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On the a.s. convergence of the one-sided ergodic Hilbert transform

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Abstract. We show that for T a Dunford–Schwartz operator on a σ-finite measure space (X, Σ, μ) and $f \in L^1(X, \mu)$, whenever the one-sided ergodic Hilbert transform $\sum_{n\geq 1}(T^n f/n)$ converges in norm, it converges μ-a.s. A similar result is obtained for any positive contraction of some fixed $L^p(X, \Sigma, \mu)$, $p > 1$. Applying our result to the case where T is the (unitary) operator induced by a measure-preserving (invertible) transformation, we obtain a positive answer to a question of Gaposhkin.

1. *Introduction*

Izumi [**13**] raised the question of the a.s. convergence of the one-sided ergodic Hilbert transform (EHT) $\sum_{k\geq 1}(f \circ \theta^k/k)$ associated to a probability preserving transformation θ and functions in $L^2(X)$. Unfortunately, Halmos [**12**] showed that Izumi's conditions are never satisfied, and proved that on any non-atomic space there always exists a centered $f \in L^2(X)$ such that the one-sided EHT fails to converge in L_2-norm. Later Dowker and Erdös [**9**] (see also Del Junco and Rosenblatt [**14**]) even obtained the existence of $f \in L^\infty(X)$, centered, such that

$$\sup_{n\geq 1}\left|\sum_{k=1}^{n}(f \circ \theta^k/k)\right| = +\infty \quad \text{(a.s.);}$$

see [**2**] for additional references.

We prove here that if $f \in L^1(X)$ and $\sum_{k\geq 1}(f \circ \theta^k/k)$ converges in norm, then it converges a.s. This yields a positive answer to the following question of Gaposhkin [**11**, p. 254]: if θ is an invertible measure-preserving transformation and $f \in L^2$, does the L^2-norm convergence of the one-sided EHT imply its a.s. convergence?

In [**11**], Gaposhkin studied the convergence of the one-sided EHT associated to a general unitary operator on L^2. He gave an example [**11**, pp. 253–254] of a unitary operator U on L^2 and $f \in L^2$ with norm convergence of the one-sided EHT but no a.s. convergence, and obtained the following conditions for the a.s. convergence (see [**11**, Theorem 3 and equation (33)]).

THEOREM 1.1. (Gaposhkin [**11**]) *Let* (X, Σ, μ) *be a probability space and* U *be a unitary operator on* $L^2(X, \mu)$. *Let* $f \in L^2(X)$ *such that*

$$\sum_{n>\mathrm{e}^{\mathrm{e}}} \frac{\|\sum_{k=1}^n U^k f\|_2^2}{n^3} \log n (\log\log\log n)^2 < +\infty. \tag{1}$$

Then $\sum_{n\geq 1}(U^n f/n)$ *converges* μ*-a.s. and in* L^2*-norm.*

It is shown in [**11**] that condition (1) is optimal for the a.s. convergence of the one-sided EHT for unitary operators; however, it is not necessary [**4**].

On the other hand, it is known that when U is induced by an invertible measure-preserving transformation, the two-sided EHT converges both a.s. (Cotlar [**6**]) and in norm (see, e.g., Campbell [**3**]). Moreover, it is known (see [**4**] or [**7**]) that the L^2-norm convergence of the one-sided EHT of any isometry (or normal contraction) V is equivalent to condition (2) below, which is strictly weaker than (1).

Sufficient conditions for the a.s. convergence of the one-sided EHT, for Dunford–Schwartz operators and functions in $L^1(X)$, were obtained by Derriennic–Lin [**8**] and Assani–Lin [**2**].

Our goal is to prove the conjecture of Gaposhkin even for Dunford–Schwartz operators or positive contractions of some L^p, $p > 1$. Moreover, our result applies to σ-finite measure spaces. The proof is based on a method introduced by Derriennic and Lin [**8**]. Our main result yields the particularly interesting following examples.

