Bull. Aust. Math. Soc. **104** (2021), 1–10
doi:10.1017/S0004972720001276

THE GENERAL POSITION NUMBER OF THE CARTESIAN PRODUCT OF TWO TREES

JING TIAN, KEXIANG XU✉ and SANDI KLAVŽAR

(Received 15 September 2020; accepted 26 September 2020; first published online 4 December 2020)

Abstract

The general position number of a connected graph is the cardinality of a largest set of vertices such that no three pairwise-distinct vertices from the set lie on a common shortest path. In this paper it is proved that the general position number is additive on the Cartesian product of two trees.

2020 *Mathematics subject classification*: primary 05C05; secondary 05C12, 05C35.

Keywords and phrases: Cartesian product, general position number, general position set, trees.

1. Introduction

Let $d_G(x, y)$ denote the number of edges on a shortest x, y-path in G. A set S of vertices of a connected graph G is a *general position set* if $d_G(x, y) \neq d_G(x, z) + d_G(z, y)$ for every $\{x, y, z\} \in \binom{S}{3}$. The *general position number* $\mathrm{gp}(G)$ of G is the cardinality of a largest general position set in G. Such a set is briefly called a *gp-set* of G.

Before the general position number was introduced in [9], an equivalent concept was proposed in [14]. Much earlier, however, the general position problem has been studied by Körner [8] in the special case of hypercubes. Following [9], the graph theory general position problem has been investigated in [1, 3, 5, 7, 10, 11, 13].

The *Cartesian product* $G \,\square\, H$ of vertex-disjoint graphs G and H is the graph with vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. In this paper we are interested in $\mathrm{gp}(G \,\square\, H)$, a problem earlier studied in [3, 5, 10, 13]. More precisely, we are interested in Cartesian products of two (finite) trees. (For some of the other investigations of the Cartesian product of trees, see [2, 12, 15].) An important reason for this interest is the fact that the general position number of products of paths is far from being trivial. First, if P_∞ denotes the two-way infinite path, one of the main results from [10] asserts that $\mathrm{gp}(P_\infty \,\square\, P_\infty) = 4$. Further, from the same paper, $10 \leq \mathrm{gp}(P_\infty^3) \leq 16$, where G^n denotes

Kexiang Xu is supported by NNSF of China (grant no. 11671202 and the China–Slovene bilateral grant 12-9). Sandi Klavžar acknowledges financial support from the Slovenian Research Agency (research core funding P1-0297, projects J1-9109, J1-1693, N1-0095 and the bilateral grant BI-CN-18-20-008).

the n-fold Cartesian product of G. The lower bound 10 was improved to 14 in [5]. Very recently, these results were superseded in [6], where it was shown that $\mathrm{gp}(P_\infty^n) = 2^{2^{n-1}}$ for an arbitrary positive integer. Let $n(G)$ denote the order of a graph G. In this paper we prove the following result.

THEOREM 1.1. *If T and T^* are trees with* $\min\{n(T), n(T^*)\} \geq 3$, *then*

$$\mathrm{gp}(T \,\square\, T^*) = \mathrm{gp}(T) + \mathrm{gp}(T^*)\,.$$

Theorem 1.1 is a significant extension of the result $\mathrm{gp}(P_\infty \,\square\, P_\infty) = 4$. Moreover, since $\mathrm{gp}(P_\infty^n) = 2^{2^{n-1}}$, there is no obvious (inductive) extension of Theorem 1.1 to Cartesian products of more than two trees. Determining the general position number of such products remains a challenging problem.

In the next section we give further definitions, recall some known results and prove several new auxiliary results. Then, in Section 3, we prove Theorem 1.1.

2. Preliminaries

Let T be a tree. We denote the set of leaves of T by $L(T)$ and set $\ell(T) = |L(T)|$. If u and v are vertices of T with $\deg(u) \geq 2$ and $\deg(v) = 1$, then the unique u, v-path is a *branching path* of T. If u is not a leaf of T, then there are exactly $\ell(T)$ branching paths starting from u; we say that u is the *root* of these branching paths and that the degree-1 vertex of a branching path P is the *leaf of* P.

LEMMA 2.1 [9]. *If T is a tree, then* $\mathrm{gp}(T) = \ell(T)$.

We next describe which vertices of a tree lie in some gp-set of the tree.

LEMMA 2.2. *A nonleaf vertex u in a tree T belongs to a gp-set of T if and only if $T - u$ has exactly two components and at least one of them is a path.*

PROOF. First, let R be a gp-set of T containing the nonleaf vertex u. Suppose that $T - u$ has at least three components, say T_1, T_2 and T_3. Since R is a gp-set containing u, R intersects at most one of T_1, T_2 and T_3. Assume without loss of generality that we have $R \cap V(T_2) = \emptyset$ and $R \cap V(T_3) = \emptyset$. Choose vertices v and w in T such that $v \in V(T_2)$ and $w \in V(T_3)$. Then $(R - \{u\}) \cup \{v, w\}$ is a larger gp-set than R in T, which is a contradiction. Hence, $T - u$ has exactly two components, say T_1 and T_2. Now suppose that neither T_1 nor T_2 is a path. Then, as before, $R \cap V(T_1) = \emptyset$ or $R \cap V(T_2) = \emptyset$. By symmetry, we assume that $R \cap V(T_2) = \emptyset$. Since T_2 is not a path, there are at least two leaves x_1 and x_2 in T_2. Again, the set $(R - \{u\}) \cup \{x_1, x_2\}$ is a larger gp-set than R in T. Therefore, at least one of T_1 and T_2 is a path.

