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Topological and almost Borel universality for systems with the weak specification property

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Abstract. We show that systems with some specification properties are topologically or almost Borel universal, in the sense that any aperiodic subshift with lower entropy may be topologically or almost Borel embedded. This improves, with elementary tools, previous results of Quas and Soo [Ergodic universality of some topological dynamical systems. *Trans. Amer. Math. Soc.* **368** (2016), 4137–4170].

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1. *Introduction*

Usually, a universal system is defined with respect to a collection $\mathcal{C}$ of measure-preserving ergodic aperiodic transformations: it is a topological system† (Y, S) such that for any system $(X, T, \mathcal{A}, \nu)$ of $\mathcal{C}$, there is a Borel S-invariant ergodic probability measure μ such that $(Y, S, \mathcal{B}, \mu)$, with $\mathcal{B}$ being the Borel σ-algebra, is isomorphic to $(X, T, \mathcal{A}, \nu)$. We will refer to such systems as *measure-theoretical $\mathcal{C}$-universal* systems. The systems are called *fully $\mathcal{C}$-universal systems* when we can assume the measure ν has full support. If one requires also that for any Borel probability S-invariant measure μ the system $(Y, S, \mathcal{B}, \mu)$ is isomorphic to some element of $\mathcal{C}$, then we say that (Y, S) is strictly $\mathcal{C}$-universal.

We focus here on the collection $\{h \in I\} = \{(X, T, \mathcal{A}, \nu), h(\nu, T) \in I\}$ for some interval I of $\mathbb{R}^+$. Krieger's generator theorem [**Kri70**] claims the full shift with K-symbols is measure-theoretical $\{h < \log K\}$-universal. More recently, Downarowicz and Serafin [**DS11**] have built strictly $\{h \in I\}$-universal systems for any non-degenerated intervals I. It was also previously established by Serafin [**Ser13**] that there is no $\{h = 0\}$-strictly universal system. Concerning fully universal systems, Quas and Soo proved, by using

† By a topological system (Y, S) we will always mean that Y is a compact metric space and S is a homeomorphism from Y into itself with finite topological entropy. The distance on Y will be denoted by d or d_Y.

Baire arguments on joinings, that some systems (Y, S) with a specification property are fully $\{h < h_{\text{top}}(S)\}$-universal. Theorems 1.2 and 4.1 below generalize their result. Lind and Thouvenot [**LT77**] proved earlier that hyperbolic (two-dimensional) toral automorphisms are fully universal.

One can also consider universal problems in the topological and Borel setting. A topological system (Y, S) is aperiodic when we have $S^n x \neq x$ for all $x \in Y$ and for all positive integers n. When Y is a zero-dimensional set, the topological system (Y, S) is said to be zero-dimensional. Such a system is expansive when there exists a partition P of Y in clopen sets such that the maximum of the diameters of atoms in $\bigvee_{k=-n}^{n} S^{-k}P$ is decreasing to zero when n goes to infinity. Let us now consider a collection $\mathcal{C}$ of expansive aperiodic zero-dimensional systems (e.a.z. systems for short). We say a topological system (Y, S) is *topologically $\mathcal{C}$-universal* when for any (X, T) in $\mathcal{C}$ there is a subsystem (Z, R) of (Y, S) topologically conjugated to (X, T); in other words, the system (X, T) may be topologically embedded into (Y, S). Here, we mainly focus on the collection $\{\mathsf{h}_{\text{top}} \in I\} = \{(X, T) \text{ e.a.z. with } \mathsf{h}_{\text{top}}(T) \in I\}$ for an interval I. Around ten years after the generator theorem, Krieger proved the full shift with K-symbols is topologically $\{\mathsf{h}_{\text{top}} < \log K\}$-universal [**Kri82**]. The proof uses similar tools (Rohklin towers), but in some sense the last theorem is easier to prove, because with the expansiveness property we get a generator in 'one step', whereas a multiscale limit approach is necessary in the generator theorem.

By a Borel system (X, T) we mean a bijection T of a standard Borel space X such that T and T^{-1} are invertible. The Borel entropy $\mathsf{h}_{\text{bor}}(T)$ is the supremum of the entropies $h(\nu, T)$ taken over all T-invariant measures ν (according to the variational principle $\mathsf{h}_{\text{bor}}(T) = \mathsf{h}_{\text{top}}(T)$ when (X, T) is a topological system). A Borel system (X, T) *almost Borel embeds* in another one (Y, S) when there is a Borel injective map $\psi : X' \to Y$ satisfying $\psi \circ T = S \circ \psi$ with X' being a full subset of X, i.e. a subset of full measure for any T-invariant probability measure. When every Borel system (X, T) in a collection $\mathcal{C}$ of Borel aperiodic systems almost Borel embeds in (Y, S), we say (Y, S) is *almost Borel $\mathcal{C}$-universal*. In particular, when every system in $\mathcal{C}$ is a topological system, a topologically $\mathcal{C}$-universal system is almost Borel $\mathcal{C}$-universal. In [**H13**], Hochman showed that a subshift of finite type (Y, S) is almost Borel $\{\mathsf{h}_{\text{bor}} < \mathsf{h}_{\text{top}}(S)\}$-universal with $\{\mathsf{h}_{\text{bor}} < \mathsf{h}_{\text{top}}(S)\}$ being the collection of aperiodic Borel systems (X, T) satisfying $\mathsf{h}_{\text{bor}}(T) < \mathsf{h}_{\text{top}}(S)\}$.

In this paper, we investigate the universality property of systems with the specification property (see [**KLO16**] for a panorama on systems with specification-like properties). For $\epsilon > 0$, a system (Y, S) is said to have the *almost ϵ-weak specification property* when for any pieces of orbits

$$T^{a_1}x_1, T^{a_1+1}x_1, \ldots, T^{b_1}x_1, \ldots, T^{a_p}x_p, T^{a_p+1}x_1, \ldots, T^{b_p}x_p$$

with $a_1 < b_1 < a_2 < \cdots < b_p$, there is a point $x \in Y$ with $d(T^k x, T^k x_i) < \epsilon$ for $k \in \bigcup_i [a_i, b_i]$ provided that $b_{i+1} - a_i \geq L(b_{i+1} - a_{i+1})$ for some function $L = L_\epsilon$ satisfying $\lim_{n\to+\infty} L(n)/n = 0$. When L is a constant function, (Y, S) is said to satisfy the *ϵ-weak specification property*.

The system (Y, S) has the *(respectively almost) weak specification property* when it satisfies the (respectively almost) ϵ-weak specification property for all $\epsilon > 0$. It is easily

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seen that a subshift satisfies the (almost) weak specification property whenever it satisfies the (almost) ϵ-weak specification property for some $\epsilon > 0$.

Our main result related to topological universality follows.

THEOREM 1.1. *Any subshift* (Y, S) *with the weak specification property is topologically* $\{\mathsf{h}_{\mathsf{top}} < \mathsf{h}_{\mathsf{top}}(S)\}$*-universal.*

In fact, we prove this universality property for any subshift with the almost coded weak specification property (see Definition 2.2). We do not know any example of subshift with the almost weak specification property which does not satisfy the almost coded weak specification property.

For general systems with the almost weak specification property, we establish almost Borel universality.

THEOREM 1.2. *Any topological system* (Y, S) *with the almost weak specification property is almost Borel* $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(S)\}$*-universal.*

By the Jewett–Krieger theorem, for any measure preserving aperiodic ergodic transformation there is a uniquely ergodic (aperiodic) subshift whose unique invariant measure is isomorphic to this given measure-preserving system. Thus, any topological system (Y, S) which is topologically $\{\mathsf{h}_{\mathsf{top}} < \mathsf{h}_{\mathsf{top}}(S)\}$-universal or almost Borel $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(S)\}$-universal is also measure-theoretical $\{h < \mathsf{h}_{\mathsf{top}}(S)\}$-universal. As a corollary, we give an affirmative answer to a question raised by Quas and Soo.

COROLLARY 1.1. *Any topological system* (Y, S) *with the almost specification property is measure-theoretical* $\{h < \mathsf{h}_{\mathsf{top}}(S)\}$*-universal.*

It was previously proved by Quas and Soo in [**QS16**] for systems satisfying also the asymptotic h-expansiveness and the small boundary property. Thus, these two additional hypotheses are not necessary.

A topological system (X, T) *almost Borel embeds finitarily* in another one (Y, S) when there is a Borel map $\psi : X \to Y$ with $\psi \circ T = S \circ \psi$ such that ψ is a finitary embedding, i.e. ψ is continuous and injective on a full subset.

Question 1.1. (See also [**QS12**, Question 1]) Is any topological system (Y, S) with the almost specification property almost Borel finitarily $\{h < \mathsf{h}_{\mathsf{top}}(S)\}$-universal, in the sense that every e.a.z. system (X, T) in $\{h < \mathsf{h}_{\mathsf{top}}(S)\}$ almost Borel embeds finitarily in (Y, S)?