THEOREM 1.2. *Let* T *be a Dunford–Schwartz operator on the* σ*-finite measure space* (X, Σ, μ). *Let* $f \in L^p(X)$ $(1 \leq p < \infty)$ *such that* $\sum_{n\geq 1}(T^n f/n)$ *converges in* $L^p(X)$. *Then* $\sum_{n\geq 1}(T^n f/n)$ *converges* μ*-almost everywhere. Moreover,*

$$\sup_{n\geq 1}\left|\sum_{k=1}^n \frac{T^k f}{k}\right|$$

is in $L^p(X)$ *if* $p > 1$, *and in weak-*$L^1(X)$ *if* $p = 1$ *and* μ *is finite.*

THEOREM 1.3. *Let* $1 < p < \infty$ *and* T *be a positive contraction of* $L^p(X, \Sigma, \mu)$, *where* μ *is* σ*-finite. Let* $f \in L^p(X)$ *such that* $\sum_{n\geq 1}(T^n f/n)$ *converges in* $L^p(X)$. *Then* $\sum_{n\geq 1}(T^n f/n)$ *converges* μ*-a.s. Moreover,*

$$\sup_{n\geq 1}\left|\sum_{k=1}^n \frac{T^k f}{k}\right| \in L^p(X).$$

COROLLARY 1.4. *Let* (X, Σ, μ) *be a probability space, and let* V *be the isometry on* $L^2(X, \mu)$ *induced by a* μ*-preserving transformation. If* $f \in L^2(X)$ *satisfies*

$$\sum_{n\geq 1} \frac{\|\sum_{k=1}^n V^k f\|_2^2}{n^3} \log n < +\infty, \tag{2}$$

then $\sum_{n\geq 1}(V^n f/n)$ *converges* μ*-a.s.*

COROLLARY 1.5. *Let (X, Σ, μ) be a σ-finite measure space, and let T be a Dunford–Schwartz operator on (X, Σ, μ) or a positive contraction of $L^2(X, \Sigma, \mu)$. Let $f \in L^2(X)$ such that the series*

$$\sum_{n\geq 1} \frac{\langle T^n f, f\rangle}{n} \log n \tag{3}$$

converges. Then $\sum_{n\geq 1}(T^n f/n)$ converges μ-almost everywhere.

2. *Main result and proof of Theorems 1.2 and 1.3*

Let (X, Σ, μ) be a σ-finite measure space and let $1 \leq r < \infty$. Let T be a linear operator on $L^r(X, \Sigma, \mu)$.

We are concerned essentially with two types of operators:

- ⋆ T is a Dunford–Schwartz operator on (X, Σ, μ) (that is, T is a contraction of every $L^p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$); recall (e.g. [**15**, p. 159]) that the modulus of T is a positive Dunford–Schwartz operator, denoted by $\mathbf{T}$, such that $|Tf| \leq \mathbf{T}|f|$ for every $f \in L^1(X)$;
- ⋆ T is a contraction of $L^1(X)$ or a *positive* contraction of $L^r(X)$.

We denote by $\mathbf{T}$ either the linear modulus of T, in cases where T is a contraction of $L^1(X)$, or the operator T itself, in cases where T is positive.

We say that $\mathbf{T}$ satisfies the pointwise ergodic theorem in $L^r(X)$ whenever $\{(1/n)\sum_{k=1}^n \mathbf{T}^k f\}$ converges μ-almost everywhere for every $f \in L^r(X)$.

For $f \in L^r$, denote $f^* := \sup_{n\geq 1}(1/n)\sum_{k=1}^n \mathbf{T}^k|f|$.

THEOREM 2.1. *Let $1 \leq r < \infty$. Let T be a power bounded operator on $L^r(X, \Sigma, \mu)$ of a σ-finite measure space, positive if $r > 1$, such that $\mathbf{T}$ satisfies the pointwise ergodic theorem in $L^r(X)$. Let $f \in L^r$ such that $\sum_{n\geq 1}(T^n f/n)$ converges in $L^r(X)$. Then, $\sum_{n\geq 1}(T^n f/n)$ converges μ-almost everywhere.*

Moreover, if f^* is in $L^r(X)$ (respectively in weak-$L^r(X)$), so is $\sup_{n\geq 1}|\sum_{k=1}^n (T^k/k)|$.

We give the proof of that theorem in the next section. We now show how to apply Theorem 2.1 to obtain Theorems 1.2 and 1.3 and their corollaries, and we discuss other possible applications.

Theorem 1.2 follows from Theorem 2.1 and the Dunford–Schwartz ergodic theorem [**10**].

Theorem 1.3 follows from Theorem 2.1 and Akcoglu's ergodic theorem for a positive contraction of $L^p(X)$ $1 < p < \infty$, see [**1**] or [**15**, p. 186].

Recall that a contraction T of $L^1(X)$ is called mean ergodic if $L^1(X) = \{f \mid Tf = f\} \oplus \overline{(I - T)L^1(X)}$. By a result of Çömez and Lin [**5**], a positive contraction of L^1 that is mean ergodic satisfies the pointwise ergodic theorem in $L^1(X)$. Hence, Theorem 2.1 applies to contractions T such that $\mathbf{T}$ is mean ergodic. For more results about ergodic theorems which allow us to apply Theorem 2.1 we refer to Krengel's book [**15**] (see also [**16**] for a result for positive contractions in $L^1(X)$).