Conversely, we observe that u is a nonleaf vertex on a pendant path in T. Thus, u belongs to a gp-set in T. □

In $G \,\square\, H$, if $h \in V(H)$, then the subgraph of $G \,\square\, H$ induced by the vertices (g, h), $g \in V(G)$, is a *G-layer*, denoted by G^h. The H-layers, ${}^gH$, are defined analogously. The G-layers and H-layers are isomorphic to G and H, respectively. The distance function

https://doi.org/10.1017/S0004972720001276 Published online by Cambridge University Press

in Cartesian products is additive, that is, if $(g_1, h_1), (g_2, h_2) \in V(G \,\square\, H)$, then

$$d_{G \,\square\, H}((g_1, h_1), (g_2, h_2)) = d_G(g_1, g_2) + d_H(h_1, h_2). \tag{2.1}$$

If $u, v \in V(G)$, then the *interval* $I_G(u, v)$ between u and v in G is the set of all vertices lying on shortest u, v-paths, that is,

$$I_G(u, v) = \{w : \ d_G(u, v) = d_G(u, w) + d_G(w, u)\}\,.$$

In what follows, the notation $d_G(u, v)$ and $I_G(u, v)$ may be simplified to $d(u, v)$ and $I(u, v)$ if G is clear from the context. Equality (2.1) implies that intervals in Cartesian products have the following nice structure (see [4, Proposition 12.4]).

LEMMA 2.3. *If G and H are connected graphs and $(g_1, h_1), (g_2, h_2) \in V(G \,\square\, H)$, then*

$$I_{G \,\square\, H}((g_1, h_1), (g_2, h_2)) = I_G(g_1, g_2) \times I_H(h_1, h_2)\,.$$

Equality (2.1) also easily implies the following fact (also proved in [13]).

LEMMA 2.4. *Let G and H be connected graphs and R a general position set of $G \,\square\, H$. If $u = (g, h) \in R$, then $V({}^gH) \cap R = \{u\}$ or $V(G^h) \cap R = \{u\}$.*

For finite paths, the result $\mathrm{gp}(P_\infty \,\square\, P_\infty) = 4$ mentioned in Section 1 reduces to the following result.

LEMMA 2.5 [10]. *If $n_1, n_2 \geq 2$, then*

$$\mathrm{gp}(P_{n_1} \,\square\, P_{n_2}) = \begin{cases} 4 & \textit{for } \min\{n_1, n_2\} \geq 3, \\ 2 & n_1 = n_2 = 2, \\ 3 & \textit{otherwise}. \end{cases}$$

To conclude the preliminaries we construct special maximal (with respect to inclusion) general position sets in products of trees.

LEMMA 2.6. *Let T and T^* be two trees with $\min\{n(T), n(T^*)\} \geq 3$, $v_i \in V(T) \setminus L(T)$ and $v_j^* \in V(T^*) \setminus L(T^*)$. Then $(L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*))$ is a maximal general position set of $T \,\square\, T^*$.*

PROOF. Set $R = (L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*))$ and let $V_0 = \{u, v, w\} \subseteq R$. We first consider the case when $V_0 \subseteq L(T) \times \{v_j^*\}$ or $V_0 \subseteq \{v_i\} \times L(T^*)$. By symmetry, assume that $V_0 \subseteq L(T) \times \{v_j^*\}$. Then each vertex of V_0 corresponds to a leaf of $L(T)$ in the layer $T^{v_j^*} \cong T$. Therefore, u, v, w do not lie on a common geodesic in $T \,\square\, T^*$.

In the following, without loss of generality, we can assume that $u, w \in L(T) \times \{v_j^*\}$ with $u = (v_k, v_j^*)$, $w = (v_s, v_j^*)$ and $v = (v_i, v_\ell^*) \in \{v_i\} \times L(T^*)$. By (2.1),

$$d(u, v) = d_T(v_k, v_i) + d_{T^*}(v_j^*, v_\ell^*), \quad d(w, v) = d_T(v_s, v_i) + d_{T^*}(v_j^*, v_\ell^*)$$

and $d(u, w) = d_T(v_k, v_s)$. Note that v_k and v_s are two distinct vertices of T in $L(T)$ and $v_i \in V(T) \setminus L(T)$. Then $d_T(v_k, v_i) < d_T(v_k, v_s) + d_T(v_s, v_i)$ whenever v_i lies on the v_k, v_s-geodesic or outside the v_k, v_s-geodesic of T. This implies that

https://doi.org/10.1017/S0004972720001276 Published online by Cambridge University Press

$d(u, v) < d(u, w) + d(w, v)$ in $T \,\square\, T^*$. Therefore, w does not lie on the u, v-geodesic in $T \,\square\, T^*$. Analogously, u does not lie on the v, w-geodesic and v does not lie on the u, w-geodesic of $T \,\square\, T^*$. Thus, u, v, w do not lie on a common geodesic in $T \,\square\, T^*$, which implies that R is a general position set in $T \,\square\, T^*$.

