega \in \Omega$. More generally, for every $A \in 2^\Omega$, we denote $e_A \in \mathbb{R}^\Omega$ as the risk with realization one in state $\omega \in A$ and zero otherwise. A risk $Y \in \mathbb{R}^\Omega$ is a *side-payment* if there exists a constant $c \in \mathbb{R}$ such that $Y = ce_\Omega$. It follows that Pareto optimality of a risk redistribution is unaffected by adding zero-sum side-payments, i.e., for all $S \subseteq N$ and $R \in \mathcal{R}_S$, it holds that $(\widetilde{X}_i)_{i\in S} \in \mathcal{PO}(R)$ if and only if $(\widetilde{X}_i + c_i e_\Omega)_{i\in S} \in \mathcal{PO}(R)$ for any $c \in \mathbb{R}^S$ such that $\sum_{i\in S} c_i = 0$ (see Jouini *et al.*, 2008). So, if we find a Pareto optimal risk redistribution $(\widetilde{X}_i)_{i\in S}$, we can construct a set of Pareto optimal risk redistributions by adding zero-sum side-payments to $(\widetilde{X}_i)_{i\in S}$. This structure of Pareto optimal risk redistributions with side-payments also holds true for the case with expected exponential utilities (Bühlmann and Jewell, 1979; Gerber and Pafumi, 1998).

Next, we provide a closed-form expression of a set of Pareto optimal risk redistributions. By straightforwardly applying Theorem 2 of Ludkovski and Young (2009) to our setting, we get for all $R \in \mathcal{R}_N$, $m \in M(R)$ and $d \in \mathbb{R}^N$ with $\sum_{i\in N} d_i = X(\omega_p)$, that $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$, where

$$\widetilde{X}_i = \sum_{k=1}^{p-1}[X(\omega_k) - X(\omega_{k+1})]1_{m(k)=i}\, e_{\{\omega_1,\ldots,\omega_k\}} + d_i e_\Omega, \qquad \text{for all } i \in N. \tag{7}$$

Here, $1_{m(k)=i} = 1$ if $m(k) = i$ and zero otherwise, and the set of functions $M(R)$ is given by

$$M(R) = \left\{ m : \{1,\ldots,p-1\} \to N \;\middle|\; \begin{array}{r} m(k) \in \operatorname{argmin}_{j\in N} \{g_j(\mathbb{P}(\{\omega_1,\ldots,\omega_k\}))\} \\ \text{for all } k \in \{1,\ldots,p-1\} \end{array} \right\}. \tag{8}$$

The Pareto optimal risk redistributions of the form (7) consist of a finite number of long and short positions on various stop-loss contracts on the aggregate risk X. The risks $d_i e_\Omega$, $i \in N$, ensure that we have $(\widetilde{X}_i)_{i\in N} \in \mathcal{F}(R)$.

For every $R \in \mathcal{R}_N$, the risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ is, up to side-payments, the unique element of $\mathcal{PO}(R)$ if

$$\mathcal{PO}(R) = \left\{ (\widetilde{X}_i + c_i e_\Omega)_{i\in N} : c \in \mathbb{R}^N, \sum_{i\in N} c_i = 0 \right\}.$$

Uniqueness up to side-payments is first introduced in the context of risk measures by Jouini *et al.* (2008). This issue is relevant since if we know that there exists a unique Pareto optimal risk redistribution up to side-payments, the only question left is to determine the size of the side-payments. We first define the following two conditions for risk redistribution problems.

Condition [SC]: The distortion function g_N^* is strictly concave.

Condition [U]: For all $k \in \{1, \ldots, p-1\}$ such that $X(\omega_k) > X(\omega_{k+1})$, there exists a firm $i \in N$ such that for all $m \in M(R)$ it holds that $m(k) = i$.

If the distortion functions g_i, $i \in N$ are all strictly concave, condition [SC] holds. Strict concavity of the distortion function is equivalent to strongly preserving second-order stochastic dominance (Chew *et al.*, 1987). A sufficient condition for condition [U] is $|M(R)| = 1$. Condition [U] states that all functions in $M(R)$ differ only for $k \in \{1, \ldots, p-1\}$ such that $X(\omega_k) = X(\omega_{k+1})$. This condition can be verified by plotting all distortion functions g_i, $i \in N$. Then, condition [U] holds if, for every value of $\mathbb{P}(\Omega_k)$, $k \in \{1, \ldots, p-1\}$, there is a unique distortion function that takes the minimum. Boonen (2015, Theorem 3.8 therein) shows that if $R \in \mathcal{R}_N$ is such that condition [SC] holds, there exists a risk redistribution that is, up to side payments, the unique element of $\mathcal{PO}(R)$ if and only if condition [U] holds. One Pareto optimal risk redistribution in $\mathcal{PO}(R)$ is given in (7).

4. The Aumann–Shapley value for risk redistribution problems

In this section, we characterize a rule to implicitly determine the side-payments of a Pareto optimal risk redistribution. The firms meet each other and trade the risk. The risk redistribution is determined via a cooperative bargaining process. We show that the risk redistribution corresponding to the Aumann–Shapley value is unique under three conditions. In this paper, we focus on risk redistribution problems in $\mathcal{R}_N$, that is defined in Definition 2.1. Recall that the state space Ω and the set of firms N are both fixed and finite.

In Subsection 4.1, we introduce how a risk redistribution problem can be formulated as an allocation problem. Subsection 4.2 defines the Aumann–Shapley value, and characterizes its existence. In Subsection 4.3, we characterize the Aumann–Shapley value via four desirable properties.

4.1. From risk redistribution problems to capital allocation problems

In general, the aim is to find a risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{F}(R)$ that is perceived as fair by the firms. We require risk redistributions to be Pareto optimal, i.e., $\sum_{i\in N} \rho_i(\widetilde{X}_i) = \rho_N^*(X)$ as in (6). In this subsection, we show how we can obtain a risk redistribution via an allocation.

Definition 4.1. *An allocation is a vector $a \in \mathbb{R}^N$ such that $\sum_{i\in N} a_i = \rho_N^*(X)$.*

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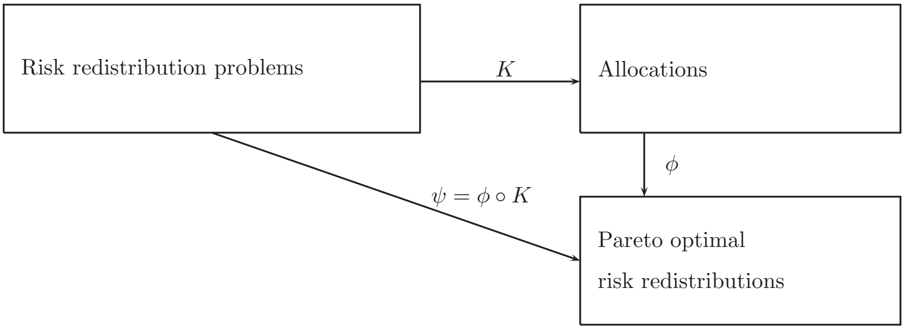

FIGURE 1: An overview of the correspondences between risk redistribution problems, allocations and risk redistributions. Every allocation rule K generates a risk redistribution rule ψ via ϕ.

An allocation assigns to every firm risk capital. In the next proposition, we show that an allocation corresponds with a Pareto optimal risk redistribution.

Proposition 4.2. *For every allocation $a \in \mathbb{R}^N$, there exists a risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ such that $a_i = \rho_i(\widetilde{X}_i)$ for all $i \in N$. Under conditions [SC] and [U], this risk redistribution is unique.*

If conditions [SC] and [U] do not hold, we need to decide which risk redistribution to pick out of a non-empty collection of risk redistributions corresponding to an allocation. For every firm, the risk capital of all the risk redistributions corresponding to an allocation is the same. For every allocation, we pick a specific corresponding Pareto optimal risk redistribution. We denote this injective mapping by ϕ, i.e., ϕ maps allocations in $\{a \in \mathbb{R}^N : \sum_{i\in N} a_i = \rho^*_N(X)\}$ to risk redistributions in $\mathcal{PO}(R)$.