Proof of Corollaries 1.4 and 1.5. It is enough to show that under the conditions of the corollaries the one-sided EHT converges in $L^2(X)$. It follows from [**4**] (see also [**7**]), that

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for every isometry T of $L^2(X)$, condition (2) implies the norm convergence of the one-sided EHT, while for every contraction of $L^2(X)$, (3) is sufficient for the norm convergence, by [**4**]. □

3. *Proof of Theorem 2.1*

Before proving Theorem 2.1 we need some preliminaries.

Let $D := \{z \in \mathbb{C} : |z| < 1\}$. For every $z \in D$, define

$$H(z) := \log\left(\frac{\mathrm{e}}{1-z}\right) = 1 + \sum_{n\geq 1} \frac{z^n}{n} = \sum_{n\geq 0} \beta_n z^n.$$

Since $1 - \mathrm{e} \notin D$, $H \neq 0$ on D. Hence, $G := 1/H$ is well-defined analytic on D. So there exists $\{\alpha_n\}_{n\geq 0}$, such that

$$G(z) = \sum_{n\geq 0} \alpha_n z^n \quad \text{for all } z \in D.$$

One can see that $\alpha_0 = 1$, and it follows from [**17**, Theorem 2.31, p. 192] that $\alpha_n \sim -1/n(\log n)^2$. For convenience we use the notation $\gamma_n := K/n(\log(n+1))^2$, $n \geq 1$, where K is such that $|\alpha_n| \leq \gamma_n$.

It follows from the identity $G(z)H(z) = 1$ that

$$\beta_n + \sum_{k=1}^{n} \alpha_k \beta_{n-k} = \beta_n + \sum_{k=0}^{n-1} \beta_k \alpha_{n-k} = 0 \quad \text{for all } n \geq 1. \tag{4}$$

Fix $1 \leq r < \infty$ as in Theorem 2.1.

Since $\sum_{n\geq 0} |\alpha_n| < \infty$, the operator series $\sum_{n\geq 0} \alpha_n T^n$ converges in $L^r(X, \mu)$ in operator norm, and defines a bounded operator, denoted by $G(T)$. Moreover, for every $f \in L^r(X, \mu)$, the series $\sum_{n\geq 0} \alpha_n T^n f$ is μ-almost everywhere absolutely convergent, by the monotone convergence theorem (since the monotone sequence $\{(\sum_{k=0}^{n} |\alpha_k T^k f|)^r\}$ is bounded in $L^1(X, \mu)$ by $(\sum_{n\geq 0} |\alpha_n| \|T^n f\|_r)^r$).

For every $f \in L^r(X, \mu)$, we define $H(T)f \in L^r$ by $f + \sum_{n\geq 1}(T^n f/n)$ whenever the series converges in L^r.

Define also, for every $n \geq 1$,

$$H_n = H_n(T) := I + \sum_{k=1}^{n} \frac{T^k}{k} = \sum_{k=0}^{n} \beta_k T^k. \tag{5}$$

Recall that for $f \in L^r$, $f^* = \sup_{n\geq 1}(1/n) \sum_{k=1}^{n} \mathbf{T}^k |f|$.

PROPOSITION 3.1. *Let T be as in Theorem 2.1. Then, there exists $K_1, K_2 > 0$, so that, for every $h \in L^r(X, \mu)$,*

$$\sup_{n\geq 1} \|H_n G(T)h\|_r \leq K_1 \|h\|_r \tag{6}$$

and

$$\sup_{n\geq 1} |H_n G(T)h(x)| \leq K_2 h^*. \tag{7}$$

Proof. We have

$$\begin{aligned} H_n G(T)(h) &= \sum_{k=0}^{n} \beta_k T^k h + \sum_{k=0}^{n} \beta_k \sum_{m\ge 1} \alpha_m T^{m+k} h \\ &= \sum_{k=0}^{n} \beta_k T^k h + \sum_{k=0}^{n} \beta_k \sum_{m\ge k+1} \alpha_{m-k} T^m h \\ &= \sum_{k=0}^{n} \beta_k T^k h + \sum_{m=1}^{n}\Big(\sum_{k=0}^{m-1} \beta_k \alpha_{m-k}\Big) T^m h + \sum_{m\ge n+1}\Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k}\Big) T^m h \\ &= h + \sum_{m\ge n+1}\Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k}\Big) T^m h, \end{aligned} \tag{8}$$

where we used (4) for the last equality.

