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ON THE DIFFERENCE OF COEFFICIENTS OF OZAKI CLOSE-TO-CONVEX FUNCTIONS

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Abstract

Let f be analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}$ be the subclass of normalised univalent functions given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. We give sharp upper and lower bounds for $|a_3| - |a_2|$ and other related functionals for the subclass $\mathcal{F}_O(\lambda)$ of Ozaki close-to-convex functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalised by $f(0) = 0 = f'(0) - 1$. Then, for $z \in \mathbb{D}$, a function $f \in \mathcal{A}$ has the representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of all univalent (that is, one-to-one) functions in $\mathcal{A}$.

In 1985, de Branges [4] solved the famous Bieberbach conjecture by showing that if $f \in \mathcal{S}$, then $|a_n| \le n$ for $n \ge 2$ with equality when $f(z) = k(z) := z/(1-z)^2$ or a rotation of it. It was therefore natural to ask if for $f \in \mathcal{S}$, the inequality $||a_{n+1}| - |a_n|| \le 1$ is true when $n \ge 2$. This was shown not to be the case even when $n = 2$ [5] and that the following sharp bounds hold.

$$-1 \le |a_3| - |a_2| \le \tfrac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.029\ldots,$$

where λ_0 is the unique value of λ in $0 < \lambda < 1$ satisfying the equation $4\lambda = e^{\lambda}$.

Hayman [7] showed that if $f \in \mathcal{S}$, then $||a_{n+1}| - |a_n|| \le C$, where C is an absolute constant. The exact value of C is unknown, the best estimate to date being $C = 3.61\ldots$ [6], which because of the sharp estimate above when $n = 2$ cannot be reduced to 1.

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Denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ consisting of starlike functions, that is, functions f which map $\mathbb{D}$ onto a set which is star-shaped with respect to the origin. Then it is well known that a function $f \in \mathcal{S}^*$ if and only if, for $z \in \mathbb{D}$,

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

We also recall the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, defined for $0 \le \alpha < 1$ by

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha.$$

It was shown in [9] that when $f \in \mathcal{S}^*$, then $||a_{n+1}| - |a_n|| \le 1$ is true when $n \ge 2$.

Next denote by $\mathcal{K}$ the subclass of $\mathcal{S}$ consisting of functions which are close-to-convex, that is, functions f which map $\mathbb{D}$ onto a close-to-convex domain. Then again it is well known that a function $f \in \mathcal{K}$ if and only if there exists $g \in \mathcal{S}^*$ such that, for $z \in \mathbb{D}$,

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0.$$

Extending the result $||a_{n+1}| - |a_n|| \le 1$ for $n \ge 2$ to close-to-convex functions remains an open problem. However, Koepf [8] has shown that if $f \in \mathcal{K}$, then $||a_3| - |a_2|| \le 1$.

The class $C(\alpha)$ for $0 \le \alpha < 1$ of convex functions of order α consisting of functions f satisfying

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for $z \in \mathbb{D}$ is well known and has been widely studied.

Finding the sharp upper and lower bounds for $|a_{n+1}| - |a_n|$, when $f \in C(0)$, that is, for the convex functions, appears to be a difficult problem, with the only significant results to date due to Ming and Sugawa [10], where sharp upper bounds have been found for $n \ge 2$ and sharp lower bounds when $n = 2$ and 3. For $n \ge 4$, finding the sharp lower bounds is an open problem.

Little attention has been given to the case when $\alpha < 0$, but several authors have considered the class $C(-1/2)$, consisting of functions satisfying

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}$$

for $z \in \mathbb{D}$, whose members are known to be close-to-convex. In particular, in a recent paper, Arora *et al.* [2] showed that if $f \in C(-1/2)$, then $|a_3| - |a_2| \le 1$, but this bound is not sharp.

The class $\mathcal{F}_O(\lambda)$ of Ozaki close-to-convex functions, defined for $z \in \mathbb{D}$ and $\frac{1}{2} \le \lambda \le 1$ by

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2} - \lambda, \tag{1.2}$$

was formally introduced in [1] and is also known to be a subclass of the close-to-convex functions. We note that clearly $\mathcal{F}_O(1/2) = C$ and $\mathcal{F}_O(1) = C(-1/2)$.