We give now an overview of the main ideas of the proof. To a given system (Y, S) with the (almost) weak specification property, we associate intermediate subshifts of finite type, called specifications. The topological entropy of specifications may be arbitrarily close to $\mathsf{h}_{\mathsf{top}}(S)$ so that it is enough to embed these particular subshifts in the system by the universality of subshifts of finite type. Under some conditions, the specification may be embedded in the system via a closed set-valued upper semicontinuous map. For subshifts with the weak specification, one can derive a (single-valued) continuous embedding proving Theorem 1.1. For systems with the almost weak specification property (proof of Theorem 1.2), we need to develop a multiscale limiting approach (we use the specification property at smaller and smaller scales ϵ) to build an almost Borel embedding.

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The present paper is organized as follows. The main tools (in particular, the admissible specifications) are introduced in §2, which ends with the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2, and in the last section we discuss the question of fully almost Borel universal systems.

2. *Topological embedding in systems with the specification property*

2.1. *Intermediate subshifts of finite type.* Let $\mathbb{N}^*$ be the set of positive integers. A non-decreasing function $L : \mathbb{N} \to \mathbb{N}^*$ with $\lim_{n\to+\infty} L(n)/n = 0$ is said to be *sublinear*. Given a topological system (Y, S) and a sublinear function L, we define an *L-specification* of (Y, S) to be any subshift of finite type (Λ, σ) as follows.

There exist positive integers m and l, a finite set $E \subset Y$, an integer-valued function $N : E \to \mathbb{N}$ on E with $l \geq L(\max_{x\in E} N(x))$, a distinguished point $o \in Y$, called the *marker point* of Λ, and an added symbol $* \notin Y$, such that $\Lambda \subset (Y \cup \{*\})^{\mathbb{Z}}$ is the subshift of finite type generated by the family of words $(w_x)_{x\in E}$, where w_x is defined for all $x \in E$ by

$$w_x := (o, So, \ldots, S^m o, *^{L(m)}, x, Sx, \ldots, S^{N(x)}x, *^l).$$

The word of length $n \in \mathbb{N}^*$ given by n consecutive symbols $*$ is here denoted by $*^n$. In other words, the elements of Λ are given by the infinite concatenations of such words. The special symbol $*$ labels the places used to *glue* the finite orbits of x and o via the specification property. When the lengths of the generating words $(w_x)_{x\in E}$ are multiplicatively independent, the L-specification of (Y, S) is a topologically mixing subshift of finite type. We recall that the shift map $\sigma : \Lambda \to \Lambda$ is defined by $\sigma((u_n)_n) = (u_{n+1})_n$ for all $(u_n)_n \in \Lambda$.

We let $\underline{N} := \min_{x\in E} N(x)$ and $\overline{N} := \max_{x\in E} N(x)$. The topological entropy of a L-specification is mostly given by the cardinality of the set E when $m/\overline{N}$ and $l/\overline{N}$ are small enough. More precisely, we have the following lemma.

LEMMA 2.1. *For any sublinear function L and any $\gamma > 0$, there is $\delta > 0$ such that any L-specification with (E, o, N, m, l) satisfying $\min(m/\overline{N}, l/\overline{N}) < \delta$ has topological entropy in $[((\log \sharp E)/\overline{N})(1-\gamma), (\log \sharp E)/\underline{N}]$.*

Proof. The topological entropy of the subshift (Λ, σ) is given by the exponential growth in n of the cardinality of the set $\mathcal{L}_n(\Lambda)$ of words of Λ with length less than or equal to n. We let $q = m + L(m) + l$ so that the length $|w_x|$ of w_x for $x \in E$ is equal to $N(x) + q$. For any positive integer P, the map from E^P to $\mathcal{L}_{P(\overline{N}+q)}(\Lambda)$ sending $(x_0, \ldots, x_{P-1})$ to the concatenation $w_{x_0} w_{x_1} \cdots w_{x_{P-1}}$ is injective, so

$$\begin{aligned} \mathsf{h}_{\mathsf{top}}(\Lambda) &= \lim_n \frac{1}{n} \log \sharp \mathcal{L}_n(\Lambda), \\ &\geq \lim_P \frac{1}{P(\overline{N}+q)} \log \sharp E^P = \frac{1}{q+\overline{N}} \log \sharp E. \end{aligned}$$

We have $q \leq m + L(\overline{N}) + l \leq m + 2l$, so for any $\gamma > 0$ we get, for $m/\overline{N}$ and $l/\overline{N}$ small enough,

$$\mathsf{h}_{\mathsf{top}}(\Lambda) \geq \frac{1}{q+\overline{N}} \log \sharp E \geq \frac{\log \sharp E}{\overline{N}}(1-\gamma).$$

On the other hand, any word in $\mathcal{L}_{n\underline{N}}(\Lambda)$ is a subword of a concatenation of $n+1$-generating words. Therefore, we have $\sharp\mathcal{L}_{n\underline{N}}(\Lambda) \leq \sup_{x\in E}|w_x|\sharp E^{n+1}$, and finally we conclude

$$\mathsf{h}_{\mathsf{top}}(\Lambda) = \lim_n \frac{1}{n\underline{N}} \log \sharp\mathcal{L}_{n\underline{N}}(\Lambda) \leq \frac{\log \sharp E}{\underline{N}}.$$

We note that this last inequality holds true for any N. □

By Krieger's theorem, which we recall below, we may embed topologically any e.a.z. system in a topologically mixing specification with larger entropy.

LEMMA 2.2. (Krieger [**Kri82**]) *Let* (X, T) *be an e.a.z. system. Then* (X, T) *embeds topologically in any topologically mixing subshift of finite type* (Y, S) *with* $\mathsf{h}_{\mathsf{top}}(S) > \mathsf{h}_{\mathsf{top}}(T)$. *In other words, any topologically mixing subshift of finite type is topologically* $\{\mathsf{h}_{\mathsf{top}} < \mathsf{h}_{\mathsf{top}}(S)\}$*-universal.*

We also recall the corresponding theorem due to Hochman for the almost Borel universality.

LEMMA 2.3. (Krieger [**H13**]) *Let* (X, T) *be an aperiodic Borel system. Then* (X, T) *almost Borel embeds in any topologically mixing subshift of finite type* (Y, S) *with* $\mathsf{h}_{\mathsf{top}}(S) > \mathsf{h}_{\mathsf{bor}}(T)$. *In other words, any topologically mixing subshift of finite type is almost Borel* $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(S)\}$*-universal.*

2.2. *Closed set-valued upper semicontinuous embeddings of admissible specifications.* Let $\epsilon > 0$ and let L be a sublinear function. For any $y \in Y$ and $\epsilon > 0$, we denote the closed ball at y of radius ϵ by $B(y, \epsilon)$. The system (Y, S) is said to have the (ϵ, L)*-specification property* when, for any L-specification (Λ, σ) and any $x = (x_k)_k \in \Lambda$, the set

$$\Delta_\epsilon^\Lambda(x) := \bigcap_{k, x_k \neq *} S^{-k}(B(x_k, \epsilon)) \text{ is non-empty.}$$

The map Δ_ϵ^Λ is a closed set-valued equivariant map from (Λ, σ) to $(\mathcal{K}(Y), S)$, where $\mathcal{K}(Y)$ is the set of closed subsets of Y endowed with the Hausdorff distance. We recall that the Hausdorff distance between $F, K \in \mathcal{K}(Y)$ is given by the infimum of the positive real numbers r such that F and K are respectively contained in the r-neighborhoods of K and F. For a finite cylinder w in Λ we let $\Delta_\epsilon^\Lambda(w) := \bigcup_{y\in w\subset\Lambda} \Delta_\epsilon^\Lambda(y)$. It is also convenient for the purpose of the next subsections to define the set $\Delta_\epsilon(v)$ for a general finite word v with letters in $Y \cup \{*\}$ as

$$\Delta_\epsilon(v) := \bigcap_{m\leq k\leq n, v_k\neq *} S^{-k}(B(v_k, \epsilon)),$$

with $v = v_m \ldots v_n$ and $v_i \in Y \cup \{*\}$ for i in the interval of integers $[\![m, n]\!]$ (called the interval of coordinates of v). Observe that when v is a word of Λ and $[v]$ is the associated cylinder given by $[v] = \{(w_k)_{k\in\mathbb{Z}} \in \Lambda,\ w_i = v_i \text{ for } i \in [\![m, n]\!]\}$, we have $\Delta_\epsilon^\Lambda([v]) \subset \Delta_\epsilon(v)$, but these two sets may a priori differ.

