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AUTOMORPHISMS OF FREE NILPOTENT LIE ALGEBRAS

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Introduction. Let F_m be the free Lie algebra of rank m over a field K of characteristic 0 freely generated by the set $\{x_1, \dots, x_m\}, m \geqq 2$. Cohn [**7**] proved that the automorphism group Aut F_m of the K-algebra F_m is generated by the following automorphisms: (i) automorphisms which are induced by the action of the general linear group GL_m $(= GL_m(K))$ on the subspace of F_m spanned by $\{x_1, \dots, x_m\}$; (ii) automorphisms of the form $x_1 \longrightarrow x_1 + f(x_2, \dots, x_m), x_k \longrightarrow x_k, k \neq 1$, where the polynomial $f(x_2, \dots, x_m)$ does not depend on x_1. This result is similar to the well-known result in group theory due to Nielsen [**16**] that the automorphism group Aut G_m of the free group G_m on $\{g_1, \dots, g_m\}, m \geqq 2$, is generated by the symmetric group acting on $\{g_1, \dots, g_m\}$ together with the automorphisms of the form $g_1 \longrightarrow g_1g_2, g_k \longrightarrow g_k, k \neq 1$, and $g_1 \longrightarrow g_1^{-1}, g_k \longrightarrow g_k, k \neq 1$. The corresponding problem for generators of the automorphism groups of free nilpotent groups and free metabelian nilpotent groups has been studied by Andreadakis [**1**], [**2**], Bachmuth [**3**], Goryaga [**11**], Gupta [**13**], Bryant and Gupta [**6**]. In this paper we study the problem of finding minimal generating sets for the automorphism groups of relatively free nilpotent Lie algebras $F_m(\mathfrak{N})$ over a field K of characteristic 0.

Let $\mathfrak{N}$ be an arbitrary subvariety of the variety $\mathfrak{N}_c$ of all nilpotent Lie algebras of class at most c and let $\mathfrak{N}$ contain non-commutative algebras. An endomorphism of $F_m(\mathfrak{N})$ is an automorphism if and only if it induces an automorphism modulo the commutator ideal $(F_m(\mathfrak{N}))'$. Let $IA = IA(F_m(\mathfrak{N}))$ be the subgroup of Aut $F_m(\mathfrak{N})$ consisting of automorphisms which are identity modulo $(F_m(\mathfrak{N}))'$. Then Aut $F_m(\mathfrak{N})$ is the split extension of IA by GL_m. If $\phi \in \operatorname{Aut} F_m(\mathfrak{N})$ can be lifted to an automorphism of F_m then we say that ϕ is a *tame* automorphism of $F_m(\mathfrak{N})$, otherwise we say that ϕ is *wild*. For $m = 2$ there is a canonical isomorphism of the groups Aut F_2 and GL_2; and it is easy to construct wild automorphisms of $F_2(\mathfrak{N})$. For example, every non-trivial IA-automorphism of $F_2(\mathfrak{N})$ is wild. The simplest example is $\phi \in \operatorname{Aut} F_2(\mathfrak{N})$ given by $\phi(x_1) = x_1 + [x_1, x_2], \phi(x_2) = x_2$.

The first result in this paper gives quantitative information for the action of GL_m on IA. Using the natural structure of $F_m(\mathfrak{N})$ as a GL_m-module, in Section 2 we prove that the algebra $F_m(\mathfrak{N})$ has wild automorphisms for all $m \geqq 2$ and all non trivial $\mathfrak{N} \neq \mathfrak{U}$, the variety of commutative Lie algebras. Next, in Section 3 we study the variety $\mathfrak{N}_c \cap \mathfrak{U}^2$ of all metabelian Lie algebras in $\mathfrak{N}_c$. We prove

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that for all $m \geqq 2, c \geqq 2$, Aut $F_m(\mathfrak{N}_c \cap \mathfrak{U}^2)$ is generated by GL_m and one more automorphism δ defined by $\delta(x_1) = x_1 + [x_1, x_2], \delta(x_k) = x_k, k \neq 1$. Finally, in Section 4 we establish that for $\mathfrak{N} \subset \mathfrak{N}_c$ and $m \geqq c \geqq 2$, Aut $F_m(\mathfrak{N})$ is generated by GL_m and δ. Our proofs are based on the representation theory of the general linear group $GL_m(K)$ and essentially make use of the fact that the characteristic of K is 0. The basic facts of the representation theory are collected in Section 1. For a background of the theory of varieties of Lie algebras we refer to Bakhturin [**4**].

1. Representations of the general linear group. In this Section we give the necessary background of the representation theory of the general linear group. The details can be found in Weyl ([**18**], Chapter 4), Green ([**12**], Chapters 2, 4) and Macdonald ([**14**], Chapter 1). For applications of the polynomial representations of GL_m to the theory of varieties of algebras, we refer to Berele [**5**] and Drensky [**8**], [**9**], and of the rational representations we refer to Formanek [**10**].

Let K be a field of characteristic 0 and $GL_m = GL_m(K)$ be the general linear group acting from the left on an m-dimensional vector space V_m spanned by $x_1, \ldots, x_m$. We can consider elements of GL_m as invertible $m \times m$ matrices over K. Let W be an s-dimensional vector space, also with a fixed basis. A homomorphism

$$\phi : GL_m \longrightarrow GL_s = GL_s(W)$$

is called a *polynomial* representation of GL_m (and W a polynomial GL_m-module) if the entries $\phi_{pq}(g)$ of the $s \times s$ matrix $\phi(g)$ are polynomial functions of the entries a_{ij} of the $m \times m$ matrix $g = (a_{ij})$, for all $g \in GL_m$. Similarly, if $\phi_{pq}(g) = \psi_{pq}(g)/\theta_{pq}(g)$ are rational functions of a_{ij}, then ϕ is called a *rational* representation. When $\psi_{pq}(g)$ and $\theta_{pq}(g)$ are homogeneous of degree n_1 and n_2 respectively, then we say that ϕ is a *homogeneous* representation of degree $n_1 - n_2$.

Let $D_m = \{d \in GL_m \mid d = d(z_1, \ldots, z_m) = z_1e_{11} + \cdots + z_me_{mm}\}$ be the subgroup of the diagonal matrices of GL_m. For any degree sequence $\alpha = (\alpha_1, \ldots, \alpha_m)$ of length m we define the α-homogeneous component of the GL_m-module W by

$$W^\alpha = \{w \in W \mid d(z_1, \ldots, z_m)w = z_1^{\alpha_1} \ldots z_m^{\alpha_m} w \text{ for all } d \in D_m\}.$$

Then we have

PROPOSITION 1.1. (see Green [**12**] and Formanek [**10**]). *Let $\phi : GL_m \longrightarrow GL_s(W)$ be a finite dimensional rational representation of GL_m. Then*

(i) *The GL_m-module W is completely reducible and is a direct sum of its homogeneous submodules.*

(ii) *As a K-vector space, W is a direct sum of its homogeneous components.*

(iii) *Define the Hilbert series of W (the character of W) to be*

$$H(W) = H(W, t_1, \ldots, t_m) = \sum_\alpha (\dim_K W^\alpha) t_1^{\alpha_1} \ldots t_m^{\alpha_m}.$$

Then $H(W)$ is a symmetric function in $t_1, \ldots, t_m$ and if W is homogeneous of degree n, so is $H(W)$.

(iv) *Two GL_m-modules W_1 and W_2 are isomorphic if and only if $H(W_1) = H(W_2)$. Furthermore,*

$$H(W_1 \oplus W_2) = H(W_1) + H(W_2) \quad \textit{and} \quad H(W_1 \otimes_K W_2) = H(W_1)H(W_2).$$

(v) *The only one-dimensional rational GL_m-representations are $\phi_n : GL_m \longrightarrow K^*$, where n is an integer and $\phi_n(g) = (\det g)^n$, $g \in GL_m$. We denote the corresponding GL_m-module by $(\det)^n$. Then $H((\det)^n) = t_1^n \ldots t_m^n$.*

(vi) *Every finite-dimensional rational GL_m-module has the form $(\det)^{-n} \otimes_K W$, where W is a polynomial GL_m-module and $(\det)^{-n} \otimes_K W$ is irreducible if and only if W is irreducible.*

The irreducible polynomial representations of GL_m are described by partitions and Young diagrams. For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_1 \geqq \cdots \geqq \lambda_m$, $\lambda_1 + \cdots + \lambda_m = n$, we consider the corresponding Young diagram $[\lambda]$ and the related irreducible GL_m-module $N_m(\lambda)$.