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In this paper we give sharp upper and lower bounds for $|a_3| - |a_2|$ when $f \in \mathcal{F}_O(\lambda)$, which in particular solves the problem when $f \in C(-1/2)$, and also find sharp upper and lower bounds for other related functions concerning coefficient differences.

2. Preliminary lemmas

Denote by $\mathcal{P}$ the class of analytic functions p with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{2.1}$$

We will use the following properties for the coefficients of functions in $\mathcal{P}$, given by (2.1).

LEMMA 2.1 [5, page 41]. *For $p \in \mathcal{P}$ and $\nu \in \mathbb{C}$,*

$$\left| p_2 - \frac{\nu}{2} p_1^2 \right| \le 2 \max\{|\nu - 1|, 1\}.$$

The inequality is sharp.

LEMMA 2.2 [3]. *If $p \in \mathcal{P}$, then*

$$p_1 = 2\zeta_1,$$
$$p_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2$$

and

$$p_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\bar{\zeta}_1\zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1, 2\}$.

For $\zeta_1 \in \mathbb{T}$, the boundary of $\mathbb{D}$, the unique function $p \in \mathcal{P}$ with p_1 as above is

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z} \quad (z \in \mathbb{D}).$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, the unique function $p \in \mathcal{P}$ with p_1 and p_2 as above is

$$p(z) = \frac{1 + (\overline{\zeta_1}\zeta_2 + \zeta_1)z + \zeta_2 z^2}{1 + (\overline{\zeta_1}\zeta_2 - \zeta_1)z - \zeta_2 z^2} \quad (z \in \mathbb{D}).$$

For $\zeta_1 \in \mathbb{D}$, $\zeta_2 \in \mathbb{T}$ and $\zeta_3 \in \mathbb{T}$, the unique function $p \in \mathcal{P}$ with p_1, p_2 and p_3 as above is

$$p(z) = \frac{1 + (\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 + \zeta_1)z + (\overline{\zeta_1}\zeta_3 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_2)z^2 + \zeta_3 z^3}{1 + (\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 - \zeta_1)z + (\overline{\zeta_1}\zeta_3 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_2)z^2 - \zeta_3 z^3} \quad (z \in \mathbb{D}).$$

We also note that from (1.2), we can write

$$1 + \frac{zf''(z)}{f'(z)} + \lambda - \frac{1}{2} = p(z) \tag{2.2}$$

for some $p \in \mathcal{P}$ and so, equating coefficients,

$$\begin{aligned} a_2 &= \tfrac{1}{4}(1 + 2\lambda)p_1, \\ a_3 &= \tfrac{1}{12}(1 + 2\lambda)(p_2 - \tfrac{1}{2}(1 - 2 - 2\lambda)p_1^2). \end{aligned} \tag{2.3}$$

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3. Main results

In order to prove the upper bound in our first result concerning the difference of coefficients, we use the following simple Fekete–Szegö inequality, which is an easy consequence of Lemma 2.1 (we omit the proof).

THEOREM 3.1. *Let $f \in \mathcal{F}_O(\lambda)$ and be given by (1.1). Then*

$$|a_3 - \mu a_2^2| \le \tfrac{1}{6}(1 + 2\lambda) \quad \textit{when } \tfrac{2}{3} \le \mu \le \tfrac{10}{9}.$$

We first prove the following sharp inequalities for $f \in \mathcal{F}_O(\lambda)$.

THEOREM 3.2. *Let $f \in \mathcal{F}_O(\lambda)$ and be given by (1.1). Then*

$$-\frac{1 + 2\lambda}{2\sqrt{3 + 2\lambda}} \le |a_3| - |a_2| \le \frac{1}{6}(1 + 2\lambda). \tag{3.1}$$

Both inequalities are sharp.

PROOF. Note first that using Theorem 3.1 when $\mu = 2/3$,

$$|a_3| - |a_2| \le |a_3 - \tfrac{2}{3}a_2^2| + \tfrac{2}{3}|a_2|^2 - |a_2| \le \tfrac{1}{6}(1 + 2\lambda) + \tfrac{2}{3}|a_2|^2 - |a_2|.$$

Since $|p_1| \le 2$, from (2.3), we have $|a_2| \le \frac{1}{2}(1 + 2\lambda)$ and a simple exercise shows that the maximum value of the right-hand side of the above is $\frac{1}{6}(1 + 2\lambda)$, as required.

