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UTILITY MAXIMIZATION IN A PURE JUMP MODEL WITH PARTIAL OBSERVATION

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This article considers the asset price movements in a financial market when risky asset prices are modeled by marked point processes. Their dynamics depend on an underlying event arrivals process—a marked point process having common jump times with the risky asset price process. The problem of utility maximization of terminal wealth is dealt with when the underlying event arrivals process is assumed to be unobserved by the market agents using, as the main tool, backward stochastic differential equations. The dual problem is studied. Explicit solutions in a particular case are given.

1. INTRODUCTION

In an incomplete market, the problem of hedging a future liability is studied using a utility maximization with exponential preferences. A rather simple model is considered over a finite time window $[0, T]$, with a single riskless money market account and a single risky asset. The price of the risk-free asset is taken equal to 1 (vit. the riskless interest rate has to be equal zero) and the liability to be hedged is assumed to be adapted to the filtration generated by the underlying price evolution. An analogous model has been considered by the same author in Gerardi and Tardelli [16,18].

The dynamics of the underlying asset price S is described by a pure jump process driven by two point processes, describing upward and downward jumps. Many authors (and, in particular, Zariphopoulou [30], for a continuous model) claim that it is sensible to assume that the price process dynamics depend on an exogenous process. According to this idea, the dynamics of the price process is assumed to depend on a pure jump process, X. Moreover, S and X may have common jump times and the exogenous process is unobservable by the market agents. Further details are provided in Section 2.

This article deals with a hedging problem in an incomplete market with partial observation, and the predictable projection of the processes involved in this model has to be found; see Section 3. Furthermore, the incompleteness of the market implies that for any given claim, a self-financing and perfect hedging strategy cannot be obtained. This means that perfect replication is not possible, and a hedging criterion under incompleteness has to be used.

In this article, a stochastic control approach is chosen since, as for many other authors (see, for instance, Kirch and Runggaldier [22]), the author of this article believes that this can be viewed as a rather general approach for problems with partial information (for models with continuous trajectories for the price process, see Mania and Santacroce [25] and references therein). Thus, the approach of minimizing the expectation of an utility function of the terminal wealth is followed, and an exponential utility function with risk aversion parameter $\alpha > 0$ is chosen.

This kind of problem was already discussed by the same author for a payoff $B = 0$ and choosing a suitable class of strategies. In that case, the classical dynamic programming approach leads to the Hamilton–Jacobi–Bellman equation for the value function. In Gerardi and Tardelli [18], the Hamilton–Jacobi–Bellman equation is written down. There, it is proven that it admits a solution for almost all t and an explicit representation of the solution is given. This result is an essential tool to provide an explicit representation for the minimal entropy martingale measure.

In Gerardi and Tardelli [17], taking into account a similar model, the stochastic factor X is supposed to be a nontradable asset, and the payoff is written on this stochastic factor. The optimal strategy is found with an explicit representation for the value function. Again, this result is reached by using the Hamilton–Jacobi–Bellman equation.

However, when the payoff is written on the price process, the Hamilton–Jacobi–Bellman approach does not allow us to find such explicit representations. Consequently, the same procedure cannot be followed. The alternative approach, as suggested by Lim and Quenez [24], is based on the Bellman principle, which studies directly the primal problem. In Sections 4 and 5, some properties of the value process are studied—in particular, it is proved that it is a solution to a suitable backward stochastic differential equation (BSDE), also making use of some results given in Kramkov and Schachermayer [23] and Schachermayer [28]. Note that these kinds of equations have been mainly used in the continuous setting (Becherer [1]; Hu, Imkeller, and Muller [20]). Existence and uniqueness of the solutions to such equations are discussed in Becherer [1], Carbone, Ferrario, and Santacroce [4], Ceci and Gerardi [7], and Morlais [26], and references therein.

The dual related problem is dealt with in Section 6, where the density of the minimal entropy martingale measure is determined. Finally a particular simplified model is studied in Section 7, assuming that the jump size of the price process depends just on the observations.

This article ends with an appendix on the filtering equation. This is used in Section 3 to obtain the predictable projections of the processes involved.

2. THE MODEL

On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions, a market model is considered with a single risky asset S and a nonrisky asset. The price of the risky asset, discounted with respect to the price of the bond, is a process S having the form

$$S_t = S_0 \exp\{Y_t\} \quad \text{with } S_0 \in \mathbb{R}^+.$$

The log-return price Y is assumed to be a nonexplosive $\mathbb{R}$-valued marked point process with initial condition $Y_0 = 0$. The dynamics of the log-return process depend on an exogenous process X, representing the amount of news reaching the market, which is unobservable by the agent. Let us assume that X is a nonexplosive marked point process, taking values in a finite set $\mathcal{X} \subset \mathbb{R}^+$, with initial condition $X_0 = 0$ and having nonnegative jump sizes.

To give the joint dynamics of the processes X and Y, the point process counting the jump times of Y up to time t is given as

$$N_t = \sum_{s\leq t} \mathbb{I}_{\{Y_s - Y_{s-} \neq 0\}}. \tag{2.1}$$

As in Centanni and Minozzo [8,9] and Gerardi and Tardelli [16,18], we assume that this process admits a $(P, \mathcal{F}_t)$-intensity λ_t, given by

$$\lambda_t = a(t) + bz_0 e^{-kt} + b\sum_{s\leq t}(X_s - X_{s-})e^{-k(t-s)}, \tag{2.2}$$

with b, k, and z_0 real positive parameters and $a(\cdot)$ a measurable $\mathbb{R}^+$-valued deterministic function, verifying

$$0 \leq a(t) \leq \overline{a} < +\infty.$$

Equation (2.2) has a natural and intuitive interpretation. The arrival of news reaching the market, represented by a positive jump size of X, produces a sudden increase in the trading activity. Successively, a progressive normalization of the market occurs, with a speed expressed by k. Finally, $a(\cdot)$ describes the activity of the market in the absence of random perturbations. By adequately choosing the function $a(\cdot)$, we would also be able to take into account deterministic features such as seasonalities.

The previous assumptions allow us to get that for Λ suitable positive constant and $\forall t$,

$$0 < \lambda_t \leq \bar{a} + bz_0 + bX_t < \Lambda < +\infty \tag{2.3}$$

since X_t is bounded by its definition. Let

$$Z_t := z_0 + \int_0^t e^{ks}\, dX_s$$

be a nonhomogeneous pure jump process, taking values in a suitable $\mathcal{Z} \subseteq \mathbb{R}^+$, having the same jump times of X and jump sizes given by

$$Z_t - Z_{t-} = e^{kt}(X_t - X_{t-}).$$

Hence,

$$\lambda_t = a(t) + be^{-kt} Z_t := \lambda(t, Z_t)$$

is a deterministic measurable function of the time t and of the process Z and the dynamics of the price process is assumed to depend just on the unobservable stochastic factor (X, Z).

Next, the dynamics of the processes X, Y, and Z is given by assuming the representation

$$\begin{aligned} X_t &:= \int_0^t \xi_u^0\, dN_u^0 + \int_0^t \xi_u^1\, dN_u^1 + \int_0^t \xi_u^2\, dN_u^2, \\ Y_t &:= \int_0^t \eta_u^1\, dN_u^1 - \int_0^t \eta_u^2\, dN_u^2, \\ Z_t &:= z_0 + \int_0^t e^{ku}\xi_u^0\, dN_u^0 + \int_0^t e^{ku}\xi_u^1\, dN_u^1 + \int_0^t e^{ku}\xi_u^2\, dN_u^2, \end{aligned} \tag{2.4}$$

where $\xi_t^k = \xi_k(t, X_{t-}, Z_{t-})$ and $\eta_t^i = \eta_i(t, X_{t-}, Z_{t-})$, and

- $\xi_k(t,x,z)$ and $\eta_i(t,x,z)$, for $k = 0, 1, 2$ and $i = 1, 2$, are measurable functions such that, for $\overline{\eta}$ and $\underline{\eta}$ real constants,

$$\begin{aligned} &\xi_k : [0,T] \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}^+ \cup \{0\}, \quad \forall x \in \mathcal{X}, \quad x + \xi_k(t,x,z) \in \mathcal{X}, \\ &\eta_i : [0,T] \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}^+, \qquad 0 < \underline{\eta} \leq \eta_i(t,x,z) \leq \overline{\eta}. \end{aligned} \tag{2.5}$$

- N^0, N^1, and N^2 are nonexplosive point processes defined as

$$\begin{aligned} N_t^0 &= \sum_{s\leq t} \mathbb{I}_{\{X_s - X_{s-} \neq 0\}}\ \mathbb{I}_{\{Y_s - Y_{s-} = 0\}}, \\ N_t^1 &= \sum_{s\leq t} \mathbb{I}_{\{Y_s - Y_{s-} > 0\}}, \qquad N_t^2 = \sum_{s\leq t} \mathbb{I}_{\{Y_s - Y_{s-} < 0\}}. \end{aligned}$$

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The process N^0 admits a $(P, \mathcal{F}_t)$-intensity given by $\lambda_t^0 := \lambda_0(t, X_{t-}, Z_{t-})$, with $\lambda_0(t, x, z)$ a bounded nonnegative measurable function, such that for the same Λ given in (2.3),

$$0 \leq \lambda_0(t, x, z) \leq \Lambda. \tag{2.6}$$

For $i = 1, 2$, N^i admits a $(P, \mathcal{F}_t)$-intensity $\lambda_t p_t^i$, where $\lambda_t := \lambda(t, Z_{t-})$ and $p_t^i := p_i(t, X_{t-}, Z_{t-})$, with $p_i(t, x, z)$ strictly positive measurable functions verifying the condition

$$p_1(t, x, z) + p_2(t, x, z) = 1.$$

This kind of dynamics allows us the possibility of common jump times between the latent process X and the log-return process Y, as well as the possibility of catastrophic events.