Definition 1.2. (see Macdonald [**14**]). Let $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\mu = (\mu_1, \ldots, \mu_m)$ and $\nu = (\nu_1, \ldots, \nu_m)$ be partitions of n_1, n_2 and $n_1 + n_2$, respectively, and let $\nu_1 \geqq \lambda_1, \ldots, \nu_m \geqq \lambda_m$.

(i) A diagram of shape $[\nu - \lambda]$ is a scheme of boxes obtained from the diagram $[\nu]$ by removing the boxes of the diagram $[\lambda]$. When $n_1 = 0, [\nu - \lambda] = [\nu]$.

(ii) A $[\nu - \lambda]$-tableau (respectively $\nu-$ tableau) with content μ is the diagram $[\nu - \lambda]$ (respectively $[\nu]$) whose boxes are filled in with μ_1 numbers $1, \ldots, \mu_m$ numbers m.

(iii) A tableau is *semistandard* if its entries do not decrease from left to right in the rows and increase from top to bottom in the columns.

(iv) The sequence $w(T)$ is obtained from a tableau T by listing the entries of T from right to left, consecutively reading the rows from top to bottom (as in Arabic).

(v) The sequence $w = a_1, a_2, \ldots, a_n$ is a *lattice permutation* if it contains the symbols $1, 2, \ldots, s$ and for each $1 \leqq k \leqq n$ and $1 \leqq i \leqq s - 1$, the number i participates in $a_1, \ldots, a_k$ no less times than $i + 1$.

Proposition 1.3. (see Macdonald [**14**]). *The coefficient*

$$a_\alpha = \dim_K (N_m(\lambda))^\alpha$$

of the Hilbert series

$$H(N_m(\lambda), t_1, \ldots, t_m) = \sum_\alpha a_\alpha t_1^{\alpha_1} \ldots t_m^{\alpha_m}$$

equals the number of semistandard λ-tableaux of content $\alpha = (\alpha_1, \ldots, \alpha_m)$.

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For example, let $\lambda = (3, 1)$ and $m = 3$. The only semistandard $[3, 1]$-tableaux of content $\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha_1 \geqq \alpha_2 \geqq \alpha_3$, are given in Table 1.

Table 1

α	(4,0,0)	(3,1,0)	(2,2,0)	(3,1,0)
		1 1 1 2	1 1 2 2	1 1 2 3 1 1 3 2
$\dim_K(N_3(3,1))^\alpha$	0	1	1	2

Since $H(N_3(3, 1))$ is a symmetric polynomial in t_1, t_2, t_3, we obtain

$$H(N_3(3, 1)) = (t_1^3 t_2 + t_1 t_2^3 + t_1^3 t_3 + t_1 t_3^3 + t_2^3 t_3 + t_2 t_3^3) + (t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2) + (t_1^2 t_2 t_3 + t_1 t_2^2 t_3 + t_1 t_2 t_3^2).$$

We shall need the following rule for the tensor product of irreducible GL_m-modules.

PROPOSITION 1.4. (The Littlewood–Richardson rule, see [**14**]). *Let λ and μ be partitions of n_1 and n_2, respectively. Then the following GL_m-module isomorphism holds:*

$$N_m(\lambda) \otimes_K N_m(\mu) \cong \sum_\nu c^\nu_{\lambda\mu} N_m(\nu),$$

where the summation runs over all partitions $\nu = (\nu_1, \dots, \nu_m)$ of $n_1 + n_2$ and the coefficient $c^\nu_{\lambda\mu}$ equals the number of semistandard tableaux T of shape $[\nu - \lambda]$ with content μ such that the sequence $w(T)$ is a lattice permutation.

For example, let $\lambda = (3, 1), \mu = (2, 1)$ and $m = 3$. Then

$$N_3(3, 1) \otimes_K N_3(2, 1) \cong N_3(5, 2) + N_3(5, 1^2) + N_3(4, 3) + 2N_3(4, 2, 1) + N_3(3^2, 1) + N_3(3, 2^2),$$

(see Fig. 1).

Finally, we shall make use of the following particular case of the Littlewood–Richardson rule.

COROLLARY 1.5. *The isomorphism $N_m(\lambda) \otimes_K N_m(1^r) \cong \Sigma_\mu N_m(\mu)$ holds, where the summation runs over all partitions $\mu = (\lambda_1 + \epsilon_1, \dots, \lambda_m + \epsilon_m)$ with $\epsilon_i = 0, 1$ and $\epsilon_1 + \dots + \epsilon_m = r$.*

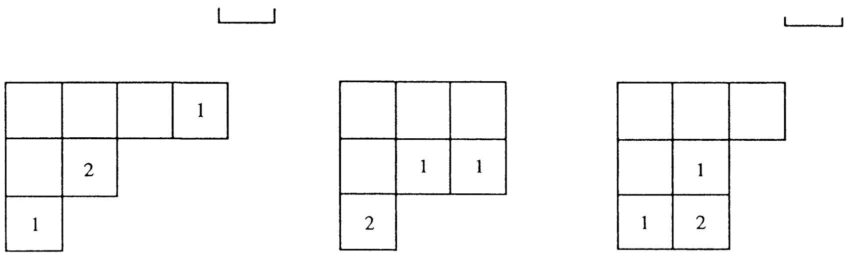

Figure 1

2. Automorphisms of relatively free nilpotent Lie algebras. Let $\mathfrak{N}_c$ denote the variety of nilpotent Lie algebras of class at most c over a field K of characteristic 0 and let $\mathfrak{N}$ be a subvariety of $\mathfrak{N}_c$. Let $F = F_m(\mathfrak{N})$ be the relatively free $\mathfrak{N}$-algebra of rank m generated by $x_1, \ldots, x_m$. The general linear group GL_m acts on the vector space V_m spanned by $x_1, \ldots, x_m$ and this action can be extended diagonally on the algebra F by

$$g[x_{i(1)}, \ldots, x_{i(n)}] = [g(x_{i(1)}), \ldots, g(x_{i(n)})], \quad g \in GL_m.$$

The automorphism group Aut F of the K-algebra F is a split extension by GL_m of the subgroup IA of Aut F consisting of automorphisms of F which induce the identity automorphism modulo the commutator ideal F' of F. We define the series

$$IA = IA_2 > IA_3 > \cdots > IA_c > IA_{c+1} = \langle \mathrm{id} \rangle,$$

where IA_s acts trivially modulo F^s, the ideal of F generated by all commutators of length s. Since the algebra F is nilpotent, every endomorphism of F which induces the identity modulo F' is an IA-automorphism of F. Thus, we have

$$IA_s = \{\phi : x_k \longrightarrow x_k + f_k \quad | \quad f_k \in F^s, k = 1, \ldots, m\}$$

and

$$IA_s/IA_{s+1} = \{\phi : x_k \longrightarrow x_k + f_k \quad | \quad f_k \in F^s/F^{s+1}, k = 1, \ldots, m\}.$$

Clearly, IA acts trivially by conjugation on IA_s/IA_{s+1} and we define a map $\sim$ which identifies any ϕ from IA_s/IA_{s+1} with the corresponding m-tuple $\tilde{\phi} = (f_1, \ldots, f_m)$. It is easy to see that $\sim$ is an isomorphism of the abelian groups IA_s/IA_{s+1} and $(F^s/F^{s+1})^{\oplus m}$. Since $(F^s/F^{s+1})^{\oplus m}$ has an additional structure of a

K-vector space, we may also consider IA_s/IA_{s+1} as a K-vector space. The general linear group GL_m acts on IA_s/IA_{s+1}, and we describe this action as follows.

