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Asymptotic behaviour as $p \to \infty$ of least energy solutions of a $(p, q(p))$-Laplacian problem

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We study the asymptotic behaviour, as $p \to \infty$, of the least energy solutions of the problem

$$\begin{cases} -(\Delta_p + \Delta_{q(p)})u = \lambda_p |u(x_u)|^{p-2} u(x_u) \delta_{x_u} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where x_u is the (unique) maximum point of $|u|$, δ_{x_u} is the Dirac delta distribution supported at x_u,

$$\lim_{p\to\infty} \frac{q(p)}{p} = Q \in \begin{cases} (0,1) & \text{if} \quad N < q(p) < p \\ (1,\infty) & \text{if} \quad N < p < q(p) \end{cases}$$

and $\lambda_p > 0$ is such that

$$\min \left\{ \frac{\|\nabla u\|_\infty}{\|u\|_\infty} : 0 \not\equiv u \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}) \right\} \leqslant \lim_{p\to\infty} (\lambda_p)^{1/p} < \infty.$$

Keywords: Asymptotic behaviour; dirac delta; infinity Laplacian; Nehari set; viscosity solution

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1. Introduction

In this paper, we first study in § 2 the existence of nonnegative least energy solutions for the Dirichlet problem

$$\begin{cases} -(\Delta_p + \Delta_q)u = \lambda \|u\|_r^{p-r} |u|^{r-2} u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 2$,

$$(\Delta_p + \Delta_q)u := \operatorname{div}[(|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u]$$

is the (p, q)-Laplacian operator, $\lambda > 0$ and $1 \leqslant r < \infty$. (In the whole paper we denote by $\|\cdot\|_s$ the standard norm of the Lebesgue space $L^s(\Omega)$, with $1 \leqslant s \leqslant \infty$).

Our main results, inspired by the recent papers [**3**, **9**], are presented in §§ 3 and 4. In § 3, we show the limit problem of (1.1) as $r \to \infty$ is the following

$$\begin{cases} -(\Delta_p + \Delta_q)u = \lambda |u(x_u)|^{p-2} u(x_u)\delta_{x_u} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where x_u is the (unique) maximum point of $|u|$ and δ_{x_u} is the Dirac delta distribution supported at x_u.

More precisely, we prove in proposition 3.6 that if $\lambda > \lambda_\infty(p)$, where

$$\lambda_\infty(p) := \min\left\{\frac{\|\nabla u\|_p^p}{\|u\|_\infty^p} : u \in W_0^{1,p}(\Omega) \setminus \{0\}\right\}, \tag{1.3}$$

and u_n denotes a nonnegative least energy solution of (1.1) for $r = r_n \to \infty$, then there exists a subsequence of $\{u_n\}$ converging strongly in $X_{p,q} := W_0^{1,\max\{p,q\}}(\Omega)$ to a nonnegative least energy solution of (1.2).

Least energy solutions for (1.2) are defined in this paper as the minimizers of the energy functional

$$J_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}\|u\|_\infty^p,$$

either on $W_0^{1,q}(\Omega)$, if $N < p < q < \infty$, or on the 'Nehari set'

$$\mathcal{N}_{\lambda,\infty} := \{u \in W_0^{1,p}(\Omega) : \|\nabla u\|_p^p + \|\nabla u\|_q^q = \lambda \|u\|_\infty^p\},$$

if $N < q < p < \infty$.

Although not differentiable, the functional $u \mapsto \|u\|_\infty^p$ has right Gateaux derivative at any $u \in C(\overline{\Omega})$. Using this fact we show in proposition 3.5 that the least energy solutions of (1.2) are weak solutions of this problem. It is simple to verify (see remark 3.2) that (1.2) cannot have weak solutions when $\lambda \leqslant \lambda_\infty(p)$.

In § 4, we consider $q = q(p)$, with

$$\lim_{p\to\infty} \frac{q(p)}{p} =: Q \in \begin{cases} (0, 1) & \text{if} \quad N < q(p) < p \\ (1, \infty) & \text{if} \quad N < p < q(p), \end{cases} \tag{1.4}$$

and fix $\Lambda \geqslant \Lambda_\infty$, where

$$\Lambda_\infty := \min\left\{\frac{\|\nabla u\|_\infty}{\|u\|_\infty} : 0 \not\equiv u \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})\right\} \tag{1.5}$$

and

$$C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u = 0 \,\text{on}\, \partial\Omega\}.$$

Then, taking $\lambda_p > 0$ satisfying

$$\lim_{p\to\infty} (\lambda_p)^{1/p} = \Lambda \geqslant \Lambda_\infty$$

we study the asymptotic behaviour, as $p \to \infty$, of the least energy solutions u_p of

$$\begin{cases} -(\Delta_p + \Delta_{q(p)})u = \lambda_p |u(x_u)|^{p-2} u(x_u)\delta_{x_u} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.6}$$

After deriving suitable estimates for u_p in $W_0^{1,m}(\Omega)$, for each $m > N$, we use the compactness of the embedding $W_0^{1,m}(\Omega) \hookrightarrow C(\overline{\Omega})$ to prove that any sequence $\{u_{p_n}\}$, with $p_n \to \infty$, admits a subsequence converging uniformly in $\overline{\Omega}$ to a function $u_\Lambda \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})$, which is strictly positive in Ω and attains its (unique) maximum point at $x_\Lambda \in \Omega$.

Moreover, we prove that u_Λ is ∞-harmonic in the punctured domain $\Omega \setminus \{x_\Lambda\}$, meaning that it satisfies, in the viscosity sense,

$$\Delta_\infty u_\Lambda = 0 \quad \text{in} \quad \Omega \setminus \{x_\Lambda\},$$

where

$$\Delta_\infty u := \frac{1}{2} \nabla u \cdot \nabla |\nabla u|^2$$

denotes the ∞-Laplacian.

In addition, we show that if either $\Lambda = \Lambda_\infty$ or $\Lambda > \Lambda_\infty$ and $Q \in (0, 1)$, then u_Λ realizes the minimum in (1.5) and satisfies

$$\|u_\Lambda\|_\infty = \frac{1}{\Lambda_\infty} \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))} \quad \text{and} \quad \|\nabla u_\Lambda\|_\infty = \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))}.$$

Hence, taking into account that $\Lambda_\infty = (\|\rho\|_\infty)^{-1}$, where $\rho : \overline{\Omega} \to [0, \infty)$ denotes the distance function to the boundary $\partial\Omega$, we conclude that

$$0 \leqslant u_\Lambda(x) \leqslant \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))} \rho(x), \quad \forall\, x \in \overline{\Omega}$$

and

$$\rho(x_\Lambda) = \|\rho\|_\infty.$$

These results are gathered in theorems 4.3 and 4.14, and their corollaries. In order to show how they fit into the recent literature, let us provide a brief review on some related problems, involving exponents p and $q(p)$, with $p \to \infty$.

We start with a case involving the p-Laplacian operator and a simpler dependence $q(p) = p$, considered by Juutinen, Lindqvist, and Manfredi in [**13**]. In that paper, the authors studied the limit problem, as $p \to \infty$, of

$$\begin{cases} -\Delta_p u = \lambda_p(p)|u|^{p-2}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.7}$$

where according to the notation we use in this paper (see (2.1)),

$$\lambda_p(p) := \min\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega) \setminus \{0\}\right\}.$$

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They first showed that,

$$\lim_{p\to\infty}(\lambda_p(p))^{1/p} = \Lambda_\infty$$

and then, denoting by u_p the positive, L^p-normalized weak solution of (1.7), proved that any sequence $\{u_{p_n}\}$, with $p_n \to \infty$, admits a subsequence converging uniformly in $\overline{\Omega}$ to a function u_∞ which is positive in Ω, L^∞-normalized and solves, in the viscosity sense, the problem

$$\begin{cases} \min\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\} = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.8}$$

These results were independently obtained by Fukagai, Ito and Narukawa in [10], where the asymptotic behaviour, as $p \to \infty$, of the higher (variational) eigenvalues of the Dirichlet p-Laplacian were also studied. Furthermore, in the recent paper [8], da Silva, Rossi and Salort showed that (1.8) has a unique (up to scalar multiplication) maximal solution $\widehat{v} \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})$ in the following sense: if u is a nonnegative, L^∞-normalized viscosity solution of (1.8), then $u \leqslant \widehat{v}$.

Charro and Peral in [4] ($q(p) < p$), and Charro and Parini in [5] ($q(p) > p$), studied the asymptotic behaviour, as $p \to \infty$, of the positive weak solutions u_p of the problem

$$\begin{cases} -\Delta_p u = \lambda_p |u|^{q(p)-2} u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda_p > 0$ is such that $\lim_{p\to\infty}(\lambda_p)^{1/p} = \Lambda \in (0, \infty)$. A consequence of the results proved in these papers is that the limit functions of the family $\{u_p\}$, as $p \to \infty$, are viscosity solutions of the problem

$$\begin{cases} \min\left\{|\nabla u| - \Lambda(u^Q), -\Delta_\infty u\right\} = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where here and in what follows Q is given by (1.4).

In [6] Charro and Parini proved that any uniform limit, as $p \to \infty$, of a sequence of positive weak solutions of the problem

$$\begin{cases} -\Delta_p u = \lambda_p |u|^{p-2} u + |u|^{q(p)-2} u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda_p > 0$ is such that $\lim_{p\to\infty}(\lambda_p)^{1/p} = \Lambda \in [0, \Lambda_\infty]$, must be a viscosity solution of the problem

$$\begin{cases} \min\{|\nabla u| - \max\{\Lambda u, (u^Q)\}, -\Delta_\infty u\} = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Bocea and Mihăilescu considered in [3] the family $\{u_p\}$ of nonnegative least energy solutions of the problem

$$\begin{cases} -(\Delta_p + \Delta_{q(p)})u = \lambda_p |u|^{p-2} u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda_p > 0$ is such that $\lim_{p\to\infty}(\lambda_p)^{1/p} = \Lambda \geqslant \Lambda_\infty$. They proved that the uniform limit, as $p \to \infty$, of a sequence of $\{u_p\}$ solves, in the viscosity sense, the problem

$$\begin{cases} \min\{\max\{|\nabla u|, |\nabla u|^Q\} - \Lambda u, -\Delta_\infty u\} = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Ercole and Pereira, in [9], showed that

$$\lim_{p\to\infty} (\lambda_\infty(p))^{1/p} = \Lambda_\infty$$

and proved that any positive minimizer u_p in (1.3) has a unique maximum point x_p and is a weak solution of the problem

$$\begin{cases} -\Delta_p u = \lambda_\infty(p)|u(x_p)|^{p-2}u_p(x_p)\delta_{x_p} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where δ_{x_p} denotes the Dirac delta distribution supported at x_p (note that $q(p) = p$). Furthermore, they proved that any normalized sequence $\{u_{p_n}/\|u_{p_n}\|_\infty\}$, with $p_n \to \infty$, admits a subsequence converging uniformly in $\overline{\Omega}$ to a function $w_\infty \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})$, which is positive in Ω and assumes its maximum value 1 at a unique point $x_* \in \Omega$. Moreover, w_∞ realizes the minimum in (1.5) and satisfies

$$\begin{cases} \Delta_\infty u = 0 & \text{in} \quad \Omega \setminus \{x_*\} \\ u = \rho/\|\rho\|_\infty & \text{on} \quad \partial(\Omega \setminus \{x_*\}) = \partial\Omega \cup \{x_*\} \end{cases}$$

in the viscosity sense.