We next introduce allocation rules and risk redistribution rules. In Subsection 4.2, we focus on an allocation rule that is not always well-defined on $\mathcal{R}_N$.

Definition 4.3. *An* allocation rule *K maps every risk redistribution problem in $\widetilde{\mathcal{R}}_N \subseteq \mathcal{R}_N$ into a unique allocation in $\{a \in \mathbb{R}^N : \sum_{i\in N} a_i = \rho^*_N(X)\}$.*

Definition 4.4. *A* risk redistribution rule *ψ maps every risk redistribution problem $R \in \widetilde{\mathcal{R}}_N \subseteq \mathcal{R}_N$ into a risk redistribution in $\mathcal{PO}(R)$.*

An allocation rule K corresponds with a risk redistribution rule ψ via the mapping ϕ, using $\psi = \phi \circ K$. A risk redistribution problem $R \in \mathcal{R}_N$ is mapped into an allocation via K. This allocation corresponds with a risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ using the mapping ϕ. To summarize, we provide an overview in Figure 1.

In this section, we focus on risk capital allocations. There is an impressive amount of literature on allocation problems within the area of game theory. The main sources for this section are Aumann and Shapley (1974), Aubin (1979, 1981), Billera and Heath (1982) and Mirman and Tauman (1982). These papers use the setting of a production problem with a given production function for multiple goods. The characterizations in these papers are formulated in terms of this production function, but are not directly transferable in a meaningful way to our setting of allocating risk capital. If all firms use the same risk measure, the allocation problem corresponds to the allocation problem discussed by Denault (2001). In the setting of Denault (2001), risk capital of a firm is to be allocated to its constituents. Denault (2001) uses a game-theoretic approach as well. The properties that we provide in this section are similar to his, but we formulate some properties in Subsection 4.3 in terms of the risk redistribution problem, while Denault (2001) focuses on a specific structure of cooperation for the allocation problem.

4.2. The Aumann–Shapley value

In Subsection 4.1, we show how the capital allocation problem can be formulated as an allocation problem. This suggests that likely candidates for solving the capital allocation problem can be solution concepts proposed in game theory for cooperative games. In this subsection, we define the allocation rule Aumann–Shapley value. We use this allocation rule to obtain a specific allocation, and thus a risk redistribution (Proposition 4.2). This allocation rule is first introduced by Aumann and Shapley (1974) for games with a continuum of players. It is based on a *fuzzy game* r. A fuzzy game is given by a mapping $r : \mathbb{R}^N_+ \to \mathbb{R}$ that is normalized such that no and full participation of firm $i \in N$ correspond with $\lambda_i = 0$ and $\lambda_i = 1$, respectively.

Let every firm consist of infinitesimally small, identical, and comonotonic portfolios that can cooperate as a separate party. Before and after risk redistribution, it follows from *Positive Homogeneity* and *Comonotonic Additivity* of ρ_i that the total risk capital of a firm will be the aggregate risk capital of its portfolios. The fuzzy game corresponding to a risk redistribution game in $\mathcal{R}_N$ is given by

$$r(\lambda) = \min\left\{ \sum_{i \in N:\lambda_i > 0} \rho_i(\hat{X}_i) : \sum_{i \in N:\lambda_i > 0} \hat{X}_i = \sum_{i \in N} \lambda_i X_i \right\}, \quad \text{for all } \lambda \in \mathbb{R}^N_+. \quad (9)$$

From (6), it straightforwardly follows that

$$r(\lambda) = \rho^*_{\{i \in N:\lambda_i > 0\}}\left(\sum_{i \in N} \lambda_i X_i \right), \quad \text{for all } \lambda \in \mathbb{R}^N_+. \quad (10)$$

Note that $r(e_N) = \rho^*_N(X)$. The fuzzy game $r : \mathbb{R}^N_+ \to \mathbb{R}$, as defined in (9), is positive homogeneous. For such fuzzy games, the Aumann–Shapley value is formally defined as follows.

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Definition 4.5. *The Aumann–Shapley value for risk redistribution problems, denoted by* $AS : \mathcal{R}'_N \to \mathbb{R}^N$*, is given by*

$$AS_i(R) = \frac{\partial r}{\partial \lambda_i}(e_N), \quad \text{for all } i \in N, \tag{11}$$

where the fuzzy game r *is defined in (9) and* $\mathcal{R}'_N \subset \mathcal{R}_N$ *is the class of all risk redistribution problems for which the fuzzy game* r *is partially differentiable at* $\lambda = e_N$.

For a more general class of fuzzy games, Aumann and Shapley (1974) define this value via a "diagonal formula". This is identical to (11) for positive homogeneous fuzzy games (see, e.g., Denault, 2001). Also for risk capital allocation problems, which resemble the problems where all firms use the same risk measure, the Aumann–Shapley value received considerable attention in the literature.[2]

Next, we state a necessary and sufficient condition to guaranty uniqueness of the Aumann–Shapley value if condition [SC] holds. We first introduce equivalent states. Two states $\omega, \omega' \in \Omega$ are *equivalent* if $X_i(\omega) = X_i(\omega')$ for all $i \in N$. The following condition requires a weak ordering of the aggregate risk.

Condition [WO]: $X(\omega) = X(\omega')$ for equivalent states $\omega, \omega' \in \Omega$ and $X(\omega) \neq X(\omega')$ otherwise.

Condition [WO] states that if the aggregate risk X is given, we are able to determine the corresponding realizations of the risks $(X_1, \ldots, X_n)$. The condition $X(\omega_1) > \cdots > X(\omega_p)$ is sufficient for condition [WO] to hold.

Theorem 4.6. *For all* $R \in \mathcal{R}_N$ *such that condition [SC] holds, it holds that* $R \in \mathcal{R}'_N$ *if and only if condition [WO] holds.*

It follows from Theorem 4.6 that if condition [SC] holds, the condition $X(\omega_1) > \cdots > X(\omega_p)$ is sufficient for the Aumann–Shapley value to exist. All other instances where the Aumann–Shapley value exists can be neglected since, without loss of generality, we can reformulate the risk redistribution problem such that there are no equivalent states. If condition [SC] holds, we have that the Aumann–Shapley value exists, and corresponds to a unique risk redistribution if and only if conditions [U] and [WO] hold simultaneously. Recall Figure 1; the three conditions for uniqueness characterize risk redistribution rule $\psi = \phi \circ K$: conditions [SC] and [U] guarantee that mapping ϕ is one-to-one, and conditions [SC] and [WO] make sure that allocation rule K is one-to-one.

Next, we provide a closed-form expression of the Aumann–Shapley value for risk redistribution problems. Tsanakas and Barnett (2003) obtain a closed-form expression of the Aumann–Shapley value in case all firms use the same risk measure and under the assumptions that the probability density function is continuous and the distortion function g is twice differentiable. Then, it holds that $AS_i(R) = E_{\mathbb{P}}[X_i g'(1 - F_X(X))]$ for all $i \in N$. In the following proposition, we show the Aumann–Shapley value for the risk redistribution problem.

Proposition 4.7. *For all $R \in \mathcal{R}'_N$, the Aumann–Shapley value is given by*

$$AS_i(R) = \sum_{k=1}^{p} [g^*_N(\mathbb{P}(\{\omega_1, \ldots, \omega_k\})) - g^*_N(\mathbb{P}(\{\omega_1, \ldots, \omega_{k-1}\}))] X_i(\omega_k),$$

$$\textit{for all } i \in N. \qquad (12)$$

For all $R \in \mathcal{R}'_N$ such that condition [SC] holds, we get from Proposition 4.7 that

$$AS_i(R) = E_{\mathbb{Q}_{g^*_N, X}}[X_i], \qquad \text{for all } i \in N, \qquad (13)$$

where $\mathbb{Q}_{g^*_N, X}$ is the probability measure in (2).