Remark 2.1: As already observed in Gerardi and Tardelli [18, Sect. 6], if the price process is strictly increasing or strictly decreasing, the model does not admit any equivalent martingale measure. The particular structure of the dynamics of the process Y is the simplest one allowing the existence of equivalent martingale measures (Bellini and Frittelli [2]).

Remark 2.2: By Theorem 7.3 in Ethier and Kurtz [12], (X, Y, Z) is a Markov process with cadlag trajectories, being the unique solution to a suitable Martingale Problem.

Fix a time window $[0, T]$, and by a little abuse of notations, let

$$\mathcal{F}_t := \sigma\{X_s, Y_s,\ 0 \leq s \leq t\}.$$

As a conclusion of this section, by a standard application of the Ito formula, the representation of the price process as a $\{P, \mathcal{F}_t\}$-local semimartingale is given by setting

$$c_u = c(u, X_{u-}, Z_{u-}) = (e^{\eta_u^1} - 1)p_u^1 + (e^{-\eta_u^2} - 1)p_u^2 \tag{2.7}$$

and

$$S_t = S_0 + \int_0^t S_u \lambda_u\, c_u\, du + M_t^S, \tag{2.8}$$

where M_t^S is a $(P, \mathcal{F}_t)$-local martingale represented as

$$M_t^S = \int_0^t S_{u-}(e^{\eta_u^1} - 1)[dN_u^1 - \lambda_u p_u^1\, du] + \int_0^t S_{u-}(e^{-\eta_u^2} - 1)[dN_u^2 - \lambda_u p_u^2\, du]. \tag{2.9}$$

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3. SETTING OF THE HEDGING PROBLEM. THE $\mathcal{F}_T^Y$-REPRESENTATION

There are many cases in which the investors acting in the market might not be able or might not want to use all available information, even if they have access to the full information and this section deals with this case.

The problem is to maximize the functional given in (4.1) in Section 4, over a suitable family Θ of strategies (see Definition 4.1). The asset price process is described as in Section 2 and the underlying event arrivals process X is assumed to be unobservable by the market agents.

First, the $\{P, \mathcal{F}_t^Y\}$-predictable projection of the processes involved in this model has to be found. The information available by the investors is $\mathcal{F}_t^Y := \sigma\{Y_s,\ 0 \le s \le t\} \subset \mathcal{F}_t$.

DEFINITION 3.1: *Given a process Γ_t, $\mathcal{F}_t$-adapted, let us denote by ${}^p\Gamma_t$ the predictable projection on $\mathcal{F}^Y$ and by ${}^o\Gamma_t$ the optional projection on $\mathcal{F}^Y$. For each τ, the $(P, \mathcal{F}_t^Y)$-predictable stopping time (Jacod [21, Thm. 1.23])*

$$ {}^p\Gamma_\tau \equiv \mathbb{E}[\Gamma_\tau | \mathcal{F}_{\tau-}^Y], $$

and for each τ, the $\mathcal{F}_t^Y$-stopping time, (Ethier and Kurtz [12, Optional Projection Theorem]) is

$$ {}^o\Gamma_\tau \equiv \mathbb{E}[\Gamma_\tau | \mathcal{F}_\tau^Y]. $$

In the continuous frame, the option projection and the predictable projection coincide. This is not the case for discontinuous models. The situation is described by the following lemma, the proof of which is along the lines of that given in Frey [13].

LEMMA 3.2: *The predictable projection and the optional projection are such that*

$$ {}^o\Gamma_{t-} = {}^p\Gamma_t. \tag{3.1} $$

PROOF: Since $\{\tau < +\infty\} \in \mathcal{F}_\tau^Y$, for each $\mathcal{F}_t^Y$-predictable stopping time τ,

$$ \mathbb{E}[{}^o\Gamma_\tau \mathbb{I}_{\tau<+\infty}] = \mathbb{E}[\Gamma_\tau \mathbb{I}_{\tau<+\infty}] = \mathbb{E}[{}^p\Gamma_\tau \mathbb{I}_{\tau<+\infty}]. $$

On the other hand, there exists a version of the process ${}^o\Gamma_t$ with cadlag trajectories and, consequently, ${}^o\Gamma_{t-}$ is a $(P, \mathcal{F}_t^Y)$-predictable process. More, the jump times of ${}^o\Gamma_{t-}$, which coincide with the jump times of the marked point process Y_t, are totally inaccessible.

Hence, for each $(P, \mathcal{F}_t^Y)$-predictable stopping time, ${}^o\Gamma_{\tau-} = {}^p\Gamma_\tau$ (see, for instance, Dellacherie-Meyer [11]) and

$$ \mathbb{E}[{}^o\Gamma_{\tau-} \mathbb{I}_{\tau<+\infty}] = \mathbb{E}[{}^p\Gamma_\tau \mathbb{I}_{\tau<+\infty}]. $$

In particular, a deterministic time $\tau \equiv t$ is a $(P, \mathcal{F}_t^Y)$-predictable stopping time and the thesis follows. ■

Some other preliminaries are given in order to prove, in Proposition 3.4, that the price process is a $\{P, \mathcal{F}_t^Y\}$-local semimartingale.

Let $m(dt, d\eta)$ denote the integer-valued random measure associated to Y_t (Bremaud [3] and Jacod [21]); for $t \in [0, T]$ and $\eta \in I = I^- \cup I^+ = [-\overline{\eta}, -\underline{\eta}] \cup [\underline{\eta}, \overline{\eta}]$,

$$\begin{aligned} m((0,t), d\eta) &= \sum_{s \le t} \mathbb{1}_{\{Y_s - Y_{s-} \neq 0\}}\, \delta_{\{s, Y_s - Y_{s-}\}}(ds, d\eta) \\ &= \sum_{s \le t} \delta_{\{Y_s - Y_{s-}\}}(d\eta)\, \Delta N_s \\ &= \sum_{s \le t} \delta_{\{\eta_s^1\}}(d\eta)\, \Delta N_s^1 + \sum_{s \le t} \delta_{\{-\eta_s^2\}}(d\eta)\, \Delta N_s^2. \end{aligned} \tag{3.2}$$

LEMMA 3.3: *Denoting by $\nu_t(d\eta)\, dt$ the $\{P, \mathcal{F}_t\}$-predictable projection and by $\widehat{\nu}_t(d\eta)\, dt$ the $\{P, \mathcal{F}_t^Y\}$-predictable projection of m, then*

$$\nu_t(d\eta) = \lambda_t \left(p_t^1\, \delta_{\{\eta_t^1\}}(d\eta) + p_t^2\, \delta_{\{-\eta_t^2\}}(d\eta)\right), \tag{3.3}$$

$$\widehat{\nu}_t(d\eta) = \pi_{t-} \left(\lambda \left(p^1\, \delta_{\{\eta^1\}}(d\eta) + p^2\, \delta_{\{-\eta^2\}}(d\eta)\right)\right), \tag{3.4}$$

where π_t is the probability measure-valued process that is the cadlag version of the conditional expectation; that is, for any bounded measurable $f(t, x, z)$,

$$\pi_t(f) = \mathbb{E}\big[f(t, X_t, Z_t) \big| \mathcal{F}_t^Y\big].$$

PROOF: As far as (3.3) is concerned, for any bounded positive $\{P, \mathcal{F}_t\}$-predictable $f(t, \eta)$, the process

$$\int_0^t \int_I f(s, \eta)\big(m(ds, d\eta) - \nu_s(d\eta)\, ds\big)$$

is a $\{P, \mathcal{F}_t\}$-martingale, recalling the definition of the $(P, \mathcal{F}_t)$-intensities of the processes N_t^1 and N_t^2.

Equation (3.4) is a consequence of Lemma 3.2, taking into account that if M_t is a $\{P, \mathcal{F}_t\}$-martingale, then $\pi_t(M) = \mathbb{E}[M_t | \mathcal{F}_t^Y]$ is a $\{P, \mathcal{F}_t^Y\}$-martingale. Hence, for any bounded positive $\{P, \mathcal{F}_t^Y\}$-predictable $\phi(t, \eta)$, the process

$$\mathbb{E}\left[\int_0^t \int_I \phi(s, \eta)\big(m(ds, d\eta) - \nu_s(d\eta)\, ds\big) \middle| \mathcal{F}_t^Y\right]$$

is a $\{P, \mathcal{F}_t^Y\}$-martingale and, as is well known,

$$\mathbb{E}\left[\int_0^t \int_I \phi(s, \eta)\, \nu_s(d\eta)\, ds \middle| \mathcal{F}_t^Y\right] - \int_0^t \int_I \mathbb{E}\left[\phi(s, \eta)\, \nu_s(d\eta) |\, \mathcal{F}_s^Y\right] ds$$

is a $\{P, \mathcal{F}_t^Y\}$-martingale. ■

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PROPOSITION 3.4: *The stock price process S_t admits the following representations as a $(P, \mathcal{F}_t^Y)$-local semimartingale:*

$$S_t = S_0 + \int_0^t S_{u-}\, \pi_{u-}(\lambda\, c)\, du + \overline{M}_t^S. \tag{3.5}$$

where c_u is defined in (2.7) and $\overline{M}_t^S$ is a $(P, \mathcal{F}_t^Y)$-local martingale represented as

$$\overline{M}_t^S = \int_0^t \int_I S_{u-}(e^\eta - 1) \Big(m(du, d\eta) - \widehat{\nu}_u(d\eta)\, du \Big). \tag{3.6}$$

PROOF: Taking into account that

$$S_t = S_0 + \int_0^t \int_I S_{u-}(e^\eta - 1) m(du, d\eta),$$

in order to obtain (3.5) and (3.6), recall the results of the previous lemma and

$$\int_0^t \int_I S_{u-}(e^\eta - 1)\, \widehat{\nu}_u(d\eta)\, du = \int_0^t S_{u-}\, \pi_{u-}(\lambda\, c)\, du. \qquad \blacksquare$$

The characterization of the filter will be given in the Appendix as the unique solution to the Kushner–stratonovich equation.