2. Existence for $1 \leqslant r < q^\star$ and $\lambda > \lambda_r(p)$

We recall that the embedding $W_0^{1,m}(\Omega) \hookrightarrow L^r(\Omega)$ is compact whenever

$$1 \leqslant r < m^\star := \begin{cases} \frac{Nm}{N-m} & \text{if} \quad 1 < m < N \\ \infty & \text{if} \quad N \leqslant m. \end{cases}$$

Thus, the Rayleigh quotient associated with this embedding assumes its minimum value, which is positive:

$$0 < \lambda_r(m) := \min\left\{ \frac{\|\nabla u\|_m^m}{\|u\|_r^m} : u \in W_0^{1,m}(\Omega) \setminus \{0\} \right\}, \quad 1 \leqslant r < m^\star. \tag{2.1}$$

In this section, we consider in the Sobolev space

$$X_{p,q} := W_0^{1,\max\{p,q\}}(\Omega),$$

the boundary value problem

$$\begin{cases} -(\Delta_p + \Delta_q)u = \lambda\|u\|_r^{p-r}|u|^{r-2}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where $1 \leqslant p, q < \infty$, $p \neq q$ and $1 \leqslant r < q^\star$.

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The energy functional $I_{\lambda,r} : X_{p,q} \to \mathbb{R}$ associated with (2.2) is given by

$$I_{\lambda,r}(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}\|u\|_r^p.$$

It belongs to $C^1(X_{p,q})$ and its Gateaux derivative is expressed as

$$\langle I'_{\lambda,r}(u), v\rangle := \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla v \mathrm{d}x - \lambda\|u\|_r^{p-r}\int_\Omega |u|^{r-2}uv\mathrm{d}x,$$
$$\forall v \in X_{p,q}.$$

Definition 2.1. We say that $u \in X_{p,q}$ is a weak solution of (2.2) if

$$\langle I'_{\lambda,r}(u), v\rangle = 0 \quad \forall v \in X_{p,q}.$$

We remark that a nontrivial weak solution of (2.2) cannot exist if $\lambda \leqslant \lambda_r(p)$. In fact, such a weak solution u would satisfy

$$\lambda\|u\|_r^p = \|\nabla u\|_p^p + \|\nabla u\|_q^q > \|\nabla u\|_p^p \geqslant \lambda_r(p)\|u\|_r^p,$$

so that $(\lambda - \lambda_r(p))\|u\|_r^p > 0$.

We show in the sequel that the functional $I_{\lambda,r}$ has a global minimizer whenever $1 < p < q < \infty$. Thus, it is clear that such a minimizer is a weak solution of (2.2), since it must be a critical point of $I_{\lambda,r}$.

In the case $1 < q < p < \infty$ the functional $I_{\lambda,r}$ is not globally bounded from below. In fact, if $e_r \in W_0^{1,p}(\Omega)$ is such that

$$\|e_r\|_r = 1 \quad \text{and} \quad \|\nabla e_r\|_p^p = \lambda_r(p), \tag{2.3}$$

then

$$I_{\lambda,r}(te_r) = \frac{t^q}{q}\|\nabla e_r\|_q^q - t^p\frac{(\lambda - \lambda_r(p))}{p} \to -\infty \quad \text{as} \quad t \to \infty.$$

However, as we will see soon, in this case the functional $I_{\lambda,r}$ assumes the minimum value on the Nehari manifold defined by

$$\mathcal{N}_{\lambda,r} := \{u \in X_{p,q} \setminus \{0\} : \langle I'_{\lambda,r}(u), u\rangle = 0\} = \{u \in X_{p,q} \setminus \{0\} : \|\nabla u\|_p^p + \|\nabla u\|_q^q$$
$$= \lambda\|u\|_r^p\}.$$

Note that if $u \in \mathcal{N}_{\lambda,r}$ then

$$\begin{aligned} I_{\lambda,r}(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}\|u\|_r^p \\ &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{1}{p}(\|\nabla u\|_p^p + \|\nabla u\|_q^q) \\ &= \left(\frac{1}{q} - \frac{1}{p}\right)\|\nabla u\|_q^q. \end{aligned}$$

Moreover, it follows from the identity

$$\langle I'_{\lambda,r}(tv), tv\rangle = t^q[\|\nabla v\|_q^q - t^{p-q}(\lambda\|v\|_r^p - \|\nabla v\|_p^p)], \quad v \in X_{p,q}, \quad t > 0,$$

that if $v \in X_{p,q} \setminus \{0\}$, then $tv \in \mathcal{N}_{\lambda,r}$ (for some $t > 0$) if, and only if,

$$\|\nabla v\|_p^p < \lambda\|v\|_r^p \quad \text{and} \quad t = \left(\frac{\|\nabla v\|_q^q}{\lambda\|v\|_r^p - \|\nabla v\|_p^p}\right)^{((1)/(p-q))}. \tag{2.4}$$

A first consequence of this fact is that $\mathcal{N}_{\lambda,r}$ is not empty, since

$$\|\nabla e_r\|_p^p = \lambda_r(p) < \lambda = \lambda\|e_r\|_r^p.$$

For the sake of completeness, we show now that a minimizer of $I_{\lambda,r}$ on $\mathcal{N}_{\lambda,r}$ is also a weak solution of (2.2) whenever $1 < q < p < \infty$.

Proposition 2.2. *Suppose that $1 < q < p < \infty$ and that $u_\lambda \in \mathcal{N}_{\lambda,r}$ is such that $I_{\lambda,r}(u_\lambda) \leqslant I_{\lambda,r}(v)$ for all $v \in \mathcal{N}_{\lambda,r}$. Then u_λ is a weak solution of* (2.2).

Proof. Since $u_\lambda \in \mathcal{N}_{\lambda,r}$ we have $\|\nabla u_\lambda\|_p^p < \|\nabla u_\lambda\|_p^p + \|\nabla u_\lambda\|_q^q = \lambda\|u_\lambda\|_r^p$. Hence, for a fixed $v \in X_{p,q}$ we can take $\delta > 0$ such that $u_\lambda + sv \not\equiv 0$ and

$$\|\nabla(u_\lambda + sv)\|_p^p < \lambda\|u_\lambda + sv\|_r^p, \quad \forall s \in (-\delta, \delta).$$

Let $\tau : (-\delta, \delta) \to (0, \infty)$ be the differentiable function given by

$$\tau(s) = \left(\frac{\|\nabla(u_\lambda + sv)\|_q^q}{\lambda\|u_\lambda + sv\|_r^p - \|\nabla(u_\lambda + sv)\|_p^p}\right)^{((1)/(p-q))}.$$

We can see from (2.4) that $\tau(s)(u_\lambda + sv) \in \mathcal{N}_{\lambda,r}$ for all $s \in (-\delta, \delta)$ and that $\tau(0) = 1$ (since $u_\lambda \in \mathcal{N}_{\lambda,r}$).

Taking into account that the differentiable function $\gamma : (-\delta, \delta) \to \mathbb{R}$, defined by

$$\gamma(s) = I_{\lambda,r}(\tau(s)(u_\lambda + sv)),$$

attains its minimum value at $s = 0$, we have

$$\begin{aligned} 0 &= \gamma'(0) \\ &= \langle I'_{\lambda,r}(u_\lambda), \tau'(0)u_\lambda + \tau(0)v)\rangle = \tau'(0)\langle I'_{\lambda,r}(u_\lambda), u_\lambda)\rangle + \tau(0)\langle I'_{\lambda,r}(u_\lambda), v)\rangle \\ &= \langle I'_{\lambda,r}(u_\lambda), v)\rangle. \end{aligned}$$

□

Definition 2.3. We say that a function $u \in X_{p,q}$ is a least energy solution of (2.2) if it minimizes the functional $I_{\lambda,r}$ either on $X_{p,q} \setminus \{0\}$ in the case $1 < p < q < \infty$, or on $\mathcal{N}_{\lambda,r}$ in the case $1 < q < p < \infty$.

Our main goal in this section is to prove that (2.2) has at least one nonnegative least energy solution. We assume that $1 \leqslant r < q^\star$ and $\lambda > \lambda_r(p)$.

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PROPOSITION 2.4. *Suppose that $1 < p, q < \infty$ $(p \neq q)$, $1 \leqslant r < q^{\star}$ and $\lambda > \lambda_r(p)$. The problem (2.2) has at least one nonnegative least energy solution u_λ.*

Proof. We start with the case $1 < p < q < \infty$, in which $X_{p,q} = W_0^{1,q}(\Omega)$. It is simple to verify that $I_{\lambda,r}$ is bounded from below and coercive. In fact,

$$\begin{aligned} I_{\lambda,r}(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}\|u\|_r^p \\ &\geqslant \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}\|u\|_r^p \\ &\geqslant \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}(\lambda_r(q))^{-p/q}\|\nabla u\|_q^p = g(\|\nabla u\|_q), \end{aligned}$$

where the function $g : [0, \infty) \longrightarrow \mathbb{R}$, given by $g(t) := 1/qt^q - ((\lambda(\lambda_r(q))^{-p/q})/(p))t^p$, satisfies

$$-\infty < \inf\{g(t) : t \in [0, \infty)\} \quad \text{and} \quad \lim_{t\to\infty} g(t) = \infty.$$

Thus, taking into account that $I_{\lambda,r}$ is also weakly sequentially lower semi-continuous, there exists $u_\lambda \in X_{p,q}$ such that

$$I_{\lambda,r}(u_\lambda) = \min\{I_{\lambda,r}(u) : u \in X_{p,q}\}.$$

Noting that $I_{\lambda,r}(u_\lambda) = I_{\lambda,r}(|u_\lambda|)$ we can assume that $u_\lambda \geqslant 0$ in Ω. In order to show that $u_\lambda \not\equiv 0$ it is sufficient to check that $I_{\lambda,r}$ assumes negative values in $X_{p,q}$ (note that $I_{\lambda,r}(0) = 0$). For this, by using a function $e_r \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \subset X_{p,q}$ satisfying (2.3), we have

$$I_{\lambda,r}(te_r) = \frac{t^q}{q}\|\nabla e_r\|_q^q - t^p\frac{(\lambda - \lambda_r(p))}{p} < 0$$

for all positive t sufficiently small.

Now, we study the case $1 < q < p < \infty$ in which $X_{p,q} = W_0^{1,p}(\Omega)$.

Since $1 \leqslant r < q^{\star} \leqslant p^{\star}$ (the latter inequality is an equality only in the case $N \leqslant q < p$) we have

$$\|u\|_r^q \leqslant \frac{1}{\lambda_r(q)}\|\nabla u\|_q^q \leqslant \frac{1}{\lambda_r(q)}(\|\nabla u\|_q^q + \|\nabla u\|_p^p) = \frac{\lambda}{\lambda_r(q)}\|u\|_r^p, \quad \forall u \in \mathcal{N}_{\lambda,r},$$

implying that

$$\|u\|_r \geqslant \left(\frac{\lambda_r(q)}{\lambda}\right)^{((1)/(p-q))} > 0, \quad \forall u \in \mathcal{N}_{\lambda,r}. \tag{2.5}$$

It follows that $I_{\lambda,r}$ restricted to $\mathcal{N}_{\lambda,r}$ is bounded from below by a positive constant:

$$\begin{aligned} I_{\lambda,r}(u) &= \left(\frac{1}{q} - \frac{1}{p}\right)\|\nabla u\|_q^q \\ &\geqslant \left(\frac{1}{q} - \frac{1}{p}\right)\lambda_r(q)\|u\|_r^q \geqslant \left(\frac{1}{q} - \frac{1}{p}\right)\lambda_r(q)\left(\frac{\lambda_r(q)}{\lambda}\right)^{((q)/(p-q))} > 0. \end{aligned}$$

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Let us show that

$$m_\lambda := \inf\{I_{\lambda,r}(u) : u \in \mathcal{N}_{\lambda,r}\}$$

is attained in $\mathcal{N}_{\lambda,r}$. Let $\{u_n\} \subset \mathcal{N}_{\lambda,r}$ be a minimizing sequence, that is,

$$I_{\lambda,r}(u_n) = \left(\frac{1}{q} - \frac{1}{p}\right) \|\nabla u_n\|_q^q \to m_\lambda.$$

It follows that $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ and hence, taking into account that

$$\|\nabla u_n\|_p^p \leqslant \|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q = \lambda \|u_n\|_r^p \leqslant \frac{\lambda}{\lambda_r(q)^{p/q}} \|\nabla u_n\|_q^p,$$

we conclude that $\{u_n\}$ is also bounded in $W_0^{1,p}(\Omega)$. Thus, we can assume that, up to a subsequence, $\{u_n\}$ converges to a function u_λ, weakly in both spaces $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, and strongly in $L^r(\Omega)$.