We next prove the lower bound in (3.1). Write

$$|a_2| - |a_3| = \tfrac{1}{6}(1 + 2\lambda)\Psi, \tag{3.2}$$

where

$$\Psi = 3|\zeta_1| - |2(1 + \lambda)\zeta_1^2 + (1 - |\zeta_1|^2)\zeta_2|.$$

Since both $\mathcal{F}_O(\lambda)$ and $\mathcal{P}$ are rotationally invariant, we may assume that $\zeta_1 \in [0, 1]$. Write $\zeta_2 = se^{i\varphi}$ with $s \in [0, 1]$ and $\varphi \in \mathbb{R}$, so that

$$\Psi = 3\zeta_1 - |2(1 + \lambda)\zeta_1^2 + (1 - \zeta_1^2)se^{i\varphi}|.$$

Then

$$\begin{aligned}\Psi &= 3\zeta_1 - \sqrt{4(1 + \lambda)^2\zeta_1^4 + 4(1 + \lambda)\zeta_1^2(1 - \zeta_1^2)s\cos\varphi + (1 - \zeta_1^2)^2 s^2} \\ &\le 3\zeta_1 - |2(1 + \lambda)\zeta_1^2 - (1 - \zeta_1^2)s| \end{aligned} \tag{3.3}$$

with equality when $\cos\varphi = -1$.

Suppose that $2(1 + \lambda)\zeta_1^2 - (1 - \zeta_1^2)s \le 0$. Then $\zeta_1 \le \sqrt{s/(2 + 2\lambda + s)} =: \eta_1$ and so, by (3.3),

$$\begin{aligned}\Psi &\le (2 + 2\lambda + s)\zeta_1^2 + 3\zeta_1 - s \le (2 + 2\lambda + s)\eta_1^2 + 3\eta_1 - s \\ &= 3\sqrt{\frac{s}{2 + 2\lambda + s}} \le \frac{3}{\sqrt{3 + 2\lambda}}\end{aligned}$$

since $s \le 1$.

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If $2(1+\lambda)\zeta_1^2 - (1-\zeta_1^2)s \geq 0$, then $\zeta_1 \geq \eta_1$. Define ϕ by

$$\phi(x) = -(2+2\lambda+s)x^2 + 3x + y$$

and let

$$\eta_2 = \frac{3}{2(2+2\lambda+s)}$$

be the unique critical point of ϕ. Then, by (3.3),

$$\Psi \leq \phi(\zeta_1). \tag{3.4}$$

The condition $\eta_1 \leq \eta_2$ is equivalent to the inequality $4s^2 + 8(1+\lambda)s - 9 \leq 0$, which holds for $0 \leq s \leq \kappa$, where

$$\kappa = \tfrac{1}{2}(\sqrt{13+8\lambda+4\lambda^2} - 2(1+\lambda)).$$

It is easily seen that $0 < \kappa < 1$.

When $0 \leq s \leq \kappa$, (3.4) implies that

$$\Psi \leq \phi(\eta_2) = \frac{9}{4(2+2\lambda+s)} + s =: h(s). \tag{3.5}$$

Differentiating h gives

$$4(2+2\lambda+y)^2 h'(s) = 4s^2 + 16(1+\lambda)s + (16\lambda^2 + 32\lambda + 7) > 0,$$

so that h is increasing on the interval $[0,\kappa]$. So, from (3.5),

$$\Psi \leq h(s) \leq h(\kappa) = \frac{9}{4(2+2\lambda+\kappa)} + \kappa = \frac{9}{4(2+2\lambda+\kappa)}. \tag{3.6}$$

Next we note that

$$\frac{9}{4(2+2\lambda+\kappa)} \leq \frac{3}{\sqrt{3+2\lambda}}. \tag{3.7}$$

Indeed, (3.7) is equivalent to

$$4(2+2\lambda+\kappa) \geq \sqrt{3+2\lambda}$$

and, since $0 < \kappa < 1$,

$$16(2+2\lambda+\kappa)^2 - 3(3+2\lambda) \geq 16(2+2\lambda)^2 - 3(3+2\lambda) = 64\lambda^2 + 122\lambda + 55 > 0$$

for all $\lambda \in [1/2, 1]$. Thus, it follows from (3.6) and (3.7) that

$$\Psi \leq \frac{3}{\sqrt{3+2\lambda}}.$$

When $s \in [\kappa, 1]$, we have $\eta_1 \geq \eta_2$ and a similar method to that used in the case $\zeta_1 \leq \eta_1$ gives

$$\Psi \leq \phi(\eta_1) = 3\sqrt{\frac{s}{2+2\lambda+s}} \leq \frac{3}{\sqrt{3+2\lambda}},$$

which completes the proof of the first inequality in (3.1).