The closed set-valued map Δ_ϵ^Λ is said to be injective when distinct points have disjoint images. We consider now a topological system (Y, S) with the (ϵ, L)-specification property, and we give sufficient conditions on the specification Λ to ensure the injectivity of Δ_ϵ^Λ. For $y \in Y$, $n \in \mathbb{N}$ and $\epsilon > 0$, we denote by $B(y, n, \epsilon)$ the n-dynamical ball at y of size ϵ defined by

$$B(y, n, \epsilon) := \bigcap_{0 \le k < n} S^{-k} B(S^k y, \epsilon).$$

We will also let $r(n, \epsilon, S)$ be the minimal number of n-dynamical balls of size ϵ that are covering Y. We recall the topological entropy of Y is given by

$$\mathsf{h}_{\mathsf{top}}(Y, S) := \lim_{\epsilon \to 0} \limsup_n \frac{1}{n} \log r(n, \epsilon, S).$$

Definition 2.1. An L-specification is said to be ϵ-admissible when the associated data (E, o, N, m, l) satisfies:

(1) $12l < m < \underline{N}$;

(2) the set E is $(N, \epsilon)-$separated, i.e. for all $x \neq y \in E$ we have either $N(x) \neq N(y)$ or $B(x, N(x), \epsilon) \cap B(y, N(y), \epsilon) = \emptyset$;

(3) $B(o, m, \epsilon) \cap S^k B(o, m, \epsilon) = \emptyset$ for any $0 < k \le \frac{3}{4}m$; and

(4) $B(o, m, \epsilon) \cap S^k B(x, N(x), \epsilon) = \emptyset$ for any $-\frac{2}{3}m \le k \le N(x) - \frac{2}{3}m$ and for any $x \in E$.

According to the fourth item, the marker point o does not belong to E when the specification is ϵ-admissible. In the next lemma, we show that the map Δ_ϵ^Λ is injective for an ϵ-admissible L-specification Λ. Conditions 3 and 4 in the above definition are used to locate the concatenated words w_x in an element of Λ, whereas condition 2 allows us to identify the points x.

LEMMA 2.4. *Assume an L-specification Λ is ϵ-admissible. Let A_1B_1 (respectively A_2B_2) be the concatenation of A_1 and B_1 (respectively A_2 and B_2) with A_i, B_i being generating words of Λ. Assume, moreover, that the intervals of coordinates $[\![m_1, n_1]\!]$ and $[\![m_2, n_2]\!]$ of A_1 and A_2 are not disjoint. Then*

$$[\Delta_\epsilon(A_1B_1) \cap \Delta_\epsilon(A_2B_2) \neq \emptyset] \Rightarrow [A_1 = A_2].$$

In particular, the closed set-valued map Δ_ϵ^Λ is injective.

Proof. Let $l \in [\![m_1, n_1]\!] \cap [\![m_2, n_2]\!]$ and $u \in \Delta_\epsilon(A_1B_1) \cap \Delta_\epsilon(A_2B_2)$. We first prove $m_1 = m_2$. We argue by contradiction: we may assume $m_1 < m_2$ by swapping m_1 and m_2 if necessary. We have the following possibilities:

- $m_1 < m_2 \le m_1 + 3m/4$ or $(n_1 - 3m/4 <) n_1 - (2m/3 + l) \le m_2 \le n_1$; then in both cases we have $S^{m_2}u \in B(o, m, \epsilon) \cap S^k B(o, m, \epsilon)$ for some k with $|k| \le 3m/4$, contradicting condition 3 of the definition; or
- $m_1 + 3m/4 < m_2 < n_1 - (2m/3 + l)$; then if we let w_x be the generating word given by A_1, we get $S^{m_2}u \in B(o, m, \epsilon) \cap S^k B(x, N(x), \epsilon)$ for some k with $-2m/3 \le k \le N(x) - 2m/3$, because we have $m/4 + L(m)$ $(<2m/3)$. This contradicts condition 4.

In a similar way, we prove now that $n_1 = n_2$ (however, note that the right boundaries of the intervals of coordinates of B_1 and B_2 may differ). In the above proof of $m_1 = m_2$, we just have used $u \in \Delta_\epsilon(A_1) \cap \Delta_\epsilon(A_2)$ and the fact that the intervals of coordinates of A_1 and A_2 were not disjoint. Therefore, to show $n_1 = n_2$, we only need to check that the intervals of coordinates of B_1 and B_2 are not disjoint. Let $w_{x'}$ and w_y be respectively the generating words given by B_1 and A_2. We argue by contradiction. Without loss of generality, we assume that the intervals of coordinates of A_1 and B_1 are contained in the interval of coordinates of A_2, which implies $n_2 - n_1 \geq \underline{N} + l + m + L(m)$, and thus

$$\begin{aligned} N(y) - (n_1 - m_1) &= N(y) + m_2 - n_1, \\ &= n_2 - m_2 - (l + m + L(m)) + m_2 - n_1, \\ &\geq \underline{N} > m. \end{aligned}$$

From $\Delta_\epsilon(A_1B_1) \cap \Delta_\epsilon(A_2B_2) \neq \emptyset$ we have $B(o, m, \epsilon) \cap S^k B(y, N(y), \epsilon) \neq \emptyset$ with $0 < k = n_1 - m_1 - (m + L(m)) < n_1 - m_1 < N(y) - m$, so we again get a contradiction with condition 4 in Definition 2.1 of ϵ-admissibility.

Finally, as the points in E are (N, ϵ)-separated (by condition 2), the sets $\Delta_\epsilon(A_1)$ and $\Delta_\epsilon(A_2)$ are disjoint if and only if A_1 and A_2 are distinct. □

We need in our proof to assume $\Delta_\epsilon(A_1B_1) \cap \Delta_\epsilon(A_2B_2) \neq \emptyset$ and not only $\Delta_\epsilon(A_1) \cap \Delta_\epsilon(A_2) \neq \emptyset$ in order to identify the last coordinates of A_1 and A_2. This comes from the fact that the marker appears only at the beginning of a generating word.

2.3. *Construction of admissible specifications.* When the integer-valued function $N : E \to \mathbb{N}^*$ satisfies $N(E) = \{n_0, n_0 + 1\}$ for some n_0 and $l = L(n_0 + 1)$, we call Λ a *simple specification.*

PROPOSITION 2.1. *Let* $0 < \alpha < \mathsf{h}_{\mathrm{top}}(S)$. *For small enough* $\epsilon > 0$ *and for any sublinear function L, there is an* ϵ*-admissible simple L-specification with topological entropy larger than* α.

The proposition will follow from the following two lemmas. We first show the existence of a marker point o satisfying condition 3 in Definition 2.1.

LEMMA 2.5. *Let* (Y, S) *be a topological system with* $\mathsf{h}_{\mathrm{top}}(S) > 0$. *For* $\epsilon > 0$ *small enough, there is, for large m, a point* $o = o(m, \epsilon) \in Y$ *with* $S^k B(o, m, \epsilon) \cap B(o, m, \epsilon) = \emptyset$ *for all* $0 < k \leq 3m/4$.

Proof. It is an easy consequence of the Ornstein–Weiss return time formula, which we recall now. For any $\epsilon > 0$ and $x \in Y$, we let

$$h^{OW}(x, \epsilon) := \limsup_n \frac{1}{n} \log \min\{k > 0, S^k x \in B(x, n, \epsilon)\},$$

and then for any ergodic S-invariant measure μ:

$$h^{OW}(\mu, \epsilon) := \int h^{OW}(x, \epsilon)\, d\mu(x).$$

Then the Ornstein–Weiss return time [**OW93**] formula states that

$$h(\mu) = \lim_{\epsilon \to 0} h^{OW}(\mu, \epsilon).$$

Let us go back to the proof of the lemma. Let μ be an ergodic measure with positive entropy and $\epsilon > 0$ with $h^{OW}(\mu, 2\epsilon) > 0$. In particular, there is $o \in Y$ with $h^{OW}(o, 2\epsilon) = \limsup_n (1/n) \log \min\{k > 0, S^k o \in B(o, n, 2\epsilon)\} > 0$. Then there are arbitrarily large

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integers m such that for any $0 \leq k \leq 3m/4$ we have $S^k o \notin B(o, m/4, 2\epsilon)$, and thus

$$S^k B(o, m, \epsilon) \cap B(o, m, \epsilon) \subset B(S^k o, m/4, \epsilon) \cap B(o, m/4, \epsilon) = \emptyset.$$

This concludes the proof of the lemma. □

We show now that we can find an (N, ϵ)-separated set E with large cardinality with $N(E) = \{n_0, n_0 + 1\}$ that satisfies condition 4 in Definition 2.1 (observe that an (n_0, ϵ)-separated set E is also (N, ϵ)-separated in this case).