It follows from (2.5) that

$$\|u_\lambda\|_r = \lim_{n\to\infty} \|u_n\|_r \geqslant \left(\frac{\lambda_r(q)}{\lambda}\right)^{((1)/(p-q))} > 0,$$

so that $u_\lambda \not\equiv 0$.

Moreover,

$$\begin{aligned}\|\nabla u_\lambda\|_p^p &< \|\nabla u_\lambda\|_p^p + \|\nabla u_\lambda\|_q^q \\ &\leqslant \liminf_{n\to\infty}(\|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q) = \liminf_{n\to\infty} \lambda \|u_n\|_r^p = \lambda \|u_\lambda\|_r^p.\end{aligned}$$

Hence, $t_\lambda u_\lambda \in \mathcal{N}_{\lambda,r}$ where

$$t_\lambda = \left(\frac{\|\nabla u_\lambda\|_q^q}{\lambda \|u_\lambda\|_r^p - \|\nabla u_\lambda\|_p^p}\right)^{((1)/(p-q))} \leqslant 1.$$

It follows that

$$\begin{aligned} m_\lambda &\leqslant I_{\lambda,r}(t_\lambda u_\lambda) \\ &= (t_\lambda)^q \left(\frac{1}{q} - \frac{1}{p}\right) \|\nabla u_\lambda\|_q^q \\ &\leqslant (t_\lambda)^q \left(\frac{1}{q} - \frac{1}{p}\right) \liminf_{n\to\infty} \|\nabla u_n\|_q^q = (t_\lambda)^q \liminf_{n\to\infty} I_{\lambda,r}(u_n) = (t_\lambda)^q m_\lambda \leqslant m_\lambda. \end{aligned}$$

Consequently, $t_\lambda = 1$, $u_\lambda \in \mathcal{N}_{\lambda,r}$ and, $I_{\lambda,r}(u_\lambda) = m_\lambda$.

Since $|u_\lambda| \in \mathcal{N}_{\lambda,r}$ and $I_{\lambda,r}(|u_\lambda|) = I_{\lambda,r}(u_\lambda) = m_\lambda$, we can assume that u_λ is nonnegative. □

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3. The limit problem as $r \to \infty$

In this section, we fix $p, q > N$, $p \neq q$, and study the following Dirichlet problem

$$\begin{cases} -(\Delta_p + \Delta_q)u = \lambda |u(x_u)|^{p-2} u(x_u) \delta_{x_u} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where x_u is a maximum point of $|u|$ (so that $|u(x_u)| = \|u\|_\infty$) and δ_{x_u} is the delta Dirac distribution supported at x_u.

As we will see in the sequel (3.1) is the limit problem of (1.1) as $r \to \infty$.

Definition 3.1. We say that $u \in X_{p,q}$ is a weak solution of (3.1) if $|u(x_u)| = \|u\|_\infty$ and

$$\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \cdot \nabla v \mathrm{d}x = \lambda |u(x_u)|^{p-2} u(x_u) v(x_u), \quad \forall\, v \in X_{p,q}. \tag{3.2}$$

Let us recall the Morrey's inequality, valid if $m > N$:

$$C\|u\|_{0,\alpha_m}^m \leqslant \|\nabla u\|_m^p, \quad \forall\, u \in W_0^{1,m}(\Omega),$$

where $\|u\|_{0,s}$ denotes the standard norm in the Hölder space $C^{0,s}(\overline{\Omega})$, $\alpha_m = 1 - N/m$ and the positive constant C depends only on Ω, m and N.

Morrey's inequality implies that the embedding $W_0^{1,m}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact and this fact guarantees that the infimum of the Rayleigh quotient $\|\nabla v\|_m^m / \|v\|_\infty^m$ is attained in $W_0^{1,m}(\Omega) \setminus \{0\}$. From now on, we make use of the additional notation

$$\lambda_\infty(m) := \min \left\{ \frac{\|\nabla v\|_m^m}{\|v\|_\infty^m} : v \in W_0^{1,m}(\Omega) \setminus \{0\} \right\}, \quad m > N.$$

As it is shown in [9],

$$\lim_{r \to \infty} \lambda_r(m) = \lambda_\infty(m). \tag{3.3}$$

Remark 3.2. A nontrivial weak solution for (3.1) cannot exist if $\lambda \leqslant \lambda_\infty(p)$. Indeed, taking $v = u$ in (3.2) one has

$$(\lambda - \lambda_\infty(p))\|u\|_\infty^p = \|\nabla u\|_p^p + \|\nabla u\|_q^q - \lambda_\infty(p)\|u\|_\infty^p > \|\nabla u\|_p^p - \lambda_\infty(p)\|u\|_\infty^p \geqslant 0.$$

So, we assume in the rest of this section that $\lambda > \lambda_\infty(p)$.

We define the energy functional $J_\lambda : X_{p,q} \to \mathbb{R}$ associated with (3.1) by

$$J_\lambda(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{p}\|u\|_\infty^p$$

and the Nehari set associated with J_λ by

$$\mathcal{N}_{\lambda,\infty} := \left\{ u \in X_{p,q} \setminus \{0\} : \|\nabla u\|_p^p + \|\nabla u\|_q^q = \lambda \|u\|_\infty^p \right\}.$$

Note that

$$u \in \mathcal{N}_{\lambda,\infty} \Longrightarrow J_\lambda(u) = \left(\frac{1}{q} - \frac{1}{p}\right) \|\nabla u\|_q^q.$$

Moreover, the identity

$$\|\nabla(tu)\|_p^p + \|\nabla(tu)\|_q^q - \lambda\|tu\|_\infty^p = t^q[\|\nabla u\|_q^q - t^{p-q}(\lambda\|u\|_\infty^p - \|\nabla u\|_p^p)],$$
$$v \in X_{p,q}, \quad t > 0,$$

allows us to derive the following equivalence, valid for the case $N < q < p < \infty$:

$$tu \in \mathcal{N}_{\lambda,\infty} \iff \lambda\|u\|_\infty^p > \|\nabla u\|_p^p \quad \text{and} \quad t = \left(\frac{\|\nabla u\|_q^q}{\lambda\|u\|_\infty^p - \|\nabla u\|_p^p}\right)^{((1)/(p-q))}. \tag{3.4}$$

Hence, by taking a function $e \in X_{p,q} \setminus \{0\}$ such that $\|\nabla e\|_p^p = \lambda_\infty(p)\|e\|_\infty^p$ we can see that $\mathcal{N}_{\lambda,\infty} \neq \varnothing$ when $N < q < p < \infty$.

Remark 3.3. In the case $N < q < p < \infty$ we also have

$$\mu_\lambda := \inf_{u\in\mathcal{N}_{\lambda,\infty}} J_\lambda(u) \geqslant \left(\frac{1}{q} - \frac{1}{p}\right)(\lambda^{-1}(\lambda_\infty(q))^{p/q})^{((q)/(p-q))} > 0.$$

Indeed, this follows from the estimates

$$\|\nabla u\|_q^q \leqslant \|\nabla u\|_p^p + \|\nabla u\|_q^q = \lambda\|u\|_\infty^p \leqslant \lambda(\lambda_\infty(q))^{-p/q}\|\nabla u\|_q^p,$$

valid for any $u \in \mathcal{N}_{\lambda,\infty}$.

Definition 3.4. We say that $u \in X_{p,q}$ is a least energy solution of (3.1) if u minimizes the functional J_λ either on $X_{p,q}$ in the case $N < p < q < \infty$ or on $\mathcal{N}_{\lambda,\infty}$ in the case $N < q < p < \infty$.

The functional J_λ is not differentiable because of the term involving the L^∞ norm. Even though we are able to show that least energy solutions are weak solutions. Indeed, this fact is a consequence of the following identity (see [**1**, Chapter 11] and [**11**]) valid for all $u \in C(\overline{\Omega})$ and that provides the right Gateaux derivative for the functional $u \mapsto \|u\|_\infty^p$:

$$\lim_{\epsilon\to 0^+} \frac{\|u+\epsilon v\|_\infty^p - \|u\|_\infty^p}{\epsilon} = p\max\{|u(x)|^{p-2}u(x)v(x) : x \in \Gamma_u\}, \quad \forall\, v \in C(\overline{\Omega}), \tag{3.5}$$

where

$$\Gamma_u := \{x \in \Omega : |u(x)| = \|u\|_\infty\}.$$

Proposition 3.5. *Least energy solutions of* (3.1) *are weak solutions of this problem.*

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Proof. First we consider the case $N < p < q < \infty$. We have, for each $v \in X_{p,q} \hookrightarrow C(\overline{\Omega})$,

$$\lim_{\epsilon \to 0^+} \frac{J_\lambda(u + \epsilon v) - J_\lambda(u)}{\epsilon} = \lim_{\epsilon \to 0^+} (A(\epsilon) - B(\epsilon)) \tag{3.6}$$

where

$$A(\epsilon) := \frac{1}{p} \frac{\|\nabla(u + \epsilon v)\|_p^p - \|\nabla u\|_p^p}{\epsilon} + \frac{1}{q} \frac{\|\nabla(u + \epsilon v)\|_q^q - \|\nabla u\|_q^q}{\epsilon}$$

and

$$B(\epsilon) := \frac{\lambda}{p} \frac{\|u + \epsilon v\|_\infty^p - \|u\|_\infty^p}{\epsilon}.$$

Taking into account that the first limit in (3.6) is nonnegative (because u minimizes J_λ) and still considering that

$$\lim_{\epsilon \to 0^+} A(\epsilon) = \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \cdot \nabla v \mathrm{d}x$$

and that, according to (3.5),

$$\lim_{\epsilon \to 0^+} B(\epsilon) = \lambda \max\{|u(x)|^{p-2} u(x) v(x) : x \in \Gamma_u\},$$

we conclude that

$$\lambda \max\{|u(x)|^{p-2} u(x) v(x) : x \in \Gamma_u\} \leqslant \int_\Omega (|\nabla u|^{p-2} \nabla u + |\nabla u|^{q-2} \nabla u) \cdot \nabla v \mathrm{d}x.$$

The arbitrariness of $v \in X_{p,q}$ allows us to replace v with $-v$ in the above inequality and also get

$$\lambda \min\left\{|u(x)|^{p-2} u(x) v(x) : x \in \Gamma_u\right\} \geqslant \int_\Omega (|\nabla u|^{p-2} \nabla u + |\nabla u|^{q-2} \nabla u) \cdot \nabla v \mathrm{d}x.$$

These last two inequalities lead us to the following identity

$$|u(x)|^{p-2} u(x) v(x) = \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \cdot \nabla v \mathrm{d}x, \quad \forall\, v \in X_{p,q} \quad \text{and} \quad \forall\, x \in \Gamma_u,$$

which implies that Γ_u is a singleton, say

$$\Gamma_u = \{x_u\}$$

for some $x_u \in \Omega$. Consequently,

$$\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \cdot \nabla v \mathrm{d}x = |u(x_u)|^{p-2} u(x_u) v(x_u), \quad \forall\, v \in X_{p,q},$$

which is (3.2) for u.

We now analyse the case $N < q < p < \infty$. Let us take an arbitrary function $v \in X_{p,q}$.

Since $u \in \mathcal{N}_{\lambda,\infty}$ we have $\|\nabla u\|_p^p < \lambda \|u\|_\infty^p$. Hence, we can take $\delta > 0$ such that $u + sv \not\equiv 0$ and

$$\|\nabla(u+sv)\|_p^p < \lambda\|u+sv\|_\infty^p, \quad \forall s \in (-\delta, \delta).$$

Let $\tau : (-\delta, \delta) \to (0, \infty)$ be the function given by

$$\tau(s) = \left(\frac{\|\nabla(u+sv)\|_q^q}{\lambda\|u+sv\|_\infty^p - \|\nabla(u+sv)\|_p^p}\right)^{((1)/(p-q))},$$

which is right differentiable at $s = 0$.