4.3. Characterization

In this subsection, we characterize the Aumann–Shapley value. To do so, we first define four properties of allocation rules. We consider the following properties of an allocation rule $K : \widetilde{\mathcal{R}}_N \to \mathbb{R}^N$ on a subclass of risk redistribution problems $\widetilde{\mathcal{R}}_N \subseteq \mathcal{R}_N$:

1. *Aggregation Invariance*: If for an $a > 0$ and $b \in \mathbb{R}^N$ it holds that $R = (X_j, \rho_j)_{j \in N} \in \widetilde{\mathcal{R}}_N$, $\widehat{R} = (\widehat{X}_j, \rho_j)_{j \in N} \in \widetilde{\mathcal{R}}_N$ and $\hat{X}_i = aX_i + b_i e_\Omega$ for all $i \in N$, then
$$K(R) = aK(\widehat{R}) + b.$$

2. *Monotonicity*: If $R \in \widetilde{\mathcal{R}}_N$ is such that there exist firms $i, j \in N$ such that $g_i(x) \leq g_j(x)$ for all $x \in [0, 1]$ and $X_i(\omega) \leq X_j(\omega)$ for all $\omega \in \Omega$, then
$$K_i(R) \leq K_j(R).$$

3. *No Split-up*: If $R = (X_i, \rho_i)_{i \in N} \in \widetilde{\mathcal{R}}_N$, $\widehat{R} = (\widehat{X}_i, \rho_i)_{i \in \widehat{N}} \in \widetilde{\mathcal{R}}_N$ and $\ell \in N$ are such that $\widehat{N} = N \cup \{n+1\}$, $\widehat{X}_j = X_j$ for all $j \in N \backslash \{\ell\}$, $\widehat{X}_\ell + \widehat{X}_{n+1} = X_\ell$ and $\rho_\ell = \rho_{n+1}$, then
$$K_j(\widehat{R}) = K_j(R) \text{ for all } j \in N \backslash \{\ell\} \text{ and } K_\ell(\widehat{R}) + K_{n+1}(\widehat{R}) \geq K_\ell(R).$$

4. *Core Selection*: For all $R \in \widetilde{\mathcal{R}}_N$, we have $K(R) \in core(R)$, where $core(R)$ denotes the *core* (Gillies, 1953) of a risk redistribution problem, which is defined as
$$core(R) = \left\{ a \in \mathbb{R}^N : \sum_{i \in S} a_i \leq \rho^*_S \left(\sum_{i \in S} X_i \right) \text{ for all } S \subset N, \sum_{i \in N} a_i = \rho^*_N(X) \right\}. \qquad (14)$$

The first three properties are based on, but weaker than, the properties in Billera and Heath (1982) and Mirman and Tauman (1982). Their properties depend only on a specific cost function in the problem to allocate production costs of a firm to the goods, whereas we define properties that are based on the potential to benefit from pooling risks from different firms.

The property *Aggregation Invariance* is necessary for an allocation rule to be compatible with the use of risk measures. If, for instance, another currency is used, the relative allocation remains the same.

The property *Monotonicity* is a standard extension of the *Monotonicity* property of risk measures. If a firm holds a portfolio of which its realization is smaller in every state of the world than another firm and is endowed with a smaller distortion function than this other firm, its allocation should be lower than the allocation of this other firm. A smaller distortion function g_i leads to a smaller value of $\rho_i(\widetilde{X}_i)$ for the same risk $\widetilde{X}_i$. Specifically, from interchanging the firms i and j in the definition of *Monotonicity*, it follows that allocation rules satisfying *Monotonicity* also satisfy the following property:

Symmetry: If $R \in \widetilde{\mathcal{R}}_N$ is such that there exist firms $i, j \in N$ where $X_i = X_j$ and $\rho_i = \rho_j$, then

$$K_i(R) = K_j(R).$$

The property *No split-up* is inspired by Tsanakas (2009) and Wang (2016) in the context of regulatory arbitrage. It implies that firms do not have any incentive to split the firm into two or more firms. If there is only one firm, it follows from *Subadditivity* of the risk measures that this firm is not willing to split-up in multiple firms. In this sense, the set of firms is stable against merging and splitting. Also, the allocation to the other firms is independent of whether a firm splitted. This property is weaker than the property *Consistency* in Mirman and Tauman (1982).

The property *Core Selection* implies that there does not exist a subgroup of firms that can strictly benefit altogether by splitting off and redistributing risk with only the firms in this subgroup. This property is widely discussed in the game-theoretic literature (see, e.g., Gillies, 1953). The conditions in (14) include all individual rationality conditions, i.e., $K_i(R) \leq \rho_i(X_i)$ for all $i \in N$. *Core Selection* implies that an allocation is in the core of the *cooperative cost game* (N, c),[3] with

$$c(S) = \rho_S^*\left(\sum_{i \in S} X_i\right) \tag{15}$$

$$= \min\left\{\sum_{i \in S} \rho_i\left(\widetilde{X}_i\right) : (\widetilde{X}_i)_{i \in S} \in \mathcal{F}(R)\right\}, \tag{16}$$

where $R \in \mathcal{R}_S$, for all $S \subseteq N$. The equality (16) follows from (6).

If firms use dual utilities, the cooperative cost game is a Transferable Utility game. This is shown by Bergstrom and Varian (1985), and follows from the fact that the Pareto optimal risk redistributions are characterized as the ones minimizing the aggregate risk capital (see (6)). The game (N, c) in (15), (16) is a special case of market games that are introduced by Shapley and Shubik (1969) for a wide class of utility functions. If all firms use the same risk measure, the game (N, c) corresponds with the cooperative cost game in Denault (2001).[4] This follows from the fact that $\rho_i = \rho$ for all $i \in N$ implies $\rho^*_S = \rho$ for all $S \subseteq N$ and, hence, $c(S) = \rho\left(\sum_{i \in S} X_i\right)$.

In the next proposition, we show that the set $core(R)$ is larger than the core of the game (N, c_N), where the cooperative cost game (N, c_N) is given by $c_N(S) = \rho^*_N\left(\sum_{i \in S} X_i\right)$. This game (N, c_N) is identical to the cooperative cost game of Denault (2001).

Proposition 4.8. *For all $R \in \mathcal{R}_N$, we have $core(R) \neq \emptyset$ and, moreover, the core of the game (N, c_N) is a subset of $core(R)$.*

In a similar way as in Csóka *et al.* (2009), we can show that the class of risk redistribution games coincides with the class of totally balanced games. This implies that the $core(R) \neq \emptyset$ for all risk redistributions $R \in \mathcal{R}_S$ with $S \subseteq N$.

In Subsection 4.1 and (15), (16), we show how the risk redistribution problem can formulated as a cooperative cost game (N, c). This suggests that likely candidates as solutions to the risk redistribution problems are well-known solution concepts for Transferable Utility games. A solution concept of cooperative Transferable Utility games that received considerable attention is the Shapley value (Shapley, 1953). The corresponding allocation rule does not satisfy all properties defined in this subsection. As discussed by Denault (2001) and Csóka and Pintér (2015), the Shapley value need not be in the core of cooperative cost games when firms use the same risk measure. Therefore, it does not satisfy *Core Selection*. Also the allocation rules corresponding to other well-known solution concepts such as the Compromise value (Tijs, 1981) and Nucleolus (Schmeidler, 1969) do not satisfy all properties defined in this subsection. The Compromise value does not satisfy *Core Selection* and *No Split-up*, while the Nucleolus does not satisfy *No Split-up*.

In the following proposition, which is derived from Mirman and Tauman (1982) and Denault (2001) in the context of fuzzy games, we show that the Aumann–Shapley value is an allocation rule satisfying all four properties that are defined in this subsection.

Proposition 4.9. *The Aumann–Shapley value satisfies the properties Aggregation Invariance, Monotonicity, No Split-up and Core Selection on $\mathcal{R}'_N$.*

Proposition 4.9 follows from Denault (2001) and Mirman and Tauman (1982), since allocation rules satisfying:

- *Additivity* and *Positivity* (see Mirman and Tauman, 1982) satisfy *Aggregation Invariance*;

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- *Additivity* (see Mirman and Tauman, 1982) satisfy *Monotonicity*;
- *Consistency* (see Mirman and Tauman, 1982) satisfy *No Split-up*,

and, moreover, *Core Selection* is shown by Denault (2001).