LEMMA 2.6. *Fix $1 > \delta > 0$ and $\epsilon > 0$. For n large enough, for any $y \in Y$ and for any $(n, 6\epsilon)$-separated set E, the subset F of E given by elements $x \in E$ satisfying $S^l x \in B(S^k y, \delta n/2, 2\epsilon)$ for some $(k, l) \in \mathcal{E}$ has cardinality less than $e^{\sqrt{1-\delta/2}\mathsf{h}_{\mathsf{top}}(S)n}$ with*

$$\mathcal{E} := \{(k', 0), 0 \leq k' \leq \delta n\} \cup \{(0, l'), 0 \leq l' \leq (1 - \delta)n\}.$$

Proof. Let $\alpha > 1$ with $\alpha(1 - \delta/2) < \sqrt{1 - \delta/2}$. There is a constant C such that $r(n, \epsilon/2, S) \leq Ce^{\alpha \mathsf{h}_{\mathsf{top}}(S)n}$ for all n. Then for any $(k, l) \in \mathcal{E}$, the set $S^{-l} B(S^k y, \delta n/2, 2\epsilon)$ may be covered by a family of n-dynamical balls of size 3ϵ with cardinality less than $C^2 e^{\gamma(1-\delta/2)\mathsf{h}_{\mathsf{top}}(S)n}$. Thus, the union of all these sets over $(k, l) \in \mathcal{E}$ may be covered by a family of such dynamical balls with cardinality less than $(1 - \delta)\delta n^2 C^2 e^{\alpha(1-\delta/2)\mathsf{h}_{\mathsf{top}}(S)n} < e^{\mathsf{h}_{\mathsf{top}}(S)\sqrt{1-\delta/2}n}$, for n large enough. This concludes the proof, as these dynamical balls contain at most one point of E. □

We now prove Proposition 2.1.

Proof of Proposition 2.1. For any $2/3 > \delta > 0$, we may take n_0 so large that $L(n_0 + 1)/n_0 < \delta/8$. We let m be the integer part of $3\delta n_0/2$. We take $\epsilon > 0$ small enough and o the marker point of Y given by Lemma 2.5. Then, for any $(n_0, 6\epsilon)$-separated set G we consider the subset F of G given by Lemma 2.6 and we put $E = G \setminus F$. We let $N(z) = n_0 + 1$ for some fixed $z \in E$ and $N(z') = n_0$ for $z' \neq z$ in E. We claim the L-specification associated to (E, y, N, m) is ϵ-admissible. Let us just check that condition 4 in Definition 2.1 is fulfilled. Arguing by contradiction, there is $x \in E$ with $B(o, m, \epsilon) \cap S^k B(x, N(x), \epsilon) \neq \emptyset$ for some $-\frac{2}{3}m \leq k \leq N(x) - \frac{2}{3}m$. Then we distinguish two cases:

- either $-\frac{2}{3}m \leq k \leq 0$; then we have $\emptyset \neq B(x, N(x), \epsilon) \cap S^{-k} B(o, m, \epsilon) \subset B(x, m/3, \epsilon) \cap B(S^{-k} o, m/3, \epsilon)$, and thus $x \in B(S^{-k} o, \delta n_0/2, 2\epsilon)$ with $0 \leq -k \leq \delta n_0$;
- or $0 \leq k \leq N(x) - \frac{2}{3}m$; then we have $\emptyset \neq B(o, m, \epsilon) \cap S^k B(x, N(x), \epsilon) \subset B(o, 2m/3, \epsilon) \cap B(S^k x, 2m/3, \epsilon))$, and thus $S^k x \in B(o, \delta n_0/2, 2\epsilon)$ with $0 \leq k \leq (1 - \delta)n_0$.

In both cases, x should belong to F, contradicting our assumption.

By taking ϵ and δ small and n_0 large enough, we may find, according to Lemma 2.1, such a specification with topological entropy arbitrarily close to $\mathsf{h}_{\mathsf{top}}(S)$. □

A closed set-valued map $\phi : X \to \mathcal{K}(Y)$ is said to be *upper semicontinuous* when for all $x \in X$ and for all neighborhoods $V_x \subset Y$ of $\phi(x)$ there is a neighborhood $U_x \subset X$ of x with $\phi(x') \subset V_x$ for all $x' \in U_x$. From Proposition 2.1 and Lemma 2.4, we get the following corollary.

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COROLLARY 2.1. *For any system* (Y, S) *with the almost weak specification property and for any e.a.z. system* (X, T)*, there is an upper semicontinous closed set-valued embedding of* (X, T) *in* $(\mathcal{K}(Y), S)$.

2.4. *Selector of a specification, the case of subshifts.* When (Y, S) has the (ϵ, L)-specification property, a map $\phi : \Lambda \to Y$ is called an *ϵ-selector of the specification* Λ if $\phi(x) \in \Delta_\epsilon^\Lambda(x)$ for all $x \in \Lambda$. By the Kuratowski–Ryll-Nardzewski measurable selection theorem, a measurable selector always exists, as the closed set-valued map Δ_ϵ^Λ is upper semicontinuous. However, we are here interested in equivariant selectors, i.e. selectors ϕ satisfying $\phi \circ \sigma = S \circ \phi$. Invoking the axiom of choice, there always exists an equivariant selector, but we do not know if there always exists a measurable one. We investigate now the existence of a continuous equivariant ϵ-selector of the specification Λ in (Y, S) when (Y, S) is a subshift.

Definition 2.2. A subshift (Y, S) is said to have the coded almost weak L-specification property for a positive sublinear function L when one can associate to any finite word w a finite word $\tilde{w}$ of the form $\tilde{w} = u_w w v_w$ such that:

- $|u_w| = c$ and $|v_w| = L(w) - c$ for some positive integer $c \leq L(0) = \inf_n L(n)$ independent of w; and
- for any finite words $w_1, w_2, \ldots, w_n$ of (Y, S), the concatenation $\tilde{w_1}\tilde{w_2} \cdots \tilde{w}_n$ is a word of (Y, S).

We also say that (Y, S) has the coded almost weak specification property when (Y, S) satisfies the coded almost weak L-specification property for some sublinear function L. Finally, when the function L may be chosen equal to a constant, we speak of coded weak specification.

We believe that any subshift with the almost weak specification property satisfies the above coded almost weak specification property, but this remains an open question. For a subshift (Y, S) and an associated specification Λ, we consider the closed set-valued map $\Delta^\Lambda : \Lambda \to \mathcal{K}(Y)$ defined by $\Delta^\Lambda((x_k)_k) = \{(z_k)_k \in Y, z_k = x_k \text{ for } x_k \neq *\}$ for any $x \in \Lambda$. It corresponds to the map Δ_ϵ^Λ for $\epsilon = 1$ and a well-chosen metric on Y.

LEMMA 2.7. *Let* (Y, S) *be a subshift satisfying the coded almost weak* L*-specification property. Then any simple* L*-specification* (Λ, σ) *with* (E, o, N, m, l) *satisfying* $L(n_0) = L(n_0 + 1)$ *with* $N(E) = \{n_0, n_0 + 1\}$ *admits a continuous equivariant selector.*

Proof. For $x \in E$ (respectively the marker point o) we let v_x (respectively v_o) be the word given by the first $N(x)$ letters of x (respectively the first M letters of o). Using the notation of Definition 2.2, the subshift of finite type Λ embeds into Y via the map sending any generating word w_x, $x \in E$, to the word $\sigma^{-c}(\tilde{v}_o\tilde{v}_x)$ of Y. Indeed, the subshift of finite type generated by these last words is a subshift of Y according to the coded almost weak L-specification property. Moreover, this map is equivariant, as the words w_x, $x \in E$ and $\sigma^{-c}(\tilde{v}_o\tilde{v}_x)$ have the same length. Finally, this embedding defines a continuous selector of Λ, as we have shifted appropriately the interval of coordinates. □

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A subshift is said to be *synchronized* when there is a word u, called the *synchronizer word*, such that for any words w and v with uw and vu being words, vuw is also a word. Any subshift with the weak specification property is synchronized [**J11**].

LEMMA 2.8. *Any subshift* (Y, S) *with the weak specification property satisfies the coded specification property.*

Proof. Let u be a synchronizer word in Y. By the weak specification property, there exists a positive integer l such that for any w in Y, there are two words a_w and b_w of length l with $ua_w wb_w u$ being a word. The property of the synchronizer clearly implies that the map $w \mapsto \widetilde{w} := ua_w wb_w$ satisfies the condition of coded weak specification (with $c = |u| + l$ and $L = l$). □

Proof of Theorem 1.1. We can now prove Theorem 1.1. Let (Y, S) be a subshift with the weak specification property. As already observed, it is enough by Lemma 2.2 to embed topologically subshifts of finite type with topological entropy arbitrarily close to $\mathsf{h}_{\mathsf{top}}(S)$. We build in Proposition 2.1 such subshifts, the admissible simple specifications (Λ, σ), for which there is an injective equivariant closed set-valued map $\Delta^\Lambda : (\Lambda, \sigma) \to (\mathcal{K}(Y), S)$. Finally, by Lemmas 2.8 and 2.7, this map has a continuous equivariant selector, which gives the desired embedding of (Λ, σ) in (Y, S). □

3. *Almost Borel embedding subshift in systems with the specification property*

In the previous section, we used the specification property at a fixed given scale. However, we were able to prove a topological embedding only for subshifts with the weak specification property. Here, we will use the specification property at an arbitrarily small scale to build an almost Borel embedding in a given system (Y, S) with the almost weak specification property.