We can see from (3.4) that $\tau(s)(u + sv) \in \mathcal{N}_{\lambda,\infty}$ for all $s \in (-\delta, \delta)$ and that $\tau(0) = 1$.

Now, let us consider the function $\gamma : (-\delta, \delta) \to \mathbb{R}$ defined by

$$\gamma(s) = J_\lambda(\tau(s)(u+sv)) = \frac{\tau(s)^p}{p}\|\nabla(u+sv)\|_p^p + \frac{\tau(s)^q}{q}\|\nabla((u+sv))\|_q^q - \frac{\lambda\tau(s)^p}{p}\|u+sv\|_\infty^p.$$

According to (3.5) this function is right differentiable at $s = 0$ and

$$\begin{aligned}\gamma'(0_+) &= \tau'(0_+)(\|\nabla u\|_p^p + \|\nabla u\|_q^q) + \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla v \mathrm{d}x \\ &\quad - \lambda \max\{|u(x)|^{p-2}u(x)v(x) : x \in \Gamma_u\} - \tau'(0_+)\lambda\|u\|_\infty^p \\ &= \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla v \mathrm{d}x - \lambda \max\{|u(x)|^{p-2}u(x)v(x) : x \in \Gamma_u\},\end{aligned}$$

where we have used that $\tau(0) = 1$ and $\|\nabla u\|_p^p + \|\nabla u\|_q^q - \lambda\|u\|_\infty^p = 0$.

Since γ attains its minimum value at $s = 0$ we have

$$\gamma'(0_+) = \lim_{s \to 0^+} \frac{\gamma(s) - \gamma(0)}{s} \geqslant 0.$$

Hence,

$$\lambda \max\{|u(x)|^{p-2}u(x)v(x) : x \in \Gamma_u\} \leqslant \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla v \mathrm{d}x.$$

Taking into account the arbitrariness of v we replace v with $-v$ to get

$$\begin{aligned}\lambda \min\{|u(x)|^{p-2}u(x)v(x) : x \in \Gamma_u\} &\geqslant \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla v \mathrm{d}x \\ &\geqslant \lambda \max\{|u(x)|^{p-2}u(x)v(x) : x \in \Gamma_u\},\end{aligned}$$

so that

$$\min\{|u(x)|^{p-2}u(x)v(x) : x \in \Gamma_u\} = \max\{|u(x)|^{p-2}u(x)v(x) : x \in \Gamma_u\},$$

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implying both that $\Gamma_u = \{x_u\}$, for some $x_u \in \Omega$, and that

$$\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla v \mathrm{d}x = \lambda |u(x_u)|^{p-2} u(x_u) v(x_u).$$

□

Now we are ready to show that in both cases $N < p < q < \infty$ and $N < q < p < \infty$ a nonnegative least energy solution of (3.1) can be obtained from the least energy solutions of (1.1) by a limit process, by making as $r \to \infty$. For this we observe from (3.3), with $m = p$, that if $\lambda > \lambda_\infty(p)$ and $r_n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\lambda_{r_n}(p) < \lambda$ for all $n \geqslant n_0$. Therefore, for each $n \geqslant n_0$ the boundary value problem

$$\begin{cases} -(\Delta_p + \Delta_q)u = \lambda \|u\|_{r_n}^{p-r_n} |u|^{r_n - 2} u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.7}$$

has at least one nonnegative least energy solution u_n. Having this in mind, we can assume that $n_0 = 1$ in the next proposition.

Proposition 3.6. *Let $\lambda > \lambda_\infty(p)$ and $r_n \to \infty$. Denote by u_n a nonnegative least energy solution of (3.7). There exists a subsequence of $\{u_n\}$ converging strongly in $X_{p,q}$ to a nonnegative least energy solution u of (3.1).*

Proof. First we consider $N < p < q < \infty$, so that $X_{p,q} = W_0^{1,q}(\Omega)$. Since

$$\begin{aligned} \|\nabla u_n\|_q^q &\leqslant \|\nabla u_n\|_q^q + \|\nabla u_n\|_p^p \\ &= \lambda \|u_n\|_{r_n}^p \leqslant \frac{\lambda}{\lambda_{r_n}(p)} \|\nabla u_n\|_p^p \leqslant \frac{\lambda}{\lambda_{r_n}(p)} \|\nabla u_n\|_q^p |\Omega|^{\frac{q-p}{q}}, \end{aligned}$$

we have

$$\|\nabla u_n\|_q \leqslant |\Omega|^{1/q} \left(\frac{\lambda}{\lambda_{r_n}(p)} \right)^{((1)/(q-p))},$$

implying thus that $\{u_n\}$ is bounded in $X_{p,q}$. Therefore, up to relabelling the sequence $\{r_n\}$, we can assume that there exists a nonnegative function $u \in X_{p,q}$ such that $u_n \rightharpoonup u$ in $X_{p,q}$ and $u_n \to u$ uniformly in $\overline{\Omega}$.

In order to prove that u minimizes J_λ globally, we fix an arbitrary function $v \in X_{p,q} \hookrightarrow C(\overline{\Omega})$. We know that

$$\frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{q}\|\nabla u_n\|_q^q - \frac{\lambda}{p}\|u_n\|_{r_n}^p \leqslant \frac{1}{p}\|\nabla v\|_p^p + \frac{1}{q}\|\nabla v\|_q^q - \frac{\lambda}{p}\|v\|_{r_n}^p,$$

so that

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{q}\|\nabla u_n\|_q^q - \frac{\lambda}{p}\|u_n\|_\infty^p \\ &\leqslant \frac{1}{p}\|\nabla v\|_p^p + \frac{1}{q}\|\nabla v\|_q^q - \frac{\lambda}{p}\|v\|_{r_n}^p + \frac{\lambda}{p}\|u_n\|_{r_n}^p - \frac{\lambda}{p}\|u_n\|_\infty^p \end{aligned}$$

$$\leqslant \frac{1}{p}\|\nabla v\|_p^p + \frac{1}{q}\|\nabla v\|_q^q - \frac{\lambda}{p}\|v\|_{r_n}^p + \frac{\lambda\|u_n\|_\infty^p}{p}(|\Omega|^{((p)/(r_n))} - 1)$$
$$= J_\lambda(v) + \frac{\lambda}{p}\|v\|_\infty^p - \frac{\lambda}{p}\|v\|_{r_n}^p + \frac{\lambda\|u_n\|_\infty^p}{p}(|\Omega|^{((p)/(r_n))} - 1).$$

Since $v \in C(\overline{\Omega})$ we have $\|v\|_{r_n}^p \to \|v\|_\infty^p$. This fact and the convergences $u_n \rightharpoonup u$ and $u_n \to u$ in $C(\overline{\Omega})$) imply that

$$J_\lambda(u) = \liminf_{n\to\infty} J_\lambda(u_n)$$
$$\leqslant J_\lambda(v) + \frac{\lambda}{p}\lim_{n\to\infty}(\|v\|_\infty^p - \|v\|_{r_n}^p) + \lim_{n\to\infty}\frac{\lambda\|u_n\|_\infty^p}{p}(|\Omega|^{((p)/(r_n))} - 1) = J_\lambda(v).$$

That is, u minimizes J_λ globally.

Now, let us consider the case $N < q < p < \infty$, so that $X_{p,q} = W_0^{1,p}(\Omega)$ and

$$\left(\frac{1}{q} - \frac{1}{p}\right)\|\nabla u_n\|_q^q = I_{\lambda,r_n}(u_n) \leqslant I_{\lambda,r_n}(v) = \left(\frac{1}{q} - \frac{1}{p}\right)\|\nabla v\|_q^q, \quad \forall\, v \in \mathcal{N}_{\lambda,r_n}. \tag{3.8}$$

In order to show that $\{u_n\}$ is bounded in $X_{p,q}$ we pick $e_n \in X_{p,q} \setminus \{0\}$ satisfying (2.3) with $r = r_n$, that is, such that

$$\|e_n\|_{r_n} = 1 \quad \text{and} \quad \|\nabla e_n\|_p^p = \lambda_{r_n}(p).$$

Since $\lambda_{r_n}(p) < \lambda$, we have $\|\nabla e_n\|_p^p < \lambda\|e_n\|_{r_n}^p$ and $t_n e_n \in \mathcal{N}_{\lambda,r_n}$, where

$$t_n = \left(\frac{\|\nabla e_n\|_q^q}{\lambda\|e_n\|_{r_n}^p - \|\nabla e_n\|_p^p}\right)^{1/(p-q)} = \frac{\|\nabla e_n\|_q^{q/(p-q)}}{(\lambda - \lambda_{r_n}(p))^{1/(p-q)}}.$$

Hence, applying (3.8), exploring the expression of t_n and using the Hölder inequality we obtain

$$\|\nabla u_n\|_q^q \leqslant \|\nabla(t_n e_n)\|_q^q$$
$$= \frac{\|\nabla e_n\|_q^{q^2/(p-q)}\|\nabla e_n\|_q^q}{(\lambda - \lambda_{r_n}(p))^{q/(p-q)}}$$
$$= \frac{(\|\nabla e_n\|_q^q)^{p/(p-q)}}{(\lambda - \lambda_{r_n}(p))^{q/(p-q)}}$$
$$\leqslant \frac{(|\Omega|^{(p-q)/p}\|\nabla e_n\|_p^q)^{p/(p-q)}}{(\lambda - \lambda_{r_n}(p))^{q/(p-q)}} = |\Omega|\left(\frac{\lambda_{r_n}(p)}{\lambda - \lambda_{r_n}(p)}\right)^{q/(p-q)}.$$

Recalling that $u_n \in \mathcal{N}_{\lambda,r_n}$ we have

$$\|\nabla u_n\|_p^p \leqslant \|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q$$
$$= \lambda\|u_n\|_{r_n}^p$$
$$\leqslant \lambda(\lambda_{r_n}(q))^{-p/q}\|\nabla u_n\|_q^p \leqslant \lambda(\lambda_{r_n}(q))^{-p/q}|\Omega|\left(\frac{\lambda_{r_n}(p)}{\lambda - \lambda_{r_n}(p)}\right)^{p/(p-q)},$$

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which gives us the boundedness of $\{u_n\}$ in $X_{p,q}$ since

$$(\lambda_{r_n}(q))^{-p/q}\left(\frac{\lambda_{r_n}(p)}{\lambda-\lambda_{r_n}(p)}\right)^{p/(p-q)}\to(\lambda_\infty(q))^{-p/q}\left(\frac{\lambda_\infty(p)}{\lambda-\lambda_\infty(p)}\right)^{p/(p-q)}.$$

Thus, up to relabelling the sequence $\{r_n\}$ we can assume that there exists a nonnegative function $u\in X_{p,q}$ such that $u_n\rightharpoonup u$ in $X_{p,q}$ and $u_n\to u$ uniformly in $\overline{\Omega}$.