Next we prove the maximality of $(L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*))$ as a general position set in $T \,\square\, T^*$. Otherwise, there is a general position set R' in $T \,\square\, T^*$ of order greater than $\ell(T) + \ell(T^*)$ such that $R \subset R'$. Then there exists a vertex $z \in R' \backslash R$, say $z = (v_p, v_q^*)$. If $p = i$, then there exist two vertices $(v_i, v_s^*), (v_i, v_t^*) \in R$ such that $z \in I_{T \square T^*}((v_i, v_s^*), (v_i, v_t^*))$ (since ${}^{v_i}T^* \cong T^*$). This is a contradiction, showing that $p \neq i$. Similarly, we have $q \neq j$. Now we consider the positions of v_p in T and v_q^* in T^*. Suppose first that $v_p \in L(T)$, $v_q^* \in L(T^*)$. Then there are two vertices $(v_p, v_j^*), (v_i, v_q^*)$ in R such that $z \in I_{T \square T^*}((v_p, v_j^*), (v_i, v_q^*))$, contradicting that $R \cup \{z\}$ is a general position set of $T \,\square\, T^*$. If $v_p \in L(T)$ and $v_q^* \notin L(T^*)$, then we select a vertex $v_{q'}^* \in L(T^*)$ such that $v_{q'}^*$ is closer to the leaf of the corresponding branching path than v_q^* in T^*. Then $z \in I_{T \square T^*}((v_p, v_j^*), (v_i, v_{q'}^*))$, which is a contradiction. Similarly, $v_p \notin L(T)$ and $v_q^* \in L(T^*)$ cannot occur. Finally, we assume that $v_p \notin L(T)$, $v_q^* \notin L(T^*)$. Now we select two vertices $v_{p'} \in L(T)$ and $v_{q'}^* \in L(T^*)$ such that $v_{p'}$ is closer to the leaf of the branching path than v_p in T and $v_{q'}^*$ is closer to the leaf of the branching path than v_q^* in T^*. But then $(v_p, v_q^*) \in I_{T \square T^*}((v_{p'}, v_j^*), (v_i, v_{q'}^*))$, which is a final contradiction. □

3. Proof of Theorem 1.1

If T and T^* are both paths, then Theorem 1.1 holds by Lemma 2.5. In what follows we may assume without loss of generality that T^* is not a path. From Lemma 2.6, $\mathrm{gp}(T \,\square\, T^*) \geq \mathrm{gp}(T) + \mathrm{gp}(T^*)$, so it remains to prove that $\mathrm{gp}(T \,\square\, T^*) \leq \mathrm{gp}(T) + \mathrm{gp}(T^*)$. Set $n = n(T)$, $n^* = n(T^*)$, $V(T) = \{v_1, \ldots, v_n\}$ and $V(T^*) = \{v_1^*, \ldots, v_{n^*}^*\}$.

Assume on the contrary that there exists a general position set R of T such that $|R| > \mathrm{gp}(T) + \mathrm{gp}(T^*)$. Since the restriction of R to a T-layer of $T \,\square\, T^*$ is a general position set of the layer (which is in turn isomorphic to T), the restriction contains at most $\mathrm{gp}(T) = \ell(T)$ elements. Similarly, the restriction of R to a T^*-layer contains at most $\mathrm{gp}(T^*) = \ell(T^*)$ elements. We now distinguish two major cases.

Case 1. There exists a T-layer $T^{v_j^*}$ with $|V(T^{v_j^*}) \cap R| = \mathrm{gp}(T)$, or a T^*-layer ${}^{v_i}T^*$ with $|V({}^{v_i}T^*) \cap R| = \mathrm{gp}(T^*)$. By the commutativity of the Cartesian product, we may assume without loss of generality that there is a T^*-layer ${}^{v_i}T^*$ with $|R \cap V({}^{v_i}T^*)| = \mathrm{gp}(T^*)$. Let $R = R_1 \cup R_2$, where $R_1 = R \cap V({}^{v_i}T^*)$ and $R_2 = R \setminus R_1 = \bigcup_{t \in [n] \setminus \{i\}} (V({}^{v_t}T^*) \cap R)$. Further, let S^* be the projection of $R \cap V({}^{v_i}T^*)$ on T^*, that is, $S^* = \{v_j^* : (v_i, v_j^*) \in R_1\}$. Since $|R_1| = \mathrm{gp}(T^*)$, our assumption implies that $|R_2| \geq \mathrm{gp}(T) + 1$. Since $\mathrm{gp}(T) = \ell(T)$, there exist two different vertices $w = (v_p, v_q^*)$ and $w' = (v_{p'}, v_{q'}^*)$ from R_2 such that v_p and $v_{p'}$ lie on the same branching path P of T. (Note that it is possible that $v_p = v_{p'}$.) We may assume that $d_T(v_{p'}, x) \leq d_T(v_p, x)$, where x is the leaf of P. We proceed by distinguishing two subcases based on the position of v_q^* and $v_{q'}^*$ in T^*.

https://doi.org/10.1017/S0004972720001276 Published online by Cambridge University Press