The properties 1.–4. do not necessarily characterize a unique allocation rule. As a solution, we adjust the property *Core Selection* to be more restrictive. This property is first introduced in the seminal works of Aubin (1979, 1981) and later imposed by Denault (2001) in the context of risk capital allocation problems. We focus on the following criterion on risk redistributions $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$:

$$\sum_{i\in N} \rho_i(\lambda_i \widetilde{X}_i) \leq \min\left\{\sum_{i\in N:\lambda_i>0} \rho_i(\hat{X}_i) : \sum_{i\in N:\lambda_i>0} \hat{X}_i = \sum_{i\in N} \lambda_i X_i\right\}, \tag{17}$$

for all $\lambda \in [0, 1]^N$. The conditions in (17) include all individual rationality conditions, because $\lambda = e_i$ yields $\rho_i(\widetilde{X}_i) \leq \rho_i(X_i)$ for all $i \in N$. In general, (17) leads to the following property of an allocation rule $K : \widetilde{\mathcal{R}}_N \to \mathbb{R}^N$:

5. *Fuzzy Core Selection*: For all $R \in \widetilde{\mathcal{R}}_N$, we have $K(R) \in Fcore(R)$, where $Fcore(R)$ denotes the *fuzzy core* (Aubin, 1979) of a risk redistribution problem $R \in \mathcal{R}_N$, which is defined as

$$Fcore(R) = \left\{a \in \mathbb{R}^N : \sum_{i\in N} \lambda_i a_i \leq r(\lambda) \text{ for all } \lambda \in [0, 1]^N, \sum_{i\in N} a_i = r\,(e_N)\right\}, \tag{18}$$

where the fuzzy game $r : \mathbb{R}^N_+ \to \mathbb{R}$ is defined in (9).[5]

Since $r(e_S) = c(S)$ for all $S \subseteq N$, we get that the fuzzy core is a subset of the core. Hence, every allocation rule satisfying *Fuzzy Core Selection* satisfies *Core Selection*.

In order to characterize the Aumann–Shapley value based on Aubin (1981), we first show that the fuzzy game r is subadditive, which implies that r is convex due to positive homogeneity of this fuzzy game.

Lemma 4.10. *For all $R \in \mathcal{R}_N$, the fuzzy game r is subadditive, i.e., $r(\lambda) + r(\lambda') \geq r(\lambda + \lambda')$ for all $\lambda, \lambda' \in \mathbb{R}^N_+$.*

Lemma 4.10 shows that the fuzzy game r is subadditive, which also implies that the cooperative cost game (N, c) is subadditive. From Aubin (1979, Proposition 3 therein on page 342) and Lemma 4.10, it follows that for all $R \in \mathcal{R}'_N$, we have $AS(R) \in Fcore(R)$ and $Fcore(R)$ is single-valued. Aubin (1981) shows this result for subadditive and positive homogeneous fuzzy games, and Lemma 4.10 states that the fuzzy game r is subadditive. From this result, we directly get the following theorem.

Theorem 4.11. *The Aumann–Shapley value is the unique allocation rule satisfying Fuzzy Core Selection on $\mathcal{R}'_N$.*

From Theorems 4.6 and 4.11, we get that condition [WO] is equivalent with existence of a single-valued fuzzy core.

In this section, we have characterized the Aumann–Shapley value by means of the property *Fuzzy Core Selection*. In other words, the Aumann–Shapley value, if existent, is the unique rule that satisfies the property *Fuzzy Core Selection*. In the next section, we show how this result relates to competitive equilibria.

5. Competitive equilibria for risk redistribution problems

In this section, we discuss results for the competitive equilibria that follow from the game-theoretic literature. We show a link with competitive equilibria, and how we can apply game-theoretic results to characterize uniqueness of competitive equilibria. In the setting with dual utilities, competitive equilibria are studied by Dana and Le Van (2010) and Boonen (2015). Let there be a linear pricing functional $\pi(\hat{p}, X) = \sum_{\omega\in\Omega} \hat{p}_\omega X(\omega)$ for all $X \in \mathbb{R}^\Omega$, where $\hat{p} \in \mathbb{R}^\Omega_{++}$ is a strictly positive price vector. We assume that the risk-free rate is zero, i.e., $\pi(\hat{p}, \mathbf{1}) = 1$, and we interpret $\hat{p}$ as the a probability measure. The economy $R \in \mathcal{R}_N$ is in equilibrium when every firm $i \in N$ solves

$$\min_{\widetilde{X}_i\in\mathbb{R}^\Omega} \rho_i(\widetilde{X}_i) \tag{19}$$

$$\text{s.t. } \pi(\hat{p}, \widetilde{X}_i) \leq \pi(\hat{p}, X_i), \tag{20}$$

where the price vector $\hat{p}$ induces market clearing, i.e.,

$$(\widetilde{X}_i)_{i\in N} \in \mathcal{F}(R). \tag{21}$$

Aumann, and Shapley (1964) and Aubin (1981) show that the fuzzy core of risk redistribution problems is equivalent to the set of allocations corresponding to the competitive equilibria, which is the set of vectors $a \in \mathbb{R}^N$ such that there exists an equilibrium risk redistribution $(\widetilde{X}_i)_{i\in N}$ with $a_i = \rho(\widetilde{X}_i)$, $i \in N$. This is originally shown by Aumann (1964) for games with a continuum of players. Aubin (1981, Theorem 4.1 therein) extends this result to the context of positive homogeneous fuzzy games. Aubin (1981, Proposition 8.1 therein) shows that the corresponding equilibrium prices are given by a probability distribution $p \in Q(g^*_N)$ such that $\sum_{k=1}^{p} p_k X_i(\omega_k) = a_i$ for all $i \in N$, where $a \in Fcore(R)$, and the set $Q(g^*_N)$ is defined in (4).[6] It follows that the allocation corresponding to the competitive equilibria is unique if and only if the fuzzy core is single-valued. From this and Theorem 4.11, we get that the allocation corresponding to the competitive equilibria is unique if and only if the Aumann–Shapley value exists. If condition [SC] holds, Proposition 4.6 shows that the Aumann–Shapley value

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is unique if and only if condition [WO] holds. From Boonen (2015, Theorem 3.8 therein), we get that the risk redistribution corresponding to this allocation is unique if and only if conditions [U] holds. Hence, we directly get the following result.

Theorem 5.1. *For all $R \in \mathcal{R}_N$ such that condition [SC] holds, the equilibrium risk redistribution $(\widetilde{X}_i)_{i \in N}$ is unique if and only if conditions [U] and [WO] hold jointly.*

Conditions [U] and [WO] hold jointly is identical to the case where the following two conditions hold jointly:

- For all $k \in \{1, \ldots, p-1\}$ such that the states ω_k and ω_{k+1} are not equivalent, there exists a firm $i \in N$ such that for all $m \in M(R)$ it holds that $m(k) = i$;[7]
- $X(\omega) = X(\omega')$ for equivalent states $\omega, \omega' \in \Omega$ only.

In an expected utility framework, Aase (1993) shows three regularity conditions that are sufficient to ensure uniqueness of the equilibrium risk redistribution. All three conditions are imposed on the utility functions only. Theorem 5.1 implies that if firms use dual utilities, it is sufficient to impose three conditions to ensure uniqueness of the equilibrium. Two conditions also depend on the aggregate risk. This result extends the main result of Boonen (2015, Theorem 4.4 therein), who only derives two jointly sufficient conditions for uniqueness of the competitive equilibria with dual utilities.