We recall from (2.5) that

$$\|u_n\|_{r_n}\geqslant\left(\frac{\lambda_{r_n}(q)}{\lambda}\right)^{1/(p-q)}.$$

Since $\|u_n\|_{r_n}\leqslant\|u_n\|_\infty|\Omega|^{1/r_n}$, we have

$$\|u\|_\infty=\lim_{n\to\infty}|\Omega|^{1/r_n}\|u_n\|_\infty\geqslant\lim_{n\to\infty}(\frac{\lambda_{r_n}(q)}{\lambda})^{1/(p-q)}=\left(\frac{\lambda_\infty(q)}{\lambda}\right)^{1/(p-q)}>0,$$

that is, $u\not\equiv 0$. Using this fact and

$$\|\nabla u_n\|_q^q+\|\nabla u_n\|_p^p=\lambda\|u_n\|_{r_n}^p\leqslant\lambda\|u_n\|_\infty^p\,|\Omega|^{p/r_n}$$

we obtain

$$\begin{aligned}\|\nabla u\|_p^p&<\|\nabla u\|_q^q+\|\nabla u\|_p^p\\&\leqslant\liminf_{n\to\infty}(\|\nabla u_n\|_q^q+\|\nabla u_n\|_p^p)\leqslant\lim_{n\to\infty}(\lambda\|u_n\|_\infty^p|\Omega|^{p/r_n})=\lambda\|u\|_\infty^p.\end{aligned}$$

It follows that $tu\in\mathcal{N}_{\lambda,\infty}$ where

$$0<t=\left(\frac{\|\nabla u\|_q^q}{\lambda\|u\|_\infty^p-\|\nabla u\|_p^p}\right)^{1/(p-q)}\leqslant 1.$$

Let us fix an arbitrary function $v\in\mathcal{N}_{\lambda,\infty}$. We know that

$$\lim_{n\to\infty}\lambda\|v\|_{r_n}^p=\lambda\|v\|_\infty^p\quad\text{and}\quad\|\nabla v\|_p^p<\|\nabla v\|_q^q+\|\nabla v\|_p^p=\lambda\|v\|_\infty^p.$$

Consequently, there exists n_0 such that

$$\|\nabla v\|_p^p<\lambda\|v\|_{r_n}^p,\quad\forall\, n\geqslant n_0.$$

This implies that $t_nv\in\mathcal{N}_{\lambda,r_n}$ for all $n\geqslant n_0$, where

$$t_n:=\left(\frac{\|\nabla v\|_q^q}{\lambda\|v\|_{r_n}^p-\|\nabla v\|_p^p}\right)^{1/(p-q)}\to\left(\frac{\|\nabla v\|_q^q}{\lambda\|v\|_\infty^p-\|\nabla v\|_p^p}\right)^{1/(p-q)}=1.$$

Thus,

$$\left(\frac{1}{q}-\frac{1}{p}\right)\|\nabla u_n\|_q^q=I_{\lambda,r_n}(u_n)\leqslant I_{\lambda,r_n}(v)=\left(\frac{1}{q}-\frac{1}{p}\right)\|\nabla(t_nv)\|_q^q,\quad\forall\, n\geqslant n_0,$$

so that

$$\|\nabla u\|_q^q \leqslant \liminf_{n\to\infty} \|\nabla u_n\|_q^q \leqslant \lim_{n\to\infty} (t_n)^q \|\nabla v\|_q^q = \|\nabla v\|_q^q.$$

Therefore,

$$J_\lambda(tu) = t^q \left(\frac{1}{q} - \frac{1}{p}\right) \|\nabla u\|_q^q \leqslant t^q \left(\frac{1}{q} - \frac{1}{p}\right) \|\nabla v\|_q^q = t^q J_\lambda(v), \quad \forall\, v \in \mathcal{N}_{\lambda,\infty}. \tag{3.9}$$

Let $\{v_n\} \subset \mathcal{N}_{\lambda,\infty}$ be such that

$$\lim_{n\to\infty} J_\lambda(v_n) = \mu_\lambda = \inf_{u\in\mathcal{N}_{\lambda,\infty}} J_\lambda(u).$$

According to remark 3.3, $\mu_\lambda > 0$. Thus, taking into account (3.9) we obtain

$$0 < \mu_\lambda \leqslant J_\lambda(tu) \leqslant \lim_{n\to\infty} t^q J_\lambda(v_n) = t^q \mu_\lambda \leqslant \mu_\lambda.$$

These inequalities imply that: $t = 1$, $u \in \mathcal{N}_{\lambda,\infty}$ and $J_\lambda(u) = \mu_\lambda$. We have then shown that u is a nonnegative least energy solution of (3.1).

In order to conclude this proof we show that, in both cases above considered, $u_n \to u$ strongly in $X_{p,q}$, up to a subsequence. In fact, recalling that

$$\int_\Omega (|\nabla u_n|^{p-2} + |\nabla u_n|^{q-2})\nabla u_n \cdot \nabla v \mathrm{d}x = \lambda \|u_n\|_{r_n}^{p-r_n} \int_\Omega |u_n|^{r_n-1} v \mathrm{d}x, \quad \forall\, v \in X_{p,q}, \tag{3.10}$$

$u_n \rightharpoonup u$ and $u_n \to u$ uniformly, we can see that

$$|\lambda \|u_n\|_{r_n}^{p-r_n} \int_\Omega |u_n|^{r_n-1}(u_n - u)\mathrm{d}x| \leqslant \lambda \|u_n\|_\infty^{p-1} |\Omega|^{((p)/(r_n))} \|u_n - u\|_\infty \to 0.$$

That is, the right-hand side of (3.10), with $v = u_n - u$, goes to zero as $n \to \infty$.

It follows that

$$A_n := \int_\Omega \left(|\nabla u_n|^{p-2} + |\nabla u_n|^{q-2}\right) \nabla u_n \cdot \nabla(u_n - u)\mathrm{d}x \to 0. \tag{3.11}$$

The weak convergence $u_n \rightharpoonup u$ in $X_{p,q}$ also implies that

$$B_n := \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla(u_n - u)\mathrm{d}x \to 0. \tag{3.12}$$

Hence, taking into account (3.11)–(3.12), noting that

$$\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u + |\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u) \cdot \nabla(u_n - u)\mathrm{d}x$$
$$= A_n - B_n$$

and recalling the following well-known inequality, valid for all $\xi, \eta \in \mathbb{R}^N$ and $m \geqslant 2$,

$$\int_\Omega (|\nabla\xi|^{m-2}\nabla\xi - |\nabla\eta|^{m-2}\nabla\eta) \cdot \nabla(\xi - \eta)\mathrm{d}x \geqslant 2^{2-m} \int_\Omega |\xi - \eta|^m \mathrm{d}x \tag{3.13}$$

we conclude that

$$\|\nabla(u_n - u)\|_q \to 0 \quad \text{and} \quad \|\nabla(u_n - u)\|_p \to 0.$$

Thus, $u_n \to u$ strongly in $X_{p,q}$. □

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4. The limit problem as $p \to \infty$

It is proved in [9] that

$$\lim_{m\to\infty} (\lambda_\infty(m))^{1/m} = \Lambda_\infty,$$

where Λ_∞ is defined in (1.5). We recall that (see [13])

$$\Lambda_\infty = \|\rho\|_\infty^{-1}$$

where $\rho : \overline{\Omega} \to \mathbb{R}$ denotes the distance function to the boundary, given by

$$\rho(x) = \inf\{|x-y| : y \in \partial\Omega\}.$$

We recall two well-known facts: $|\nabla \rho| = 1$ almost everywhere in Ω and $\rho \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}) \subset W_0^{1,m}(\Omega)$ for all $m \in [1, \infty)$.

Lemma 4.1. *Let $\lambda > \lambda_\infty(p)$ and consider u a nonnegative least energy solution of the boundary value problem*

$$\begin{cases} -(\Delta_p + \Delta_q)u = \lambda \|u\|_\infty^{p-1}\delta_{x_u} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then

$$\|\nabla u\|_q \leqslant |\Omega|^{1/q} \left(\frac{\lambda_\infty(p)}{\lambda - \lambda_\infty(p)}\right)^{((1)/(p-q))}, \quad \text{if } N < q < p, \tag{4.1}$$

and

$$\|\nabla u\|_q \leqslant |\Omega|^{1/q} \left(\frac{\lambda}{\lambda_\infty(p)}\right)^{((1)/(q-p))}, \quad \text{if } N < p < q. \tag{4.2}$$

Proof. First we consider the case $N < q < p$. Let $e \in X_{p,q} = W_0^{1,p}(\Omega)$ be such that

$$\|e\|_\infty = 1 \quad \text{and} \quad \|\nabla e\|_p^p = \lambda_\infty(p).$$

Since

$$\lambda\|e\|_\infty^p - \|\nabla e\|_p^p = \lambda - \lambda_\infty(p) > 0$$

we have $te \in N_{\lambda,\infty}$, where

$$t := \left(\frac{\|\nabla e\|_q^q}{\lambda\|e\|_\infty^p - \|\nabla e\|_p^p}\right)^{1/(p-q)} = \left(\frac{\|\nabla e\|_q^q}{\lambda - \lambda_\infty(p)}\right)^{1/(p-q)}.$$

Noting that

$$0 < \left(\frac{1}{q} - \frac{1}{p}\right)\|\nabla u\|_q^q = I_{\lambda,\infty}(u) \leqslant I_{\lambda,\infty}(te) = \left(\frac{1}{q} - \frac{1}{p}\right)\|\nabla(te)\|_q^q$$

we obtain (by exploring the expression of t and using the Hölder inequality)

$$\begin{aligned}\|\nabla u\|_q^q &\leqslant \|\nabla(te)\|_q^q \\ &= \frac{\|\nabla e\|_q^{q^2/(p-q)}\|\nabla e\|_q^q}{(\lambda - \lambda_\infty(p))^{q/(p-q)}} \\ &= \frac{(\|\nabla e\|_q^q)^{p/(p-q)}}{(\lambda - \lambda_\infty(p))^{q/(p-q)}} \leqslant \frac{(|\Omega|^{(p-q)/p}\|\nabla e\|_p^q)^{p/(p-q)}}{(\lambda - \lambda_\infty(p))^{q/(p-q)}} \\ &= |\Omega| \left(\frac{\lambda_\infty(p)}{\lambda - \lambda_\infty(p)}\right)^{q/(p-q)}.\end{aligned}$$

This leads to the estimate in (4.1).

The estimate in (4.2) is a direct consequence of the following

$$\begin{aligned}\|\nabla u\|_q^q &\leqslant \|\nabla u\|_q^q + \|\nabla u\|_p^p \\ &= \lambda\|u\|_\infty^p \leqslant \frac{\lambda}{\lambda_\infty(p)}\|\nabla u\|_p^p \leqslant \frac{\lambda}{\lambda_\infty(p)}|\Omega|^{((q-p)/(q))}\|\nabla u\|_q^p.\end{aligned}$$

□

We recall that

$$\lim_{p\to\infty} \frac{q(p)}{p} = \begin{cases} Q \in (0,1) & \text{if} \quad N < q < p \\ Q \in (1,\infty) & \text{if} \quad N < p < q. \end{cases}$$

Lemma 4.2. *Let $\Lambda > \Lambda_\infty$ and $m > N$ be fixed. Take $\lambda_p > 0$ satisfying*

$$\lim_{p\to\infty} (\lambda_p)^{1/p} = \Lambda$$

and denote by u_p a nonnegative least energy solution of

$$\begin{cases} -(\Delta_p + \Delta_{q(p)})u = \lambda_p\|u\|_\infty^{p-1}\delta_{x_u} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.3}$$

We affirm that

$$\limsup_{p\to\infty} \|\nabla u_p\|_m \leqslant |\Omega|^{\frac{1}{m}} \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))} \tag{4.4}$$

and

$$\liminf_{p\to\infty} \|u_p\|_\infty \geqslant \begin{cases} (\Lambda_\infty)^{-1}(\Lambda_\infty/\Lambda)^{((1)/(1-Q))} & \text{if} \quad Q \in (0,1) \\ (\Lambda_\infty)^{-1} & \text{if} \quad Q \in (1,\infty). \end{cases} \tag{4.5}$$

Proof. Since

$$\lim_{p\to\infty} (\lambda_\infty(p))^{1/p} = \Lambda_\infty < \Lambda = \lim_{p\to\infty} (\lambda_p)^{1/p},$$

we can see that $\lambda_\infty(p) < \lambda_p$ for all p large enough. Therefore, the existence of a least energy solution u_p follows from proposition 3.6.