4. SETTING OF THE HEDGING PROBLEM. PRELIMINARIES

As a consequence of Proposition 3.4, from now on the price process is a totally observed marked point process, studied with respect to its internal filtration $\mathcal{F}_t^Y$.

In this article, the exponential utility function is $U(x) = 1 - e^{-\alpha x}$, with risk aversion parameter $\alpha > 0$, and the payoff B is a $\mathcal{F}_T^Y$-measurable random variable $B = B(Y_T)$, with $B(y)$ a bounded, nonnegative, measurable function $B(y) \le \overline{B}$.

DEFINITION 4.1: *The family of strategies Θ is given by the real-valued processes θ_t, which are $(P, \mathcal{F}_t^Y)$-predictable, S-integrable, and self-financing. Furthermore, $\forall t \in [0, T]$ and $\forall \theta \in \Theta$, the wealth process, defined as*

$$W_t^\theta = w + \int_0^t \theta_r\, dS_r = w + \int_0^t \int_I \theta_r S_{r-} \Big(e^\eta - 1\Big)\, m(dr, d\eta),$$

is assumed to be bounded from below.

This definition implies that $\forall t \in [0,T]$ and $\forall \theta \in \Theta$,

$$\mathbb{E}\Big[\exp\Big\{-\alpha\int_t^T \theta_r\, dS_r\Big\}\Big] < \infty.$$

Hence, the criterion to maximize is well defined and is given by

$$J^w(\theta) = \mathbb{E}\Big[1 - \exp\Big\{-\alpha\big(W_T - B(Y_T)\big)\Big\}\Big], \tag{4.1}$$

or, equivalently, the criterion to minimize is

$$J(\theta) = \mathbb{E}\Big[\exp\Big\{-\alpha\Big(\int_0^T \theta_r\, dS_r - B(Y_T)\Big)\Big\}\Big].$$

The value process is given by

$$\mathcal{V}_t(w) = 1 - e^{-\alpha w} V_t, \tag{4.2}$$

where

$$V_t = \operatorname*{ess\,inf}_{\theta\in\Theta_t} \mathbb{E}\Big[\exp\Big\{-\alpha\Big(\int_t^T \theta_r\, dS_r - B(Y_T)\Big)\Big\}\Big|\mathcal{F}_t^Y\Big] \tag{4.3}$$

and Θ_t denotes the set of the admissible strategies on the interval $[t,T]$.

The approach followed in this section is basically related to dynamic programming. This method will allow us to characterize the value function (more precisely, the process V_t defined in (4.3)) as the unique solution to a suitable BSDE (see (5.2) below).

The following properties of the process V_t are a slight modification of arguments discussed in Lim and Quenez [24, Sect. 3]. We recall them for the sake of completeness.

PROPOSITION 4.2: *The process V_t is strictly positive and bounded. Moreover, $V_T = e^{\alpha B(Y_T)}$.*

PROOF: Noting that the strategy $\theta \equiv 0$ belongs to Θ, then

$$V_t \le \mathbb{E}\Big[\exp\Big\{\alpha B(Y_T)\Big\}\Big|\mathcal{F}_t^Y\Big] \le e^{\alpha\overline{B}}.$$

Furthermore, by Theorem 2.2 given in Schachermayer [28], there exists an optimal strategy $\theta^* \in \Theta$. As a consequence,

$$e^{\alpha\overline{B}} \ge V_t = \mathbb{E}\Big[\exp\Big\{-\alpha\Big(\int_t^T \theta_r^*\, dS_r - B(Y_T)\Big)\Big\}\Big|\mathcal{F}_t^Y\Big] > 0. \tag{4.4}$$

■

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The next result gives a first characterization of the value process. In what follows, let $\Gamma_t(\theta)$ denote the wealth process defined for $w = 0$, that is,

$$\Gamma_t(\theta) = W_t^\theta - w = \int_0^t \theta_r \, dS_r = \int_0^t \int_I \theta_r S_{r-}\Big(e^\eta - 1\Big) \, m(dr, d\eta). \tag{4.5}$$

PROPOSITION 4.3: *The following hold true:*

(i) For any $\theta \in \Theta$, the process $V_t e^{-\alpha\Gamma_t(\theta)}$ is a $\{P, \mathcal{F}_t^Y\}$-submartingale.

(ii) V_t is the largest process $\mathcal{F}_t^Y$-adapted verifying (i) such that $V_T = e^{\alpha B(Y_T)}$.

(iii) The process $\theta^* \in \Theta$ is an optimal strategy iff the process $V_t e^{-\alpha\Gamma_t(\theta^*)}$ is a $\{P, \mathcal{F}_t^Y\}$-martingale.

PROOF: For any $\theta \in \Theta$, by Definition 4.1 and by Proposition 4.2,

$$\mathbb{E}[V_t e^{-\alpha\Gamma_t(\theta)}] < \infty.$$

For $s \le t \le T$, by (4.4),

$$\begin{aligned}
&\mathbb{E}\left[e^{-\alpha(\Gamma_t(\theta) - \Gamma_s(\theta))} V_t \middle| \mathcal{F}_s^Y\right] \\
&\quad = \mathbb{E}\left[\exp\left\{-\alpha\Big(\int_s^t \theta_r \, dS_r + \int_t^T \theta_r^* \, dS_r - B(Y_T)\Big)\right\}\middle| \mathcal{F}_s^Y\right] \\
&\quad = \mathbb{E}\left[\exp\left\{-\alpha\Big(\int_s^T \widetilde{\theta}_r \, dS_r - B(Y_T)\Big)\right\}\middle| \mathcal{F}_s^Y\right],
\end{aligned}$$

where $\widetilde{\theta}_u$ is a strategy belonging to Θ defined by gluing θ and θ^* at time t. Hence,

$$\mathbb{E}\left[e^{-\alpha(\Gamma_t(\theta) - \Gamma_s(\theta))} V_t | \mathcal{F}_s^Y\right] \ge V_s$$

and part (i) is proved.

Next, let us assume that $\widetilde{V}_t$ is another process $\mathcal{F}_t^Y$-adapted verifying part (i) and such that $\widetilde{V}_T = e^{\alpha B(Y_T)}$. Since $\widetilde{V}_t e^{-\alpha\Gamma_t(\theta)}$ is a $\{P, \mathcal{F}_t^Y\}$-submartingale, successively

$$\mathbb{E}\left[\widetilde{V}_T e^{-\alpha\Gamma_T(\theta)} \middle| \mathcal{F}_t^Y\right] \ge \widetilde{V}_t e^{-\alpha\Gamma_t(\theta)},$$

$$\mathbb{E}\left[e^{\alpha B(Y_T)} e^{-\alpha(\Gamma_T(\theta) - \Gamma_t(\theta))} \middle| \mathcal{F}_t^Y\right] \ge \widetilde{V}_t,$$

and finally, taking the infimum over Θ_t,

$$V_t = \operatorname*{ess\,inf}_{\theta \in \Theta} \mathbb{E}\left[e^{\alpha B(Y_T)} e^{-\alpha(\Gamma_T(\theta) - \Gamma_t(\theta))} \middle| \mathcal{F}_t^Y\right] \ge \widetilde{V}_t.$$

Regarding part (iii), if θ_t^* is an optimal strategy, then

$$V_0 = \mathbb{E}\left[\exp\left\{-\alpha\Big(\Gamma_T(\theta^*) - B(Y_T)\Big)\right\}\right].$$

By the submartingale property

$$\mathbb{E}\left[V_T e^{-\alpha\Gamma_T(\theta^*)} \middle| \mathcal{F}_t^Y\right] \ge V_t e^{-\alpha\Gamma_t(\theta^*)}$$

and taking the mean value of both sizes, we get the desired martingale property.