From Aubin (1981, Theorem 4.1 therein), we can find all equilibria from the fuzzy core even when the function g_N^* is not strictly concave or when the fuzzy game r is not partially differentiable at $\lambda = e_N$. The following representation of the fuzzy core follows from Aubin (1979, Proposition 4 therein on page 343):

$$Fcore(R) = \{(E_{\mathbb{Q}}[X_i])_{i \in N} : \mathbb{Q} \in Q^*\}, \qquad \text{for all } R \in \mathcal{R}_N, \tag{22}$$

where the set Q^* is given by

$$Q^* = \left\{\mathbb{Q} \in Q(g_N^*) : \rho_N^*(X) = E_{\mathbb{Q}}[X]\right\}, \tag{23}$$

and where $Q(g_N^*)$ is defined in (4). From (4), we get that $Q(g_N^*)$ is a finite-dimensional bounded space, and the intersection of a finite number of closed half-spaces. Therefore, $Q(g_N^*)$ is a convex polytope. Hence, it is compact and so it holds that $Q^* \neq \emptyset$. Aubin (1981, Proposition 8.1 therein) shows that the set of all equilibrium prices is given by Q^*. This leads to the following theorem, which extends the result of Boonen (2015, Theorem 4.2 therein) to the case where condition [SC] does not hold.

Theorem 5.2. *For all $R \in \mathcal{R}_N$ such that $X(\omega_1) > \cdots > X(\omega_p)$, the equilibrium prices are unique.*

If the fuzzy core is not single-valued, one can select an element based on Mertens (1988). He characterizes a generalization of the Aumann–Shapley value which is well-defined also in case of non-differentiability of the fuzzy game r in $\lambda = e_N$.

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This allocation rule is given by a convex combination of "nearby" Aumann–Shapley values $(E_{\mathbb{Q}}[X_i])_{i\in N}$ for $\mathbb{Q} \in Q^*$.

6. NO-TRADE

In this section, we characterize no-trade in the risk redistribution problem. This means that the market is already in an optimum, and there exists no risk redistribution that is welfare-improving for all firms, and strictly welfare-improving for at least one firm. The aggregate hedge benefit of a risk redistribution $(\widetilde{X}_i)_{i\in N}$ is given by

$$\sum_{i\in N} \left[\rho_i(X_i) - \rho_i(\widetilde{X}_i)\right].$$

From (6), it follows that

$$\min\left\{\sum_{i\in N} \rho_i(\widetilde{X}_i) : (\widetilde{X}_i)_{i\in N} \in \mathcal{F}(R)\right\} = \rho_N^*(X).$$

From this, it follows that the maximum aggregate hedge benefit over all feasible risk redistributions is given by

$$\begin{aligned}\max\left\{\sum_{i\in N}[\rho_i(X_i) - \rho_i(\widetilde{X}_i)] : (\widetilde{X}_i)_{i\in N} \in \mathcal{F}(R)\right\} &= \sum_{i\in N} \rho_i(X_i) - \rho_N^*(X)\\ &= \sum_{i\in N} c(\{i\}) - c(N).\end{aligned}$$

Since $(X_i)_{i\in N} \in \mathcal{F}(R)$, we derive from (6) that $\sum_{i\in N} \rho_i(X_i) \geq \rho_N^*(X)$, and, so, the maximum aggregate hedge benefit is non-negative. We analyze whether this amount is zero or positive. This is zero if and only if the cooperative cost game (N, c) is additive. If the maximum aggregate hedge benefit is positive, there is an opportunity to obtain welfare gains from trading for all firms. This maximum aggregate hedge benefit can be allocated freely to every firm. This needs to be subtracted from the vector $(\rho_i(X_i))_{i\in N}$. Due to Proposition 4.2, there exists a risk redistribution to this allocation. If $\rho_N^*(X) = \sum_{i\in N} \rho_i(X_i)$, it follows that no firm can benefit from risk redistribution. Then, a Pareto optimal risk redistribution for firms is to keep their prior risk. We call this situation no-trade. The following proposition characterize no-trade by two restrictive conditions.

Proposition 6.1. *If $R \in \mathcal{R}_N$ is such that condition [SC] holds, we have $\rho_N^*(X) = \sum_{i\in N} \rho_i(X_i)$ if and only if the following two conditions hold jointly:*

- *all X_i, $i \in N$ are comonotonic with each other;*[8]
- *$\rho_i(X_i) = \rho_N^*(X_i)$, for all $i \in N$.*

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If condition [SC] holds and there is no hedge potential, i.e., if $\rho_N^*(X) = \sum_{i\in N}\rho_i(X_i)$, all risks are comonotonic according to Proposition 6.1. Moreover, $\rho_i(X_i) = \rho_N^*(X_i)$ for all $i \in N$. This implies that for all $i \in N$, we have

$$g_i(x) = g_N^*(x), \quad \text{for all } x \in \{\mathbb{P}(\{\omega_1,\ldots,\omega_k\}), k \in \{1,\ldots,p-1\} \text{ s.t. } X_i(\omega_k) > X_i(\omega_{k+1})\}. \tag{24}$$

Hence, it holds that $\rho_N^*(X) = \sum_{i\in N}\rho_i(X_i)$ only under two strong conditions. If all firms are expected utility maximizers, it is a necessary and sufficient condition that $u_i'(X_i)$ is proportional with $u_j'(X_j)$ for all $i, j \in N$ (see Borch, 1962), where $u_i, i \in N$ are the expected utility functions.

7. Illustration with conditional Value-at-Risk

In this section, we provide an extensive example of the Aumann–Shapley value for risk redistributions. We assume that every firm $i \in N$ is risk-neutral, but faces costs of holding capital given by $CoC_i(\rho_i(Y) - E_{\mathbb{P}}[Y])$ for all $Y \in \mathbb{R}^\Omega$, where $CoC_i \in [0,1]$ and $\rho_i := CVaR_{\alpha_i}$. Here, CoC_i represents the cost of capital for holding a buffer, and $CVaR_{\alpha_i}$ with $\alpha_i \in (0,1)$ is the conditional Value-at-Risk which is the distortion risk measure with distortion function $g_i(x) = \min\{\frac{x}{1-\alpha_i}, 1\}$ (see Dhaene *et al.*, 2006). The conditional Value-at-Risk, also called Expected Shortfall, received considerable attention after the introduction of the Basel III regulations and the Swiss Solvency Test (see, e.g., Eling *et al.*, 2008; Basel Committee on Banking Supervision, 2012; Chen, 2014). Also in reinsurance contract design, there is a substantial literature that studies CVaR (see, e.g., Chi and Tan, 2011; Chi, 2012; Asimit *et al.*, 2013; Chi and Weng, 2013; Boonen *et al.*, 2016).

Define $\beta_i = 1 - CoC_i$. Let the preferences of firm $i \in N$ be given by a distortion risk measure *mean-CVaR*, which is given by

$$MCVaR_{\alpha_i,\beta_i}(Y) := \beta_i E_{\mathbb{P}}[Y] + (1-\beta_i)CVaR_{\alpha_i}(Y), \qquad \text{for all } Y \in \mathbb{R}^\Omega. \tag{25}$$

These preferences are generated by the distortion function $g_i(x) = \beta_i x + (1-\beta_i)\min\{\frac{x}{1-\alpha_i}, 1\}$ for $x \in [0,1]$.

For every $S \subseteq N$, we define firms $i_S^* \in \operatorname{argmin}\{\frac{\alpha_i\beta_i}{1-\alpha_i} : i \in S\}$ and $j_S^* \in \operatorname{argmin}\{\beta_j : j \in S\}$, where it is possible that $i_S^* = j_S^*$. Moreover, we define

$$x_S = \frac{\beta_{j_S^*}}{\beta_{j_S^*} + \alpha_{i_S^*}\beta_{i_S^*}/(1-\alpha_{i_S^*})}.$$

Then, we derive that the minimum of the distortion functions for the firms in S is given by

$$g_S^*(x) = \begin{cases} (1-\beta_{i_S^*} + \frac{\beta_{i_S^*}}{1-\alpha_{i_S^*}})x & \text{if } 0 \le x \le x_S, \\ (1-\beta_{j_S^*})x + \beta_{j_S^*} & \text{if } x_S < x \le 1, \end{cases} \tag{26}$$

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TABLE 1

THE SELECTION OF THE FIRMS i_S^*, j_S^*, $S \subseteq N$, CORRESPONDING TO EXAMPLE 7.2.