THEOREM 2.1. *Let $F = F_m(\mathfrak{N})$ be the relatively free algebra of rank m of a variety $\mathfrak{N}$ of nilpotent Lie algebras. The group GL_m acts on the factors IA_s/IA_{s+1} of IA by $(g \cdot \phi) = g\phi g^{-1}, g \in GL_m, \phi \in IA_s/IA_{s+1}$ and, as a GL_m-module, IA_s/IA_{s+1} is isomorphic to the tensor product*

$$(det)^{-1} \otimes_K N_m(1^{m-1}) \otimes_K (F^s/F^{s+1}).$$

Proof. Let $\phi \in IA_s/IA_{s+1}$ be an arbitrary element with

$$\phi(x_k) = x_k + f_k(x_1, \dots, x_m), \quad f_k \in F^s/F^{s+1}, k = 1, \dots, m$$

and let $g = (a_{ij}) \in GL_m$. Denote $g^{-1} = (b_{ij})$. Then we have

$$\begin{aligned}(g \cdot \phi)(x_k) &= g\phi g^{-1}(x_k) = g\phi\left(\sum_i b_{ik}x_i\right)\\ &= g\left(\sum_i b_{ik}\phi(x_i)\right) = g\left(\sum_i b_{ik}(x_i + f_i)\right)\\ &= g\left(\sum_i b_{ik}(x_i)\right) + \sum_i b_{ik}g(f_i)\\ &= gg^{-1}(x_k) + \sum_i b_{ik}f_i(g(x_1), \dots, g(x_m))\\ &= x_k + \sum_i b_{ik}f_i(g(x_1), \dots, g(x_m)).\end{aligned}$$

We restate the left action of g on $\tilde{\phi} \in (F^s/F^{s+1})^{\oplus m}$ as

$$\begin{aligned}g \cdot (f_1, \dots, f_m) &= (f_1(g(x_1), \dots, g(x_m)), \dots, f_m(g(x_1), \dots, g(x_m)))(b_{ij})\\ &= (g(f_1), \dots, g(f_m))(b_{ij})\\ &= (g(f_1), \dots, g(f_m))g^{-1}.\end{aligned}$$

Here $g(f_i)$ means the canonical action of $g \in GL_m$ on F and the multiplication on the right with g^{-1} is the usual multiplication of two $1 \times m$ and $m \times m$ matrices.

In order to obtain the GL_m-module structure of IA_s/IA_{s+1} we next compute its Hilbert series. Let $u = [x_{i(1)}, \dots, x_{i(s)}]$ be an arbitrary commutator in F^s with $\deg_{x(i)}(u) = \alpha_i$. Then for the automorphism $\varphi_1 \in IA_s/IA_{s+1}$ such that $\varphi_1(x_1) = x_1 + u, \varphi_1(x_k) = x_k, k \neq 1$, and for the diagonal matrix $d = d(z_1, \dots, z_m)$, we obtain

$$d \cdot (u, 0, \dots, 0) = (d(u), 0, \dots, 0)d^{-1} = z_1^{\alpha_1} \dots z_m^{\alpha_m}(u/z_1, 0, \dots, 0).$$

Thus φ_1 belongs to the homogeneous component $(IA_s/IA_{s+1})^{\beta}$, where $\beta = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_m)$. Similar considerations hold for automorphisms φ_i fixing the variables $x_k, k \neq i$. Therefore, for the Hilbert series of IA_s/IA_{s+1}, we obtain

$$H(IA_s/IA_{s+1}, t_1, \ldots, t_m) = (1/t_1 + \cdots + 1/t_m)H(F^s/F^{s+1}, t_1, \ldots, t_m).$$

Clearly,

$$(1/t_1 + \cdots + 1/t_m) = (t_1, \ldots, t_m)^{-1} \cdot e_{m-1}$$

where $e_i = e_i(t_1, \ldots, t_m)$ is the i-th elementary symmetric polynomial. By Proposition 1.3, it is easy to verify that e_{m-1} coincides with the Hilbert series of the GL_m-module $N_m(1^{m-1})$. Thus the GL_m-modules IA_s/IA_{s+1} and $(\det)^{-1} \otimes_K N_m(1^{m-1}) \otimes_K (F^s/F^{s+1})$ have the same Hilbert series. By virtue of Proposition 1.1(iv), this completes the proof of the theorem.

THEOREM 2.2. *Let $\mathfrak{N}$ be a nilpotent variety of Lie algebras over a field K of characteristic* 0 *and let $\mathfrak{N}$ contain non-commutative algebras. Then for all $m \geqq 2$, the automorphism group* $\operatorname{Aut} F_m(\mathfrak{N})$ *contains wild automorphisms.*

Proof. Since $\mathfrak{N}_2$ is the only minimal non-abelian variety and $\mathfrak{N}_2 \subset \mathfrak{N}$, it suffices to prove the theorem for $\mathfrak{N}_2 = \mathfrak{N}$ only. In this case

$$IA = IA_2 \geqq IA_3 = \langle \mathrm{id} \rangle,$$

i.e., IA is an abelian group isomorphic to $(F')^{\oplus m}$. The K-vector space $F' = F_{m^2}(\mathfrak{N}_2)$ has a basis consisting of all the commutators $[x_i, x_j], i > j$. It is well-known (and can be easily obtained by comparing the corresponding Hilbert series) that, as GL_m-modules, F' and $N_m(1^2)$ are isomorphic. Therefore, by Theorem 2.1 we have

$$IA \cong (\det)^{-1} \otimes_K N_m(1^{m-1}) \otimes_K N_m(1^2).$$

When $m = 2$, we have

$$IA \cong (\det)^{-1} \otimes_K N_2(1) \otimes_K N_2(1^2) \cong N_2(1),$$

since $(\det)^{-1} \otimes_K N_2(1^2)$ is isomorphic to the trivial GL_2-module K. This immediately gives the proof of the theorem for $m = 2$ because in this case Aut $(F_2) \cong GL_2$ and all non-trivial IA-automorphisms are wild.

When $m > 2$ we have, using the Littlewood–Richardson rule,

$$\begin{aligned} IA &\cong (\det)^{-1} \otimes_K N_m(1^{m-1}) \otimes_K N_m(1^2) \\ &\cong (\det)^{-1} \otimes_K (N_m(2^2, 1^{m-3}) \oplus N_m(2, 1^{m-1})). \end{aligned}$$

Since $(\det)^{-1} \otimes_K N_m(2, 1^{m-1})$ and $N_m(1)$ have the same Hilbert series, they are isomorphic and

$$IA \cong (\det)^{-1} \otimes_K N_m(2^2, 1^{m-3}) \oplus N_m(1).$$

Since $(\det)^{-1} \otimes_K N_m(2^2, 1^{m-3})$ is a proper submodule of the GL_m-submodule IA, it suffices to observe that the tame automorphisms from IA belong only to the GL_m-submodule $(\det)^{-1} \otimes_K N_m(2^2, 1^{m-3})$. The subgroup of all tame automorphisms is generated by GL_m together with all automorphisms $\phi = \phi_{1,a}$ defined by

$$\phi(x_1) = x_1 + a[x_2, x_3], \quad a \in K, \quad \phi(x_k) = x_k, \quad k \neq 1.$$

Since $\phi_{1,a}$ corresponds to the element

$$\tilde{\phi}_{1,a} = (a[x_2, x_3], 0, \ldots, 0)$$

of $(F')^{\oplus m}$, it is homogeneous of degree $(-1, 1, 1, 0, \ldots, 0)$. By Proposition 1.3 the Hilbert series of the GL_m-module $N_m(1)$ has a trivial coefficient of t_2t_3/t_1 and, hence the homogeneous component of degree $(-1, 1, 1, 0, \ldots, 0)$ of $N_m(1)$ equals zero. This gives immediately that the tame automorphisms from IA belong to the GL_m-submodule $(\det)^{-1} \otimes_K N_m(2^2, 1^{m-3})$. This completes the proof of the theorem.