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Next, assuming that $V_t e^{-\alpha\Gamma_t(\theta^*)}$ is a $\{P, \mathcal{F}_t^Y\}$-martingale,

$$V_0 = \inf_{\theta\in\Theta} \mathbb{E}\left[\exp\left\{-\alpha\Big(\Gamma_T(\theta) - B(Y_T)\Big)\right\}\right] = \mathbb{E}\left[V_T e^{-\alpha\Gamma_T(\theta^*)}\right] \tag{4.6}$$

and θ_t^* is an optimal strategy. ■

5. INTRODUCING THE BSDE

As a consequence of Proposition 4.3, for a vanishing strategy, V_t is a bounded $\{P, \mathcal{F}_t^Y\}$-submartingale. Then its Doob–Meyer decomposition is

$$dV_t = dM_t^V + dA_t^V,$$

with M_t^V a square integrable martingale and A_t^V an increasing $\{P, \mathcal{F}_t^Y\}$-predictable process such that $A_0^V = 0$. Furthermore, a classical representation of M_t^V allows us to write

$$V_t = V_0 + \int_0^t \int_I R_r(\eta) \Big(m(dr, d\eta) - \widehat{\nu}(d\eta)\, dr\Big) + A_t^V, \tag{5.1}$$

where $R_t(\eta)$ is a η-indexed process $\{P, \mathcal{F}_t^Y\}$-predictable, jointly measurable and such that

$$\mathbb{E}\Big[\int_0^t \int_I R_r^2(\eta)\, \widehat{\nu}(d\eta)\, dr\Big] < +\infty.$$

The next theorem is the main result of this section.

THEOREM 5.1: *The process (V_t, R_t) verifies the BSDE*

$$\begin{aligned} V_t = e^{\alpha B(Y_T)} &+ \int_t^T \int_I R_r(\eta) \Big(m(dr, d\eta) - \widehat{\nu}(d\eta)dr\Big) \\ &- \int_t^T \operatorname*{ess\,inf}_{\theta\in\Theta_t} \int_I \Big(V_r + R_r(\eta)\Big)\Big(\exp\{-\alpha\theta_r S_r(e^\eta - 1)\} - 1\Big)\, \widehat{\nu}(d\eta)\, dr. \end{aligned} \tag{5.2}$$

Furthermore, V_t is the largest solution to (5.2) and R_t is uniquely determined by the martingale representation theorem.

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PROOF: By (4.5), the Ito formula provides

$$e^{-\alpha\Gamma_t(\theta)} = \int_0^t \int_I e^{-\alpha\Gamma_{r-}(\theta)} K_r^\theta(\eta)\, m(dr, d\eta),$$

where, for notational convenience,

$$K_r^\theta(\eta) = \exp\left\{-\alpha\theta_r S_{r-}\left(e^\eta - 1\right)\right\} - 1.$$

The product formula with (5.1) gives

$$\begin{aligned} &e^{-\alpha\Gamma_t(\theta)}\ V_t \\ &\quad = \int_0^t\int_I e^{-\alpha\Gamma_{r-}(\theta)}\Big[R_r(\eta) + \Big(V_{r-} + R_r(\eta)\Big)K_r^\theta(\eta)\Big]\Big(m(dr, d\eta) - \widehat{\nu}(d\eta)\, dr\Big) \\ &\qquad + \int_0^t \int_I e^{-\alpha\Gamma_r(\theta)}\ \Big(V_r + R_r(\eta)\Big)\ K_r^\theta(\eta)\,\widehat{\nu}(d\eta)\, dr + \int_0^t e^{-\alpha\Gamma_r(\theta)}\ dA_r^V. \end{aligned}$$

Since $e^{-\alpha\Gamma_t(\theta)}\ V_t$ is a $\{P, \mathcal{F}_t^Y\}$-submartingale, the bounded variation term has to be increasing for any strategy, which implies that

$$dA_r^V = \operatorname*{ess\,sup}_{\theta\in\Theta}\left\{-\int_I \Big(V_r + R_r(\eta)\Big)\ K_r^\theta(\eta)\,\widehat{\nu}(d\eta)\right\}.$$

Finally, the last assertion is a consequence of Proposition 4.3. ■

On the other hand, if the investor wants to maximize the expected utility of his terminal wealth without considering the claim, the criterion becomes

$$J_0^w(\theta) = \mathbb{E}\Big[1 - \exp\Big\{-\alpha\big(W_T\big)\Big\}\Big] \tag{5.3}$$

with value process

$$\begin{aligned} V_t^0(w) &= \operatorname*{ess\,sup}_{\theta\in\Theta_t} \mathbb{E}\Big[1 - \exp\Big\{-\alpha\Big(w + \int_t^T \theta_r\, dS_r\Big)\Big\}\Big|\mathcal{F}_t^Y\Big] \\ &= 1 - e^{-\alpha w} V_t^0. \end{aligned} \tag{5.4}$$

Again, Θ_t denotes the set of the admissible strategies on the interval $[t, T]$ and

$$V_t^0 = \operatorname*{ess\,inf}_{\theta\in\Theta_t} \mathbb{E}\Big[\exp\Big\{-\alpha\Big(\int_t^T \theta_r\, dS_r\Big)\Big\}\Big|\mathcal{F}_t^Y\Big]. \tag{5.5}$$

Results analogous to that discussed in Section 4 and a representation analogous to (5.1) can be obtained with the same procedure in this particular case. Finally, as a consequence of Theorem 5.1, we get the following corollary.

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COROLLARY 5.2: *The process (V_t^0, R_t^0) is characterized by the BSDE*

$$V_t^0 = 1 + \int_t^T \int_I R_r^0(\eta) \Big(m(dr, d\eta) - \widehat{\nu}(d\eta)\, dr \Big)$$
$$- \int_t^T \operatorname*{ess\,inf}_{\theta \in \Theta_t} \int_I \Big(V_r^0 + R_r^0(\eta) \Big) \Big(\exp\{-\alpha\theta_r S_r(e^\eta - 1)\} - 1 \Big) \widehat{\nu}(d\eta)\, dr. \quad \textbf{(5.6)}$$

Remark 5.3: Let us recall that the writer's indifference price is the price p of the claim such that the agent is indifferent between optimizing the expected utility with and without the derivative at hand (see, e.g., Becherer [1], Frittelli [15], Mania and Santacroce [25], and Rouge and El Karoui [27], and references therein).

This means that the agent is indifferent between optimizing the expected utility without the contingent claim or optimizing the expected utility including the payoff derivative $B(S_T)$ at time T with the compensation p_t. Thus, it is defined implicitly by

$$V_t^0(w) = V_t(w + p_t), \quad \textbf{(5.7)}$$

where $V_t^0(w)$ and $V_t(w)$ are defined in (5.4) and in (4.2), so that

$$p_t = \frac{1}{\alpha} \log \frac{V_t}{V_t^0}.$$

6. DUAL PROBLEMS

This section deals with the problem of determining a probability measure realizing the maximum in the relation

$$\sup_{Q \in \widehat{\mathcal{M}}_f} \Big(\mathbb{E}^Q[\alpha B(Y_T)] - H(Q|P) \Big),$$

where $\widehat{\mathcal{M}}_f$ denotes the set of martingale measures defined on $(\Omega, \mathcal{F}_T^Y, P)$ equivalent to P (EMM for short) with finite entropy, and $H(Q|P)$, the relative entropy of a probability measure Q w.r.t. P, is defined in (6.1). Here and in what follows, by a little abuse of notations, let P denote the restriction of the probability measure P on $\mathcal{F}_T^Y$. At first we have to show that the set $\widehat{\mathcal{M}}_f$ is not empty.

6.1. Equivalent Martingale Measures With Finite Entropy

Let $\mathcal{M}$ be the set of the the probability measures Q on $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0 \le t \le T}, P)$ under which the price process S_t is a $\{Q, \mathcal{F}_t\}$ local martingale, and for any $Q \in \mathcal{M}$, let $\widehat{Q}$ denote its restriction on $\mathcal{F}_T^Y$.

Next, as usual, the relative entropy of a probability measure Q on $(\Omega, \mathcal{F}_T, P)$ w.r.t. P is given by

$$H(Q|P) = \begin{cases} \mathbb{E}^P\Big[\frac{dQ}{dP}\ \log\Big(\frac{dQ}{dP}\Big)\Big], & Q \ll P \\ +\infty & \text{otherwise.} \end{cases} \tag{6.1}$$

Let $\widehat{H}(Q|P)$ denote the relative entropy of a probability measure Q on $(\Omega, \mathcal{F}_T^Y, P)$ w.r.t. P, defined by an analogous rule.

By the Girsanov theorem, any probability measure $\widetilde{P}$ equivalent to P on $\mathcal{F}_T^Y$ has a density

$$\frac{d\widetilde{P}}{dP}\bigg|_{\mathcal{F}_t^Y} = \exp\left\{\int_0^t \int_I \log\big(1 + \widetilde{A}_r(\eta)\big)\, m(dr, d\eta) - \int_0^t \int_I \widetilde{A}_r(\eta)\, \widehat{\nu}_r(d\eta)\, dr\right\}$$

with $\widetilde{A}_r(\eta)$ an η-indexed predictable process such that the right-hand side is well defined and is a 1-mean martingale.