3.1. *Inverse limits.*

Let $(X_k, T_k)_{k\in\mathbb{N}}$ be a sequence of topological systems with semiconjugacies $\pi_k : (X_{k+1}, T_{k+1}) \twoheadrightarrow (X_k, T_k)$ for all k (we use the usual notation $\twoheadrightarrow$ to denote a surjective map). The inverse limit $(\varprojlim X_k, T)$ is the topological system given by

$$\varprojlim_k X_k := \left\{ (x_k)_k \in \prod_k X_k,\ \pi_k(x_{k+1}) = x_k \text{ for all } k \right\}$$

with T acting as T_k on the kth coordinate for all k.

For any $k \in \mathbb{N}$ we let $\eta_k : (\{0, 1, \ldots, k+1\}^{\mathbb{Z}}, \sigma) \to (\{0, 1, \ldots, k\}^{\mathbb{Z}}, \sigma)$ be the surjective equivariant map that replaces the letter $k+1$ with the letter k, i.e. the lth term of the sequence $\eta_k((x_l)_l)$ is equal to k if $x_l = k+1$ and to x_l if not, or equivalently, $\eta_k((x_l)_l) = (\min(x_l, k))_l$. Fix a non-decreasing sequence $\underline{n} = (n_k)_{k\geq 1}$ of positive integers. Let $Z^0_{\underline{n}}$ be the sequence with all terms equal to zero. We define by induction on $k \geq 1$ the subshift $Z^k_{\underline{n}}$ of $\{0, 1, \ldots, k\}^{\mathbb{Z}}$ as the set of $x = (x_l)_l$ satisfying:

- $\eta_{k-1}(x) \in Z^{k-1}_{\underline{n}}$; and
- for two consecutive integers $p < q$ in $\{i \in \mathbb{Z}, x_i = k\}$, the cardinality of $\{i \in [\![p, q-1]\!]$ with $x_i = k-1\}$ is either n_k or $n_k + 1$.

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We let $(Z_{\underline{n}}, \sigma)$ be the inverse limit $\varprojlim Z_{\underline{n}}^k$ with semiconjugacies given by $(\eta_k)_k$ endowed with the shift map on each coordinate. Clearly, the topological entropy of $Z_{\underline{n}}$ is arbitrarily small for fast enough increasing sequence $\underline{n}$.

LEMMA 3.1. *Let $\underline{n} = (n_k)$ be a non-decreasing sequence of positive integers. For any aperiodic Borel system (X, T), there is a Borel map $\psi : X' \to Z_{\underline{n}}$ satisfying $\psi \circ T = \sigma \circ \psi$ with X' being a full subset of X.*

Proof. For an aperiodic Borel system (X, T), there is for any positive integer n_1 a Borel set $U_1 \subset X$ such that the first return in U_1 takes a value in $\{n_1, n_1 + 1\}$ and $X' = \bigcup_{k=0}^{n_1+1} T^k U_1$ is a full invariant set. Indeed, firstly, any aperiodic Borel system almost Borel embeds on a full set into the shift on $\mathbb{N}^{\mathbb{Z}}$ [**W84**], and finally, we may use for this shift a Borel version of Alpern's towers lemma [**GW06**]. This defines a Borel equivariant map $\psi_1 : X \to Z_{\underline{n}}^1$ by letting $\psi_1(x) = (1_{U_1}(T^k x))_k$ for all $x \in X'$. Then we may consider the Borel system (U_1, T_{U_1}) given by the first return map in U_1. There is now a Borel set $U_2 \subset U_1$ such that the first return in U_2 takes a value in $\{n_2, n_2 + 1\}$ and we now define a Borel equivariant map $\psi_2 : X \to Z_{\underline{n}}^2$ with $\eta_2 \circ \psi_2 = \psi_1$ by letting $\psi_2(x) = ((1_{U_1} + 1_{U_2})(T^k x))_k$ for all $x \in X$. By pursuing the process, we build a sequence of Borel equivariant maps $(\psi_k)_k$ compatible with the factor maps $(\eta_k)_k$ of the inverse limit $Z_{\underline{n}}$ so that this sequence defines a Borel equivariant map from (X, T) to $(Z_{\underline{n}}, \sigma)$. □

The lemma below follows easily from the above lemma and Hochman's theorem (Lemma 2.3).

LEMMA 3.2. *For any non-decreasing sequence $\underline{n}$ of positive integers and any topologically mixing subshift of finite type (Y, S), the product system $(Y \times Z_{\underline{n}}, S \times \sigma)$ is almost Borel $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(S)\}$-universal.*

Thus, it is enough to almost Borel embed in (Y, S) such a product system $(X \times Z_{\underline{n}}, T \times \sigma)$ with $\mathsf{h}_{\mathsf{top}}(T)$ arbitrarily close to $\mathsf{h}_{\mathsf{top}}(S)$. Observe that we may also write $X \times Z_{\underline{n}}$ as the inverse limit of $(X \times Z_{\underline{n}}^k)_k$. In fact, we will build in the next subsection an inverse limit of specifications $(\Lambda_k, \sigma)_k$ almost Borel embeddable in (Y, S) such that $\varprojlim_k \Lambda_k$ is topologically conjugated to $\Lambda_0 \times Z_{\underline{n}}$.

Remark 3.1. By using Krieger's marker lemma, see [**D11**, Lemma 8.5.4], one proves similarly for an e.a.z. system the existence of a continuous map $\psi : Y \to Z_{\underline{n}}$ satisfying $\psi \circ S = \sigma \circ \psi$. Combined with Krieger's theorem (Lemma 2.2), we get that for any topologically mixing subshift of finite type (Y, S), the product system $(Y \times Z_{\underline{n}}, S \times \sigma)$ is topologically $\{\mathsf{h}_{\mathsf{top}} < \mathsf{h}_{\mathsf{top}}(S)\}$-universal.

3.2. *Almost Borel embedding of admissible inverse limit of specifications.*

3.2.1. *Nature of the almost Borel embedding.* To any specification (Λ, σ) of (Y, S) given by a marker point o and a separated set E we may associate the function $\psi_\Lambda : \Lambda \to Y$, which maps $z = (z_k)_k \in \Lambda$ to z_0 whenever $z_0 = S^k x$ for some $x \in E$ and maps z to o otherwise. Note that this function is clearly continuous; in fact, ψ_Λ is locally constant

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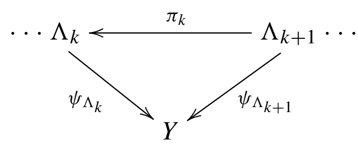
and orbitwise injective, i.e. for any $z \neq z' \in \Lambda$ there is $k \in \mathbb{Z}$ with $\psi_\Lambda(\sigma^k z) \neq \psi_\Lambda(\sigma^k z')$. However, ψ_Λ is not equivariant, i.e. $\psi_\Lambda \circ \sigma \neq S \circ \psi_\Lambda$.

The desired embedding of $(\varprojlim_k \Lambda_k, \sigma)$, where σ acts again as the shift map on each coordinate, into (Y, S) will be obtained as the pointwise limit of $(\phi_k)_k$ where for all k we let $\phi_k := \psi_{\Lambda_k} \circ p_k$ with $p_k : \varprojlim_k \Lambda_k \to \Lambda_k$ being the projection on Λ_k. Of course, there is a priori no reason that such a sequence is converging pointwisely and its limit is a Borel embedding. In the next paragraph, we introduce sufficient conditions on the inverse limit to ensure these properties. Inverse limits of specifications satisfying these conditions will be said to be *admissible*.

3.2.2. *Admissible inverse limit.* An admissible inverse limit of specifications $\varprojlim_k \Lambda_k$ almost embeds in (Y, S) via the family of continuous maps $(\psi_{\Lambda_k})_k$, in the sense that the following diagram commutes on a larger and larger set within a smaller and smaller error as k is increasing.



To be more precise, we first recall the notion of orbit capacity introduced in [**SW91**]. For a topological system (X, T), the *orbit capacity* of a subset C of X is defined as

$$\mathsf{ocap}(C) := \lim_n \frac{1}{n} \sup_{x \in X} \sharp\{0 \leq k < n,\, T^k x \in C\}.$$

For a Borel subset C, one easily checks by using the pointwise ergodic theorem that the orbit capacity of C is larger than $\mu(C)$ for any T-invariant probability measure μ. We call a Borel subset C of X a *full set* if $\mu(C) = 1$ for all μ.