In order to show that the inequalities are sharp, first let the function f_1 be defined by (2.2) with $p(z) = (1+z^2)/(1-z^2)$. Then $f_1 \in \mathcal{F}_O(\lambda)$ with

$$f(z) = z + \tfrac{1}{6}(1+2\lambda)z^3 + \cdots .$$

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Next put $\zeta_1 = 1/\sqrt{2\lambda + 3}$ and $\zeta_2 = -1$. Then

$$\Psi_2 = 3|\zeta_1| - |2(1 + \lambda)\zeta_1^2 + (1 - |\zeta_1|^2)\zeta_2| = \frac{3}{\sqrt{2\lambda + 3}}. \tag{3.8}$$

Since $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, it follows from Lemma 2.2 that the function $\hat{p}$ defined by

$$\hat{p}(z) = \frac{1 - z^2}{1 - 2\zeta_1 z + z^2}$$

belongs to $\mathcal{P}$. Now let the function f_2 be defined by (2.2) with $p = \hat{p}$. Then $f_2 \in \mathcal{F}_O(\lambda)$ and so, from (3.2) and (3.8),

$$|a_2| - |a_3| = \frac{1}{6}(1 + 2\lambda)\Psi_2 = \frac{1 + 2\lambda}{2\sqrt{2\lambda + 3}},$$

which shows that the left-hand equality in (3.1) is sharp. This completes the proof of Theorem 3.2. □

We next note that $f \in C(\alpha)$ for $\alpha \in \mathbb{R}$ if and only if $zf' \in \mathcal{S}^*(\alpha)$. In [2], it was shown that if $f \in \mathcal{S}^*(\alpha)$ for $\alpha \le 0$, then

$$||a_{n+1}| - |a_n|| \le \frac{\Gamma(1 - 2\alpha + n)}{\Gamma(1 - 2\alpha)\Gamma(n + 1)} \tag{3.9}$$

with equality for $f(z) = z(1 - z)^{2(\alpha-1)}$. Using (3.9), we are now able to deduce the following theorem, thereby extending a result for $C(-1/2)$ proved in [2] to $\mathcal{F}_O(\lambda)$.

Theorem 3.3. *Let $f \in \mathcal{F}_O(\lambda)$ and be given by (1.1). Then*

$$|n|a_n| - m|a_m|| \le \frac{1}{\Gamma(2\lambda)} \sum_{k=m}^{n-1} \frac{\Gamma(2\lambda + k)}{\Gamma(1 + k)}.$$

The inequalities are sharp.

Proof. Let $f \in \mathcal{F}_O(\lambda)$. Then, since $zf' \in \mathcal{S}^*((1/2) - \lambda)$, we deduce from (3.9) that

$$|(k + 1)|a_{k+1}| - k|a_k|| \le \frac{\Gamma(2\lambda + k)}{\Gamma(2\lambda)\Gamma(k + 1)}, \quad k \in \mathbb{N}.$$

Here $a_1 = 1$. Using the triangle inequality, it follows that for $n \ge m$,

$$\begin{aligned} |n|a_n| - m|a_m|| &= \left|\sum_{k=m}^{n-1} (k + 1)|a_{k+1}| - k|a_k|\right| \\ &\le \sum_{k=m}^{n-1} \left|(k + 1)|a_{k+1}| - k|a_k|\right| \\ &\le \frac{1}{\Gamma(2\lambda)} \sum_{k=m}^{n-1} \frac{\Gamma(2\lambda + k)}{\Gamma(1 + k)}. \end{aligned}$$

Clearly equality holds for $f \in \mathcal{F}_O(\lambda)$ defined by

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + 2\lambda z}{1 - z}, \quad z \in \mathbb{D}.$$

□

THEOREM 3.4. *Let $f \in \mathcal{F}_O(\lambda)$ and be given by (1.1). Then*

$$|a_3 - a_2| \le \tfrac{1}{6}(1 + 2\lambda)(5 + 2\lambda). \tag{3.10}$$

The inequality is sharp.