Case 1.1. There exists a branching path P^* of T^* that contains both v_q^* and $v_{q'}^*$. Recall that T^* is not a path. Lemma 2.2 implies that a vertex of a tree belongs to a gp-set if and only if it lies on a pendant path and has degree 1 or 2. Therefore, we can select P^* with the root of degree at least 3. Assume that $d_{T^*}(v_{q'}^*, y) \leq d_{T^*}(v_q^*, y)$, where y is the leaf of P^*. (The reverse case can be treated analogously.) Since S^* is a gp-set of T^*, which is not isomorphic to a path, there is a vertex $v_k^* \in S^*$ lying on P^*. So, we may assume that P^* is a branching path that contains v_q^*, $v_{q'}^*$ and a vertex $v_k^* \in S^*$. (It is possible that some of these vertices are the same.) Let $z = (v_i, v_k^*)$. Then $z \in R_1$. We proceed by distinguishing four subcases based on the position of v_p, $v_{p'}$ and v_i in T.

Subcase 1.1.1. $v_{p'} \in I(v_i, v_p)$. If v_k^* is closer than v_q^*, $v_{q'}^*$ to the leaf y of P^*, then, by Lemma 2.3, $w' \in I_{T \,\square\, T^*}(w, z)$, which is a contradiction. On the other hand, if $v_k^* \in I(v_q^*, v_{q'}^*)$, then, since $\ell(T^*) \geq 3$, there exists $z' = (v_i, v_{k'}^*) \in \{v_i\} \times S^*$ such that $v_k^*, v_q^* \in I(v_{q'}^*, v_{k'}^*)$ in T^*. Then

$$
\begin{aligned}
d(w', z') &= d_T(v_{p'}, v_i) + d_{T^*}(v_{q'}^*, v_{k'}^*) \\
&= d_T(v_{p'}, v_i) + d_{T^*}(v_{q'}^*, v_k^*) + d_{T^*}(v_k^*, v_{k'}^*) \\
&= d(w', z) + d(z, z'),
\end{aligned}
$$

which implies that $z \in I_{T \,\square\, T^*}(w', z')$, which is a contradiction.

Subcase 1.1.2. $v_i \in I(v_p, v_{p'})$. If $v_k^* \in I(v_q^*, v_{q'}^*)$ in P^*, then $z \in I_{T \,\square\, T^*}(w, w')$ by Lemma 2.3, which is a contradiction. Assume instead that v_k^* is closer than v_q^*, $v_{q'}^*$ to the leaf of P^*. Since $|S^*| = \ell(T^*) \geq 3$, there is a vertex $z' = (v_i, v_{k'}^*) \in \{v_i\} \times S^*$ such that v_q^*, $v_{q'}^* \in I(v_k^*, v_{k'}^*)$ in T^*. Let $v_{k'}^*$ be on a branching path P'^* in T^*, where $P'^* \neq P^*$. Note that $\ell(T) + 1 \geq 3$. There exists at least one vertex $a = (v_x, v_y^*) \in R_2 \setminus \{w, w'\}$. Next we consider the positions of v_x, v_y^* in T, T^*, respectively.

Suppose first that $v_y^* \in V(P^* \cup P'^*)$. If v_x, v_p, $v_{p'}$ and v_i lie on a path in T, then there are five vertices w, w', z, z' and a in R_2, three of which lie on a common geodesic in $T \,\square\, T^*$, which is a contradiction. Note that if T is a path, then we are done as above. Therefore, we can assume that T is not isomorphic to a path and the root of P has degree at least 3. (Otherwise, $v_x \notin P$ and v_x, v_p lie on a common branching path in T.) Let V_s be the set of vertices of T not contained in $T_{ip'}$, where $T_{ip'}$ is the subtree of $T - v_p$ containing v_i and $v_{p'}$. If there is a vertex $a' = (v_s, v_l^*) \in R_2$ with $v_s \in V_s$, then R_2 contains w, w', z, z' and a', three of which are on a common geodesic, which is a contradiction. Therefore, the first coordinate of any vertex in R_2 cannot be in V_s. Assume that $P' \neq P$ is any branching path containing v_p and a leaf in both $T_{ip'}$ and T. Then, besides w, $P' \,\square\, T^*$ contains at most one vertex in R_2 of $T \,\square\, T^*$. Otherwise, $P' \,\square\, T^*$ contains two vertices h, h' in R_2. Then there exist two vertices $h_0, h_0' \in \{v_i\} \times S^*$ such that three vertices from $\{h, h', h_0, h_0', w\}$ lie on some geodesic in $T \,\square\, T^*$, which is a contradiction. (Here h_0 may be equal to h_0'.) Note that V_s contains at least two leaves of T since the root of P (just in V_s) has degree at least 3. Then $T_{ip'}$ has at most $\ell(T) - 2$ leaves in T. Since $P \,\square\, T^*$ contains two vertices w and w'

https://doi.org/10.1017/S0004972720001276 Published online by Cambridge University Press

in R_2, we have $|R_2| \leq \ell(T) - 2 + 1 < \ell(T) = \mathrm{gp}(T)$, which is a contradiction with the assumption.