S	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}	N
i_S^*	1	2	3	1	3	3	3
j_S^*	1	2	3	2	1	2	2

and, hence, we get $\rho_S^*(Y) = MCVaR_{\hat{\alpha},\beta_{j_S^*}}(Y)$ for all $Y \in \mathbb{R}^\Omega$, where

$$\hat{\alpha} = \frac{-\beta_{i_S^*} + \beta_{i_S^*}/(1-\alpha_{i_S^*})}{\beta_{j_S^*} - \beta_{i_S^*} + \beta_{i_S^*}/(1-\alpha_{i_S^*})}.$$

We illustrate this distortion function in Example 7.2.

A Pareto optimal risk redistribution is given by a stop-loss contract on the aggregate risk: firm i_N^* bears the risk $\max\{X - X(\omega_{k^*}), 0\}$ and firm j_N^* bears the risk $\min\{X, X(\omega_{k^*})\}$, where k^* is the largest index in $\{1, \ldots, p\}$ such that $\mathbb{P}(X > X(\omega_{k^*})) \leq x_N$. Hence, there exists a Pareto optimal risk redistribution such that only the firms i_N^* and j_N^* bear non-degenerate risk. This leads to the following proposition.

Proposition 7.1. *Let every firm $i \in N$ endowed with risk measure $MCVaR_{\alpha_i,\beta_i}$. The following two statements hold true:*

- *For all $R \in \mathcal{R}_N$, there exists a Pareto optimal risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ such that there are at most two firms $i, j \in N$ for which $\widetilde{X}_i$ and $\widetilde{X}_j$ are not degenerate.*
- *For all $R \in \mathcal{R}_N$ such that there exists a firm $i \in N$ such that $\alpha_i \geq \alpha_j$ and $\beta_i \geq \beta_j$ for all $j \in N$, there exists a Pareto optimal risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ such that $\widetilde{X}_j$ is degenerate for all $j \neq i$.*

The second statement of Proposition 7.1 is also shown by Asimit *et al.* (2013) in the context of risk transfers between two divisions within an insurance company. It follows from the fact that there exist i_N^*, j_N^* such that $i_N^* = j_N^*$.

In this paper, we assume that the side-payments follow from the Aumann–Shapley value, for which the prices are generated by the distortion function of $MCVaR_{\hat{\alpha},\beta_N^*}$. We conclude this section with an example.

Example 7.2. *Let $N = \{1, 2, 3\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\{\omega\}) = \frac{1}{3}$ for all $\omega \in \Omega$, $\rho_1 = MCVAR_{0.5,0.5}$, $\rho_2 = MCVAR_{0.8,0.2}$ and $\rho_3 = MCVAR_{0.2,0.8}$. Moreover, let $X(\omega_1) = 1$, $X(\omega_2) = 0$, $X(\omega_3) = -1$ and $X_1 = X_2 = -X_3 = X$.*

In this example, the firms i_S^* and j_S^* are uniquely determined, and given in Table 1. From (26), we get $x_N = 0.5$, and the distortion function g_N^* is given by

$$g_N^*(x) = \begin{cases} 1.2x & \text{if } 0 \leq x \leq 0.5, \\ 0.8x + 0.2 & \text{if } 0.5 < x \leq 1, \end{cases}$$

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TABLE 2

THE COOPERATIVE COST GAME (N, c) CORRESPONDING TO EXAMPLE 7.2.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c(S)$	1/3	1/3	1/3	14/30	0	0	4/30

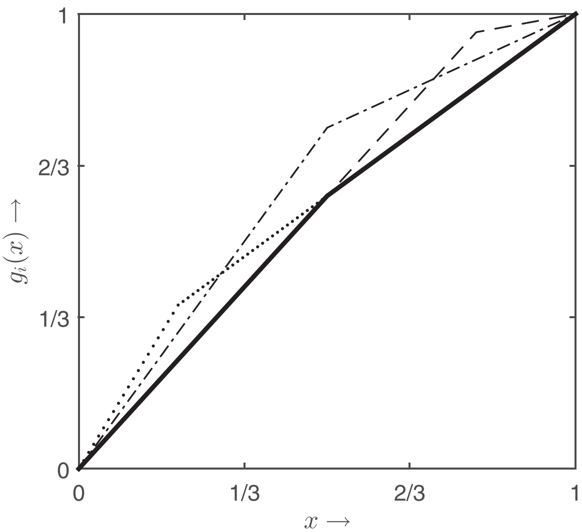

FIGURE 2: Construction of the function g_N^* via the distortion functions g_1, g_2 and g_3 corresponding to Example 7.2. The function g_1 is the dashed-dotted line, g_2 is the dotted line, g_3 is the dashed line and g_N^* is the solid line.

so that $\rho_N^* = MCVaR_{0.5,0.2}$. *The distortion functions* g_1, g_2, g_3 *and* g_N^* *are displayed in Figure 2. From this figure, we get that argmin*$\{g_j(x) : j \in N\}$ *is single-valued at* $x = \mathbb{P}(\{\omega_1\}) = \frac{1}{3}$ *and at* $x = \mathbb{P}(\{\omega_1, \omega_2\}) = \frac{2}{3}$. *Hence, it holds that* $|M(R)| = 1$ *and, therefore, condition [U] holds. Moreover, it holds that* $m(1) = 3$ *and* $m(2) = 2$. *According to* (7) *with* $d_1 = -1$ *and* $d_2 = d_3 = 0$, *a Pareto optimal risk redistribution is given by* $(\widetilde{X}_i)_{i\in N}$ *such that* $\widetilde{X}_1 = -1 \cdot e_\Omega$, $\widetilde{X}_2(\omega_1) = 1$, $\widetilde{X}_2(\omega_2) = 1$, $\widetilde{X}_1(\omega_3) = 0$, $\widetilde{X}_3(\omega_1) = 1$, $\widetilde{X}_3(\omega_2) = 0$ *and* $\widetilde{X}_3(\omega_3) = 0$. *We proceed with determining the side-payments.*

The cooperative cost game (N, c), *as defined in* (15), *for this example is given in Table 2. The core is defined in* (14). *We derive that the core in this example is given by the polytope*

$$
\begin{aligned}
core(R) = \{a \in \mathbb{R}^N : a_1 + a_2 + a_3 = 4/30, a_i \leq 1/3, a_1 + a_2 \leq 14/30, a_1 \\
+ a_3 \leq 0, a_2 + a_3 \leq 0\} \\
= conv\{(4/30, 1/3, -1/3), (1/3, 4/30, -1/3), (4/30, 4/30, -4/30)\},
\end{aligned}
$$

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where conv *denotes the convex hull operator. We proceed with discussion the Aumann–Shapley value, which is an element of this set due to Proposition 4.8.*

We get that the equilibrium price vector is given by $p = \mathbb{Q}_{g^*_N, X} = \left(0.4, \frac{1}{3}, \frac{8}{30}\right)$. *This leads to the Aumann–Shapley value, that is given by* $AS_1(R) = AS_2(R) = E_{\mathbb{Q}_{g^*_N, X}}[X] = \frac{4}{30}$, *and* $AS_3(R) = E_{\mathbb{Q}_{g^*_N, X}}[-X] = -\frac{4}{30}$. *It is easy to verify this is a core element. Then, the risk redistribution* $\hat{X}_1$, $\hat{X}_2$ *and* $\hat{X}_3$ *corresponding to the Aumann–Shapley value is given by* $\hat{X}_1 = \frac{4}{30} \cdot e_\Omega$, $\hat{X}_2(\omega_1) = 0.4$, $\hat{X}_2(\omega_2) = 0.4$, $\hat{X}_2(\omega_3) = -0.6$, $\hat{X}_3(\omega_1) = 0.4$, $\hat{X}_3(\omega_2) = -0.6$ *and* $\hat{X}_3(\omega_3) = -0.6$. *So, we see that Firm 3 gets a smaller risk in the risk redistribution corresponding to the Aumann–Shapley value. This is due to the fact that* X_3 *is a negatively correlated to the aggregate risk, and therefore a good hedge for the other firms.*

8. Conclusion

In this paper, we analyze optimal risk sharing with dual utility maximizing firms. We characterize a game-theoretic solution concept that is in line with the solution concept of Denault (2001) for risk capital allocation problems. Whereas the characterization is similar, the underlying problem is fundamentally different from risk capital allocations. Risk capital allocations generally serve as performance measure (see, e.g., Tasche, 1999), whereas in this paper we aim to share risk. By doing so, we also contribute to the literature on uniqueness of competitive equilibria, and we characterize no-trade.