PROOF. Let $f \in \mathcal{F}_O(\lambda)$ and be given by (1.1). Then, since

$$a_2 = \tfrac{1}{4}p_1(1 + 2\lambda) \quad \text{and} \quad a_3 = \tfrac{1}{12}(1 + 2\lambda)[p_2 + \tfrac{1}{2}p_1^2(1 + 2\lambda)],$$

from Lemma 2.2,

$$\begin{aligned} a_3 - a_2 &= \tfrac{1}{12}(1 + 2\lambda)[p_2 + \tfrac{1}{2}p_1^2(1 + 2\lambda) - 3p_1] \\ &= \tfrac{1}{6}(1 + 2\lambda)\Psi(\zeta_1, \zeta_2), \quad \text{with } \zeta_1, \zeta_2 \in \overline{\mathbb{D}}, \end{aligned} \tag{3.11}$$

where

$$\Psi(\zeta_1, \zeta_2) = 2(1 + \lambda)\zeta_1^2 - 3\zeta_1 + (1 - |\zeta_1|^2)\zeta_2.$$

Since $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}$,

$$\begin{aligned} |\Psi(\zeta_1, \zeta_2)| &\le 2(1 + \lambda)|\zeta_1|^2 + 3|\zeta_1| + (1 - |\zeta_1|^2) \\ &= (1 + 2\lambda)|\zeta_1|^2 + 3|\zeta_1| + 1 \\ &\le 5 + 2\lambda. \end{aligned} \tag{3.12}$$

Thus, from (3.11) and (3.12), we obtain (3.10).

To see that (3.10) is sharp, consider $f_1 : \mathbb{D} \to \mathbb{R}$ defined so that

$$1 + \frac{zf_1''(z)}{f_1'(z)} = \left(\frac{1}{2} + \lambda\right)\frac{1 - z}{1 + z} + \frac{1}{2} - \lambda, \quad z \in \mathbb{D}.$$

Then $f_1 \in \mathcal{F}_O(\lambda)$, with expansion

$$f_1(z) = z - (\tfrac{1}{2} + \lambda)z^2 + \tfrac{1}{3}(1 + \lambda)(1 + 2\lambda)z^3 + \cdots, \quad z \in \mathbb{D},$$

which gives equality in (3.10) for f_1. □

References

[1] V. Allu, N. Tuneski and D. K. Thomas, ‘On Ozaki close-to-convex functions’, *Bull. Aust. Math. Soc.* **99**(1) (2019), 89–100.

[2] V. Arora, S. Ponnusamy and S. Sahoo, ‘Successive coefficients for spirallike and related functions’, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **113** (2019), 2969–2979.

[3] N. E. Cho, B. Kowalczyk and A. Lecko, ‘Sharp bounds of some coefficient functionals over the class of functions convex in the direction of the imaginary axis’, *Bull. Aust. Math. Soc.* **100**(1) (2019), 86–96.

[4] L. De Branges, 'A proof of the Bieberbach conjecture', *Acta Math.* **154**(1–2) (1985), 137–152.
[5] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften, 259 (Springer, New York, 1983).
[6] A. Z. Grinspan, 'The sharpening of the difference of the moduli of adjacent coefficients of schlicht functions', in: *Some Problems in Modern Function Theory, Proc. Conf. Modern Problems of Geometric Theory of Functions* (Institute of Mathematics, Academy of Sciences of the USSR, Novosibirsk, 1976), 41–45. (in Russian).
[7] W. K. Hayman, 'On successive coefficients of univalent functions', *J. Lond. Math. Soc.* **38** (1963), 228–243.
[8] W. Koepf, 'On the Fekete–Szego problem for close to convex functions', *Proc. Amer. Math. Soc.* **101** (1987), 89–95.
[9] Y. Leung, 'Successive coefficients of starlike functions', *Bull. Lond. Math. Soc.* **10** (1978), 193–196.
[10] L. Ming and T. Sugawa, 'A note on successive coefficients of convex functions', *Comput. Methods Funct. Theory* **17**(2) (2017), 17900193.

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