Assume now that $v_y^* \notin V(P^* \cup P'^*)$. Then there is a vertex $z'' = (v_i, v_{k''}^*)$ in $\{v_i\} \times S^*$ such that $v_y^*, v_{k''}^*$ lie on a common branching path in T^*. If v_y^* is closer to the leaf of the branching path than $v_{k''}^*$ in T^*, then $v_i \in I(v_x, v_i)$ and $v_{k''}^* \in I(v_y^*, v_k^*)$. Therefore, by Lemma 2.3, $z'' \in I_{T \square T^*}(a, z)$, which is a contradiction. If $v_{k''}^*$ is closer to the leaf of the branching path than v_y^* in T^*, we consider the positions of v_x, v_p, $v_{p'}$ and v_i in T. Let $V_1 = \{z, z', w, w', a, z''\}$. Then $V_1 \subseteq R_2$. If v_x, v_p, $v_{p'}$ and v_i lie on a path in T, then there exist three vertices in V_1 lying on a common geodesic in $T \square T^*$, which is again a contradiction. Otherwise, $v_x \notin P$ and v_x, v_p lie on a common branching path in T. In the same way as above, this leads to a a contradiction.

Subcase 1.1.3. $v_p \in I(v_i, v_{p'})$. Since $\ell(T^*) \geq 3$, there is a vertex $z' = (v_i, v_{k'}^*) \in \{v_i\} \times S^*$ such that $v_{k'}^* \notin P^*$ and $v_q^* \in I(v_{k'}^*, v_{q'}^*)$ in T^*. Since

$$\begin{aligned} d(z', w') &= d_T(v_i, v_{p'}) + d_{T^*}(v_{k'}^*, v_{q'}^*) \\ &= d_T(v_i, v_p) + d_{T^*}(v_{k'}^*, v_q^*) + d_T(v_p, v_{p'}) + d_{T^*}(v_q^*, v_{q'}^*) \\ &= d(z', w) + d(w, w'), \end{aligned}$$

we have $w \in I_{T \square T^*}(z', w')$, which is a contradiction.

Subcase 1.1.4. $v_i \notin V(P)$ *such that* v_i, v_p *lie on the same branching path in* T. Since $\ell(T^*) \geq 3$, there is a vertex $z' = (v_i, v_{k'}^*) \in \{v_i\} \times S^*$ such that $v_q^* \in I(v_{k'}^*, v_k^*)$ in T^*. If $v_k^* \in I(v_q^*, v_{q'}^*)$, then obviously $v_k^* \in I(v_q^*, v_{k'}^*)$ and, therefore,

$$\begin{aligned} d(w', z') &= d_T(v_{p'}, v_i) + d_{T^*}(v_{q'}^*, v_{k'}^*) \\ &= d_T(v_{p'}, v_i) + d_{T^*}(v_{q'}^*, v_k^*) + d_{T^*}(v_k^*, v_{k'}^*) \\ &= d(w', z) + d(z, z')\,. \end{aligned}$$

We conclude that $z \in I_{T \square T^*}(w', z')$, which is a contradiction. On the other hand, if v_k^* is closer to the leaf of P^* than $v_q^*, v_{q'}^*$, then we get a contradiction as in Subcase 1.1.2.

Case 1.2. v_q^* *and* $v_{q'}^*$ *do not lie on the same branching path in* T^*. We may assume that v_q^* and $v_{q'}^*$ lie on distinct branching paths P^* and P'^* in T^*, respectively. Since $\ell(T^*) \geq 3$ and T^* is not isomorphic to a path, there exist two vertices $z = (v_i, v_k^*)$ and $z' = (v_i, v_{k'}^*)$ from $\{v_i\} \times S^*$ such that $v_k^* \in P^*$ and $v_{k'}^* \in P'^*$. We consider four subcases based on the positions of v_p, $v_{p'}$ and v_i in T.

Subcase 1.2.1. $v_{p'} \in I(v_i, v_p)$. If $v_{k'}^*$ is closer than $v_{q'}^*$ to the leaf of P'^*, then $v_{p'} \in I(v_p, v_i)$ and $v_{q'}^* \in I(v_q^*, v_{k'}^*)$. Lemma 2.3 gives $w' \in I_{T \square T^*}(w, z')$, which is a contradiction. On the other hand, if $v_{q'}^*$ is closer than $v_{k'}^*$ to the leaf of P'^*, then $v_i \in I(v_i, v_{p'})$ and $v_{k'}^* \in I(v_k^*, v_{q'}^*)$ and hence Lemma 2.3 gives $z' \in I_{T \square T^*}(w', z)$, which is again a contradiction.

Subcase 1.2.2. $v_i \in I(v_p, v_{p'})$. We first assume that $v_{q'}^*$ is closer than $v_{k'}^*$ to the leaf of P'^*. Then $v_i \in I(v_i, v_{p'})$ and $v_{k'}^* \in I(v_k^*, v_{q'}^*)$. Therefore, by Lemma 2.3, $z' \in I_{T \square T^*}(z, w')$

which is a contradiction. Otherwise, $v^*_{k'}$ is closer than $v^*_{q'}$ to the leaf of P'^*. If v^*_q is closer than v^*_k to the leaf of P^*, then $v_i \in I(v_p, v_i)$ and $v^*_k \in I(v^*_q, v^*_{k'})$. Therefore, by Lemma 2.3, $z \in I_{T\,\square\,T^*}(w, z')$, which is a contradiction. If v^*_k is closer than v^*_q to the leaf of P^*, we reach a contradiction in the same way as in the proof of Subcase 1.1.2.