Remark 6.1: Assuming that $\big(1 + \widetilde{A}_r(\eta)\big) > 0$, for any $\eta \in I$, setting

$$\widetilde{M}_t = \frac{d\widetilde{P}}{dP}\bigg|_{\mathcal{F}_t^Y},$$

the condition

$$\mathbb{E}\left[\int_0^T \int_I \big|\widetilde{A}_r(\eta)\big| \widehat{\nu}_r(d\eta)\, dr\right] < +\infty$$

implies that $\widetilde{M}_t$ is a strictly positive supermartingale and $\mathbb{E}[\widetilde{M}_t] \leq 1$. It is a 1-mean martingale if

$$\mathbb{E}\left[\int_0^t \int_I M_r\, \big|\widetilde{A}_r(\eta)\big|\, \widehat{\nu}_r(d\eta)\, dr\right] < +\infty.$$

The $\{\widetilde{P}, \mathcal{F}_t^Y\}$-predictable projection of m is given by $\widetilde{\nu}_t = \widehat{\nu}_t\ (1 + \widetilde{A}_t(\eta))$. Thus, a sufficient condition under which $\widetilde{P}$ is an EMM is

$$\int_0^t \int_I S_u(e^\eta - 1)\, \widehat{\nu}_u(d\eta)\, du + \int_0^t \int_I S_u(e^\eta - 1)\, \widetilde{A}_u(\eta)\, \widehat{\nu}_u(d\eta)\, du = 0. \tag{6.2}$$

As a particular case, let $\overline{P}$ be the probability measure obtained by choosing

$$\overline{A}_t(\eta) = -\frac{\pi_{t-}(\lambda c)}{(e^\eta - 1)\, \pi_{t-}(\lambda)}, \tag{6.3}$$

which verifies (6.2) since $\widehat{\nu}_u(I) = \pi_u\big(\nu(I)\big) = \pi_u(\lambda)$.

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PROPOSITION 6.2: *Under the assumption*

$$e^{-\underline{\eta}} - 1 \leq c \leq e^{\underline{\eta}} - 1, \tag{6.4}$$

a probability martingale measure $\overline{P}$ equivalent to P with finite entropy w.r.t. P is defined by

$$\left.\frac{d\overline{P}}{dP}\right|_{\mathcal{F}_t^Y} = \exp\left\{\int_0^t \int_I \log\left(1 - \frac{\pi_{r-}(\lambda c)}{(e^{\eta}-1)\,\pi_{r-}(\lambda)}\right) m(dr, d\eta) + \int_0^t \int_I \frac{\pi_{r-}(\lambda c)}{(e^{\eta}-1)\,\pi_{r-}(\lambda)}\, \widehat{\nu}_r(d\eta)\, dr\right\}. \tag{6.5}$$

PROOF: By observing that the process $\overline{A}_t(\eta)$ is bounded and verifies the inequality $1 + \overline{A}_t(\eta) > 0$ a.s., the first claim is achieved.

Under the assumptions of this article, there exists a positive constant C, depending on $\underline{\eta}$ and $\overline{\eta}$, such that successively $|c| \leq e^{\overline{\eta}}$ and

$$|\overline{A}_t(\eta)| = \left|-\frac{\pi_{t-}(\lambda c)}{(e^{\eta}-1)\,\pi_{t-}(\lambda)}\right| \leq \frac{e^{\overline{\eta}}\pi_{t-}(\lambda)}{|e^{\eta}-1|\,\pi_{t-}(\lambda)} \leq \frac{e^{\overline{\eta}}}{1-e^{-\underline{\eta}}} < C(\underline{\eta}, \overline{\eta}),$$

$$\int_I \overline{A}_r(\eta)\widehat{\nu}_r(d\eta) \leq \int_I \bar{|}A_r(\eta)|\widehat{\nu}_r(d\eta) = C(\underline{\eta}, \overline{\eta})\pi_r(\lambda) \leq C(\underline{\eta}, \overline{\eta})\Lambda.$$

Therefore,

$$\log\frac{d\overline{P}}{dP} \leq \int_0^T \int_I \left|\overline{A}_t(\eta)\right| m(dr, d\eta) + \int_0^T \int_I \left|\overline{A}_t(\eta)\right| \widehat{\nu}_r(d\eta)\, dr \leq C(\underline{\eta}, \overline{\eta})(N_T + \Lambda T),$$

which implies

$$\widehat{H}(\overline{P}|P) = \mathbb{E}^{\overline{P}}\left[\log\frac{d\overline{P}}{dP}\right] \leq C(\underline{\eta}, \overline{\eta})\,(\mathbb{E}^{\overline{P}}[N_T] + \Lambda T) < +\infty.$$

■

Another method for finding the EMM with finite entropy, according to Remark 4.8 in Ceci and Gerardi [6], is given by the following lemma.

LEMMA 6.3: *Let Q be a probability measure on $(\Omega, \mathcal{F}_T, P)$. If $Q \in \mathcal{M}$, then $\widehat{Q} \in \widehat{\mathcal{M}}$, and if $H(Q|P) < \infty$, then $\widehat{H}(\widehat{Q}|P) < \infty$.*

Thus, we find an EMM on $(\Omega, \mathcal{F}, P)$ with finite entropy, recalling Lemma 4.1 in Ceci and Gerardi [7].

Proposition 6.4: *An EMM on $(\Omega, \mathcal{F}, P)$ with finite entropy is given by the probability measure $\overline{Q}$ defined by the density*

$$\left.\frac{d\overline{Q}}{dP}\right|_{\mathcal{F}_t} = \prod_{i=1,2} \exp\left\{\int_0^t \log(1+\overline{A}_r^i)\, dN_r^i - \int_0^t \lambda_r p_r^i \overline{A}_r^i\, dr\right\},$$

where

$$1+\overline{A}_r^1 = \left[\frac{p_s^1\left(e^{\eta_s^1}-1\right)}{p_s^2\left(1-e^{-\eta_s^2}\right)}\right]^{(1-e^{\eta_s^1})/(e^{\eta_s^1}-e^{-\eta_s^2})},$$

$$1+\overline{A}_r^2 = \left[\frac{p_s^1\left(e^{\eta_s^1}-1\right)}{p_s^2\left(1-e^{-\eta_s^2}\right)}\right]^{(1-e^{-\eta_s^2})/(e^{\eta_s^1}-e^{-\eta_s^2})}.$$

6.2. Duality

The link between the primal problem and the dual one is given by the Duality Principle (see Delbaen, Grandits, Rheinlander, Samperi, Schweizer, and Stricker [10] and Bellini and Frittelli [2])

$$\inf_{\theta\in\widetilde{\Theta}} \mathbb{E}\left[\exp\left\{-\alpha\left(\int_0^T \theta_r\, dS_r - B(Y_T)\right)\right\}\right] = \exp\left\{\sup_{Q\in\widehat{\mathcal{M}}_f}\left(\mathbb{E}^Q[\alpha B(Y_T)] - \widehat{H}(Q|P)\right)\right\} \quad \textbf{(6.6)}$$

for $\widetilde{\Theta}$, suitable family of strategies. Relation (6.6) is robust for different choices of the set $\widetilde{\Theta}$. Chose $\widetilde{\Theta} = \Theta$, as given in Section 4; this choice implies that the left-hand side of (6.6) has the value V_0.

The aim is to find the solution to the dual problem and to prove that the Duality Principle holds true.

The next theorem provides the main result of this section. Let us note that the assumption of B bounded can be weakened. See Delbaen et al. [10] for further details.

Theorem 6.5: *Assume $B(y)$ bounded. The probability measure Q_B (see (4.6)), defined as*

$$\frac{dQ_B}{dP} = \frac{1}{V_0} e^{-\alpha\left(\int_0^T \theta_r^*\, dS_r - B(Y_T)\right)} = \frac{\exp\left\{-\alpha\left(\int_0^T \theta_r^*\, dS_r - B(Y_T)\right)\right\}}{\mathbb{E}\left[\exp\left\{-\alpha\left(\int_0^T \theta_r^*\, dS_r - B(Y_T)\right)\right\}\right]},$$

is the solution to the dual problem, where θ^* is the optimal strategy of the primal problem. Furthermore, the Duality Principle holds.

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PROOF: Following a method suggested in Delbaen et al. [10], on $(\Omega, \mathcal{F}_T^Y, P)$ a change of probability measure is introduced by setting

$$\frac{dP_B}{dP} = \frac{e^{\alpha B(Y_T)}}{\mathbb{E}[e^{\alpha B(Y_T)}]}.$$

By a direct computation, for any Q equivalent to P,

$$\widehat{H}(Q|P) = \widehat{H}(Q|P_B) + \mathbb{E}^Q[\alpha B(Y_T)] - \log \mathbb{E}[e^{\alpha B(Y_T)}];$$

thus, the set $\widehat{\mathcal{M}}_f$ is the same under P or under P_B.

The Duality Principle becomes

$$\inf_{\theta\in\Theta} \mathbb{E}^{P_B}\Big[\exp\Big\{-\int_0^T \alpha\theta_r\, dS_r\Big\}\Big] = \exp\Big\{-\inf_{Q\in\widehat{\mathcal{M}}_f} \widehat{H}(Q|P_B)\Big\} \tag{6.7}$$

and the infimum in the right-hand side is attained by the minimal entropy martingale measure (MEMM) Q_B under P_B. Existence and uniqueness of the MEMM is guaranteed, since S is locally bounded and there exists a martingale measure with finite entropy (Frittelli [14] and Grandits and Rheinlander [19]). In this frame there exists a predictable process θ_r^B such that

$$\widehat{H}(Q_B|P_B) = -\log \mathbb{E}^{P_B}\left[\exp\left(\int_0^T \theta_r^B\, dS_r\right)\right]$$

and the infimum in the left-hand side of (6.7) is attained choosing the same predictable process $\theta^B = -\alpha\theta^*$, according to the discussion in Frittelli [14]. Thus,

$$\widehat{H}(Q_B|P_B) = -\log \mathbb{E}^{P_B}\left[\exp\left(\int_0^T -\alpha\theta_r^*\, dS_r\right)\right]$$

and (6.7) follows, which, in turn, implies that (6.6) holds true.