We characterize a solution by means of cooperative game theory. The cooperative game that we derive in this paper can be seen as a generalization of the cooperative game of Denault (2001), where the risk measures are firm-specific. Moreover, it is a market game (Shapley and Shubik, 1969) with preferences given by dual utilities. We characterize one specific solution concept that happens to coincide with the competitive equilibrium. For future research, we suggest to study alternative solution concepts for this cooperative game with dual utilities.

In this paper, we assume that there is a finite space. This allows us to characterize the set of Pareto optimal risk redistributions as being unique up to side-payments (Boonen, 2015, Theorem 3.8 therein). Moreover, we use a finite state space to characterize existence of the Aumann–Shapley value (Theorem 4.6). Generalizing the setting to an infinite state space would be an interesting topic for future research. Whereas condition [U] has a straightforward translation to the class of continuous risks, condition [WO] has not.

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NOTES

1. For notational convenience, we write g_i instead of g^{ρ_i}.
2. For risk capital allocation problems, there is a wide range of game-theoretic (e.g., Denault, 2001; Tsanakas and Barnett, 2003; Kalkbrener, 2005), financial (e.g., Tasche, 1999) and economic (e.g., Myers and Read, 2001) approaches described in the literature. In this literature, it is called the Euler rule.
3. Cooperative cost games consist of the set of firms N and a characteristic function $c : 2^N \to \mathbb{R}$. In the context of the risk redistribution problem considered in this paper, the characteristic function yields for any subset $S \subseteq N$ the minimal aggregate risk capital if only the firms in S decide to redistribute.
4. Literally, this is not true as Denault (2001) defines this game for all coherent risk measures and allows for a continuous state space.
5. Note that the fuzzy game satisfies $r(e_S) = c(S)$ for all $S \subseteq N$, where the cooperative cost game (N, c) is as defined in (15) and e_S is the vector with ones for firms in S and zeros for firms in $N \backslash S$.
6. Note that $Q(g_N^*)$ contains probability measures whereas p is a vector. Here, we mean that $p_k = \mathbb{Q}(\{\omega_k\})$ for some $\mathbb{Q} \in Q(g_N^*)$ and all $k \in \{1, \dots, p\}$.
7. Alternative formulation: for all $m, m' \in M(R)$ such that $m \neq m'$, it holds that $m(k) = m'(k)$ only if $k \in \{1, \dots, p-1\}$ is such that the states ω_k and ω_{k+1} are equivalent.
8. Risks $X_i, i \in N$ are comonotonic with each other if there exists an ordering $(\omega_1, \dots, \omega_p)$ on the state space Ω such that $X_i(\omega_1) \geq \dots \geq X_i(\omega_p)$ for all $i \in N$.

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APPENDIX A. PROOFS

Proof of Theorem 4.2. One can determine a Pareto optimal risk redistribution $(\hat{X}_i)_{i\in N} \in \mathcal{PO}(R)$ via (7) for any choice of $m \in M(R)$ and $d \in \mathbb{R}^N$ such that $\sum_{i\in N} d_i = X(\omega_p)$. For every allocation $a \in \mathbb{R}^N$, it holds for the side-payments $c_i \cdot e_\Omega$, $i \in N$ with $c_i = a_i - \rho_i(\hat{X}_i)$ that $\sum_{i\in N} c_i = 0$, $(\widetilde{X}_i)_{i\in N} = (\hat{X}_i + c_i \cdot e_\Omega)_{i\in N} \in \mathcal{PO}(R)$ and $a = (\rho_i(\widetilde{X}_i))_{i\in N}$.

If the conditions [SC] and [U] hold, it follows from Boonen (2015, Theorem 3.8 therein) that the choice of the Pareto optimal risk redistribution $(\hat{X}_i)_{i\in N}$ is unique up to side-payments. Hence, for every allocation a, there is a unique risk redistribution $(\widetilde{X}_i)_{i\in N}$ such that $a_i = \rho_i(\widetilde{X}_i)$ for all $i \in N$. ■

Proof of Theorem 4.6. First, we show "⇐" (if) part of the proof. Let $R \in \mathcal{R}_N$ be such that $X(\omega) = X(\omega')$ for equivalent states $\omega, \omega' \in \Omega$ only. Then, it holds that $\sum_{i \in N} \lambda_i X_i(\omega) = \sum_{i \in N} \lambda_i X_i(\omega')$ for all equivalent states $\omega, \omega' \in \Omega$ and $\lambda \in \mathbb{R}^N_{++}$. For all other states $\omega, \omega' \in \Omega$ such that $X(\omega) > X(\omega')$, it follows from continuity that there exists a neighborhood $\hat{U} \subset \mathbb{R}^N_{++}$ of e_N such that $\sum_{i \in N} \lambda_i X_i(\omega) > \sum_{i \in N} \lambda_i X_i(\omega')$ for all $\lambda \in \hat{U}$. Hence, there exists a neighborhood $U \subset \mathbb{R}^N_{++}$ of e_N such that $\sum_{i \in N} \lambda_i X_i(\omega_1) \geq \cdots \geq \sum_{i \in N} \lambda_i X_i(\omega_p)$ for all $\lambda \in U$. From this, (1) and (2), it follows that

$$r(\lambda) = E_{\mathbb{Q}_{g^*_N, X}}\left[\sum_{i \in N} \lambda_i X_i\right], \qquad \text{for all } \lambda \in U. \tag{A1}$$

This linear function is partially differentiable and, so, the Aumann–Shapley value $\nabla r(e_N)$ exists.

We continue by showing the "⇒" (only if) part of the proof. Let $R \in \mathcal{R}_N$ be such that condition [SC] holds and the Aumann–Shapley value $\nabla r(e_N)$ exists. From (4), we get that $Q(g^*_N)$ is a finite-dimensional bounded space, and the intersection of a finite number of half-spaces. Therefore, $Q(g^*_N)$ is a convex polytope. Define $\hat{Q} \subseteq Q(g^*_N)$ as all extreme points of the convex polytope $Q(g^*_N)$ such that $r(e_N) = E_{\mathbb{Q}}[X]$ for all $\mathbb{Q} \in \hat{Q}$, where the set $Q(g^*_N)$ is defined in (4). Note that the set $\hat{Q}$ is non-empty since $Q(g^*_N)$ is compact. Since the fuzzy game r is piecewise linear, there exists a neighborhood $U \subset \mathbb{R}^N_{++}$ of e_N and a $\hat{\mathbb{Q}} \in \hat{Q}$ such that

$$r(\lambda) = E_{\hat{\mathbb{Q}}}\left[\sum_{i \in N} \lambda_i X_i\right], \qquad \text{for all } \lambda \in U. \tag{A2}$$

For all $\mathbb{Q} \in \hat{Q}$, it holds by definition that

$$E_{\mathbb{Q}}\left[\sum_{i \in N} \lambda_i X_i\right] \leq r(\lambda), \qquad \text{for all } \lambda \in U, \tag{A3}$$

and, by local linearity of the fuzzy game r on U, it holds for all $\mathbb{Q} \in \hat{Q}$ that

$$E_{\mathbb{Q}}\left[\sum_{i \in N} \lambda_i X_i\right] = r(\lambda), \qquad \text{for all } \lambda \in U. \tag{A4}$$

Since the gradient $\nabla r(e_N)$ exists, it follows that $(E_{\mathbb{Q}}[X_i])_{i \in N}$ is constant for all $\mathbb{Q} \in \hat{Q}$.