Subcase 1.2.3. $v_p \in I(v_i, v_{p'})$. If v^*_k is closer than v^*_q to the leaf of P^*, then $v_p \in I(v_i, v_{p'})$ and $v^*_q \in I(v^*_k, v^*_{q'})$. So, Lemma 2.3 gives $w \in I_{T\,\square\,T^*}(z, w')$, which is a contradiction. If v^*_q is closer than v^*_k to the leaf of P^*, then $v_i \in I(v_i, v_p)$ and $v^*_k \in I(v^*_{k'}, v^*_q)$ and so $z \in I_{T\,\square\,T^*}(z', w)$.

Subcase 1.2.4. $v_i \notin V(P)$ such that v_i, v_p lie on the same branching path in T. First suppose that v^*_q is closer to the leaf than v^*_k in P^*. Then we have $v_i \in I(v_i, v_p)$ and $v^*_k \in I(v^*_q, v^*_{k'})$. Thus, by Lemma 2.3, $z \in I_{T\,\square\,T^*}(w, z')$.

Assume that v^*_k is closer than v^*_q to the leaf of P^*. If $v^*_{q'}$ is closer to the leaf than $v^*_{k'}$, then $v_i \in I(v_i, v_{p'})$ and $v^*_{k'} \in I(v^*_k, v^*_{q'})$, which gives $z' \in I_{T\,\square\,T^*}(z, w')$. If $v^*_{k'}$ is closer than $v^*_{q'}$ to the leaf of P'^*, we can proceed in the same way as in Subcase 1.1.4.

Case 2. $|R \cap V({}^{v_k}T^*)| < \ell(T^*)$ for any $k \in [n]$, and $|R \cap V(T^{v^*_t})| < \ell(T)$ for any $t \in [n^*]$. Let ${}^{v_i}T^*$ be a layer with $|R \cap V({}^{v_i}T^*)| = \max\{|R \cap V({}^{v_k}T^*)| : k \in [n]\}$. Let $R = R_1 \cup R_2$, where $R_1 = R \cap V({}^{v_i}T^*)$ and $R_2 = R \setminus R_1 = \bigcup_{k\in[n]\setminus\{i\}}(V({}^{v_k}T^*) \cap R)$. Further, set $S^* = \{v^*_j : (v_i, v^*_j) \in R_1\}$. Then $1 \le |S^*| \le \ell(T^*) - 1$.

Assume first that $|S^*| = 1$. Therefore, $|R \cap V({}^{v_k}T^*)| \le 1$ for any $k \in [n]$. Next we only need to consider $|R \cap V(T^{v^*_j})| \le 1$ for any $j \in [n^*]$. (If $|R \cap V(T^{v^*_j})| \ge 2$ for some $j \in [n^*]$, by commutativity of $T\,\square\,T^*$, the proof is similar to the subcase in which $2 \le |S^*| \le \ell(T^*) - 1$.) So, suppose $|R \cap V(T^{v^*_j})| \le 1$ for any $j \in [n^*]$. Then $|R| \le \min\{n, n^*\}$. We now claim that $|R| \le \ell(T) + \ell(T^*)$. If not, then, since $|R| \ge \ell(T) + \ell(T^*) + 1 \ge 6$, there exist three vertices $u = (v_p, v^*_j)$, $v = (v_{p'}, v^*_q)$ and $w = (v_s, v^*_\ell)$ from R such that $v_p, v_{p'}$ lie on the same branching path in T, and v^*_j, v^*_ℓ lie on a common branching path in T^*. Note that we may have $p' = s, q = \ell$. But we can always select a vertex $h \in R \setminus \{u, v, w\}$ such that u, v, h or u, w, h lie on the same geodesic in $T\,\square\,T^*$, which is a contradiction. So, our result holds when $|S^*| = 1$.

Suppose next that $2 \le |S^*| \le \ell(T^*) - 1$. Since $|R_1| = |S^*|$, we need to prove that $|R_2| \le \ell(T) + \ell(T^*) - |S^*|$. Assume on the contrary that $|R_2| \ge \ell(T) + \ell(T^*) - |S^*| + 1$. Since $|S^*| \ge 2$, there are two distinct vertices $w = (v_i, v^*_j)$ and $w' = (v_i, v^*_{j'})$ from $\{v_i\} \times S^*$. We distinguish the following cases based on the positions of v^*_j, $v^*_{j'}$ in T^*.

Case 2.1. v^*_j and $v^*_{j'}$ lie on the same branching path P^* of T^*. Without loss of generality, we may assume that $v^*_{j'}$ is closer than v^*_j to the leaf of P^*. Let $T^*_{v^*_{j'}}$ be the maximal subtree of $T^* - v^*_j$ containing $v^*_{j'}$ and let $V_{s^*} = V(T^*) \setminus V(T^*_{v^*_{j'}})$. In addition, introduce $S^*_1 = \{v^*_q : v^*_q \in I(v^*_j, v^*_\ell), v^*_\ell \in S^* \cap V(T^*_{v^*_{j'}})\}$. Now we prove the following claim.