Let us fix $p_n \to \infty$ and simplify the notation by defining

$$u_n := u_{p_n}, \quad q_n := q(p_n) \quad \text{and} \quad \lambda_n := \lambda_{p_n}.$$

Let $n_0 \in \mathbb{N}$ such that $m < \min\{q_n, p_n\}$ for all $n \geqslant n_0$. Now, fix $0 < \epsilon < (\Lambda/\Lambda_\infty) - 1$ and consider $n_1 \geqslant n_0$ such that

$$1 < a_\epsilon := \frac{\Lambda}{\Lambda_\infty} - \epsilon \leqslant \left(\frac{\lambda_n}{\lambda_\infty(p_n)}\right)^{((1)/(p_n))} \leqslant \frac{\Lambda}{\Lambda_\infty} + \epsilon =: b_\epsilon, \quad \forall\, n \geqslant n_1.$$

First we prove (4.4) in the case $Q \in (0, 1)$, so that $N < q_n < p_n$. Thus, according to (4.1), with $\lambda = \lambda_n$, we have

$$\begin{aligned} \|\nabla u_n\|_{q_n} &\leqslant |\Omega|^{1/q_n} \left(\frac{\lambda_\infty(p_n)}{\lambda_n - \lambda_\infty(p_n)}\right)^{1/(p_n - q_n)} \\ &= |\Omega|^{1/q_n} \left(\frac{1}{(\lambda_n/\lambda_\infty(p_n)) - 1}\right)^{1/(p_n - q_n)}. \end{aligned} \tag{4.6}$$

Applying the Hölder inequality in (4.6)

$$\begin{aligned} \|\nabla u_n\|_m &\leqslant |\Omega|^{1/m - 1/q_n} \|\nabla u_n\|_{q_n} \\ &\leqslant |\Omega|^{1/m - 1/q_n} |\Omega|^{1/q_n} \left(\frac{1}{(\lambda_n/\lambda_\infty(p_n)) - 1}\right)^{1/(p_n - q_n)} \\ &\leqslant |\Omega|^{1/m} \left(\frac{1}{(a_\epsilon)^{p_n} - 1}\right)^{1/(p_n - q_n)}, \quad \forall\, n \geqslant n_1, \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{n\to\infty} \|\nabla u_n\|_m &\leqslant |\Omega|^{1/m} \lim_{p\to\infty} ((a_\epsilon)^{p_n} - 1)^{-1/(p_n - q_n)} \\ &= |\Omega|^{1/m} \lim_{p\to\infty} ((a_\epsilon)^{p_n} - 1)^{-((1)/(p_n))((1)/(1-(q_n/p_n)))} \\ &= |\Omega|^{1/m} (a_\epsilon)^{1/(1-Q)} \end{aligned}$$

since

$$\lim_{p\to\infty} ((a_\epsilon)^p - 1)^{-1/p} = \lim_{p\to\infty} \exp\left(-\frac{1}{p}\log((a_\epsilon)^p - 1)\right) = a_\epsilon.$$

Letting $\epsilon \to 0$, we obtain (4.4) when $Q \in (0, 1)$.

Now, we prove (4.4) when $Q \in (1, \infty)$, in which case $N < p_n < q_n$. By the Hölder inequality and (4.2), with $\lambda = \lambda_n$, we have

$$\begin{aligned}\|\nabla u_n\|_m &\leqslant |\Omega|^{1/m-1/q_n} \|\nabla u_n\|_{q_n} \\ &\leqslant |\Omega|^{1/m-1/q_n} |\Omega|^{1/q_n} \left(\frac{\lambda_n}{\lambda_\infty(p_n)}\right)^{((1)/(q_n-p_n))} \\ &\leqslant |\Omega|^{1/m} (b_\epsilon)^{p_n((1)/(q_n-p_n))}, \quad \forall\, n \geqslant n_1.\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty} \|\nabla u_n\|_m &\leqslant |\Omega|^{1/m} \lim_{n\to\infty} (b_\epsilon)^{p_n((1)/(q_n-p_n))} = |\Omega|^{1/m} \lim_{n\to\infty} (b_\epsilon)^{((1)/((q_n/p_n)-1))} \\ &= |\Omega|^{1/m} (b_\epsilon)^{1/Q-1}.\end{aligned}$$

Letting $\epsilon \to 0$, we also obtain (4.4) when $Q \in (1, \infty)$.

Let us pass to the proof of (4.5). In the case $Q \in (0, 1)$, in which $N < q_n < p_n$, we have

$$\begin{aligned}\|u_n\|_\infty^{q_n} &\leqslant \frac{1}{\lambda_\infty(q_n)} \|\nabla u_n\|_{q_n}^{q_n} \\ &\leqslant \frac{1}{\lambda_\infty(q_n)} (\|\nabla u_n\|_{q_n}^{q_n} + \|\nabla u_n\|_{p_n}^{p_n}) \\ &= \frac{\lambda_n}{\lambda_\infty(q_n)} \|u_n\|_\infty^{p_n} \leqslant (b_\epsilon)^{p_n} \frac{\lambda_\infty(p_n)}{\lambda_\infty(q_n)} \|u_n\|_\infty^{p_n}.\end{aligned}$$

It follows that

$$\begin{aligned}\liminf_{n\to\infty} \|u_n\|_\infty &\geqslant \lim_{n\to\infty} \left((b_\epsilon)^{-p_n} \frac{\lambda_\infty(q_n)}{\lambda_\infty(p_n)}\right)^{1/(p_n-q_n)} \\ &= \lim_{n\to\infty} (b_\epsilon)^{-((1)/(1-(q_n/p_n)))} \lim_{n\to\infty} (\lambda_\infty(q_n)^{((1)/(q_n))})^{q_n/(p_n-q_n)} \\ &\quad \lim_{n\to\infty} (\lambda_\infty(p_n)^{-((1)/(p_n))})^{p_n/(p_n-q_n)} \\ &= (b_\epsilon)^{-((1)/(1-Q))} (\Lambda_\infty)^{Q/(1-Q)} (\Lambda_\infty)^{-1/(1-Q)} \\ &= (b_\epsilon)^{-((1)/(1-Q))} (\Lambda_\infty)^{-1}.\end{aligned}$$

Thus, making $\epsilon \to 0$ we obtain (4.5) in the case $Q \in (0, 1)$.

As for the case $Q \in (1, \infty)$, in which $N < p_n < q_n$, we have

$$\left(\frac{1}{q_n} - \frac{1}{p_n}\right) \|\nabla u_n\|_{q_n}^{q_n} = I_{\lambda_n,\infty}(u_n) \leqslant I_{\lambda_n,\infty}(\rho) = \frac{|\Omega|}{q_n} + \frac{|\Omega|}{p_n} - \frac{\lambda_n}{p_n} \|\rho\|_\infty^{p_n}$$

since $\rho \in X_{p_n,q_n} = W_0^{1,q_n}(\Omega)$ and $|\nabla \rho| = 1$ almost everywhere.

Since $\|\nabla u_n\|_{q_n}^{q_n} \leqslant \|\nabla u_n\|_{p_n}^{p_n} + \|\nabla u_n\|_{q_n}^{q_n} = \lambda_n \|u_n\|_\infty^{p_n}$ and $\|\rho\|_\infty^{-1} = \Lambda_\infty$, we obtain

$$\frac{\lambda_n}{p_n(\Lambda_\infty)^{p_n}} \leqslant |\Omega| \left(\frac{1}{p_n} + \frac{1}{q_n}\right) + \left(\frac{1}{p_n} - \frac{1}{q_n}\right) \lambda_n \|u_n\|_\infty^{p_n},$$

so that

$$\|u_n\|_\infty^{p_n} \geqslant \left(1 - \frac{p_n}{q_n}\right)^{-1} \left[\frac{1}{(\Lambda_\infty)^{p_n}} - \frac{|\Omega|}{\lambda_n}\left(1 + \frac{p_n}{q_n}\right)\right].$$

Since $p_n/q_n \to Q^{-1} \in (0,1)$ and $(|\Omega|/\lambda_n)^{1/p_n} \to \Lambda^{-1}$ we can assume that

$$\frac{p_n}{q_n} + 1 \leqslant 2 \quad \text{and} \quad \frac{|\Omega|}{\lambda_n} \leqslant \frac{2}{\Lambda}, \quad \forall\, n \geqslant n_1.$$

Hence, redefining n_1 if necessary we conclude that

$$\|u_n\|_\infty^{p_n} \geqslant \left(1 - \frac{p_n}{q_n}\right)^{-1} \left(\frac{1}{(\Lambda_\infty)^{p_n}} - \frac{4}{\Lambda^{p_n}}\right) > 0, \quad \forall\, n \geqslant n_1.$$

Therefore,

$$\begin{aligned} \liminf_{p\to\infty} \|u_n\|_\infty &\geqslant \lim_{p\to\infty} \left(1 - \frac{p_n}{q_n}\right)^{-((1)/(p_n))} \left(\frac{1}{(\Lambda_\infty)^{p_n}} - \frac{4}{\Lambda^{p_n}}\right)^{((1)/(p_n))} \\ &= \frac{1}{\Lambda} \lim_{p\to\infty} \left[\left(\frac{\Lambda}{\Lambda_\infty}\right)^{p_n} - 4\right]^{((1)/(p_n))} = \frac{1}{\Lambda}\frac{\Lambda}{\Lambda_\infty} = \frac{1}{\Lambda_\infty}. \end{aligned}$$

□

THEOREM 4.3. *Let $\Lambda > \Lambda_\infty$ be fixed and take $\lambda_p > 0$ satisfying*

$$\lim_{p\to\infty} (\lambda_p)^{1/p} = \Lambda.$$

Denote by u_p a nonnegative least energy solution of (4.3) and by x_p the only maximum point of u_p (that is $x_p := x_{u_p}$). There exists a sequence $p_n \to \infty$, a point $x_\Lambda \in \Omega$ and a function $u_\Lambda \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})$ such that $x_{p_n} \to x_\Lambda$ and $u_{p_n} \to u_\Lambda$ uniformly in $\overline{\Omega}$. Moreover,

$$\|\nabla u_\Lambda\|_\infty \leqslant \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))} \tag{4.7}$$

and

$$u_\Lambda(x_\Lambda) = \|u_\Lambda\|_\infty \geqslant \begin{cases} (\Lambda_\infty)^{-1}(\Lambda_\infty/\Lambda)^{((1)/(1-Q))} & \text{if} \quad Q \in (0,1) \\ (\Lambda_\infty)^{-1} & \text{if} \quad Q \in (1,\infty). \end{cases} \tag{4.8}$$

Proof. Let $p_n \to \infty$ and $N < m < \infty$. It follows from the previous lemma that $\{u_{p_n}\}$ is bounded in $W_0^{1,m}(\Omega)$. Thus, up to a subsequence, u_{p_n} converges weakly

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in $W_0^{1,m}(\Omega)$ and uniformly in $\overline{\Omega}$ to a nonnegative function $u_\Lambda \in W_0^{1,m}(\Omega) \cap C(\overline{\Omega})$. Therefore, in view of (4.4) we have

$$\|\nabla u_\Lambda\|_m \leqslant \liminf_{n\to\infty} \|\nabla u_{p_n}\|_m \leqslant \limsup_{n\to\infty} \|\nabla u_{p_n}\|_m \leqslant |\Omega|^{((1)/(m))} \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))}.$$

Hence, noting that $m \in (N, \infty)$ is arbitrary, we conclude that $u_\Lambda \in W^{1,\infty}(\Omega)$ and

$$\|\nabla u_\Lambda\|_\infty \leqslant \lim_{m\to\infty} |\Omega|^{1/m} \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(Q-1))} = \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))},$$

which is (4.7).

The uniform convergence and (4.5) imply (4.8), which in turn, shows that $\|u_\Lambda\|_\infty > 0$. Taking into account that $\{x_{p_n}\}$ is bounded, we can assume (up to relabelling the sequence $\{p_n\}$) that $x_{p_n} \to x_\Lambda$ for some $x_\Lambda \in \overline{\Omega}$. The uniform convergence also implies that $u_\Lambda(x_\Lambda) = \|u_\Lambda\|_\infty > 0$ so that $x_\Lambda \in \Omega$ (note that $u_\Lambda \equiv 0$ on $\partial\Omega$). □

The next corollary shows that in the case $Q \in (0, 1)$ the function u_Λ, such as ρ, minimizes the Rayleigh quotient $\|\nabla v\|_\infty / \|v\|_\infty$ in $(W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})) \setminus \{0\}$.