Remark 2.3. For $F = F_m(\mathfrak{N}_2)$ it is possible to obtain the tame automorphisms explicitly. For this purpose it suffices to find a K-basis of the submodule $(\det)^{-1} \otimes_K N_m(2^2, 1^{m-3})$ of the GL_m-module IA. For example, for $m = 3$, a direct verification shows that $\phi \in IA$ is tame if and only if

$$\begin{aligned}
\phi(x_1) &= x_1 + a_2[x_1, x_2] + a_3[x_1, x_3] + a_{23}[x_2, x_3], \\
\phi(x_2) &= x_2 + a_1[x_1, x_2] + a_{13}[x_1, x_3] - a_3[x_2, x_3], \\
\phi(x_3) &= x_3 + a_{12}[x_1, x_2] - a_1[x_1, x_3] - a_2[x_2, x_3],
\end{aligned}$$

where a_i, a_{jk} are arbitrary elements of K.

Remark 2.4. If the GL_m-module structure of a relatively free Lie algebra $F_m(\mathfrak{M})$ is known then we also know the GL_m-module structure of $F_m(\mathfrak{M} \cap \mathfrak{N}_c)$. In particular, Thrall [**17**] has obtained the decomposition of L^s/L^{s+1} for the free Lie algebra $L = L(x_1, \ldots, x_m)$ with $s \leqq 10$; the descriptions of $F_m(\mathfrak{N}_2\mathfrak{U} \cap \mathfrak{U}\mathfrak{N}_2)$ and $F_m([\mathfrak{U}^2, \mathfrak{E}, \mathfrak{E}])$ are obtained in Drensky [**8**] and Mishchenko [**15**], etc.

3. Free nilpotent metabelian algebras. In this section we shall obtain generators for the automorphism group of the relatively free nilpotent of class c and metabelian Lie algebra $F_m(\mathfrak{N}_c \cap \mathfrak{U}^2)$. The main result is that $\operatorname{Aut} F_m(\mathfrak{N}_c \cap \mathfrak{U}^2)$ is generated by GL_m and a single automorphism δ, defined by

$$\delta(x_1) = x_1 + [x_1, x_2], \quad \delta(x_k) = x_k \quad \text{for } k > 1.$$

We also establish some results for Aut $F_m(\mathfrak{N}_c)$. For $m, c \geqq 2$, we denote by F the algebra $F_m(\mathfrak{N}_c)$. Let G be the subgroup of Aut F generated by GL_m together with the automorphism δ, defined above. Clearly,

$$IA_{c+1} = \langle \mathrm{id} \rangle \quad \text{and} \quad IA_c/IA_{c+1} = IA_c \cong (F^c)^{\oplus m}.$$

We denote by $\tilde{G}$ the image of $G \cap IA_c$ under the isomorphism $\sim$ from IA_c to $(F^c)^{\oplus m}$. By definition, $x_1(\mathrm{ad}\ x_2) = [x_1, x_2]$ and all the commutators are left-normed, i.e.,

$$[x_1, x_2, x_3] = [[x_1, x_2], x_3].$$

Additionally, we use the same notation G for the subgroup of Aut $F_m(\mathfrak{N})$ generated by GL_m and δ for all varieties $\mathfrak{N} \subset \mathfrak{N}_c$.

LEMMA 3.1. *For $a \in K^*$ and $f \in F^c$, let $\phi_{a,f} \in IA_c$ be given by*

$$\phi_{a,f}(x_1) = x_1 + af, \quad \phi_{a,f}(x_k) = x_k, \quad k > 1.$$

Then, if $\phi = \phi_{1,f} \in G$, then $\phi_{a,f} \in G$ for all $a \in K^*$.

Proof. We shall prove the lemma in two steps. First, let $a = p/q$ be a rational number. With $n = pq^{c-2}$, we have $\phi^n \in G$ with

$$\phi^n(x_1) = x_1 + pq^{c-2}f, \quad \phi^n(x_k) = x_k, \quad k \neq 1.$$

Conjugating ϕ^n with the diagonal matrix $d = d(1/q, \ldots, 1/q) \in GL_m$ yields

$$d \cdot \phi^n = d\phi^n d^{-1} = \phi_{p/q,f} \in G.$$

This gives the proof for the case when a is rational. Now, let $a \in K^*$ be arbitrary. Conjugating ϕ with the diagonal matrices $d_i = d(a+i, \ldots, a+i), i = 0, 1, \ldots, c-1$, we obtain $d_i\phi d_i^{-1} \in G$. For each $i = 0, 1, \ldots, c-1, d_i\phi d_i^{-1}$ corresponds to the equation

$$(a+i)^{c-1}f = \sum_r i^r(\tbinom{c-1}{r}a^{c-r-1}f).$$

Considering these equations as a system of linear equations with $\binom{c-1}{r}a^{c-r-1}f$ as indeterminates yields a $c \times c$ Vandermonde matrix. It follows that each $\binom{c-1}{r}a^{c-r-1}f$ can be expressed as a rational linear combination of $(a+i)^{c-1}f, i = 0, 1, \ldots, c-1$. In particular, there exists a rational number p/q such that the automorphism $\phi_{(p/q)a,f}$ also belongs to G. Applying once again the first step, we establish that the desired automorphism $\phi_{a,f} \in G$.

LEMMA 3.2. *Let $R = K[t]/(t^{s+1}), s \geqq 1$, be the algebra of polynomials in one variable modulo the ideal generated by t^{s+1} and let $a \cdot f(t) = f(at), a \in K^*$, define*

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the action of K^* on R. Let $H = 1 + tR$ be the subgroup of the multiplicative group R^* consisting of all polynomials of the form $1 + a_1 t + \cdots + a_s t^s$. Then $H = \langle a \cdot (1 + t) \mid a \in K^* \rangle$, i.e., H coincides with the K^*-invariant subgroup generated by the single element $1 + t$.

Proof. The logarithmic map

$$\log : 1 + tf \longrightarrow (-t)f/1 + (tf)^2/2 + \cdots + (-1)^s (tf)^s/s$$

gives an isomorphism of the multiplicative group H and the additive group

$$tR = \{b_1 t + \cdots + b_s t^s \quad | \quad b_i \in K\}.$$

We consider the equalities

$$\log(1 - kt) = k(t/1) + k^2(t^2/2) + \ldots + k^s(t^s/s), \quad k = 1, \ldots, s,$$

as a system of linear equations with $t^i/i, i = 1, 2, \ldots, s$, as indeterminates. Then as in the proof of Lemma 3.1, each t^i/i is a rational linear combination of $\{\log(1-kt) \mid k = 1, \ldots, s\}$. In particular, since $\log(1+t^s) = (-s)t^s/s$, it follows that $\log(1+t^s)$ is a rational linear combination of $\{\log(1-kt) \mid k = 1, 2, \ldots, s\}$. Thus, for a suitable n, $(1+t^s)^n$ belongs to the multiplicative subgroup $\langle 1-kt \mid k = 1, \ldots, s\rangle$. Since $1 - kt = (-k) \cdot (1 + t)$, it follows that $(1 + t^s)^n$ belongs to the K^*-invariant subgroup generated by $(1 + t)$. Similar arguments as in the proof of Lemma 3.1 show that for any $a \in K^*, (1 + at^s) \in H = K^* \cdot \langle 1 + t \rangle$. Now, the proof of the lemma is completed by induction on s. The case $s = 1$ being trivial, we assume that the lemma holds for $s - 1 \geqq 1$. Let $1 + a_1 t + \cdots + a_s t^s$ be an arbitrary element of H. The inductive assumption implies that there exists b in K, such that

$$g(t) = 1 + a_1 t + \cdots + a_{s-1} t^{s-1} + bt^s$$

lies in H. Since $f(t)g^{-1}(t)$ is of the form $1 + at^s$ which belongs to H, it follows that $f(t)$ belongs to H. This completes the proof of the lemma.