We now define precisely admissible inverse limits of specifications. For any $\epsilon > 0$, we let L_ϵ be the sublinear function involved in the definition of the ϵ-almost specification property of (Y, S). For a given specification (Λ, σ) and for all $x = (x^q)_q \in \Lambda$, we let $I_\Lambda(x)$ be the interval of integers $[\![m, \ldots, 0, \ldots, n]\!]$ such that $B_\Lambda(x) := x^m \ldots x^0 \ldots x^n$ is a generating word of Λ (thus, $I_\Lambda(x)$ is the interval of coordinates of $B_\Lambda(x)$). We also let $J_\Lambda(x) \supset I_\Lambda(x)$ be the interval of coordinates of the concatenation $\tilde{B}_\Lambda(x)$ of $B_\Lambda(x)$ with the next generating word of Λ used to form x (which is just $B_\Lambda(\sigma^{n+1}(x))$ using the previous notation, so that we have $\tilde{B}_\Lambda(x) = B_\Lambda(x) B_\Lambda(\sigma^{n+1}(x))$). Finally, for a finite subset I of $\mathbb{Z}$, we let d_I be the distance on Y given by $d_I(x, y) := \max_{k \in I} d(S^k x, S^k y)$ for $x, y \in Y$.

Let $(\epsilon_k)_k$ be a fixed decreasing sequence of positive real numbers with $\sum_{k>0} \epsilon_k < \epsilon_0/2$, and let $(\epsilon'_k)_k$ be the sequence given for all $k \geq 1$ by $\epsilon'_k := \epsilon_0 - \sum_{l=1}^{k} \epsilon_k > \epsilon_0/2 > \epsilon_{k+1}$ (we let $\epsilon'_0 := \epsilon_0$). For a fixed k, the specification Λ_k will be an $L_{\epsilon_{k+1}}$-specification. We recall that by the almost weak specification property, the system (Y, S) satisfies the $(\epsilon_k, L_{\epsilon_k})$ specification property for all k. Moreover, the specification Λ_k is ϵ'_k-admissible. We will denote by $(E_k, o_k, N_k, m_k, l_k)$ the data associated to the specification Λ_k.

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Definition 3.1. Using the previous notation, an inverse limit $\varprojlim_k \Lambda_k$ of specifications with factor maps $(\pi_k)_k$ is said to be *admissible* whenever there exists a sequence $(P_k)_k$ of positive integers with the following properties for all $k \geq 0$:

(1) Λ_k is a $L_{\epsilon_{k+1}}$-specification which is ϵ'_k-admissible;

(2) $\pi_k : \Lambda_{k+1} \to \Lambda_k$ is a block code, i.e. the π_k-image of every generating word in Λ_{k+1} is a word with the same length given by the finite concatenation of generating words in Λ_k;

(3) (Λ_k, σ) is topologically conjugated to $(\Lambda_0 \times Z_{\underline{n}}^k, \sigma)$ via a conjugacy map $\theta_k : \Lambda_k \to \Lambda_0 \times Z_{\underline{n}}^k$ satisfying $\theta_k \circ \pi_k = (\mathsf{Id}_{\Lambda_0} \times \eta_k) \circ \theta_{k+1}$, with Id_{Λ_0} being the identity map on Λ_0;

(4) for $x = (x_{k'})_{k'} \in \varprojlim_{k'} \Lambda_{k'}$ and $(\phi_k)_k$ as in §3.2.1,

$$[d(\phi_{k+1}(x), \phi_k(x)) \geq \epsilon_k] \Rightarrow [\exists l \in \mathbb{Z}\sigma^l(x_{k+1}) = o_{k+1} \text{ with } |l| \leq P_{k+1}]; \quad \text{and}$$

(5)

$$P_k \geq m_k + L(m_k) + l_k + \overline{N_{k-1}}$$

and

$$P_k/\underline{N_k} < \frac{1}{2^{k+1}}.$$

From the third item, it follows that an admissible inverse limit $\varprojlim_k \Lambda_k$ is topologically conjugated to $\Lambda_0 \times Z_{\underline{n}}$ through the conjugacy map $(x_k)_k \in \varprojlim_k \Lambda_k \mapsto (\theta_k(x_k))_k$.

We consider for $k \in \mathbb{N}$ the following subsets F_k and G_k of $\varprojlim_{k'} \Lambda_{k'}$:

$$F_k := \left\{x = (x_{k'})_{k'} \in \varprojlim_{k'} \Lambda_{k'},\ \exists l \in \mathbb{Z}\sigma^l(x_k) = o_k \text{ with } |l| \leq P_k\right\}$$

and

$$G_k := \left\{x = (x_{k'})_{k'} \in \varprojlim_{k'} \Lambda_{k'},\ \exists l \in \mathbb{Z}\sigma^l(x_k) = o_k \text{ with } |l| \leq 3P_k\right\}.$$

According to the fourth item in the above definition, we have $d(\phi_{k+1}(x), \phi_k(x)) < \epsilon_k$ for $x \notin F_{k+1}$. It follows also from the inequality $P_k \geq m_k + L(m_k) + l_k$ that for $x = (x_{k'})_{k'} \notin F_k$, the point $x_k \in \Lambda_k$ belongs to the piece of orbit of some point in E_k (with E_k being the set associated to the specification Λ_k). Thus, we have $\phi_k \circ \sigma(x) = S \circ \phi_k(x)$. If $x \notin G_{k+1}$ then we have $\sigma^j(x) \notin F_{k+1}$ for any $j \in J_{\Lambda_k}(x_k)$, because P_{k+1} is larger than the length of any generating word of Λ_k. Therefore, $d_{J_{\Lambda_k}(x_k)}(\phi_{k+1}(x), \phi_k(x)) < \epsilon_k$. Finally, we observe that

$$\mathsf{ocap}(G_k) \leq \frac{6P_k}{\underline{N_k}} < 1/2^k.$$

In the next lemma, we show by a Borel–Cantelli argument that the sequence $(\phi_k)_k$ is converging on a full set. The maps $(\phi_k)_k$ are more and more equivariant as they send finite pieces of orbits of the inverse limit with longer and longer lengths to pieces of orbit of (Y, S) with the same length. It follows that the limit map is equivariant. Moreover, it is injective, as a consequence of the ϵ'_k-admissibility of the $L_{\epsilon_{k+1}}$-specification Λ_k.

LEMMA 3.3. *Let $(\varprojlim_k \Lambda_k, \sigma)$ be an admissible inverse limit of specifications. Then the continuous maps $(\phi_k)_k$, as defined in §3.2.1, are converging on a full set G to an injective function ϕ satisfying $\phi \circ T = S \circ \phi$.*

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The map ϕ may be then extended equivariantly on the whole set $\varprojlim_k \Lambda_k$ (see [**BD16**, Remark 14] for details).

Proof of Lemma 3.3. By the Borel–Cantelli lemma we get $\mu(\limsup_k G_k) = 0$ for any σ-invariant probability measure μ. We let G be the full set given by the complement of $\limsup_k G_k$. For $x \in G$ we have $d(\phi_k(x), \phi_{k+1}(x)) < \epsilon_k$ and $\phi_k \circ \sigma(x) = S \circ \phi_k(x)$ when k is large enough. Consequently, the sequence $(\phi_k)_k$ is converging pointwisely on G to a limit ϕ satisfying $\phi \circ T = S \circ \phi$. It remains to show that ϕ is injective on G. For $z = (z_k)_k$ in G, the sequence of intervals $(I_{\Lambda_k}(z_k))_k$ is going to the whole set $\mathbb{Z}$ of integers; in other words, any bounded interval K is contained in $I_{\Lambda_k}(z_k)$ for large enough k. Moreover, as π_k is a block code, the word $B_{\Lambda_k}(z_k)$ completely determines z_i for $k \geq i$ on $I_{\Lambda_k}(z_k)$. Thus, for $x = (x_k)_k$ and $y = (y_k)_k$ two distinct points in G, there exists an integer $i > 1$ such that $B_{\Lambda_k}(x_k) \neq B_{\Lambda_k}(y_k)$ for $k \geq i$. By taking i large enough, we may also assume that for $k \geq i$ we have $d_{J_{\Lambda_k}(x_k)}(\phi_k(x), \phi_{k+1}(x)) < \epsilon_k$ and $d_{J_{\Lambda_k}(y_k)}(\phi_k(y), \phi_{k+1}(y)) < \epsilon_k$. In particular, $\phi(x) = \lim_k \phi_k(x) \in \Delta_{\epsilon_i''}(\tilde{B}_{\Lambda_i}(x_i))$ and $\phi(y) = \lim_k \phi_k(y) \in \Delta_{\epsilon_i''}(\tilde{B}_{\Lambda_i}(y_i))$ with $\epsilon_i'' := \sum_{i<k} \epsilon_k \leq \epsilon_i'$. By definition of B_Λ, the interval of coordinates of $B_{\Lambda_i}(x_i)$ and $B_{\Lambda_i}(y_i)$ both contain 0. As Λ_i is ϵ_i'-admissible, it follows from Lemma 2.4 that $\Delta_{\epsilon_i'}(\tilde{B}_{\Lambda_i}(x_i)) \cap \Delta_{\epsilon_i'}(\tilde{B}_{\Lambda_i}(y_i)) = \emptyset$, since $B_{\Lambda_i}(x_i)$ and $B_{\Lambda_i}(y_i)$ are distinct. Therefore, we conclude $\phi(x) \neq \phi(y)$. □

3.3. *Construction of admissible inverse limit of specifications.* We consider a system (Y, S) with the almost specification property.