Claim 1. If $z = (v_p, v^*_t) \in R_2$, then $v^*_t \in S^*_1$.

If not, suppose first that $v^*_t \in V(P^*)$ is closer than $v^*_{j'}$ to the leaf of P^*. Then $v_i \in I(v_i, v_p)$ and $v^*_{j'} \in I(v^*_t, v^*_j)$. Hence, $w' \in I_{T\,\square\,T^*}(w, z)$. On the other hand, if $v^*_t \in V_{s^*}$, then

$v_j^* \in I(v_t^*, v_{j'}^*)$. Combining this fact with $v_i \in I(v_i, v_p)$, we have $w \in I_{T \,\square\, T^*}(w', z)$. This proves Claim 1.

By Claim 1, $|\bigcup_{v_t^* \in S_1^*}(V(T^{v_t^*}) \cap R)| \geq \ell(T) + \ell(T^*) - |S^*| + 1 \geq \ell(T) + 1$. So, there exist two vertices $z = (v_p, v_\ell^*)$ and $z' = (v_{p'}, v_{\ell'}^*)$ from $\bigcup_{v_t^* \in S_1^*}(V(T^{v_t^*}) \cap R)$ such that $v_\ell^*, v_{\ell'}^* \in S_1^*$ and $v_p, v_{p'}$ lie on the same branching path P in T. Without loss of generality, let $v_{p'}$ be closer than v_p to the leaf of P and let $v_\ell^*, v_{\ell'}^* \in I(v_j^*, v_{j'}^*)$ (by the definition of S_1^*). We consider four subcases according to the positions of $v_i, v_p, v_{p'}$ in T.

Subcase 2.1.1. $v_{p'} \in I(v_i, v_p)$. If $v_{\ell'}^*$ is closer than v_ℓ^* to $v_{j'}^*$ in P^*, then $v_{p'} \in I(v_i, v_p)$ and $v_{\ell'}^* \in I(v_\ell^*, v_{j'}^*)$. Therefore, $z' \in I_{T \,\square\, T^*}(z, w')$. And, if v_ℓ^* is closer than $v_{\ell'}^*$ to $v_{j'}^*$ in P^*, then $v_{p'} \in I(v_i, v_p)$ and $v_{\ell'}^* \in I(v_\ell^*, v_j^*)$ and so $z' \in I_{T \,\square\, T^*}(z, w)$.

Subcase 2.1.2. $v_i \in I(v_p, v_{p'})$. Note that $\ell(T) + \ell(T^*) - |S^*| + 1 \geq 4$, so there is at least one vertex $a = (v_x, v_y^*)$ in $\bigcup_{v_t^* \in S_1^*}(V(T^{v_t^*}) \cap R)$ different from z and z'. Based on the position of v_y^* ($v_y^* \in P^*$ or $v_y^* \notin P^*$) in T^*, and the positions of v_x, v_i, v_p and $v_{p'}$ in T, we get contradictions in the same way as in Subcase 1.1.2.

Subcase 2.1.3. $v_p \in I(v_i, v_{p'})$. If $v_{\ell'}^*$ is closer than v_ℓ^* to $v_{j'}^*$ in T^*, then $v_p \in I(v_i, v_{p'})$ and $v_\ell^* \in I(v_j^*, v_{\ell'}^*)$; therefore, $z \in I_{T \,\square\, T^*}(w, z')$. And, if v_ℓ^* is closer than $v_{\ell'}^*$ to $v_{j'}^*$ in T^*, then $v_p \in I(v_i, v_{p'})$ and $v_\ell^* \in I(v_{j'}^*, v_{\ell'}^*)$ and hence $z \in I_{T \,\square\, T^*}(w, z')$.

Subcase 2.1.4. $v_i \notin V(P)$ such that v_i, v_p lie on the same branching path in T. Since $\ell(T) + \ell(T^*) - |S^*| + 1 \geq 4$, there exists a vertex $(v_x, v_y^*) \in \bigcup_{v_t^* \in S_1^*}(V(T^{v_t^*}) \cap R)$. By arguments similar to those in Subcase 1.1.4, we reach a contradiction. But this implies that $|\bigcup_{v_t^* \in S_1^*}(V(T^{v_t^*}) \cap R)| \leq \ell(T) + \ell(T^*) - |S^*|$, contrary to the assumption.

Case 2.2. v_j^*, $v_{j'}^*$ lie on different branching paths P^*, P'^* in T^*, respectively. Let S_2^* be the set of vertices of ${}^{v_i}T^*$ closer to the leaf of a branching path than v_g^* for any $v_g^* \in S^*$. Note that $S^* \cap S_2^* = \emptyset$. We prove the following claim.

Claim 2. If (v_p, v_t^*) in R_2, then $v_t^* \in V(T^*) \setminus (S^* \cup S_2^*)$.

Lemma 2.4 implies that $v_t^* \notin S^*$. Assume that $v_t^* \in S_2^*$ lies on the same branching path as some v_g^* in T^*. Note that $|S^*| \geq 2$. Then there exists another vertex $v_{g'}^*$ such that $v_g^* \in I(v_t^*, v_{g'}^*)$. Combining this fact with $v_i \in I(v_i, v_p)$, we arrive at a contradiction: $w \in I_{T \,\square\, T^*}(z, w')$. This proves Claim 2.