It suffices to deal with the series of the last equality. We have

$$\begin{aligned} \sum_{m\ge n+1}\Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k}\Big) T^m h = \sum_{m=n+1}^{2n}\Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k}\Big) T^m h \\ + \sum_{m\ge 2n+1}\Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k}\Big) T^m h. \end{aligned} \tag{9}$$

Let us prove (6). Since T is power bounded in L^r and, by monotonicity of $\{\gamma_n\}$, there exists $L_1 > 0$ such that

$$\begin{aligned} \Big\| \sum_{m\ge 2n+1}\Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k}\Big) T^m h \Big\|_r &\le L_1 \|h\|_r \sum_{m\ge 2n+1}\Big(\sum_{k=0}^{n} \beta_k \gamma_{m-n}\Big) \\ &\le L_2 \|h\|_r \log n \sum_{m\ge n+1} \gamma_m \le L_3 \|h\|_r. \end{aligned}$$

For the first sum in (9), we have

$$\begin{aligned} \Big\| \sum_{m=n+1}^{2n}\Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k}\Big) T^m h \Big\|_r &\le L_1 \|h\|_r \sum_{m=n+1}^{2n}\Big[\sum_{k=0}^{[n/2]} \beta_k \gamma_{m-[n/2]} + \sum_{k=[n/2]+1}^{n} \beta_{[n/2]} \gamma_{m-k}\Big] \\ &\le L_4 \|h\|_r \Big(\log n \sum_{m\ge [n/2]} \gamma_m + \frac{1}{n} \sum_{m=n+1}^{2n} \sum_{k\ge 0} \gamma_k\Big) \\ &\le L_5 \|h\|_r. \end{aligned}$$

We now prove (7). Write $S_m := \sum_{k=1}^{m} \mathbf{T}^k |h|$. For the second sum in (9), monotonicity of $\{\gamma_n\}$ yields

$$\begin{aligned} \Big| \sum_{m\ge 2n+1}\Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k}\Big) T^m h \Big| &\le \sum_{m\ge 2n+1} \gamma_{m-n}\Big(\sum_{k=0}^{n} \beta_k\Big) \mathbf{T}^m |h| \\ &\le C \log n \sum_{m\ge 2n+1} \gamma_{m-n} (S_m - S_{m-1}) \\ &\le C \log n \Big[\sum_{m\ge 2n+1} (\gamma_{m-n} - \gamma_{m+1-n}) S_m\Big]. \end{aligned}$$

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Hence, using Abel summation again, we obtain

$$\Big|\sum_{m\ge 2n+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m h\Big| \le C\log n\Big[\sum_{m\ge 2n+1}(\gamma_{m-n}-\gamma_{m+1-n})mh^*\Big]$$
$$\le C\log n h^*\Big(4n\gamma_{n+1}+\sum_{m\ge n+1}\gamma_m\Big)\le C'h^*.$$

Let us deal with the first sum of (9). By (4), we have

$$\Big|\sum_{m=n+1}^{2n}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m h\Big| = \Big|\sum_{m=n+1}^{2n}\beta_m T^m h+\sum_{m=n+1}^{2n}\Big(\sum_{k=n+1}^{m-1}\beta_k\alpha_{m-k}\Big)T^m h\Big|$$
$$\le \sum_{m=n+1}^{2n}\beta_m \mathbf{T}^m|h|+\sum_{m=n+1}^{2n}\Big(\sum_{k=n+1}^{m-1}\beta_k\gamma_{m-k}\Big)\mathbf{T}^m|h|$$
$$\le \frac{1}{n}S_{2n}+\sum_{m=n+1}^{2n}\Big(\frac{1}{n}\sum_{k\ge 1}\gamma_k\Big)\mathbf{T}^m|h|\le 2\Big(1+\sum_{k\ge 1}\gamma_k\Big)h^*.$$

We deduce the following proposition.

PROPOSITION 3.2. *Let T be as in Theorem 2.1 and $h\in\overline{(I-T)L^r(X)}$. Then $h=\lim_{n\to+\infty}H_nG(T)h$ both in L^r and almost everywhere.*

Proof. Let us first prove the convergence in L^r. By (6) it suffices to show the result for $h\in(I-T)L^r(X)$. By (8), the assertion is that for $h\in(I-T)L^r(X)$

$$\Big\|\sum_{m\ge n+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m h\Big\|_r \underset{n\to+\infty}{\longrightarrow} 0.$$