For every $\mathbb{Q} \in \hat{Q}$, there exists an ordering on the state space $\Omega = \{\omega_1, \ldots, \omega_p\}$ such that $\mathbb{Q}(\omega_k) = g^*_N(\mathbb{P}(\{\omega_1, \ldots, \omega_k\})) - g^*_N(\mathbb{P}(\{\omega_1, \ldots, \omega_{k-1}\}))$ for all $k \in \{1, \ldots, p\}$. Since condition [SC] holds, we get from Boonen (2015, equation (36) therein) that for all $\mathbb{Q}_1, \mathbb{Q}_2 \in \hat{Q}$ we have $\mathbb{Q}_1(\omega_k) \neq \mathbb{Q}_2(\omega_k)$ only if $X(\omega_{k-1}) = X(\omega_k)$ or $X(\omega_k) = X(\omega_{k+1})$. Let $X(\omega_k) = X(\omega_{k+1})$ and let $\mathbb{Q}_1, \mathbb{Q}_2 \in \hat{Q}$ both be generated by a different ordering on the state space $\Omega = \{\omega_1, \ldots, \omega_p\}$ such that $X(\omega_1) \geq \cdots \geq X(\omega_p)$ only via interchanging the states ω_k and ω_{k+1}. So, it holds that $\mathbb{Q}_1(\omega) = \mathbb{Q}_2(\omega)$ for all $\omega \in \Omega \backslash \{\omega_k, \omega_{k+1}\}$. From strict concavity of the function g^*_N, we get $\mathbb{Q}_1(\omega_k) \neq \mathbb{Q}_2(\omega_k)$. Hence, $(E_{\mathbb{Q}}[X_i])_{i \in N}$ is constant for $\mathbb{Q} \in \{\mathbb{Q}_1, \mathbb{Q}_2\}$ only if $X_i(\omega_k) = X_i(\omega_{k+1})$ for all $i \in N$. Continuing this procedure for all states $\omega_k, \omega_{k+1} \in \Omega$ such that $X(\omega_k) = X(\omega_{k+1})$ yields that the Aumann–Shapley value exists if for all $\omega_k, \omega_{k+1} \in \Omega$ such that $X(\omega_k) = X(\omega_{k+1})$ the states ω_k and ω_{k+1} are equivalent. This concludes the proof. ■

Proof of Proposition 4.7. Let $R \in \mathcal{R}'_N$. Then, the fuzzy game r on a neighborhood of e_N is given in (A1). Partial differentiating the fuzzy game r in $\lambda = e_N$ yields

$$\frac{\partial r}{\partial \lambda_i}(e_N) = E_{\mathbb{Q}_{g_N^*,X}}[X_i], \qquad \text{for all } i \in N.$$

Hence, the Aumann–Shapley value is given by

$$AS_i(R) = \sum_{k=1}^{p} [g_N^*(\mathbb{P}(\{\omega_1, \ldots, \omega_k\})) - g_N^*(\mathbb{P}(\{\omega_1, \ldots, \omega_{k-1}\}))] X_i(\omega_k), \qquad \text{for all } i \in N.$$

This concludes the proof. ■

In order to prove, e.g., Proposition 4.8, Lemma 4.10 and Proposition 6.1, we first show the following technical result.

Lemma A.1. *For all $R \in \mathcal{R}_N$ and $S \subset T \subseteq N$, it holds that*

$$\rho_S(Y) \geq \rho_T^*(Y), \qquad \text{for all } Y \in \mathbb{R}^\Omega.$$

Proof. This result follows directly from the fact that $g_S(x) = \min\{g_j(x) : j \in S\} \geq \min\{g_j(x) : j \in T\} = g_T(x)$ for all $x \in [0, 1]$ and, therefore, $Q(g_T^*) \subseteq Q(g_S^*)$, where the set $Q(g)$ is defined in (4). ■

Proof of Proposition 4.8. Since ρ_N^* is coherent, it is shown by Denault (2001) that the core of (N, c_N) is non-empty for all $R \in \mathcal{R}_N$.

Let a be an element of the core of (N, c_N). Then, from

$$\sum_{i \in S} a_i \leq \rho_N^*\left(\sum_{i \in S} X_i\right) \tag{A5}$$

$$\leq \rho_S^*\left(\sum_{i \in S} X_i\right) \tag{A6}$$

$$= c(S), \tag{A7}$$

for all $S \subseteq N$ and $c_N(N) = c(N)$ follow that $a \in core(R)$, where (A5) follows from that a is in the core of (N, c_N), (A6) follows from Lemma A.1 and (A7) follows from (16). Hence, we get that the core of (N, c_N) is a subset of $core(R)$, which concludes the proof. ■

Proof of Lemma 4.10. Let $R \in \mathcal{R}_N$. Subadditivity of the fuzzy game r follows directly from

$$r(\lambda) + r(\lambda') = \rho^*_{\{i \in N : \lambda_i > 0\}}\left(\sum_{i \in N} \lambda_i X_i\right) + \rho^*_{\{i \in N : \lambda'_i > 0\}}\left(\sum_{i \in N} \lambda'_i X_i\right) \tag{A8}$$

$$\geq \rho^*_{\{i \in N : \lambda_i + \lambda'_i > 0\}}\left(\sum_{i \in N} \lambda_i X_i\right) + \rho^*_{\{i \in N : \lambda_i + \lambda'_i > 0\}}\left(\sum_{i \in N} \lambda'_i X_i\right) \tag{A9}$$

$$\geq \rho^*_{\{i \in N : \lambda_i + \lambda'_i > 0\}}\left(\sum_{i \in N} (\lambda_i + \lambda'_i) X_i\right) \tag{A10}$$

$$= r(\lambda + \lambda'), \tag{A11}$$

for all $\lambda, \lambda' \in \mathbb{R}_+^N$. Here, (A8) follows from (10), (A9) follows from Lemma A.1, (A10) follows from *Subadditivity* of $\rho^*_{\{i\in N:\lambda_i+\lambda_i'>0\}}$ and (A11) follows from (10). This concludes the proof. ■

Proof of Proposition 6.1. Let $R \in \mathcal{R}$ be such that condition [SC] holds. First, we show the "⇐" (if) part of the proof. Let $R \in \mathcal{R}_N$ be such that $X_i, i \in N$ are all comonotonic with each other and let $\rho_i(X_i) = \rho_N^*(X_i)$ for all $i \in N$. Then, $\rho_N^*(X) = \sum_{i\in N} \rho_i(X_i)$ follows directly from

$$\sum_{i\in N} \rho_i(X_i) = \sum_{i\in N} \rho_N^*(X_i) \tag{A12}$$

$$= \rho_N^*(X). \tag{A13}$$

Here, (A12) follows from $\rho_i(X_i) = \rho_N^*(X_i)$ for all $i \in N$ and (A13) follows from that all $X_i, i \in N$ are comonotonic with each other and *Comonotonic Additivity* of ρ_N^*.

Next, we show the "⇒" (only if) part of the proof. Let $R \in \mathcal{R}_N$ be such that $\rho_N^*(X) = \sum_{i\in N} \rho_i(X_i)$. Generally, it follows from *Subadditivity* of ρ_N^* and Lemma A.1 that

$$\rho_N^*(X) \leq \sum_{i\in N} \rho_N^*(X_i) \tag{A14}$$

$$\leq \sum_{i\in N} \rho_i(X_i). \tag{A15}$$

Since it holds that $\rho_N^*(X) = \sum_{i\in N} \rho_i(X_i)$, the inequalities turn into equalities in (A14)–(A15). From (6), we get that the equality $\rho_N^*(X) = \sum_{i\in N} \rho_N^*(X_i)$ implies that $X_i, i \in N$ is Pareto optimal. Since condition [SC] holds, Boonen (2015, Proposition 3.7 therein) shows that this implies comonotonicity of the risks $X_i, i \in N$ with each other. From Lemma A.1, it follows that the equality $\sum_{i\in N} \rho_N^*(X_i) = \sum_{i\in N} \rho_i(X_i)$ implies $\rho_i(X_i) = \rho_N^*(X_i)$ for every $i \in N$. This concludes the proof. ■