Moreover (Frittelli [14] and Grandits and Rheinlander [19]),

$$\frac{dQ_B}{dP_B} = \frac{\exp\Big\{-\alpha\int_0^T \theta_t^*\, dS_t\Big\}}{\mathbb{E}^{P_B}\Big[\exp\Big\{-\alpha\int_0^T \theta_t^*\, dS_t\Big\}\Big]}$$

and

$$\mathbb{E}^{P_B}\Big[e^{-\alpha\int_0^T \theta_t^*\, dS_t}\Big] = \frac{\mathbb{E}\Big[\exp\Big\{-\alpha\Big(\int_0^T \theta_t^*\, dS_t - B(Y_T)\Big)\Big\}\Big]}{\mathbb{E}[e^{\alpha B(Y_T)}]} = \frac{V_0}{\mathbb{E}[e^{\alpha B(Y_T)}]}.$$

■

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The solution to the dual problem of the utility maximization discussed in Corollary 5.2 is achieved by writing down the density of the MEMM $\widehat{Q}^*$ under P as a particular case for $B(y) \equiv 0$ and $\alpha = 1$. Actually, in this case, the Duality Principle becomes

$$\inf_{\theta\in\Theta} \mathbb{E}\Big[\exp\Big\{-\int_0^T \theta_r\, dS_r\Big\}\Big] = \exp\Big\{-\inf_{Q\in\widehat{\mathcal{M}}_f} \widehat{H}(Q|P)\Big\} \tag{6.8}$$

and the density of the MEMM is

$$\frac{d\widehat{Q}^*}{dP} = \frac{1}{V_0^0}\exp\Big\{-\int_0^T \widehat{\theta}_r^0\, dS_r\Big\},$$

where $\widehat{\theta}^0$ is the optimal strategy of the primal problem.

7. A PARTICULAR MODEL

In this section there is a modification of the model described in Section 2, a modification that strongly simplifies the problem, allowing us to apply a procedure that is along the lines of Becherer [1], Ceci and Gerardi [7], Hu et al. [20], and Mania and Santacroce [25]. This procedure did not seem to be appropriate for the general model– studied in the previous sections — but it allows us to find more explicit expressions of the quantities of interest.

The main assumption made in this section is that the jump sizes of the price process are $\mathcal{F}_t^Y$-adapted; that is, for $i = 1, 2$,

$$\eta_t^i = \eta^i(t, Y_{t-}), \tag{7.1}$$

with $\eta(t, y)$ real-valued measurable functions verifying the inequality given in (2.5):

$$0 < \underline{\eta} \leq \eta_i(t, y) \leq \overline{\eta}$$

for $\overline{\eta}$ and $\underline{\eta}$, being real constants. In this case, the processes $\widehat{\nu}_t$ and $\overline{M}_t^S$ given in (3.4) and (3.6) become

$$\widehat{\nu}_t(d\eta) = \pi_{t-}\left(\lambda p^1\right)\delta_{\{\eta_t^1\}}(d\eta) + \pi_{t-}\left(\lambda p^2\right)\delta_{\{-\eta_t^2\}}(d\eta)$$

and

$$\overline{M}_t^S = \int_0^t S_{u-}(e^{\eta_u^1} - 1)\, dN_u^1 + \int_0^t S_{u-}(e^{-\eta_u^2} - 1)\, dN_u^2 - \int_0^t S_{u-}\, \pi_{u-}(\lambda\, c)\, du.$$

since

$$S_t = S_0 + \int_0^t S_{u-}(e^{\eta_u^1} - 1)\, dN_u^1 + \int_0^t S_{u-}(e^{-\eta_u^2} - 1)\, dN_u^2.$$

Note that under (7.1) the model becomes formally strictly similar to that studied in Ceci and Gerardi [7], even if in that work a full information case is considered. Lemma 7.1 and Theorem 7.2 are analogous to Proposition 3.1 and Theorem 3.6 in Ceci and Gerardi [7].

In this frame, the set Θ of admissible strategies is given by the real-valued processes, $(P, \mathcal{F}_t^Y)$-predictable, S-integrable, self-financing and taking values in a compact set. Moreover, for $i = 1, 2$, let $p_t^i \geq \underline{p} > 0$.

In what follows, for notational convenience, set

$$\pi_{s-}(\lambda) = \widehat{\lambda}_s, \qquad \pi_{s-}(\lambda p^1) = \widehat{\lambda}_s^1, \qquad \pi_{s-}(\lambda p^2) = \widehat{\lambda}_s^2, \quad \text{and} \quad \overline{\Gamma}(\theta) = e^{-\alpha \Gamma_t(\theta)}$$

for $\Gamma_t(\theta)$ defined in (4.5).

LEMMA 7.1: *Assume that there exists for any $\theta \in \Theta$, a $\{P, \mathcal{F}_t^Y\}$-submartingale $\Phi_t(\theta)$ verifying the following:*

(i) $\Phi_T(\theta) = \overline{\Gamma}_T(\theta)\, e^{\alpha B(Y_T)}$,

(ii) $\exists \theta^* \in \Theta$ *such that* $\Phi_t(\theta^*)$ *is a* $\{P, \mathcal{F}_t^Y\}$*-martingale.*

Then

$$V_t = \frac{\Phi_t(\theta^*)}{\overline{\Gamma}_t(\theta^*)} = \Phi_t(\theta^*)\, e^{\alpha \Gamma_t(\theta^*)} \tag{7.2}$$

and θ^* is the optimal control.

PROOF: For any $\theta \in \Theta$, the inequality

$$\frac{\Phi_t(\theta)}{\overline{\Gamma}_t(\theta)} \leq \frac{\mathbb{E}[\Phi_T(\theta) \big| \mathcal{F}_t^Y]}{\overline{\Gamma}_t(\theta)} = \mathbb{E}\Big[\frac{\overline{\Gamma}_T(\theta)\, e^{\alpha B(Y_T)}}{\overline{\Gamma}_t(\theta)} \Big| \mathcal{F}_t^Y\Big] \quad \text{implies that } \frac{\Phi_t(\theta)}{\overline{\Gamma}_t(\theta)} \leq V_t.$$

For $\theta = \theta^*$,

$$\frac{\Phi_t(\theta^*)}{\overline{\Gamma}_t(\theta^*)} = \frac{\mathbb{E}[\Phi_T(\theta^*) \big| \mathcal{F}_t^Y]}{\overline{\Gamma}_t(\theta^*)} \geq \operatorname*{ess\,inf}_{\theta \in \Theta} E\Big[\frac{\overline{\Gamma}_T(\theta)\, e^{\alpha B(Y_T)}}{\overline{\Gamma}_t(\theta)} \Big| \mathcal{F}_t^Y\Big] = V_t.$$

■

THEOREM 7.2: *Let us consider the BSDE*

$$\xi_t = B(Y_T) - \int_t^T \zeta_s^1\, dN_s^1 + \int_t^T \zeta_s^2\, dN_s^2 - \int_t^T \varphi_s(\zeta_s^1, \zeta_s^2)\, ds, \tag{7.3}$$

with

$$\varphi_s(\zeta_s^1, \zeta_s^2) = \frac{\widehat{\lambda}_s}{\alpha} - \frac{\widehat{\lambda}_s^2 \left(e^{\eta_s^1} - e^{-\eta_s^2}\right)}{\alpha \left(e^{\eta_s^1} - 1\right)} \left(\frac{\widehat{\lambda}_s^1 \, (e^{\eta_s^1} - 1)}{\widehat{\lambda}_s^2 \, (1 - e^{-\eta_s^2})}\right)^{(1-e^{-\eta_s^2})/(e^{\eta_s^1} - e^{-\eta_s^2})}$$
$$\times \cdot \exp\left\{\alpha \frac{\zeta_s^1 \left(1 - e^{-\eta_s^2}\right) - \zeta_s^2 \left(e^{\eta_s^1} - 1\right)}{\left(e^{\eta_s^1} - e^{-\eta_s^2}\right)}\right\}. \tag{7.4}$$

(i) Equation (7.3) admits a unique bounded solution $\xi_t, \zeta_t^1, \zeta_t^2$, where ξ_t is $\mathcal{F}_t^Y$-adapted, ζ_t^1andζ_t^2 are $\{P, \mathcal{F}_t^Y\}$-predictable.

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(ii) *The optimal strategy is*

$$\theta_r^* = \frac{1}{S_{r-}\left(e^{\eta_r^1} - e^{-\eta_r^2}\right)}\left(\zeta_r^1 + \zeta_r^2 + \frac{1}{\alpha}\log\frac{\widehat{\lambda}_r^1\left(e^{\eta_r^1} - 1\right)}{\widehat{\lambda}_r^2\left(1 - e^{-\eta_r^2}\right)}\right).$$

(iii) *The value process is given by* $V_t(w) = 1 - e^{\alpha\xi_t - \alpha w}$.

Proof: The first claim is a direct consequence of Theorem 3.6 in Ceci and Gerardi [7] and can be also deduced by the results given in Morlais [26].