Corollary 4.4. *If $Q \in (0, 1)$, then*

$$\|u_\Lambda\|_\infty = \frac{1}{\Lambda_\infty}\left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))} \quad \text{and} \quad \Lambda_\infty = \frac{\|\nabla u_\Lambda\|_\infty}{\|u_\Lambda\|_\infty}, \quad \forall\, \Lambda > \Lambda_\infty. \tag{4.9}$$

Therefore, x_Λ is also a maximum point of the distance function to the boundary ρ and

$$0 \leqslant u_\Lambda(x) \leqslant \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))} \rho(x) \quad \forall\, x \in \overline{\Omega}, \tag{4.10}$$

with the equality holding in $\partial\Omega \cup \{x_\Lambda\}$.

Proof. According to (4.8) and (4.7) we have,

$$\frac{1}{\Lambda_\infty}\left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))} \leqslant \|u_\Lambda\|_\infty \leqslant \frac{\|\nabla u_\Lambda\|_\infty}{\Lambda_\infty} \leqslant \frac{1}{\Lambda_\infty}\left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))},$$

which gives (4.9).

Taking into account that $\|\nabla u_\Lambda\|_\infty = \Lambda_\infty \|u_\Lambda\|_\infty = \|\rho\|_\infty^{-1} \|u_\Lambda\|_\infty$ we have

$$0 \leqslant u_\Lambda(x) = u_\Lambda(x) - u_\Lambda(y) \leqslant \|\nabla u_\Lambda\|_\infty |x - y| = \|\rho\|_\infty^{-1} \|u_\Lambda\|_\infty |x - y|$$

for each $x \in \overline{\Omega}$ and $y \in \partial\Omega$. It follows that

$$0 \leqslant \frac{\|\rho\|_\infty}{\|u_\Lambda\|_\infty} u_\Lambda(x) \leqslant \rho(x) \leqslant \|\rho\|_\infty, \quad \forall\, x \in \overline{\Omega}.$$

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Since $u(x_\Lambda) = \|u_\Lambda\|_\infty$ we conclude that $\rho(x_\Lambda) = \|\rho\|_\infty$. Noting that

$$\frac{\|u_\Lambda\|_\infty}{\|\rho\|_\infty} = \|u_\Lambda\|_\infty \Lambda_\infty = \left(\frac{\Lambda_\infty}{\Lambda}\right)^{((1)/(1-Q))}$$

we obtain (4.10), with the equality holding at x_Λ and also on $\partial\Omega$ (since $u_\Lambda = \rho = 0$ on $\partial\Omega$). □

Corollary 4.5. *Lemma (4.2), theorem (4.3) and corollary (4.4) remain true for* $\Lambda = \Lambda_\infty$ *in both cases* $Q \in (0, 1)$ *and* $Q \in (1, \infty)$*, if one takes* $\lambda_p = c|\Omega|(\Lambda_\infty)^p$*, with* $c > 1$.

Proof. It is proved in [**9**] that the function $(N, \infty) \ni m \mapsto (|\Omega|^{-1}\lambda_\infty(m))^{1/m}$ is increasing. It follows that

$$(|\Omega|^{-1}\lambda_\infty(p))^{1/p} \leqslant \lim_{m\to\infty} (|\Omega|^{-1}\lambda_\infty(m))^{1/m} = \Lambda_\infty.$$

Hence, by taking $\lambda_p = c|\Omega|(\Lambda_\infty)^p$ with $c > 1$ we have $\lim_{p\to\infty}(\lambda_p)^{1/p} = \Lambda_\infty$ and

$$(|\Omega|^{-1}\lambda_\infty(p))^{1/p} \leqslant \Lambda_\infty < c^{1/p}\Lambda_\infty,$$

so that $\lambda_\infty(p) < \lambda_p$. Proposition 3.6 then guarantees that (4.3) has a nonnegative least energy solution u_p. Following the proofs of lemma 4.2, theorem 4.3 and corollary 4.4, we obtain a nonnegative function $u_{\Lambda_\infty} \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})$ as the uniform limit in $\overline{\Omega}$ of a sequence $\{u_{p_n}\}$, with $p_n \to \infty$. Moreover, such a function satisfies

$$u_{\Lambda_\infty}(x_{\Lambda_\infty}) = \|u_{\Lambda_\infty}\|_\infty = \frac{1}{\Lambda_\infty}, \quad \|\nabla u_{\Lambda_\infty}\|_\infty = 1$$

and

$$0 \leqslant u_{\Lambda_\infty}(x) \leqslant \rho(x) \quad \forall\, x \in \overline{\Omega},$$

so that x_{Λ_∞} is also a maximum point of ρ. □

Remark 4.6. Recalling that $\lim_{p\to\infty}(\lambda_\infty(p))^{1/p} = \Lambda_\infty$, one can see that if λ_p is such that $\lim_{p\to\infty}(\lambda_p)^{1/p} = \Lambda < \Lambda_\infty$, then $\lambda_p < \lambda_\infty(p)$ for all p large enough. Thus, according to remark 3.2, if $\Lambda < \Lambda_\infty$ the problem (4.3) has no weak solution for all p large enough.

Before determining the equation satisfied by u_Λ, let us recall some definitions. In what follows D denotes a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$. Further up we will take $D = \Omega \setminus \{x_\Lambda\}$.

Definition 4.7. Let $u \in C(\overline{D})$, $\phi \in C^2(\Omega)$ and $x_0 \in D$. We say that ϕ touches u at x_0 from below if

$$\phi(x) - u(x) < 0 = \phi(x_0) - u(x_0), \quad \forall\, x \in D \setminus \{x_0\}.$$

Analogously, we say that ϕ touches u at x_0 from above if

$$\phi(x) - u(x) > 0 = \phi(x_0) - u(x_0), \quad \forall\, x \in D \setminus \{x_0\}.$$

In the sequel we recall the concept of viscosity solution for an equation in the form

$$F(u, \nabla u, D^2 u) = 0 \quad \text{in } D. \tag{4.11}$$

The differential operator $F(u, \nabla u, D^2 u)$ includes two operators we are interested in, which are the ∞-Laplacian

$$\Delta_\infty u := \frac{1}{2} \nabla u \cdot \nabla |\nabla u|^2 = \sum_{i,j=1}^{N} u_{x_i} u_{xj} u_{x_i x_i}$$

and the (p, q)-Laplacian

$$(\Delta_p + \Delta_q) u := (|\nabla u|^{p-4} + |\nabla u|^{q-4}) |\nabla u|^2 \Delta u + ((p-2)|\nabla u|^{p-4} + (q-2)|\nabla u|^{q-4}) \Delta_\infty u,$$

where $\Delta u = \sum_{i=1}^{N} u_{x_i x_i}$ is the Laplacian.

Definition 4.8. We say that $u \in C(\overline{D})$ is a viscosity subsolution of (4.11) if

$$F(\phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) \geqslant 0$$

whenever $x_0 \in D$ and $\phi \in C^2(D)$ are such that ϕ touches u from above at x_0. Analogously, we say that u is a viscosity supersolution of (4.11) if

$$F(\phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) \leqslant 0$$

whenever $x_0 \in D$ and $\phi \in C^2(D)$ are such that ϕ touches u from below at x_0.

Definition 4.9. Let $u \in C(\overline{D})$. We say that u is viscosity solution of (4.11) if u is both a viscosity subsolution and a viscosity supersolution of (4.11).

Definition 4.10. We say that $u \in C(\overline{D})$ is (p, q)-subharmonic (respectively, (p, q)-superharmonic and (p, q)-harmonic) in D if u is a viscosity subsolution (respectively, supersolution and solution) of

$$(\Delta_p + \Delta_q) u = 0 \quad \text{in } D.$$

Definition 4.11. We say that $u \in C(\overline{D})$ is ∞-subharmonic (respectively, ∞-superharmonic and ∞-harmonic) in D if u is a viscosity subsolution (respectively, supersolution and solution) of

$$\Delta_\infty u = 0 \quad \text{in } D.$$

The next lemma is adapted from [**14**].

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Lemma 4.12. *Let $N < m < p, q < \infty$ and suppose that $u \in C(D) \cap W_0^{1,m}(D)$ is a weak solution of*

$$(\Delta_p + \Delta_q)u = 0 \quad \text{in}\, D,$$

that is,

$$\int_D (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \cdot \nabla \eta \mathrm{d}x = 0, \quad \forall\, \eta \in C_0^\infty(D). \tag{4.12}$$

Then u is (p,q)-harmonic in D.

Proof. Suppose, by contradiction, that u is not (p,q)-superharmonic in D. Then, there exist $x_0 \in D$ and $\phi \in C^2(D)$ touching u at x_0 from below such that $(\Delta_p + \Delta_q)\phi(x_0) > 0$. By continuity, this strict inequality holds in ball $B_{2\epsilon}(x_0) \subset D$, that is,

$$\begin{aligned}(|\nabla\phi|^{p-4} + |\nabla\phi|^{q-4})|\nabla\phi vert^2\Delta\phi + ((p-2)|\nabla\phi|^{p-4}\\ + (q-2)|\nabla\phi|^{q-4})\Delta_\infty\phi > 0 \quad \text{in}\, B_{2\epsilon}(x_0).\end{aligned} \tag{4.13}$$

Define

$$\psi(x) = \phi(x) + \frac{\alpha}{2}, \quad x \in B_\epsilon(x_0),$$

where

$$\alpha := \min\{u(x) - \phi(x) : x \in \partial B_\epsilon(x_0)\}.$$

Note that $\alpha > 0$ since $u(x) > \phi(x)$ for all $x \in D \setminus \{x_0\}$. Hence, $\psi(x_0) = u(x_0) + \alpha/2 > u(x_0)$ and

$$\psi(x) = u(x) - (u(x) - \phi(x)) + \frac{\alpha}{2} \leqslant u(x) - \frac{\alpha}{2} < u(x) \quad \forall\, x \in \partial B_\epsilon(x_0).$$

Let D_ϵ be a subdomain of $B_\epsilon(x_0)$ such that $\psi > u$ in D_ϵ and $\psi = u$ on ∂D_ϵ. In view of (4.13) we have

$$\operatorname{div}[(|\nabla\psi|^{p-2} + |\nabla\psi|^{q-2})\nabla\psi] = \operatorname{div}[(|\nabla\phi|^{p-2} + |\nabla\phi|^{q-2})\nabla\phi] > 0 \quad \text{in}\, B_{2\epsilon}(x_0),$$

so that

$$\int_{D_\epsilon} (|\nabla\psi|^{p-2} + |\nabla\psi|^{q-2})\nabla\psi \cdot \nabla\eta \mathrm{d}x \leqslant 0, \quad \forall\, \eta \in C_0^\infty(B_\epsilon(x_0)), \quad \eta \geqslant 0.$$

Combining this inequality with (4.12) and recalling that $(\psi - u)_+ \in W_0^{1,m}(B_\epsilon(x_0))$ can be approximated in $W_0^{1,m}(B_\epsilon(x_0))$ by functions in $C_0^\infty(B_\epsilon(x_0))$ we obtain

$$\begin{aligned}\int_{B_\epsilon(x_0)} [(|\nabla\psi|^{p-2}\nabla\psi - |\nabla u|^{p-2}\nabla u) + (|\nabla\psi|^{q-2}\nabla\psi - |\nabla u|^{q-2}\nabla u)]\\ \cdot \nabla(\psi - u)_+ \mathrm{d}x \leqslant 0.\end{aligned}$$

Taking (3.13) into account, we conclude that $\psi \leqslant u$ in $B_\epsilon(x_0)$, which contradicts the fact that $\psi > u$ in a neighbourhood of x_0 (recall that $\psi(x_0) > u(x_0)$).

Analogously, we arrive at a contradiction if we assume that u is not (p, q)-subharmonic in D. $\square$

The following lemma is taken from [**14**].