PROPOSITION 3.1. *Let $0 < \alpha < \mathsf{h}_{\mathsf{top}}(S)$. There is an admissible inverse limit of specifications $\varprojlim_k \Lambda_k$ with $\mathsf{h}_{\mathsf{top}}(\Lambda_0)$ larger than α.*

Proof. We recall that $(E_k, o_k, N_k, m_k, l_k)$ denotes the data associated to the specification Λ_k. We first consider an ϵ_0-admissible simple L_{ϵ_1}-specification Λ_0' with topological entropy larger than α, as built in Proposition 2.1. We let $n_0, n_0 + 1$ be the length of the generating words in Λ_0'. In Proposition 2.1, we may choose the set E_0' of the specification Λ_0' having the form $E_0' = E_0 \coprod E_0'' \coprod \{a_0, b_0\}$ (where $\coprod$ denotes the disjoint union of sets) with $\sharp E_0'' \geq e^{\alpha'' n_0}$ and $\sharp E_0 \geq e^{\alpha' n_0}$ for $\alpha'' > \alpha' > \alpha$. Indeed, one only needs to take E_0' with $\sharp E_0' \geq e^{\alpha'' n_0} + e^{\alpha' n_0} + 2$. We may also assume the specifications Λ_0'', $\Lambda_0 \subset \Lambda_0'$ associated to E_0'', E_0 are also simple and therefore topologically mixing. Finally, for large enough n_0 we have $\mathsf{h}_{\mathsf{top}}(\Lambda_0'') > \mathsf{h}_{\mathsf{top}}(\Lambda_0) > \alpha$ according to Lemma 2.1. In particular, any inverse limit with Λ_0 as a first term has topological entropy larger than α. We let $n_0, n_0 + 1$ be the length of the generating words in Λ_0', Λ_0'' and Λ_0.

By induction, we build for all k an ϵ_k'-admissible L_{ϵ_k}-specification Λ_k' such that the associated set E_k' may be written $E_k' = E_k \coprod E_k'' \coprod \{a_k, b_k\}$ and the specifications Λ_k'', $\Lambda_k \subset \Lambda_k'$ associated to E_k'', E_k satisfy $\mathsf{h}_{\mathsf{top}}(\Lambda_{k-1}'') \geq \mathsf{h}_{\mathsf{top}}(\Lambda_k'') > \mathsf{h}_{\mathsf{top}}(\Lambda_k) \geq \mathsf{h}_{\mathsf{top}}(\Lambda_{k-1})$. The sequence of specifications $(\Lambda_k)_k$ associated to E_k will define an admissible inverse limit.

The points $a_k, b_k \in E_k$ will be used to define the marker point o_{k+1} of the specification Λ_{k+1}. Let us explain briefly the role of Λ_k'' in the construction: to get the specification Λ_k we will need to compress the information given by Λ_{k-1}. As the topological entropy of

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Λ''_{k-1} is larger than that of Λ_{k-1}, it can be achieved by building generating words in Λ_k from words in Λ''_{k-1}.

We assume Λ_{i-1} and π_{i-2} already built for $i \leq k$. In a given specification, a concatenation of n-generating words is called an n-block. Generating words in Λ_k are obtained from every n_k-block and (n_k+1)-block in Λ_{k-1} after a series of modifications, where $(n_k)_k$ is a fast increasing sequence. In particular, the generating words in Λ_k have length in the interval of integers $[\![\prod_{i=0}^k n_i, \prod_{i=0}^k (n_i+1)]\!]$ and all values are achieved (it implies Λ_k is topologically mixing). The conjugacy map $\theta_k : \Lambda_k \to \Lambda_0 \times Z_{\underline{n}}^k$ will be obtained for its first coordinate by decompressing the information from Λ_k to Λ_0, whereas the second coordinate gives the structure of blocks in Λ_l for $l \leq k$.

The construction of Λ_k, Λ'_k and Λ''_k is divided into four steps. In the first two steps we describe these specifications by identifying the marker word and the generating words. We define the factor map $\pi_{k-1} : \Lambda_k \to \Lambda_k$ in the third step, and the last step is devoted to the proof of the admissibility of Λ'_k.

Step 1: Marker word in Λ_k, Λ'_k and Λ''_k. We let $A = A_{k-1}$ and $B = B_{k-1}$ be the generating words in Λ'_{k-1} associated to a_{k-1} and b_{k-1}. We consider a block C_k in Λ'_{k-1} of the form $C_k \in \{A^q B^q,\ q \in \mathbb{N}^*\}$ with $A^q = \underbrace{A \cdots A}_{q \text{ times}}$.

If $\Delta_{\epsilon'_{k-1}}^{\Lambda_{k-1}}(C_k) \cap \Delta_{\epsilon'_{k-1}}^{\Lambda_{k-1}}(\sigma^l C_k) \neq \emptyset$ for some integer l, the blocks $\sigma^l C_k$ and C_k should coincide on the intersection of their interval of coordinates (which may be empty) by ϵ'_{k-1}-admissibility of Λ'_{k-1}, according to Lemma 2.4. As A and B are distinct, this intersection is non-empty only for $l = 0$. We put m_k equal to the length of C_k and we pick $o_k \in \Delta_{\epsilon_k}^{\Lambda_{k-1}}(C_k)$. For $0 < l \leq \frac{3}{4} m_k$, the intervals of coordinates of C_k and $\sigma^l C_k$ are not disjoint, and therefore $\Delta_{\epsilon'_{k-1}}^{\Lambda_{k-1}}(C_k) \cap \Delta_{\epsilon'_{k-1}}^{\Lambda_{k-1}}(\sigma^l C_k) = \emptyset$. Thus, we have

$$B(o_k, m_k, \epsilon'_k) \cap S^l B(o_k, m_k, \epsilon'_k) \subset \Delta_{\epsilon'_{k-1}}^{\Lambda_{k-1}}(C_k) \cap \Delta_{\epsilon'_{k-1}}^{\Lambda_{k-1}}(\sigma^l C_k) = \emptyset.$$

The point o_k will play the role of the marker in the specifications Λ_k, Λ'_k and Λ''_k.

Step 2: Generating words in Λ_k, Λ'_k and Λ''_k. For an n_k- or (n_k+1)-block D_k in Λ_{k-1} or Λ''_{k-1}, we replace the first letters by $o_k, \ldots, S^{m_k} o_k, *^{L_{\epsilon_k}(m_k)}$ and the last letters by $*^{l_k}$ with $l_k = L_{\epsilon_k}(\prod_{l=0}^k (n_l+1))$. We consider the subword D'_k given by the remaining middle part of D_k.

We associate to the n_k- or (n_k+1)-block D_k a word D''_k in Λ'_{k-1} with the same length as D'_k (see Step 3). Then, we pick up $x_k \in \Delta_{\epsilon_k}^{\Lambda_{k-1}}(D''_k)$ and we replace the subword D'_k of D_k by $(x_k, Sx_k, \ldots, S^{|D'_k|-1} x_k)$. The set E_k (respectively $E''_k \coprod \{a_k, b_k\}$) is defined as the set of all points x_k obtained in this way from all n_k- and (n_k+1)-blocks D_k in Λ_{k-1} (respectively in Λ''_{k-1}). We let Λ_k (respectively Λ''_k) be the specification with respect to $(E_k$ (respectively E''_k), $o_k, N_k, m_k, l_k)$. The map $D_k \mapsto D''_k$, defined below, will be injective on n_k- and (n_k+1)-blocks of Λ_{k-1} so that we will obtain a bijective map π_{k-1} from the 1-blocks of the specification Λ_k to the n_k- and (n_k+1)-blocks of Λ_{k-1}. This map induces the required topological conjugacy θ_k between Λ_k and $\Lambda_0 \times Z_{\underline{n}}^k$ (define θ_k such that we have $\theta_{k-1} \circ \pi_{k-1} = (\mathrm{Id}_{\Lambda_0} \times \eta_{k-1}) \circ \theta_k$ and the nth term of the second component of θ_k in $\{0, 1, \ldots, k\}^{\mathbb{Z}}$ is equal to k if and only if n is the first coordinate of a 1-block in Λ_k).

Moreover, when $\prod_{i=0}^{k} n_i \gg m_k \gg \prod_{i=0}^{k-1}(n_i+1)$ we will get $\mathsf{h}_{\mathsf{top}}(\Lambda_{k-1}) \leq \mathsf{h}_{\mathsf{top}}(\Lambda_k) < \mathsf{h}_{\mathsf{top}}(\Lambda''_k) \sim \mathsf{h}_{\mathsf{top}}(\Lambda''_{k-1})$.