Next, for any $\theta \in \Theta$, we are going to define the structure of the processes $\Phi_t(\theta)$ by setting

$$\Phi_t(\theta) = e^{\alpha\xi_t}\overline{\Gamma}_t(\theta).$$

Noting that (7.3) is equivalent to

$$\xi_t = \xi_0 + \int_0^t \zeta_s^1\, dN_s^1 - \int_0^t \zeta_s^2\, dN_s^2 + \int_0^t \varphi_s(\zeta_s^1, \zeta_s^2)\, ds,$$

then

$$\Phi_t(\theta) = e^{\alpha\xi_0}\exp\left\{\alpha\int_0^t \varphi_r(\zeta_r^1, \zeta_r^2)\, dr\right\} \times \exp\left\{\alpha\sum_{i=1,2}\int_0^t\left[(-1)^{i+1}\zeta_r^i - \theta_r S_{r-}(e^{(-1)^{i+1}\eta_r^i} - 1)\right]dN_r^i\right\}.$$

On the other hand, if M_t^θ is a $\{P, \mathcal{F}_t^Y\}$-martingale and D_t^θ is a nondecreasing process, the product $M_t^\theta D_t^\theta$ is a $\{P, \mathcal{F}_t^Y\}$-submartingale. Thus, by setting

$$M_t(\theta) = e^{\alpha\xi_0}\prod_{i=1,2}\exp\left\{\int_0^t \log(1 + A_r^i(\theta))\, dN_r^i - \int_0^t \widehat{\lambda}_r^i A_r^i(\theta)\, dr\right\},$$

$$D_t(\theta) = \exp\left\{\int_0^t \beta_r(\theta)\, dr\right\}$$

and choosing

$$A_r^1(\theta) = \exp\left\{\alpha\left(\zeta_r^1 - \theta_r S_{r-}(e^{\eta_r^1} - 1)\right)\right\} - 1,$$

$$A_r^2(\theta) = \exp\left\{-\alpha\left(\zeta_r^2 - \theta_r S_{r-}(1 - e^{-\eta_r^2})\right)\right\} - 1,$$

$M_t(\theta)$ is a $\{P, \mathcal{F}_t^Y\}$-martingale. Furthermore, setting

$$\beta_r(\theta) = \alpha\varphi_s(\zeta_s^1, \zeta_s^2) + \left(\widehat{\lambda}_r^1 A_r^1(\theta) + \widehat{\lambda}_r^2 A_r^2(\theta)\right),$$

then $\beta_r(\theta) \geq 0$ for any $\theta \in \Theta$, whereas $\beta_r(\theta^*) = 0$.

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Finally, by direct computations, $\Phi_r(\theta) = M_t(\theta)D_t(\theta)$ so that it is a $\{P, \mathcal{F}_t^Y\}$-submartingale. Hence, the thesis is a consequence of Lemma 7.1. ∎

When $B = 0$, we have the following proposition.

PROPOSITION 7.3: *Let us consider the BSDE*

$$\xi_t^0 = -\int_t^T \zeta_s^{0,1}\, dN_s^1 + \int_t^T \zeta_s^{0,2}\, dN_s^2 - \int_t^T \varphi_s^0\, ds, \tag{7.5}$$

with

$$\varphi_s^0(\zeta_s^{0,1}, \zeta_s^{0,2}) = \frac{\widehat{\lambda}_s}{\alpha} - \frac{\widehat{\lambda}_s^2\left(e^{\eta_s^1} - e^{-\eta_s^2}\right)}{\alpha\left(e^{\eta_s^1} - 1\right)} \left(\frac{\widehat{\lambda}_s^1(e^{\eta_r^1} - 1)}{\widehat{\lambda}_s^2(1 - e^{-\eta_r^2})}\right)^{(1-e^{-\eta_r^2})/(e^{\eta_r^1} - e^{-\eta_r^2})}$$
$$\times \exp\left\{\alpha \frac{\zeta_s^{0,1}\left(1 - e^{-\eta_s^1}\right) - \zeta_s^{0,2}\left(e^{\eta_s^1} - 1\right)}{\left(e^{\eta_s^1} - e^{-\eta_s^2}\right)}\right\}. \tag{7.6}$$

(i) Equation (7.5) admits a unique bounded solution $\xi_t^0, \zeta_t^{0,1}, \zeta_t^{0,2}$, where ξ_t^0 is $\mathcal{F}_t^Y$-adapted, $\zeta_t^{0,1}$ and $\zeta_t^{0,2}$ are $\{P, \mathcal{F}_t^Y\}$-predictable.

(ii) The optimal strategy is

$$\theta_r^{*,0} = \frac{1}{S_{r-}\left(e^{\eta_r^1} - e^{-\eta_r^2}\right)} \left(\zeta_r^{0,1} + \zeta_r^{0,2} + \frac{1}{\alpha} \log \frac{\widehat{\lambda}_r^1\left(e^{\eta_r^1} - 1\right)}{\widehat{\lambda}_r^2\left(1 - e^{-\eta_r^2}\right)}\right).$$

(iii) The value process is given by

$$V_t^0(w) = 1 - e^{\alpha \xi_t^0 - \alpha w}.$$

The solution to the dual problem, in this frame, can be proved to be the probability measure $\widehat{Q}$ defined by the density

$$\left.\frac{d\widehat{Q}}{dP}\right|_{\mathcal{F}_t^Y} = \prod_{i=1,2} \exp\left\{\int_0^t \log(1 + \widehat{A}_r^i)\, dN_r^i - \int_0^t \widehat{\lambda}_r^i \widehat{A}_r^i\, dr\right\},$$

where

$$1 + \widehat{A}_r^1 = \exp\left\{-\alpha\left(\theta_r^*\, S_{r-}\left(e^{\eta_r^1} - 1\right) - \zeta_r^{0,1}\right)\right\},$$

$$1 + \widehat{A}_r^2 = \exp\left\{-\alpha\left(\theta_r^*\, S_{r-}\left(e^{-\eta_r^2} - 1\right) + \zeta_r^{0,2}\right)\right\}.$$

The proof is similar to that of Theorem 6.5 or Theorem 4.3 in Ceci and Gerardi [7] and will be omitted, even if an alternative expression of the density of $\widehat{Q}$ is given observing that for $t = T$,

$$\frac{d\widehat{Q}}{dP} = e^{-\alpha\xi_0}\, M_T(\theta_T^*) = e^{-\alpha\xi_0}\, \Phi_T(\theta_T^*)$$

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and

$$\frac{d\widehat{Q}}{dP} = e^{-\alpha\left(\xi_0 - B(Y_T)\right)} \exp\left\{ -\alpha \int_0^T \theta_r^* \, S_{r-} \, dS_r \right\}. \qquad \textbf{(7.7)}$$

As a consequence, the density of the MEMM, obtained setting $B = 0$ and $\alpha = 1$, is

$$\frac{d\widehat{Q}^*}{dP} = e^{-\alpha \, \xi_0^1} \exp\left\{ -\alpha \int_0^T \theta_r^{*,1} \, S_{r-} \, dS_r \right\}, \qquad \textbf{(7.8)}$$

where $\theta^{*,1} = \theta^{*,0}\big|_{\alpha=1}$; that is,

$$\theta^{*,1} = \frac{1}{S_{r-}\left(e^{\eta_r^1} - e^{-\eta_r^2}\right)} \left(\zeta_r^{1,1} + \zeta_r^{1,2} + \log \frac{\widehat{\lambda}_r^1 \left(e^{\eta_r^1} - 1\right)}{\widehat{\lambda}_r^2 \left(1 - e^{-\eta_r^2}\right)} \right)$$

being the $(\xi^1, \zeta^{1,1}, \zeta^{1,2})$ solution of (7.5) written for $\alpha = 1$.

Finally, by (5.7), the indifference price is given by $p_t = \xi_t - \xi_t^0$.

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APPENDIX

Filtering

The filter π_t is the probability measure-valued process that is the cadlag version of the conditional expectation; that is, for any bounded measurable $f(t,x,z)$,

$$\pi_t(f) = \mathbb{E}\big[f(t,X_t,Z_t)\big|\mathcal{F}_t^Y\big].$$

In this section we will write down the filtering equation, obtained by the innovation method (Bremaud [3]) and we will discuss its uniqueness property.

Under the assumptions made on the model in Section 2, for $F(t,x,y,z)$ belonging to a suitable class of real-valued measurable functions, $t \geq 0$, $x \in \mathcal{X}$, $y \in \mathbb{R}$, and $z \in \mathcal{Z}$, the dynamics of the process (X,Y,Z) is described by the operator

$$\mathcal{L}F(t,x,y,z) = \frac{\partial}{\partial t}F(t,x,y,z) + \mathcal{L}_tF(t,x,y,z), \tag{A.1}$$

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where

$$\mathcal{L}_t F(t,x,y,z) = \mathcal{L}_t^0 F(t,x,y,z) + \lambda(t,z) \sum_{i=1,2} p_i(t,x,z) \mathcal{L}_t^i F(t,x,y,z),$$

$$\mathcal{L}_t^0 F(t,x,y,z) = \lambda_0(t,x,z) \left[F(t, x + \xi^0(t,x,z), y, z + e^{kt}\xi^0(t,x,z)) - F(t,x,y,z) \right],$$

$$\mathcal{L}_t^i F(t,x,y,z) = F(t, x + \xi^i(t,x,z), y + (-1)^{i-1}\eta_i(t,x,z), z + e^{kt}\xi^i(t,x,z)) - F(t,x,y,z).$$

This means that for a bounded real-valued measurable function F,

$$F(t, X_t, Y_t, Z_t) - F(0, X_0, Y_0, Z_0) - \int_0^t \mathcal{L}F(s, X_{s-}, Y_{s-}, Z_{s-}))\, ds$$

is a $(P, \mathcal{F}_t)$-martingale. Let us consider the case of a bounded measurable function $f(x,z)$, for which, defining $\lambda_i(t,x,z) = \lambda(t,z)\, p_i(t,x,z)$, the operator $\mathcal{L}$ reduces to

$$Lf(t,x,z) = \sum_{i=0,1,2} \lambda_i(t,x,z) \Big[f\big(x + \xi^i(t,x,z), z + e^{kt}\xi^i(t,x,z)\big) - f(x,z) \Big]$$

and the process $f(X_t, Z_t)$ admits the $(P, \mathcal{F}_t)$-semimartingale representation

$$f(X_t, Z_t) = f(0, z_0) + \int_0^t Lf(s, X_s, Z_s)\, ds + m_t^f,$$

where m_t^f is a 0-mean $(P, \mathcal{F}_t)$-martingale.