LEMMA 3.3. *Let ψ and φ be automorphisms of F defined by*

$$\psi(x_1) = x_1 + x_1(\mathrm{ad}\, x_2)^{c-1}, \varphi(x_1) = x_1 + \sum [x_1, x_{\sigma(2)}, \ldots, x_{\sigma(c)}],$$
$$\psi(x_k) = \varphi(x_k) = x_k, \quad k \neq 1,$$

where the summation is taken over all permutations of $\{2, \ldots, c\}$. Then ψ and φ are elements of the subgroup G of $\mathrm{Aut}\, F$ *generated by GL_m and δ, where*

$$\delta(x_1) = x_1 + [x_1, x_2] = x_1 + x_1(ad\, x_2) \quad and \quad \delta(x_k) = x_k, \quad k \neq 1.$$

Proof. By Lemma 3.2, there exist rational numbers $a_1, \ldots, a_p, b_1, \ldots, b_q$ such that

$$1 + t^{c-1} \equiv \prod(1 + a_i t)/\prod(1 + b_j t)(\text{mod } t^c).$$

Thus

$$\begin{aligned} x_1 + x_1(\text{ad } x_2)^{c-1} &= x_1(1 + (\text{ad } x_2)^{c-1}) \\ &= x_1 \prod(1 + a_i(\text{ad } x_2))/\prod(1 + b_j(\text{ad } x_2)), \end{aligned}$$

and it follows that the automorphism ψ belongs to the subgroup of IA-automorphisms generated by $\{\delta_{a,[x_1,x_2]} \mid a \in K^*\}$, where $\delta_a = \delta_{a,[x_1,x_2]}$ is defined by

$$\delta_a(x_1) = x_1 + a[x_1, x_2] \quad \text{and} \quad \delta_a(x_k) = x_k, \quad k \neq 1.$$

Since $\delta_a = d^{-1}\delta d$, where $d = d(1, a, 1, \ldots, 1) \in GL_m$, it follows that $\psi \in G$. It remains to prove that $\varphi \in G$. To achieve this we apply to the automorphism $\psi \in G$ the standard process of linearization as follows. Let $g \in GL_m \subset G$ be such that $g(x_2) = x_2 + \cdots + x_c$, $g(x_k) = x_k, k \neq 2$. Then $g\psi g^{-1}$ sends x_1 to $x_1 + x_1$ $(\text{ad}(x_2 + \cdots + x_c))^{c-1}$. Since φ is the homogeneous component of degree $(0, 1, \ldots, 1)$ of the automorphism $g\psi g^{-1}$, standard Vandermonde arguments show that φ belongs to G.

LEMMA 3.4. *The GL_m-module $F^c/(F^c \cap F'')$ is isomorphic to $N_m(c-1, 1)$, $c \geqq 2$.*

Proof. Since $F_m(\mathfrak{N}_c \cap \mathfrak{U}^2) \cong F/F''$, it follows that

$$F_m^c(\mathfrak{N}_c \cap \mathfrak{U}^2) \cong F^c/(F^c \cap F'').$$

Therefore, it suffices to prove that

$$F_m^c(\mathfrak{N}_c \cap \mathfrak{U}^2) \cong N_m(c-1, 1).$$

This GL_m-module isomorphism is well-known. For example, this can be obtained in the following way. Bearing in mind that $F^c/F^c \cap F''$ has a basis of left-normed commutators $[x_{i_1}, x_{i_2}, \ldots, x_{i_c}]$, $i_1 > i_2 \leqq \ldots \leqq i_c$ and applying Proposition 1.3 we obtain that the Hilbert series of $F^c/F^c \cap F''$ and $N_m(n-1, 1)$ coincide. Therefore,

$$F^c/(F^c \cap F'') \cong F_m^c(\mathfrak{N}_c \cap \mathfrak{U}^2) \cong N_m(c-1, 1).$$

LEMMA 3.5. *Let $\mathfrak{N}$ be the variety of Lie algebras with a verbal ideal $L^{c+1} + (L^c \cap L''), L$ being the free Lie algebra (i.e., $\mathfrak{N}_{c-1} \subset \mathfrak{N} \subset \mathfrak{N}_c$ and $F_m^c(\mathfrak{N}) \cong$*

$F^c/(F^c \cap F''))$. Then for the GL_m-module structure of the subgroup IA_c of the group of automorphisms of $F = F_m(\mathfrak{N})$ one has

$$IA_c \cong N_m(c-1) \oplus N_m(c-2,1) \oplus ((det)^{-1} \otimes_K N_m(c,2,1^{m-3})),$$

where the third summand appears in the case $m > 2$ only.

Proof. The proof is a direct consequence of Theorem 2.1, the Littlewood-Richardson rule from Corollary 1.5 and Lemma 3.4.

For the variety $\mathfrak{N}$ of Lemma 3.5 the following identity holds:

$$\begin{aligned} 0 &= [x_1,\dots,x_s,[x_{s+1},x_{s+2}],x_{s+3},\dots,x_c] \\ &= [x_1,\dots,x_s,x_{s+1},x_{s+2},x_{s+3},\dots,x_c] \\ &- [x_1,\dots,x_s,x_{s+2},x_{s+1},x_{s+3},\dots,x_c]. \end{aligned}$$

Therefore we obtain the identity

$$[x_1,x_2,x_{\sigma(3)},\dots,x_{\sigma(c)}] = [x_1,x_2,x_3,\dots,x_c]$$

for all permutations σ of $\{3,\dots,c\}$. We shall make repeated use of this identity in the sequel.

PROPOSITION 3.6. *Let $\mathfrak{N}$ be the variety of Lie algebras with a verbal ideal $L^{c+1} + (L^c \cap L'')$. Then $IA_c = IA_c(\mathfrak{N})$ is a subgroup of the group G generated by GL_m and δ.*

Proof. Let $\tilde{G}$ be the image of $G \cap IA_c$ in $(F_m^c(\mathfrak{N}))^{\oplus m}$. We have to show that

$$\tilde{G} = (F_m^c(\mathfrak{N}))^{\oplus m}.$$

First, let $m = 2$. We use induction on c. The base of the induction $c = 2$, when $IA_2 \cong N_2(1)$, was considered in the proof of Theorem 2.2. Since $\mathrm{id} \neq \delta \in IA_2$ and IA_2 is an irreducible GL_2-module, we obtain that δ generates IA_2, i.e.,

$$\tilde{G} = (F_2^2(\mathfrak{N}))^{\oplus 2}.$$

We assume $c > 2$. In this case

$$IA_c \cong N_2(c-1) \oplus N_2(c-2,1).$$

Applying Proposition 1.3 for $\alpha = (1, c-2)$ we obtain that

$$\dim_K N_2^{(1,c-2)}(c-1) = \dim_K N_2^{(1,c-2)}(c-2,1) = 1.$$

Therefore, if we establish that $\dim_K \tilde{G}^{(1,c-2)} = 2$, this will give that

$$\tilde{G} \supset N_2^{(1,c-2)}(c-1) \oplus N_2^{(1,c-2)}(c-2,1).$$

Since the GL_2-modules $N_2(c-1)$ and $N_2(c-2,1)$ are irreducible, this will mean that $G \supset IA_c$. By Lemma 3.3, the automorphism ψ defined by

$$\psi(x_1) = x_1 + x_1(\mathrm{ad}\, x_2)^{c-1}, \quad \psi(x_2) = x_2,$$

belongs to G, i.e.,

$$\tilde{\psi} = (x_1(\mathrm{ad}\, x_2)^{c-1}, 0) \in \tilde{G}.$$

Let $g \in GL_2, g(x_1) = x_1, g(x_2) = x_1 + x_2$. Then

$$\begin{aligned} g \cdot \tilde{\psi} &= (g(x_1(\mathrm{ad}\, x_2)^{c-1}), 0)g^{-1} \\ &= (x_1(\mathrm{ad}(x_1 + x_2))^{c-1}, -x_1(\mathrm{ad}(x_1 + x_2))^{c-1}) \in \tilde{G}. \end{aligned}$$

The Vandermonde arguments give that the homogeneous components of $g \cdot \tilde{\psi}$ also belong to $\tilde{G}$. Since in $F_2(\mathfrak{N})$ we work modulo $F^c \cap F''$, the component of degree $(1, c-2)$ equals

$$\tilde{\rho}_1 = ((c-2)[x_1, x_2, x_1](\mathrm{ad}\, x_2)^{c-3}, -x_1(\mathrm{ad}\, x_2)^{c-1}) \in \tilde{G}.$$