Now let $S_{1'}^* = \{v_q^* : \; v_q^* \in I(v_g^*, v_{g'}^*), v_g^*, v_{g'}^* \in S^*\}$. Reasoning as in Subcase 2.1 and using Claim 2, we infer that $|\bigcup_{v_t^* \in S_{1'}^*}(V(T^{v_t^*}) \cap R)| \leq \ell(T)$.

Let $S = \{v_k : \; (v_k, v_t^*) \in \bigcup_{v_t^* \in S_{1'}^*}(V(T^{v_t^*}) \cap R)\}$ and set $S^{**} = V(T^*) \setminus (S^* \cup S_{1'}^*)$. From the assumption, $|\bigcup_{v_t^* \in S^{**}}(V(T^{v_t^*}) \cap R)| \geq \ell(T) + \ell(T^*) - |S| - |S^*| + 1$. So, there exists a vertex $z = (v_p, v_\ell^*) \in \bigcup_{v_t^* \in S^{**}}(V(T^{v_t^*}) \cap R)$ and we can always select two distinct vertices $u = (v_h, v_g^*)$ and $v = (v_{h'}, v_{g'}^*)$ from R such that v_p and v_h lie on the same branching path in T, while v_ℓ^* and $v_{g'}^*$ lie on a common branching path in T^*. But then we can choose another vertex $w \in R$ such that either u, w, z or u, v, z lie on the same geodesic in $T \,\square\, T^*$,

https://doi.org/10.1017/S0004972720001276 Published online by Cambridge University Press

which is a contradiction. Therefore, $|\bigcup_{v_i^* \in S^{**}}(V(T^{v_i^*}) \cap R)| \leq \ell(T) + \ell(T^*) - |S| - |S^*|$. This completes the proof.

References

[1] B. S. Anand, S. V. Ullas Chandran, M. Changat, S. Klavžar and E. J. Thomas, 'Characterization of general position sets and its applications to cographs and bipartite graphs', *Appl. Math. Comput.* **359** (2019), 84–89.

[2] R. Balakrishnan, S. F. Raj and T. Kavaskar, '*b*-coloring of Cartesian product of trees', *Taiwanese J. Math.* **20** (2016), 1–11.

[3] M. Ghorbani, S. Klavžar, H. R. Maimani, M. Momeni, F. Rahimi-Mahid and G. Rus, 'The general position problem on Kneser graphs and on some graph operations', *Discuss. Math. Graph Theory*, to appear. Published online (24 January 2020).

[4] W. Imrich, S. Klavžar and D. F. Rall, *Topics in Graph Theory: Graphs and their Cartesian Product* (A. K. Peters, Wellesley, MA, 2008).

[5] S. Klavžar, B. Patkós, G. Rus and I. G. Yero, 'On general position sets in Cartesian grids', Preprint, 2019, arXiv:1907.04535 [math.CO].

[6] S. Klavžar and G. Rus, 'The general position number of integer lattices', *Appl. Math. Comput.* **390** (2021), Article ID 125664.

[7] S. Klavžar and I. G. Yero, 'The general position problem and strong resolving graphs', *Open Math. J.* **17** (2019), 1126–1135.

[8] J. Körner, 'On the extremal combinatorics of the Hamming space', *J. Combin. Theory Ser. A* **71** (1995), 112–126.

[9] P. Manuel and S. Klavžar, 'A general position problem in graph theory', *Bull. Aust. Math. Soc.* **98** (2018), 177–187.

[10] P. Manuel and S. Klavžar, 'The graph theory general position problem on some interconnection networks', *Fund. Inform.* **163** (2018), 339–350.

[11] B. Patkós, 'On the general position problem on Kneser graphs', *Ars Math. Contemp.*, to appear.

[12] W. C. Shiu and R. M. Low, 'The integer-magic spectra and null sets of the Cartesian product of trees', *Australas. J. Combin.* **70** (2018), 157–167.

[13] J. Tian and K. Xu, 'The general position number of Cartesian products of trees or cycles with general graphs', Preprint, 2020.

[14] S. V. Ullas Chandran and G. Jaya Parthasarathy, 'The geodesic irredundant sets in graphs', *Int. J. Math. Combin.* **4** (2016), 135–143.

[15] D. R. Wood, 'Colouring the square of the Cartesian product of trees', *Discrete Math. Theor. Comput. Sci.* **13** (2011), 109–111.

JING TIAN, College of Science,
Nanjing University of Aeronautics and Astronautics,
Nanjing, Jiangsu 210016, PR China
e-mail: jingtian526@126.com

KEXIANG XU, College of Science,
Nanjing University of Aeronautics and Astronautics,
Nanjing, Jiangsu 210016, PR China
e-mail: kexxu1221@126.com

https://doi.org/10.1017/S0004972720001276 Published online by Cambridge University Press

SANDI KLAVŽAR,
Faculty of Mathematics and Physics, University of Ljubljana, Slovenia;
Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
and
Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
e-mail: sandi.klavzar@fmf.uni-lj.si