Step 3: Recoding and factor map π_{k-1}. When D_k is an n_k- or (n_k+1)-block in Λ''_{k-1} we just let $D''_k = D'_k$ (there is no compression of information in this case). We now examine the case when D_k is an n_k- or (n_k+1)-block in Λ_k. For a specification Λ and an integer p, we let $\mathcal{W}^{\Lambda}(p)$ be the set of words of Λ of length p and $\mathcal{V}^{\Lambda}(p)$ be the subset of $\mathcal{W}^{\Lambda}(p)$ given by words which end with a generating word. A word $V \in \mathcal{V}^{\Lambda}(p)$ may be written (uniquely) as the concatenation of a prefix, which contains no generating words, with a (complete) block $s(V)$ of Λ, called the suffix of V.

As the topological entropy of Λ''_{k-1} is strictly larger than the topological entropy of Λ_{k-1}, there is an integer p_k with $\prod_{i=0}^{k} n_i \gg p_k \gg \max(l_k, m_k)$ such that

$$\sharp\mathcal{V}^{\Lambda''_{k-1}}(p_k) \geq \sharp(\mathcal{W}^{\Lambda_{k-1}}(m_k + L(m_k) + p_k) \times \mathcal{W}^{\Lambda_{k-1}}(l_k)).$$

In fact, in the above inequality we may replace the set $\mathcal{V}^{\Lambda''_{k-1}}(p_k)$ by a subset $\mathcal{U}^{\Lambda''_{k-1}}(p_k) \subset \mathcal{V}^{\Lambda''_{k-1}}(p_k)$ such that any two distinct elements in this subset have distinct suffixes (indeed, the length of the prefixes is of order $\prod_{i=0}^{k-1} n_i$ and we choose $p_k \gg m_k \gg \prod_{i=0}^{k-1} n_i$). We may thus consider a surjective map Θ_k from $\mathcal{U}^{\Lambda''_{k-1}}(p_k)$ onto the product set $\mathcal{W}^{\Lambda_{k-1}}(m_k + L(m_k) + p_k) \times \mathcal{W}^{\Lambda_{k-1}}(l_k)$. As explained above, we replace the first letters of D_k by $o_k, \ldots, S^{m_k} o_k, *^{L_{\epsilon_k}(m_k)}$ and the last letters by $*^{l_k}$ with $l_k = L_{\epsilon_k}(\prod_{l=0}^{k}(n_l+1))$. We write D_k as the concatenation $D_k = D_k^1 D_k^2 D_k^3 D_k^4$ with $D'_k = D_k^2 D_k^3$ and $|D_k^2| = p_k$. To get D''_k we modify only D_k^2. We re-encode the information lost in D_k^1, D_k^2 and D_k^4 in a word of length p_k via Θ_k. More precisely, we let D''_k be the concatenation $\Theta_k^{-1}(D_k^1 D_k^2, D_k^4) D_k^3$ for some fixed right inverse Θ_k^{-1} of Θ_k. Recall that we then take $x_k \in \Delta_{\epsilon_k}^{\Lambda_{k-1}}(D''_k)$ and we replace the subword D'_k of D_k by $(x_k, Sx_k, \ldots, S^{|D'_k|-1} x_k)$ to get a generating word of Λ_k. Any coordinate in D_k lying in D_k^3 and the corresponding coordinate in the generating word of Λ_k are ϵ_k-close. In particular, the second-last item of Definition 3.1 is satisfied for $P_k := \max(m_k + L(m_k) + p_k, l_k)$. Finally, the condition on P_k, in the last item of the same definition, holds true for $\prod_{i=0}^{k} n_i \gg p_k \gg \max(l_k, m_k)$.

The 1-blocks of the specification Λ_k are in bijection with the n_k- and (n_k+1)-blocks of Λ_{k-1}. The (surjective) factor map $\pi_{k-1} : \Lambda_k \to \Lambda_{k-1}$ is the associated block code.

Step 4: Admissibility of Λ'_k. We now prove by induction on k that Λ'_k (and therefore Λ_k) is ϵ'_k-admissible. We first check $E'_k = E_k \coprod E''_k \coprod \{a_k, b_k\}$ is (N, ϵ'_k)-separated. Let $x \neq y \in E'_k$ with $N_k(x) = N_k(y)$. The points x and y belong respectively to $\Delta_{\epsilon_k}^{\Lambda_{k-1}}(D''_k)$ and $\Delta_{\epsilon_k}^{\Lambda_{k-1}}(\tilde{D}''_k)$ with D''_k and $\tilde{D}''_k$ being words on the same interval of coordinates with distinct suffixes. By ϵ'_{k-1}-admissibility of Λ'_{k-1} the sets $\Delta_{\epsilon'_{k-1}}(D'_k)$ and $\Delta_{\epsilon'_{k-1}}(\tilde{D}'_k)$ are disjoint according to Lemma 2.4, and thus it is also the case for $B(x, N_k(x), \epsilon'_k)$ and $B(y, N_k(y), \epsilon'_k)$.

Having already checked the required property on the marker point, it is enough to see $B(o_k, m_k, \epsilon'_k) \cap S^i B(x, N_k(x), \epsilon'_k) = \emptyset$ for any $-\frac{2}{3}m_k \leq i \leq N_k(x) - \frac{2}{3}m_k$ and for any $x \in E'_k$. But this follows, as above, from the ϵ'_{k-1}-admissibility of Λ'_{k-1} and the fact that o_k and elements of E'_k are issued from disjoint families of 1-blocks in Λ'_{k-1} (associated to

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$\{a_{k-1}, b_{k-1}\}$ for o_k and to $E_{k-1} \coprod E''_{k-1}$ for elements in E'_k). This concludes the proof of Proposition 3.1. □

3.4. *Almost Borel universality for systems with the almost weak specification.*

Proof of Theorem 1.2. Let (Y, S) be a topological system with the almost weak specification property. By Proposition 3.1, there is an admissible inverse limit $(\varprojlim_k \Lambda_k, \sigma)$ of specifications of (Y, S) with $\mathsf{h}_{\mathsf{top}}(\Lambda_0)$ arbitrarily close to $\mathsf{h}_{\mathsf{top}}(S)$. By Lemma 3.2, such an inverse limit is almost Borel $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(\Lambda_0)\}$-universal. Finally, by Lemma 3.3, this inverse limit almost Borel embeds in (Y, S) so that (Y, S) is also almost Borel $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(\Lambda_0)\}$-universal. As it holds for $\mathsf{h}_{\mathsf{top}}(\Lambda_0)$ arbitrarily close to $\mathsf{h}_{\mathsf{top}}(S)$, the system is $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(S)\}$-universal. □

4. *Fully topologically and almost Borel universal systems*

We now discuss the relevance of fully embedding in our context. One can not embed topologically an e.a.z. system (X, T) in a system (Y, S) with the weak specification property and larger topological entropy in such a way the invariant measures of the embedded system (X', S) have full support. Indeed, $Y \setminus X'$ being open, it should be the empty set, and therefore (X, T) is topologically conjugated to (Y, S). In particular, we have the equality $\mathsf{h}_{\mathsf{top}}(X, T) = \mathsf{h}_{\mathsf{top}}(Y, S)$, which contradicts our assumption.

In the almost Borel setting, the concept of fully embedding makes sense and an easy adaptation of the construction in the above section gives the following statement. A system (Y, S) is said to be fully almost Borel $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(S)\}$-universal when any Borel system (X, T) with $\mathsf{h}_{\mathsf{bor}}(X, T) < \mathsf{h}_{\mathsf{top}}(Y, S)$ almost Borel embeds in (Y, S) such that the pushforward of T-invariant measures by the embedding have full support.

THEOREM 4.1. *Any topological system (Y, S) with the almost weak specification property is fully almost Borel $\{\mathsf{h}_{\mathsf{bor}} < \mathsf{h}_{\mathsf{top}}(S)\}$-universal.*

Proof. A topological system (Y, S) with the almost weak specification property is topologically transitive. Let z be a point in Y with a dense forward S-orbit. For all k we fix an integer r_k such that $Z, Sz, \ldots, S^{r_k-1}z$ is ϵ_k-dense, i.e. this finite piece of orbit meets all balls of radius ϵ_k. Then, in the construction of the specifications $(\Lambda_k)_k$ we replace the final subword $*^{l_k}$ of the generating words by $*^{l_k}, z, Sz, \ldots, S^{r_k-1}z, *^{L_{\epsilon_k}(r_k)}$. The construction remains valid provided one chooses the length of the marker word $m_k \gg r_k$. Finally, for any ergodic T-invariant measure μ, the induced measure $\phi_k^*\mu$ on Y gives positive weight to any ball of radius ϵ_k, and by admissibility of the inverse limit $\varprojlim_k \Lambda_k$, so does the measure $\phi^*\mu$ for any ball of radius $\epsilon_k + \epsilon''_k = \epsilon''_{k-1}$. As k may be chosen arbitrarily, we conclude that $\phi^*\mu$ has full support. □

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