For $h \in GL_2, h(x_1) = x_2, h(x_2) = x_1$, we obtain

$$h \cdot \delta(x_1) = x_1, \quad h \cdot \delta(x_2) = x_2 - [x_1, x_2] \quad \text{and} \quad h \cdot \delta \in G.$$

By the inductive assumption, there is an automorphism $\theta \in G$ such that

$$\theta(x_1) = x_1 + x_1(\mathrm{ad}\, x_2)^{c-2} + p_1(x_1, x_2), \quad \theta(x_2) = x_2 + p_2(x_1, x_2),$$

$p_1, p_2 \in (F^{c-1} \cap F'') + F^c$. We calculate

$$\rho_2 = (h \cdot \delta, \theta) = (\theta(h \cdot \delta))^{-1}((h \cdot \delta)\theta),$$

bearing in mind that

$$\begin{aligned} &p_i(x_1 + f_1, x_2 + f_2) \equiv p_i(x_1, x_2) \\ &(\mathrm{mod}\, F^c \cap F'') \quad \text{for all } f_1, f_2 \in F^2, i = 1, 2, \\ &x_1(\mathrm{ad}(x_2 - [x_1, x_2]))^{c-2} \equiv x_1(\mathrm{ad}\, x_2)^{c-2} + [x_1, x_2, x_1](\mathrm{ad}\, x_2)^{c-3} \end{aligned}$$

and that IA acts trivially on $F_2^c(\mathfrak{N})$:

$$\begin{aligned} \theta(h \cdot \delta)(x_1) &= \theta(x_1) = x_1 + x_1(\mathrm{ad}\, x_2)^{c-2} + p_1(x_1, x_2), \\ (h \cdot \delta)\theta(x_1) &= (h \cdot \delta)(\theta(x_1)) = (h \cdot \delta)(x_1 + x_1(\mathrm{ad}\, x_2)^{c-2} + p_1(x_1, x_2)) \\ &= x_1 + x_1(\mathrm{ad}(x_2 - [x_1, x_2]))^{c-2} + p_1(x_1, x_2 - [x_1, x_2]) \\ &= x_1 + x_1(\mathrm{ad}\, x_2)^{c-2} + [x_1, x_2, x_1](\mathrm{ad}\, x_2)^{c-3} + p_1(x_1, x_2) \\ &= \theta(h \cdot \delta)(x_1) + [x_1, x_2, x_1](\mathrm{ad}\, x_2)^{c-3}. \end{aligned}$$

Since $[x_1, x_2, x_1](\mathrm{ad}\, x_2)^{c-3} \in F_2^c(\mathfrak{N})$, we obtain

$$\theta(h\cdot\delta)(p_1(x_1,x_2)) = p_1(x_1,x_2) \quad \text{and}$$

$$\begin{aligned}\rho_2(x_1) &= (h\cdot\delta,\theta)(x_1) = (\theta(h\cdot\delta))^{-1}((h\cdot\delta)\theta)(x_1)\\ &= x_1 + [x_1,x_2,x_1](\mathrm{ad}\, x_2)^{c-3}.\end{aligned}$$

Similarly,

$$\begin{aligned}\theta(h\cdot\delta)(x_2) &= \theta(x_2 - [x_1,x_2])\\ &= x_2 + p_2(x_1,x_2) - [x_1,x_2] - x_1(\mathrm{ad}\, x_2)^{c-1},\\ (h\cdot\delta)\theta(x_2) &= (h\cdot\delta)(x_2 + p_2(x_1,x_2))\\ &= (x_2 - [x_1,x_2] + p_2(x_1,x_2) - x_1(\mathrm{ad}\, x_2)^{c-1}) + x_1(\mathrm{ad}\, x_2)^{c-1}\\ &= \theta(h\cdot\delta)(x_2) + x_1(\mathrm{ad}\, x_2)^{c-1},\\ \rho_2(x_2) &= (h\cdot\delta,\theta)(x_2) = (\theta(h\cdot\delta))^{-1}((h\cdot\delta)\theta)(x_2) = x_2 + x_1(\mathrm{ad}\, x_2)^{c-1}.\end{aligned}$$

Therefore,

$$\tilde\rho_2 = ([x_1,x_2,x_3](\mathrm{ad}\, x_2)^{c-3}, x_1(\mathrm{ad}\, x_2)^{c-1}) \in \tilde G.$$

Since $c > 2$, the elements $\tilde\rho_1$ and $\tilde\rho_2$ are linearly independent, $\dim_K \tilde G^{(1,c-2)} = 2$, and this completes the proof for $m = 2$.

Now, let $m > 2$. First we shall consider the case $c = 2$, when $F_m(\mathfrak{N})$ is isomorphic to the free nilpotent algebra $F = F_m(\mathfrak{N}_2)$. Clearly, the GL_m-module $(F^2)^{\oplus m}$ is generated by $\tilde\delta = ([x_1,x_2],0,\dots,0)$ and $([x_3,x_2],0,\dots,0)$. But for $g \in GL_m$,

$$g(x_1) = x_1 + x_2, \quad g(x_k) = x_k, \quad k > 1,$$

$$g\cdot\tilde\delta = ([x_1 + x_3, x_2], 0,\dots,0) \quad \text{and}$$

$$([x_3,x_2],0,\dots,0) = g\cdot\tilde\delta - \tilde\delta \in \tilde G.$$

Therefore $\tilde G = (F^2)^{\oplus m}$. Now, let $c > 2$. By Lemma 3.5,

$$IA_c \cong N_m(c-1) \oplus N_m(c-2,1) \oplus ((\det)^{-1} \otimes_K N_m(c,2,1^{m-3})).$$

Applying Proposition 1.3 for the irreducible components of IA_c and for $\alpha = (1, c-2, 0, \dots, 0)$ we obtain

$$\dim_K N_m^\alpha(c-1) = \dim_K N_m^\alpha(c-2,1) = 1.$$

The most difficult case is $(\det)^{-1} \otimes_K N_m(c,2,1^{m-3})$. Since the GL_m-module $(\det)^{-1}$ is homogeneous of degree $(-1,-1,\dots,-1)$, in this case we have to calculate the number of semistandard $(c, 2, 1^{m-3})$-tableaux of content $\beta = (2, c-1, 1, \ldots, 1)$. All these tableaux are given in Fig. 2.

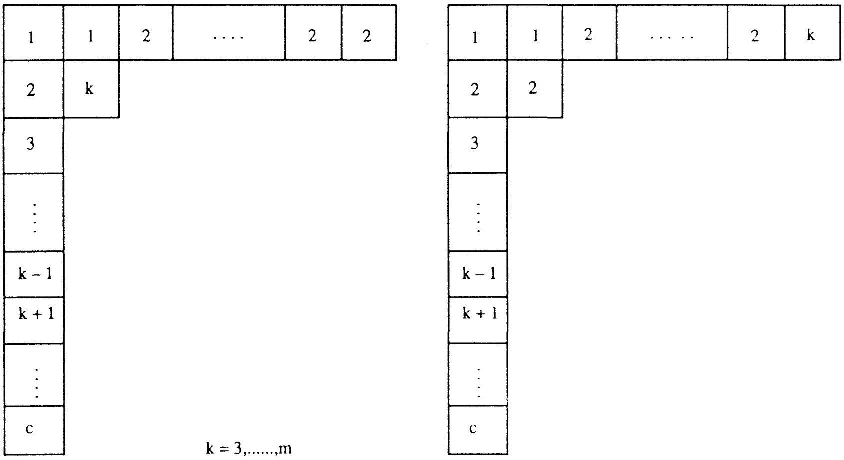

Figure 2

Therefore

$$\dim_K((\det)^{-1} \otimes_K N_m(c, 2, 1^{m-3}))^{\alpha} = 2(m-2).$$

As in the case $m = 2$ we have to show that

$$\dim_K \tilde{G}^{\alpha} = 1 + 1 + 2(m-2) = 2(m-1).$$

From the case $m = 2$ we know that the elements

$$\tilde{\pi}_1 = ([x_1, x_2, x_1](\operatorname{ad} x_2)^{c-3}, 0, 0, \ldots, 0),$$
$$\tilde{\pi}_2 = (0, x_1(\operatorname{ad} x_2)^{c-1}, 0, \ldots, 0)$$

belong to $\tilde{G}$. If we obtain $\sigma_i, \tau_i \in G, i = 3, \ldots, m$, such that

$$\sigma_i(x_1) = x_i + [x_1, x_i](\operatorname{ad} x_2)^{c-2},$$
$$\tau_i(x_i) = x_i + [x_1(\operatorname{ad} x_2)^{c-2}, x_i],$$
$$\sigma_i(x_k) = \tau_i(x_k), \quad k \neq i,$$

we shall find out $2(m-2)$ more linearly independent elements of degree $(1, c-2, 0, \ldots, 0)$ in $\tilde{G}$ and this will complete the proof. So, without loss of generality we assume $m = 3$.