LEMMA 4.13. *Suppose that $f_n \to f$ uniformly in $\overline{D}$, f_n, $f \in C(\overline{D})$. If $\phi \in C^2(D)$ touches f from below at x_0, then there exists $x_{n_j} \to x_0$ such that*

$$f(x_{n_j}) - \phi(x_{n_j}) = \min_D \{f_{n_j} - \phi\}.$$

In the sequel, u_Λ denotes the function obtained in theorem 4.3, for $\Lambda > \Lambda_\infty$, and u_{Λ_∞} denotes the function described in corollary 4.5 (for $\Lambda = \Lambda_\infty$).

THEOREM 4.14. *The function u_Λ is ∞-harmonic in $D = \Omega \setminus \{x_\Lambda\}$. Therefore, u_Λ is strictly positive in Ω and attains its maximum point only at x_Λ.*

Proof. Let $x_0 \in D$ and take $\phi \in C^2(D)$ touching u_Λ from below at x_0. Thus,

$$\phi(x) - u_\Lambda(x) < 0 = \phi(x_0) - u_\Lambda(x_0), \quad \text{if } x \neq x_0.$$

If $|\nabla\phi(x_0)| = 0$ then we trivially have

$$\Delta_\infty \phi(x_0) = \sum_{i,j=1}^{N} \frac{\partial \phi}{\partial x_i}(x_0) \frac{\partial \phi}{\partial x_j}(x_0) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_0) = 0.$$

So, we assume that $|\nabla\phi(x_0)| \neq 0$. Let $B_\epsilon(x_0) \subset D$ be a ball centred at x_0 with radius $\epsilon > 0$ such that $|\nabla\phi| > 0$ in $B_\epsilon(x_0)$.

Let u_n, p_n and x_{p_n} given in theorem 4.3. Since $x_{p_n} \to x_\Lambda \neq x_0$ we can take $n_0 > \mathbb{N}$ such that $x_{p_n} \notin B_\epsilon(x_0)$ for all $n > n_0$. Consequently,

$$\int_{B_\epsilon(x_0)} (|\nabla u_{p_n}|^{p_n - 2} + |\nabla u_{p_n}|^{q(p_n)-2}) \nabla u_{p_n} \cdot \nabla \varphi \, dx = 0,$$
$$\forall\, \varphi \in C_0^\infty(B_\epsilon(x_0)) \quad \text{and} \quad n \geqslant n_0. \tag{4.14}$$

We recall that $u_{p_n} \in W_0^{1,m}(\Omega)$ for all n sufficiently large, where $m > N$ is fixed. Thus, combining (4.14) and lemma 4.12 we conclude that u_{p_n} is a viscosity solution of

$$(\Delta_{p_n} + \Delta_{q(p_n)})u = 0 \quad \text{in } B_\epsilon(x_0), \quad \forall\, n \geqslant n_0.$$

Applying lemma 4.13 we can take $\{x_{n_j}\} \subset B_\epsilon(x_0)$ such that $x_{n_j} \to x_0$ and

$$\alpha_j := \min_{B_\epsilon(x_0)} \{u_{p_{n_j}} - \phi\} = u_\Lambda(x_{n_j}) - \phi(x_{n_j}) \leqslant u_{p_{n_j}}(x) - \phi(x), \quad x \neq x_{n_j}.$$

The function $\psi(x) := \phi(x) + \alpha_j - |x - x_{n_j}|^4$ belongs to $C^2(B_\epsilon(x_0))$ and

$$\begin{aligned} \psi(x) - u_{p_{n_j}}(x) &= \phi(x) - u_{p_{n_j}}(x) + \alpha_j - |x - x_{n_j}| \\ &\leqslant -|x - x_{n_j}|^4 < 0 = \psi(x_{n_j}) - u_{p_{n_j}}(x_{n_j}), \quad x \neq x_{n_j}. \end{aligned}$$

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That is, ψ touches u_{n_j} from below at x_{n_j}. It follows that

$$(\Delta_{p_{n_j}} + \Delta_{q(p_{n_j})})\psi(x_{n_j}) \leqslant 0.$$

Since $|\nabla\psi(x_{n_j})| = |\nabla\phi(x_{n_j})| > 0$ and

$$\begin{aligned}(\Delta_{p_{n_j}} + \Delta_{q(p_{n_j})})\psi(x_{n_j}) = {} & (|\nabla\psi(x_{n_j})|^{p_{n_j}-4} + |\nabla\psi(x_{n_j})|^{q(p_{n_j})-4}) \\ & |\nabla\psi(x_{n_j})|^2\Delta\psi(x_{n_j}) \\ & + ((p_{n_j}-2)|\nabla\psi(x_{n_j})|^{p_{n_j}-4} + (q(p_{n_j})-2) \\ & |\nabla\psi(x_{n_j})|^{q(p_{n_j})-4})\Delta_\infty\psi(x_{n_j})\end{aligned}$$

we obtain

$$\Delta_\infty\psi(x_{n_j}) \leqslant -\frac{(|\nabla\psi(x_{n_j})|^{p_{n_j}-4} + |\nabla\psi(x_{n_j})|^{q(p_{n_j})-4})|\nabla\psi(x_{n_j})|^2\Delta\psi(x_{n_j})}{(p_{n_j}-2)|\nabla\psi(x_{n_j})|^{p_{n_j}-4} + (q(p_{n_j})-2)|\nabla\psi(x_{n_j})|^{q(p_{n_j})-4}}. \quad (4.15)$$

Noting that

$$\lim_{j\to\infty} |\nabla\psi(x_{n_j})|^2\Delta\psi(x_{n_j}) = \lim_{j\to\infty} |\nabla\phi(x_{n_j})|^2\Delta\phi(x_{n_j}) = |\nabla\phi(x_0)|^2\Delta\phi(x_0)$$

and

$$\begin{aligned}0 & \leqslant \frac{(|\nabla\psi(x_{n_j})|^{p_{n_j}-4} + |\nabla\psi(x_{n_j})|^{q(p_{n_j})-4})}{(p_{n_j}-2)|\nabla\psi(x_{n_j})|^{p_{n_j}-4} + (q(p_{n_j})-2)|\nabla\psi(x_{n_j})|^{q(p_{n_j})-4}} \\ & \leqslant \max\left\{\frac{1}{p_{n_j}-2}, \frac{1}{q(p_{n_j})-2}\right\}\end{aligned}$$

we can see that the right-hand side of (4.15) tends to zero as $j \to \infty$. Therefore, letting $j \to \infty$ in (4.15) we arrive at

$$\Delta_\infty\phi(x_0) = \lim_{j\to\infty} \Delta_\infty\psi(x_{n_j}) \leqslant 0,$$

concluding thus that u_Λ is ∞-superharmonic in D.

Analogously, we can prove that u_Λ is also ∞-subharmonic in D.

As in [**9**] we can apply the Harnack inequality (see [**15**]) and the comparison principle (see [**2**, **7**, **12**]), both for ∞-harmonic functions, to prove, respectively, that u_Λ is strictly positive in Ω and that its maximum point is attained only at x_Λ. The comparison principle is used to compare u_Λ with the function $v(x) := \|u_\Lambda\|_\infty(1 - ((1)/(\beta))|x - x_\Lambda|)$, where $\beta = \max\{|x - x_\Lambda| : x \in \partial\Omega\}$. This function is ∞-harmonic in $D = \Omega \setminus \{x_\Lambda\}$ and such that $v \geqslant u_\Lambda$ on $\partial D = \partial\Omega \cup \{x_\Lambda\}$. Hence,

$$u_\Lambda(x) \leqslant v(x) = \|u_\Lambda\|_\infty\left(1 - \frac{1}{\beta}|x - x_\Lambda|\right) < \|u_\Lambda\|_\infty, \quad \forall\, x \in D.$$

□

The following result applies when Ω is a ball, a square and many other symmetric domains, even nonconvex ones.

COROLLARY 4.15. *Suppose that Ω is such that the distance function to its boundary has a unique maximum point x_ρ. If $\Lambda > \Lambda_\infty$ and $Q \in (0, 1)$, then*

$$u_\Lambda = (\Lambda_\infty/\Lambda)^{((1)/(1-Q))} u_{\Lambda_\infty}.$$

Proof. Let $v := (\Lambda_\infty/\Lambda)^{1/(1-Q)} u_{\Lambda_\infty}$ where u_{Λ_∞} is the function described in the corollary 4.5. Taking into account corollaries 4.4 and 4.5 we have $x_\Lambda = x_\rho$ and

$$\begin{aligned} v(x_\rho) = \|v\|_\infty &= (\Lambda_\infty/\Lambda)^{((1)/(1-Q))} \|u_{\Lambda_\infty}\|_\infty \\ &= (\Lambda_\infty/\Lambda)^{((1)/(1-Q))} (1/\Lambda_\infty) = u_\Lambda(x_\rho), \quad \Lambda \geqslant \Lambda_\infty. \end{aligned}$$

It follows that both v and u_Λ are functions in $C(\overline{\Omega})$ that solve, in the viscosity sense, the problem

$$\begin{cases} \Delta_\infty u = 0 & \text{in} \quad \Omega \setminus \{x_\rho\} \\ u = 0 & \text{on} \quad \partial\Omega \\ u(x_\rho) = (\Lambda_\infty/\Lambda)^{1/(1-Q)} (1/\Lambda_\infty). \end{cases}$$

Therefore, by uniqueness (see [**2**, **7**, **12**]) we have $v \equiv u_\Lambda$. □

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References

1 S. Banach. *Theory of linear operations*, vol. 38 (Amsterdam: Elsevier, 1987).
2 G. Barles and J. Busca. Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term. *Comm. Part. Diff. Equ.* **26** (2001), 2323–2337.
3 M. Bocea and M. Mihăilescu. Existence of nonnegative viscosity solutions for a class of problems involving the ∞-Laplacian. *NoDEA Nonlinear Differ. Equ. Appl.* **23** (2016), 21 Art. 11.
4 F. Charro and I. Peral. Limits branch of solutions as $p \to \infty$ for a family of subdiffusive problems related to the p-Laplacian. *Comm. Part. Diff. Equ.* **32** (2007), 1965–1981.
5 F. Charro and E. Parini. Limits as $p \to \infty$ of p-Laplacian problems with a superdiffusive power-type nonlinearity: positive and sign-changing solutions. *J. Math. Anal. Appl.* **372** (2010), 629–644.
6 F. Charro and E. Parini. Limits as $p \to \infty$ of p-Laplacian eigenvalue problems perturbed with a concave or convex term. *Calc. Var. Partial Differ. Equ.* **46** (2013), 403–425.
7 M. G. Crandall, L. C. Evans and R. F. Gariepy. Optimal Lipschitz extensions and the infinity Laplacian. *Calc. Var. Partial Differ. Equ.* **13** (2001), 123–139.
8 J. V. da Silva, J. D. Rossi and A. M. Salort. Maximal solutions for the ∞-eigenvalue problem. *Adv. Calc. Var.* (to appear). https://doi.org/10.1515/acv-2017-0024
9 G. Ercole and G. Pereira. Asymptotics for the best Sobolev constants and their extremal functions. *Math. Nachr.* **289** (2016), 1433–1449.
10 N. Fukagai, M. Ito and K. Narukawa. Limit as $p \to \infty$ of p-Laplace eigenvalue problems and L^∞-inequality of the Poincaré type. *Differ. Integral Equ.* **12** (1999), 183–206.
11 R. Hynd and E. Lindgren. Extremal functions for Morrey's inequality in convex domains. *Math. Ann.* (2018) https://doi.org/10.1007/s00208-018-1775-8.
12 R. Jensen. Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. *Arch. Rational Mech. Anal.* **123** (1993), 51–74.

13 P. Juutinen, P. Lindqvist and J. Manfredi. The ∞-eigenvalue problem. *Arch. Ration. Mech. Anal.* **148** (1999), 89–105.

14 P. Lindqvist. *Notes on the infinity Laplace equation*. BCAM SpringerBriefs in Mathematics (Bilbao: Springer, 2016).

15 P. Lindqvist and J. Manfredi. The Harnack inequality for ∞-harmonic functions. *Electron. J. Differ. Equ.* **4** (1995), 1–5.