Theorem A.1: *The probability measure-valued process $\pi_t(f)$, which is the cadlag version of $\mathbb{E}\big[f(X_t, Z_t) \big| \mathcal{F}_t^Y\big]$, is the unique solution to the Kushner–Stratonovich equation*

$$\pi_t(f) = f(0, z_0) + \int_0^t \pi_s(Lf)\, ds + \int_0^t \int_I K_s(\eta) \Big(m(ds, d\eta) - \widehat{\nu}_s(d\eta)\, ds \Big), \qquad \textbf{(A.2)}$$

with

$$K_s(\eta) = \frac{d\pi_{s-}\big(f\ \nu(d\eta)\big)}{d\widehat{\nu}(d\eta)} - \pi_{s-}\big(f\big) + \frac{d\pi_{s-}(B^f(d\eta))}{d\widehat{\nu}(d\eta)}.$$

Here, $d\mu(\eta)/d\nu(\eta)$ denotes the Radon–Nykodim derivative of the measure μ w.r.t. the measure ν, both measures being defined on the family of Borel subsets of I.

Proof: By Theorem VIII-T9 in Bremaud [3], the $(P, \mathcal{F}_t^Y)$-semimartingale representation is

$$\pi_t(f) = f(0, z_0) + \int_0^t \pi_s(Lf)\, ds + M_t^f.$$

Since M_t^f is $(P, \mathcal{F}_t^Y)$-martingale, there exists an η-indexed $(P, \mathcal{F}_t)$-predictable process $K_t(\eta)$ such that

$$M_t^f = \int_0^t \int_I K_s(\eta) \Big(m(ds, d\eta) - \widehat{\nu}_s(d\eta)\, ds \Big)$$

and, by the same theorem already quoted,

$$K_s(\eta) = \Psi^1_s(\eta) - \Psi^2_s(\eta) + \Psi^3_s(\eta),$$

where

$$\Psi^2_s(\eta) = \pi_{s-}(f),$$

and, for $i = 1, 3$, $\Psi^i_s(\eta)$ are η-indexed $(P, \mathcal{F}^Y_t)$-predictable processes $P \times m$ uniquely determined by the following equalities holding for any η-indexed non-negative $(P, \mathcal{F}^Y_t)$-predictable process $C_s(\eta)$:

$$\mathbb{E}\left[\int_0^t \int_I \Psi^1_s(\eta) C_s(\eta) \widehat{\nu}_s(d\eta) ds\right] = \mathbb{E}\left[\int_0^t \int_I f(X_s, Z_s) C_s(\eta) \nu_s(d\eta) ds\right], \tag{A.3}$$

$$\mathbb{E}\left[\int_0^t \int_I \Psi^3_s(\eta)\, C_s(\eta)\, \widehat{\nu}_s(d\eta)\, ds\right] = \mathbb{E}\left[\int_0^t \int_I \Big(f(X_s, Z_s) - f(X_{s-}, Z_{s-})\Big)\, C_s(\eta)\, m(ds, d\eta)\right]. \tag{A.4}$$

In order to determine the structure of the processes $\Psi^i_s(\eta)$, for $i = 1, 3$ taking into account the structure of the model discussed in this article, by using an idea suggested in Ceci [5], in (A.3) and (A.4), let $C_s(\eta) = \gamma_s\, \mathbb{I}_A(\eta)$, with γ_s be a nonnegative $(P, \mathcal{F}^Y_t)$-predictable process and A Borel subset of I. Thus, (A.3) provides

$$\begin{aligned}\mathbb{E}\left[\int_0^t \gamma_s \int_A \Psi^1_s(\eta)\, \widehat{\nu}_s(d\eta)\, ds\right] &= \mathbb{E}\left[\int_0^t \gamma_s \int_A f(X_s, Z_s)\, \nu_s(d\eta)\, ds\right] \\ &= \mathbb{E}\left[\int_0^t \gamma_s \mathbb{E}\left[\int_A f(X_s, Z_s)\, \nu_s(d\eta) \middle| \mathcal{F}^Y_s\right] ds\right] \\ &= \mathbb{E}\left[\int_0^t \gamma_s \int_A \pi_s\Big(f\; \nu(d\eta)\Big)\, ds\right].\end{aligned}$$

This implies that $\Psi^1_s(\eta)$ coincides with the Radon–Nykodim derivative of the measure $\pi_{s-}(f\; \nu(d\eta))$ w.r.t. $\widehat{\nu}_s(d\eta)$.

As far as $\Psi^3_s(\eta)$ is concerned, let us consider the right-hand side of (A.4) and successively

$$\begin{aligned}&f(X_s, Z_s) - f(X_{s-}, Z_{s-}) \\ &\quad = \Big[\sum_{i=0,1,2} f(X_{s-} + \xi^i_s, Z_{s-} + e^{ks}\xi^i_s) - f(X_{s-}, Z_{s-})\Big]\, \Delta N^i_s \\ &\quad \times \mathbb{E}\Big[\int_0^t \gamma_s \int_A \Big(f(X_s, Z_s) - f(X_{s-}, Z_{s-})\Big)\, m(ds, d\eta)\Big]\end{aligned}$$

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$$= \mathbb{E}\Big[\int_0^t \gamma_s \sum_{i=0,1,2} \Big(f(X_{s-}+\xi_s^i, Z_{s-}+e^{ks}\xi_s^i) - f(X_{s-}, Z_{s-})\Big)\, \mathbb{I}_{\{\xi_s^i \in A\}}\, dN_s^i\Big]$$

$$= \mathbb{E}\Big[\int_0^t \gamma_s \sum_{i=0,1,2} \Big(f(X_{s-}+\xi_s^i, Z_{s-}+e^{ks}\xi_s^i) - f(X_{s-}, Z_{s-})\Big)\, \mathbb{I}_{\{\xi_s^i \in A\}}\, \lambda_s^i\, ds\Big]$$

$$= \mathbb{E}\Big[\int_0^t \gamma_s\, \pi_s\Big(\sum_{i=0,1,2} \Big(f(X+\xi^i, Z+e^{k\cdot}\xi^i) - f(X,Z)\Big)\, \mathbb{I}_{\{\xi^i \in A\}}\, \lambda^i\Big)\, ds\Big].$$

Hence, defining, on the family of borel subsets of I, the measure

$$B_s^f(A) = \sum_{i=0,1,2} \Big(f(X_{s-}+\xi_s^i, Z_{s-}+e^{ks}\xi_s^i) - f(X_{s-}, Z_{s-})\Big)\, \mathbb{I}_{\{\xi_s^i \in A\}}\, \lambda_s^i$$

$\Psi_s^3(\eta)$ coincides with the Radon–Nykodim derivative of the measure $\pi_{s-}(B^f(d\eta))$ w.r.t. $\widehat{\nu}_s(d\eta)$, and (A.2) is obtained.

Equation (A.2) admits a unique solution. Actually, denote by $\{T_i\}$ the sequence of jump times of Y. At any jump time $t = T_i$, (A.2) reduces to

$$\pi_{T_i}(f) = \pi_{T_i-}(f) + K_{T_i}(Y_{T_i} - Y_{T_i-});$$

thus, $\pi_{T_i}(f)$ is completely determined by the knowledge of

$$\pi_{T_i-}(f) = \lim_{t \to T_i^-} \pi_t(f).$$

On the other hand, for $t \in [T_i, T_{i+1})$, (A.2) becomes

$$\pi_t(f) = \pi_{T_i}(f) + \int_{T_i}^t \pi_s(Lf)\, ds - \int_{T_i}^t \int_I K_s(\eta)\, \widehat{\nu}_s(d\eta)\, ds,$$

where

$$\int_{T_i}^t \int_I K_s(\eta)\, \widehat{\nu}_s(d\eta)\, ds$$
$$= \int_{T_i}^t \pi_s(Lf)\, ds - \int_{T_i}^t \int_I \left(\frac{d\pi_{s-}\big(f\ \nu(d\eta)\big)}{d\widehat{\nu}(d\eta)} - \pi_{s-}(f) + \frac{d\pi_{s-}(B^f(d\eta))}{d\widehat{\nu}(d\eta)} \right) \widehat{\nu}_s(d\eta)\, ds.$$

Therefore,

$$\pi_t(f) = \pi_{T_i}(f) + \int_{T_i}^t \Big(\pi_s(Lf)\, ds - \pi_s\big(f\ \nu(I)\big) + \pi_s(f)\, \pi_s\big(\nu(I)\big) - \pi_{s-}(B^f(I))\Big)\, ds,$$

which is an ordinary differential equation Lipschitz w.r.t. the bounded variation norm. ■

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