Let $g, h \in GL_3, g(x_3) = x_1 + x_3, g(x_k) = x_k, k \neq 3, h(x_1) = x_1 + x_3, h(x_k) = x_k, k \neq 1$. Then

$$\begin{aligned}
&g \cdot \tilde{\pi}_1 = ([x_1, x_2, x_1](\operatorname{ad} x_2)^{c-3}, 0, -[x_1, x_2, x_1](\operatorname{ad} x_2)^{c-3}) \in \tilde{G},\\
&\tilde{\pi}_1 - g \cdot \tilde{\pi}_1 = (0, 0, [x_1, x_2, x_1](\operatorname{ad} x_2)^{c-3}) \in \tilde{G},\\
&h \cdot (\tilde{\pi}_1 - g \cdot \tilde{\pi}_1) = (-[x_1 + x_3, x_2, x_1 + x_3](\operatorname{ad} x_2)^{c-3}, 0,\\
&[x_1 + x_3, x_2, x_1 + x_3](\operatorname{ad} x_2)^{c-3}) \in \tilde{G}
\end{aligned}$$

and for the homogeneous component of degree $(1, c-2, 0)$ we get

$$\begin{aligned}
&\tilde{\theta}_1 = (-[x_1, x_2, x_1](\operatorname{ad} x_2)^{c-3}, 0,\\
&([x_1, x_2, x_3] + [x_3, x_2, x_1])(\operatorname{ad} x_2)^{c-3}) \in \tilde{G}.
\end{aligned}$$

Therefore

$$\tilde{\pi}_1 + \tilde{\theta}_1 = (0, 0, ([x_1, x_2, x_3] + [x_3, x_2, x_1])(\operatorname{ad} x_2)^{c-3}) \in \tilde{G}.$$

Applying the Jacobi identity and the anticommutative law we establish that

$$\tilde{\pi}_1 + \tilde{\theta}_1 = (0, 0, 2[x_1(\operatorname{ad} x_2)^{c-2}, x_3] - [x_1, x_3](\operatorname{ad} x_2)^{c-2}) \in \tilde{G}.$$

Now, for $g', h' \in GL_3, g'(x_3) = x_2 + x_3, g'(x_k) = x_k, k \neq 3, h'(x_2) = x_2 + x_3, h'(x_k) = x_k, k \neq 2$, we obtain

$$\begin{aligned}
&\tilde{\pi}_2 - g' \cdot \tilde{\pi}_2 = (0, 0, x_1(\operatorname{ad} x_2)^{c-1}) \in \tilde{G},\\
&h' \cdot (\tilde{\pi}_2 - g' \cdot \tilde{\pi}_2) = (0, -x_1(\operatorname{ad}(x_2 + x_3))^{c-1}, 0,\\
&x_1(\operatorname{ad}(x_2 + x_3))^{c-1}) \in \tilde{G}
\end{aligned}$$

and for the homogeneous component of degree $(1, c-2, 0)$ we get

$$\begin{aligned}
&\tilde{\theta}_2 = (0, -x_1(\operatorname{ad} x_2)^{c-1},\\
&(c-2)[x_1(\operatorname{ad} x_2)^{c-2}, x_3] + [x_1, x_3](\operatorname{ad} x_2)^{c-2}) \in \tilde{G}.
\end{aligned}$$

Therefore

$$\tilde{\pi}_2 + \tilde{\theta}_2 = (0, 0, (c-2)[x_1(\operatorname{ad} x_2)^{c-2}, x_3] + [x_1, x_3](\operatorname{ad} x_2)^{c-2}) \in \tilde{G}.$$

Since $\tilde{\pi}_1 + \tilde{\theta}_1$ and $\tilde{\pi}_2 + \tilde{\theta}_2$ are linearly independent, we can obtain $\tilde{\sigma}_3$ and $\tilde{\tau}_3$ as their linear combination. Hence σ_3 and τ_3 belong to G and this completes the proof of the proposition.

THEOREM 3.7. *Let $\mathfrak{N}_c \cap \mathfrak{U}^2$ be the variety of all metabelian and nilpotent of class $\leqq c$ Lie algebras over a field of characteristic* 0. *Then the group of automorphisms of the relatively free algebra $F_m(\mathfrak{N}_c \cap \mathfrak{U}^2), m \geqq 2$, is generated* *by the general linear group GL_m with its canonical action on the free generators $x_1, \ldots, x_m$ and by one more automorphism δ defined by*

$$\delta(x_1) = x_1 + [x_1, x_2], \quad \delta(x_k) = x_k, \quad k > 1.$$

Proof. The theorem follows immediately from Proposition 3.6 using an induction on c: let $\varphi \in \operatorname{Aut} F_m(\mathfrak{N}_c \cap \mathfrak{U}^2)$. By the inductive assumption, there exists an automorphism $\psi \in G$ such that ψ and φ induce the same automorphism on $F_m(\mathfrak{N}_{c-1} \cap \mathfrak{U}^2)$. Therefore $\varphi\psi^{-1} \in IA_c$. By Proposition 3.6 $IA_c \subset G$, hence φ also belongs to G and $G = \operatorname{Aut}(F_m(\mathfrak{N}_c \cap \mathfrak{U}^2)$.

4. Nilpotent algebras of large rank. In this section we shall study the automorphism group of the free nilpotent Lie algebra $F_m(\mathfrak{N}_c)$ when the rank m is at least c. Throughout this section we fix the integers m and c assuming that $m \geqq c \geqq 2$. All the considerations will be in the free nilpotent algebra $F = F_m(\mathfrak{N}_c)$ and in Aut F. Clearly, in this case $IA_{c+1} = \langle \mathrm{id} \rangle$ and in the notation of Section 2,

$$IA_c / IA_{c+1} = IA_c \cong (F^c)^{\oplus m}.$$

Besides, G is the subgroup of Aut F generated by GL_m and by the automorphism δ, defined by $\delta(x_1) = x_1 + [x_1, x_2], \delta(x_k) = x_k$ for $k > 1$. We shall establish that $G = \operatorname{Aut} F$.

PROPOSITION 4.1. *For $m \geqq c$,* Aut F *is generated by GL_m and by the automorphisms $\rho_s, s = 2, \ldots, c$, defined by*

$$\rho_s(x_1) = x_1 + [x_1, x_2, \ldots, x_s], \quad \rho_s(x_k) = x_k, \quad k > 1.$$

Proof. We make use of an induction on c bearing in mind that every automorphism of $F_m(\mathfrak{N}_{c-1})$ can be lifted to an automorphism of $F_m(\mathfrak{N}_c)$. The base of the induction $c = 1$ is trivial. In virtue of Lemma 3.1 it suffices to show that the GL_m-module IA_c is generated by the automorphism ρ_c. Equivalently, we have to establish that the GL_m-module $(F^c)^{\oplus m}$ (with the action of GL_m described in Section 2) coincides with its submodule N generated by the element $([x_1, x_2, \ldots, x_c], 0, \ldots, 0)$. Applying the Jacobi identity and the anticommutative law, every element of F^c can be expressed as a linear combination of left-normed commutators $[x_{i_1}, \ldots, x_{i_c}]$ such that $i_1 = \min\{i_1, \ldots, i_c\}$. In what follows we consider such commutators only. We shall prove the proposition in several steps.