For $u\in L^r(X)$, we have

$$\begin{aligned}&\sum_{m\ge n+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m(u-Tu)\\
&=\sum_{m\ge n+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m u-\sum_{m\ge n+2}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k-1}\Big)T^m u\\
&=\sum_{m\ge n+2}\Big(\sum_{k=1}^{n}(\beta_k-\beta_{k-1})\alpha_{m-k}\Big)T^m u\\
&\quad+\Big(\sum_{k=0}^{n}\beta_k\alpha_{n+1-k}\Big)T^{n+1}u+\sum_{m\ge n+1}\alpha_m T^m u-\sum_{m\ge n+2}\beta_n\alpha_{m-n-1}T^m u.\end{aligned}\tag{10}$$

Now, using (4), we have

$$\Big\|\sum_{k=0}^{n}\beta_k\alpha_{n+1-k}T^{n+1}u+\sum_{m\ge n+1}\alpha_m T^m u-\sum_{m\ge n+2}\beta_n\alpha_{m-n}T^m u\Big\|_r$$
$$\le\frac{\|u\|_r}{n}+\sum_{m\ge n+1}\gamma_m\|u\|_r+\frac{1}{n}\sum_{m\ge 2}\gamma_m\|u\|_r \underset{n\to+\infty}{\longrightarrow} 0.$$

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It remains to deal with the term in (10). We have, splitting the sum according to $k \le [n/2]$,

$$\begin{aligned}\Big\| \sum_{m \ge n+2} \Big(\sum_{k=1}^{n} (\beta_k - \beta_{k-1}) \alpha_{m-k} \Big) T^m u \Big\|_r &\le \sum_{m \ge n+2} \Big(\sum_{k=1}^{n} |\beta_k - \beta_{k-1}| \gamma_{m-k} \Big) \|u\|_r \\ &\le \sum_{k=1}^{[n/2]} (\beta_{k-1} - \beta_k) \|u\|_r \sum_{m \ge n+2} \gamma_{m-[n/2]} + \frac{\|u\|_r}{[n/2]} \sum_{m \ge n+2} \gamma_{m-n} \\ &\le \frac{K}{\log n} \|u\|_r,\end{aligned}$$

for a constant $K > 0$ independent of u.

To show the almost everywhere convergence, by (7) and the Banach principle (see, e.g., [**10**, p. 332]) it suffices to show the result for $h \in (I - T)L^r(X)$. We could proceed as above, but since we have identified the limit, it suffices to show only the almost everywhere convergence of $H_n G(T)h$ for every $h \in (I - T)L^r(X)$.

Let $u \in L^r(X)$. We have

$$H_n G(T)(u - Tu) = H_n(G(T)u - TG(T)u) = H_n(v - Tv),$$

where $v := G(T)u \in L^r(X)$. Hence,

$$\begin{aligned} H_n G(T)(u - Tu) &= v - Tv + \sum_{k=1}^{n} \frac{T^k v - T^{k+1} v}{k} \\ &= v - \sum_{k=2}^{n} \frac{T^k v}{k(k-1)} - \frac{T^{n+1} v}{n+1} \underset{n \to +\infty}{\longrightarrow} v - \sum_{k \ge 2} \frac{T^k v}{k(k-1)},\end{aligned}$$

since $\sum_{k \ge 2} (T^k v / k(k-1))$ converges a.s. by Beppo Levi's theorem and $|T^n v|/n \le \mathbf{T}^n v/n$ goes to zero, as $\mathbf{T}$ satisfies the pointwise ergodic theorem in L^r. □

Proof of Theorem 2.1. Since $\sum_{n \ge 1} (T^n f/n)$ converges in $L^1(X)$, we have

$$\lim_{n \to +\infty} \frac{1}{n} \Big\| \sum_{k=1}^{n} T^k f \Big\|_1 = 0;$$

hence, $f \in \overline{(I - T)L^1(X)}$ (see, e.g., [**15**, Theorem 2.1.3, p. 73]).

Since $H(T)f = \lim_n H_n f$ exists in L^r by assumption, Proposition 3.2 and continuity of $G(T)$ yield, in L^r

$$f = \lim_{n \to +\infty} H_n G(T) f = \lim_{n \to +\infty} G(T) H_n f = G(T)H(T)f.$$

Define $g := H(T)f$. Then, we have $f = G(T)g$ and $g \in \overline{(I - T)L^r(X)}$. By the a.s. part of Proposition 3.2, we then obtain

$$H_n f = H_n G(T) g \underset{n \to +\infty}{\longrightarrow} g \quad \mu\text{-almost everywhere.}$$

The fact that the maximal function belongs to weak $L^1(X)$ when $r = 1$, or to $L^r(X)$ when $r > 1$, follows from Proposition 3.1. □

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