Step 1. Denote by M the subspace of $(F^c)^{\oplus m}$ spanned by all elements $([x_1, x_{i_2}, \ldots, x_{i_c}], 0, \ldots, 0)$, where $\{i_2, \ldots, i_c\} \subset \{2, \ldots, m\}$. We consider the group GL_{m-1} as the subgroup of GL_m fixing x_1. Then GL_{m-1} acts on M in the same way as on the tensor power $(W_{m-1})^{\otimes c-1}$, where the vector space

W_{m-1} has a basis $x_2,\dots,x_m$. Since $m \geqq c$, the GL_{m-1}-module $(W_{m-1})^{\otimes c-1}$ is generated by $x_2 \otimes \cdots \otimes x_c$; similarly the GL_{m-1}-module M is generated by $([x_1,x_2,\dots,x_c],0,\dots,0)$, i.e., $M \subset N$.

Step 2. Let $g \in GL_m, g(x_1)=x_2, g(x_2)=x_1, g(x_k)=x_k, k>2$. Applying g to $([x_1,x_2,\dots,x_c],0,\dots,0)$ we obtain $(0,[x_2,x_1,x_3,\dots,x_c],0,\dots,0) \in N$ and by Step 1 we obtain also $(0,[x_2,x_{i_2},\dots,x_{i_c}],0,\dots,0) \in N$ when $i_2,\dots,i_c \neq 2$. In the same way we establish that $(0,\dots,0,[x_k,x_{i_2},\dots,x_{i_c}],0,\dots,0) \in N$, where the only non-zero coordinate is the k-th and $i_2,\dots,i_c \neq k$.

Step 3. Assume that $[x_{i_1},x_{i_2},\dots,x_{i_c}]$ does not depend on x_1. Then by Step 1 we obtain that

$$f=([x_1,x_{i_2},\dots,x_{i_c}],0,\dots,0) \in N.$$

Let $g \in GL_m, g(x_1)=x_1+x_{i_1}, g(x_k)=x_k, k>1$. Straightforward calculations show that

$$g\cdot f-f=([x_{i_1},x_{i_2},\dots,x_{i_c}],0,\dots,0) \in N.$$

Similarly, $(u_1,\dots,u_m) \in N$ when all commutators u_k are of length c and do not depend on x_k.

Step 4. Let

$$u=[x_1,\dots,x_{p-1},x_1,x_{p+1},\dots,x_{q-1},x_1,x_{q+1},\dots,x_c].$$

We illustrate by considering only the case $p=m=c=3$; the general case can be handled in a similar manner. Let $g \in GL_3, g(x_3)=x_1+x_3, g(x_k)=x_k, k \neq 3$. Then we have in N

$$\begin{aligned} g([x_1,x_2,x_3],0,0) &= ([x_1,x_2,x_1],0,0)+([x_1,x_2,x_3],0,0)\\ &\quad -(0,0,[x_1,x_2,x_3])-(0,0,[x_1,x_2,x_1]). \end{aligned}$$

By virtue of Steps 1 and 2 the second and the third summands belong to N because they are linear in x_1 and x_3. By Step 3 the fourth summand also belongs to N. Therefore the same holds for $([x_1,x_2,x_1],0,0)$. As a consequence, we obtain $(u,0,\dots,0) \in N$ for all commutators u which depend on x_1 and are linear in the other varibles.

Step 5. Let u be an arbitrary commutator of length c and let $\deg_{x_1} u>1$, i.e.,

$$u=[x_1,x_{i_2},\dots,x_{i_{p-1}},x_1,x_{i_{p+1}},\dots,x_{i_c}].$$

Since by Step 4 $([x_1,\dots,x_{p-1},x_1,x_{p+1},\dots,x_c],0,\dots,0) \in N$, as in Step 1 we obtain that $(u,0,\dots,0) \in N$. Hence we obtain that all the elements $(u_1,\dots,u_m)$ belong to N, u_k being commutators of length c, i.e., $(F_c)^{\oplus m}=N$. This completes the proof of the proposition.

PROPOSITION 4.2. *Let $m \geqq c$ and let $\sigma_s \in \operatorname{Aut} F$ be defined by*

$$\sigma_s(x_1) = x_1 + [x_1, \ldots, x_s, [x_{s+1}, \ldots, x_c]], \quad s = 2, 3, \ldots, c-2.$$

Let $\phi \in \operatorname{Aut} F$ be such that it induces identity automorphism modulo $F^c \cap F''$. Then ϕ belongs to the subgroup of $\operatorname{Aut} F$ generated by GL_m and $\sigma_2, \sigma_3, \ldots, \sigma_{c-2}$.

Proof. Every element of $F^c \cap F''$ is a linear combination of commutators

$$[x_{i_1}, \ldots, x_{i_s}, [x_{i_{s+1}}, \ldots, x_{i_c}]],$$

where

$$i_1 = \min\{i_1, \ldots, i_s\}, \quad i_{s+1} = \min\{i_{s+1}, \ldots, i_c\}, \quad s = 2, 3, \ldots, c-2.$$

Then the proof can be completed by repeating verbatim the arguments of Proposition 4.1.

We can now prove the following main result of this section.

THEOREM 4.3. *Let $m \geqq c \geqq 2$. Then the group of automorphisms of the free nilpotent Lie algebra $F_m(\mathfrak{N}_c)$ is generated by the general linear group GL_m with its canonical action on the free generators $x_1, \ldots, x_m$ and by one more automorphism δ defined by*

$$\delta(x_1) = x_1 + [x_1, x_2], \quad \delta(x_k) = x_k, \quad k > 1.$$

Proof. We use induction on c; the base of the induction $c = 2$ follows from Proposition 3.6. It suffices to establish that $IA_c \subset G = \langle GL_m, \delta \rangle$. By the inductive assumption there exist automorphisms $\theta, \pi \in G$ such that

$$\begin{aligned}
\theta(x_1) &= x_1 + [x_1, \ldots, x_{s+1}] + p_1, \\
\theta(x_k) &= x_k + p_k, k \neq 1, p_i \in F^c, i = 1, \ldots, m, \\
\pi(x_{s+1}) &= x_{s+1} + [x_{s+1}, \ldots, x_c] + q_{s+1}, \\
\pi(x_k) &= x_k + q_k, k \neq s+1, q_i \in F^c, i = 1, \ldots, m.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\theta\pi(x_1) &= x_1 + [x_1, \ldots, x_{s+1}] + p_1 + q_1, \\
\pi\theta(x_1) &= x_1 + [x_1, \ldots, x_{s+1}] + p_1 + q_1 + [x_1, \ldots, x_s, [x_{s+1}, \ldots, x_c]] \\
&= (\theta\pi)^{-1}(x_1 + [x_1, \ldots, x_s, [x_{s+1}, \ldots, x_c]])
\end{aligned}$$

and with $(\pi, \theta) = \pi^{-1}\theta^{-1}\pi\theta$,

$$(\pi, \theta)(x_1) = x_1 + [x_1, \ldots, x_s, [x_{s+1}, \ldots, x_c]], (\pi, \theta)(x_k) = x_k, \quad k \neq 1.$$

Therefore $(\pi, \theta) = \sigma_s \in G$.

Let $\phi \in IA_c$. By Proposition 3.6, there exists $\psi \in G$ such that ϕ and ψ induce the same automorphism modulo $F^c \cap F''$. Hence $\phi\psi^{-1}$ induces the identity automorphism modulo $F^c \cap F''$. In virtue of Proposition 4.2 $\phi\psi^{-1} \in G$, i.e., ϕ also belongs to G and

$$IA_c \subset G = \langle GL_m, \delta \rangle.$$

This completes the proof of the theorem.

As an immediate consequence of Theorem 4.3 we obtain the following assertion.

COROLLARY 4.4. *Let* $\mathfrak{N}$ *be a subvariety of* $\mathfrak{N}_c$ *and let* $m \geqq c \geqq 2$. *Then* $\operatorname{Aut} F_m(\mathfrak{N})$ *is generated by* GL_m *and by the automorphism* δ.

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